Geometric Transitions, Large N duality and Integrable Systems

Geometric transitions:

\[ X_0 \leftarrow X \]
\[ \text{family of CYs parameterized by } \mu \]
\[ \text{singular crepant resolution of } X_0 \]

Large N duality provides a correspondence:

closed strings \( \leftrightarrow \) open strings on \( X_0 \) \( \leftrightarrow \) open strings on \( X \)

We can enlarge the picture by adding mirror symmetry to it:

\[ \text{II} \text{B closed } X_0 \leftarrow X \text{ II} \text{B open} \]
\[ \text{MS} \uparrow \text{MS + branes} \]
\[ \text{II} \text{A closed } Y \rightarrow Y_0 \text{ II} \text{A open} \]

The horizontal maps are large N duality
Recall that: $M_5$ converts period integrals on cycles in $X$ to closed string $G_2$ invariants on $Y$.

$M_5$ and branes convert:
period integrals on cycles in $X$ to open string $G_2$ invariants in $Y$.

New ingredient introduced by Dijkgraaf-Vafa: they showed that on a local CY the open $\mathcal{N}=2$ theory is related to the matrix model. We would like to understand this in algebraic-geometric terms.

Typical setup for the DV story:

\[ X^a \quad \longrightarrow \quad \mathbb{C}^4 : -y^2 + uv + z^2 = 0 \]

\[ \mathbb{C} \quad \longrightarrow \quad \mathbb{C} \quad \leftarrow \quad \text{coordinate } z \]

\[ \text{coordinate } x \]

\[ W = W_a(x) = \sum_{i=0}^{n_i} a_i x^i \quad \text{polynomial in } x \]

(depending on the parameters $(a_0, a_1, \ldots)$.)
\[ x_a \text{ has } n \text{ isolated } A_1 \text{ singularities given by } \]
\[
\begin{align*}
    u &= v = y = 0 \\
    w'(x) &= 0
\end{align*}
\]

Also \( x_a = CY_3 \) with a holomorphic volume form

\[ 2x_a = \frac{du \wedge dv}{u} \]

When \( A_3 = 0 \Rightarrow x_0 \text{ is singular along a curve } \]
\[ C = C_0 = \{ u = v = y = 0 \} \]

When \( a \to 0 \Rightarrow x_a \text{ has } n \text{ singularities.} \]

**Note:** \( W \) can be reconstructed by computing a period of \( \omega_a \) on a \( a > 0 \) cycles on \( x_a \). However, these periods vary transversally holomorphically in \( a \).

**DV:** combine the superpotential deformation (parameter \( a \)) with smoothing deformation (parameter \( b \)).

\[ x_{a, b} = -y^2 + uv + (w'(x))^2 + f_p (x) = 0 \]

\[ f_p (x) = \sum_{i=0}^{n-1} \frac{m_i x^i}{i} \]
Matrix model $\Rightarrow$ produces quantized superpotential

$$W = W_{\phi} (\vec{x})$$

$\vec{x}$: coordinate on the hyper elliptic curve

$$\tilde{C}_{\phi} : y^2 = W'(X)^2 + f_{\mu}(X)$$

Coefficients of $W$ with special coordinates produce open string GW invariants in $\tilde{C}_{\phi}$.

Goal: (joint with Diacrescu, Flores, Grassi, Pandey, Hoffman)

Want to understand geometrically the matrix model deformation and to extend it to incorporate compact CYrs.

A simple compact example (Katz-Morrison-Plesser)

$$\chi_{\phi, \mu} = Q \cap R \subset \mathbb{P}_1$$
\[ a = p = 0 \implies \text{rank } Q = 3 \implies \text{Sing } (Q) = \mathbb{C}^3 \subset \mathbb{C}^5 \]
\[ p = 0 \implies \text{rank } Q = 4 \implies \text{Sing } (Q) = \mathbb{C}^4 \subset \mathbb{C}^5 \]
\[ \text{Sing } (X_{a,p}) = 4 \text{ pt} \]
\[ \implies a \in H^0 (K_X) \]

General case: \[ a / p \implies \text{rank } Q \leq 6 \implies X \text{ generically non-singular} \]

This picture gives rise three moduli spaces:

\[ \text{MCML} \]

- Moduli of CYs with a curve of isolated singularities
- Moduli of CYs with a curve of isolated singularities
- Moduli of smooth CYs

There are two CY moduli spaces:

- \[ S = \text{moduli of the resolution } \xi \text{ of the CYs in } M \]
- \[ (\phi^*)^{-1} (c_1) = \text{moduli of the resolution of the CYs in } M \]
\text{dim } g = 83, \text{ dim } m = 86, \text{ dim } l = 89, j = m = 3.

1. To use the large $N$-duality to compute open and \textit{invariants} of the mirror we need to compute periods.

Using Mixed Hodge theory and the Clemens-Schmid exact sequence one gets that to order $0$

\[ \lim_{p \to 0} \delta \chi_p = \delta \chi \]

\[ \Gamma = 3 \text{-cycle in } X \]

\[ (\exists \Gamma = \text{sum of exceptional } \mathbb{P}^1's) \]

\[ \Gamma_0 = \text{image of } \Gamma \text{ in } X_0 \]

\[ \Gamma_0 \text{ and } \Gamma_1 \text{ and } \Gamma_2 \text{ and } \Gamma_m \text{ are } 3 \text{-cycle in } X_0 \]

\[ \Gamma_m = \text{def} \text{ of } \Gamma_0 \text{ to an } \text{cycle in } X_m \]

\textbf{Note:} The usual interpretation of } \mathcal{W} \text{ is that it is a \textit{universal function}, i.e., a section of a family of the intermediate Jacobians } J(C | X_m).
Integrable systems:

Recall that an algebraic (analytic) family (analytic family)

\[ T \rightarrow X \]

\[ \downarrow \]

\[ B \]

with \( T \) - abelian varieties (complex tori)

\( X \) has an algebraic (holomorphic) symplectic form

\( T \rightarrow \text{lagrangian} \)

\[ \dim B = \dim T \]

Basic examples:

- \( C \) curve \( \rightarrow J(C) \) an algebraic variety

\( X = \text{CY} \Rightarrow J(X) \) a polarized complex torus

These form in integrable systems

\( \text{CY case: If } \mathcal{M} \equiv \text{moduli of complex CY} \)

\[ \tilde{\mathcal{M}} = \text{moduli of CY} \]

holomorphic volume form
Then \[ \tilde{J} \to \tilde{M} \]
\[ \begin{array}{c}
\nu \to \Phi \\
\text{If } \delta' = T^N \mathcal{C} \text{ and } \mathcal{B} = \text{linear system of compact curves } \Sigma \\
\text{Then } \mathcal{N} \to \mathcal{B} \end{array} \]
\[ \bigcup (\Sigma) \to \Sigma \]
(Kit and Hori)

Criterion for a family of polarized tori to be an integrable system:

Let \( \mathcal{C} \) be a parameter space of polarized (complex) tori. Then the following are equivalent:

1. There exists a complex symplectic manifold \((X, \omega)\) with a map \( X \to \mathcal{C} \) which is Lagrangian and \( \mathcal{B} \mapsto X \).
$\bar{\nu}$ induces the identity

$$\begin{align*}
\tilde{T}/B & \xrightarrow{\sim} \tilde{T}/B \\
\Pi & \xrightarrow{} \Pi \\
\tilde{\tau} & \xrightarrow{} \tilde{\tau} \\
\Pi & \xrightarrow{} \Pi
\end{align*}$$

Here $V \cong B \cong \mathbb{C}^n$.

Define $p : B \rightarrow \text{Sym}^2(V)$ (imaginary part vanishing).

1. $p : B \rightarrow \text{Sym}^2(V)$ is locally in $B$.
2. The Hessian is a projection from $B$ to $B$.
3. $dp + \text{Hom}(\tilde{T}/B, \text{Sym}^2(V)) = V \otimes \text{Sym}^2(V)$.

Examples: In the $\mathbb{C}$-case, take the Yukawa cubic.

In the elliptic case (e.g., for $SU(2)$), we have

$$\text{cusp } B = H^0(K_C)$$

and $d \mapsto \text{Res}^2_C \left( \frac{dz}{z^2} \right)$.
Going back to our setup

$s \subset M \subset L$

$x_{0,10}, x_{a,1}, x_{a,j}$

C$_{0}$ with C$_{0}$ with smooth CY

a curve C flate #

of singularities of singularities

We want to understand CYIS(L) near $a = b = 0$

Claim: To first order

$$CYIS(L) \cong CYIS(S) \times Hodge(C, G)$$

where $G$ = ABSE group of singularities of $x_{0,10}$.

AG argument: degenerate to the normal case

Analogous: If $g = U(3)$ one can do this to the Minkowski system on $S$

$$B = |u||v|$$

- the Minkowski system
and we can degenerate \( D \) to the Hitchin system on \( T^* C \)

(Donagi - F. - Lazarsfeld)

The idea is construct \( \mathcal{D} \)

\[
\mathcal{D}_t = \begin{cases} 
\mathcal{D} & \text{for } t > 0 \\
T^* C & \text{for } t = 0
\end{cases}
\]

For the \( \mathcal{C}s \), we get the following picture.

If \( X_{0,0} \in S \) then

\[
\left( \mathcal{N}_{\mathcal{S}} / \mathcal{M} \right)_{X_{0,0}} = \mathcal{H}^0 \left( \mathcal{K}_C \right) \]

The deformation to the normal cone replaces the resolution of \( X_{0,0} \) by \( \text{tot } (K_f) \) where \( F \to C \) - fibered surface over \( C \).

We have a linearization of our moduli spaces:

\[
\begin{align*}
\mathcal{F} & \to \mathcal{M} \\
\mathcal{C} & \to \mathcal{G} \\
\mathcal{T} & \to \mathcal{C} \\
\mathcal{L} & \to \mathcal{C}
\end{align*}
\]
Corresponding to the deformation to the normal cone

\[ \tilde{X}_{0,0} \to \text{tot}(K_F) \] (F = exceptional divisor in \( \tilde{F}_{0,0} \to X_{0,0} \))

\( X_{0,0} \) maps to \( \text{tot}(\text{affine bundle over } K_F) \)

(affine bundle \( = H^1(K_F) = H^0(K_F) \))

Note \( \text{tot}(K_F) \) contains a rank \( 1 \) surface

\( \text{tot}(K_F \cdot x) \) contains \( 2g - 2 \) lines

(when \( a = 0 \))

\[ X_a = \mathbb{P}(\mathbb{P}^1 \to C = \cup_{x \in C} \mathbb{P}(\mathbb{P}^1_{x})) \]

was known as the \( \text{mirror} \) mirror symmetry.

\( \mathbb{P}_{a, \frac{1}{2}} \) is a \( \text{limit} \) of \( \mathbb{P}_{a, 0} \)’s

generically \( \mathbb{P}_{a, \frac{1}{2}} \)’s at the zeros of \( a \).

In fact, \( \mathbb{P}_{a, 0} = \mathbb{P}(\mathbb{P}^1 \to C) \)

\( \mathbb{P}^1 \to C \) bundle over \( C \)