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Lecture 2

Yesterday: considered the geometry of $T \oplus T^*$.

We defined the notion of a generalized complex manifold: manifold M of dimension n together with $E \subset (T \oplus T^*) \otimes \mathbb{C}$ so that

- $E \subset (T \oplus T^*) \otimes \mathbb{C}$ is maximally isotropic
- $E \cap \bar{E} = 0$
- sections of E are closed under the Courant bracket.

Basic examples:
- symplectic manifold: E is the transform of $T \subset T \oplus T^*$ by $i\omega$
- complex manifold
$$E = T^{1,0} \oplus T^{*\,0,1}$$
- transform either of the above by a closed B-field

We would like to have an easy way to check the closeness of E w.r.t. Courant bracket. This is possible if we use some facts about pure spinors.

Pure spinors: We interpreted $\Lambda^{\bullet} T^*$ as spinors:
$$(X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi$$
Clifford multiplication

Suppose

$$(X+\xi)\cdot\varphi = 0 \quad \Rightarrow \quad (X+\xi)^2\cdot\varphi = 0$$
$$\Rightarrow \quad (X+\xi, X+\xi) = 0$$

Thus the annihilator of any spinor $\varphi \in \Lambda^\bullet T^*$ is an isotropic subspace in $T \oplus T^*$ (or $(T\oplus T^*)\otimes\mathbb{C}$).

<u>Def:</u> A spinor φ is called a <u>pure spinor</u> if its annihilator is maximally isotropic

<u>Note:</u> The property of being pure is invariant under the action of the Spin group.

<u>Examples:</u>

- $1 \in \Lambda^0 T^* \subset \Lambda^\bullet T^*$

  $\Rightarrow \quad i_X 1 + \xi\wedge 1 = 0 \iff \xi = 0$

  hence annihilator$(1) = T \subset T\oplus T^*$

  i.e. 1 is a pure spinor

- $e^B\cdot 1$ is pure

- If $\dim M = m$, then decomposable m-forms are pure spinors:

  $$i_X(dx_1\wedge\cdots\wedge dx_m) + \xi\wedge(dx_1\wedge\cdots\wedge dx_m) = 0$$
  $$X = \sum_{i=m+1}^{n} a_i \frac{\partial}{\partial x_i}, \qquad \xi = \sum_{i=1}^{m} b_i\, dx_i$$

- $e^B\cdot(dx_1\wedge\cdots\wedge dx_m)$ - pure

Proposition There is a 1-to-1 correspondence between pure spinors (taken up to scalar multiplication) and maximally isotropic subspaces

$$\varphi \longrightarrow E_\varphi \subset T \oplus T^*$$

Proof: E. Cartan, C. Chevalley $\square$

Fact: $\dim(E_\varphi \cap E_\psi) > 0$ iff $\langle \varphi, \psi \rangle = 0$
Here $\langle\ ,\ \rangle$ is the 'Mukai pairing' on spinors.

If now M - generalized complex manifold then $E \subset (T \oplus T^*) \otimes \mathbb{C}$ corresponds to some pure spinor φ and the Fact above implies

$$E \cap \bar{E} = 0 \iff \langle \varphi, \bar{\varphi} \rangle \neq 0$$

Note that if M is a generalized complex manifold, then $\forall\ m \in M$ $\exists$ a 1-dim space of pure spinors $\varphi_m \in \Lambda^{\bullet} T^*_m$ s.t. $\mathrm{Ann}(\varphi_m) = E_m$.

This spaces L_m fit together into a line bundle $L \to M$ - the canonical bundle of the generalized complex structure.

<u>Example</u>: If $E = T^{0,1} \oplus T^{1,0*}$
(gives a usual complex structure)
then $L = K_M$.

<u>Proposition</u>: If φ is a pure spinor, then E_φ is closed under the Courant bracket iff $d\varphi = (X+\xi)\varphi$ for some X, ξ.

In particular if φ is a pure spinor and $d\varphi = 0$, then E_φ is closed under the Courant bracket.

<u>Def</u>: A <u>generalized Calabi-Yau manifold</u> is a manifold M^{2n} together with a closed form $\varphi \in \Lambda^\bullet T^*$ s.t.

- φ is a pure form
- $\langle \varphi, \bar{\varphi} \rangle \neq 0$

<u>Examples</u>: (1) If M is an ordinary CY manifold (here this means a complex manifold M together with a section $\Omega \in H^0(K_M)$ trivializing K_M).

Ω - top degree holomorphic form $\Rightarrow$ decomposable
$\Rightarrow$ Ω - pure spinor

$$\langle \Omega, \bar{\Omega} \rangle = \pm \Omega \wedge \bar{\Omega} \neq 0.$$

2. If (M, ω) symplectic manifold then

$$\varphi = e^{i\omega} \quad \text{is a pure spinor}$$

and

$$\langle e^{i\omega}, e^{-i\omega} \rangle = \langle e^{2i\omega}, 1 \rangle = \pm (2i\omega)^m \neq 0$$

3. A B-field transform of a symplectic manifold is also a generalized CY:

$$e^{B+i\omega} = e^{B} \varphi \quad \text{- pure spinor.}$$

Comment: At a first glance it sounds strange that one can treat symplectic and CY manifolds on an equal footing.

Indeed:

- If (M, ω) - compact symplectic

$\Rightarrow$ Lie algebra automorphism group $= C^{\infty}(M)/\mathbb{R}$ is infinite dimensional

- If (M, Ω) - compact CY with $b_1 = 0$

$\Rightarrow$ M has no vector fields (holomorphic).

However if we add the action of B-fields we get similar pictures:

If (M, ω) symplectic and we consider the action of $\mathrm{Diff}(M) \ltimes \Omega^2_{closed}$
$\Rightarrow$ on the level of Lie algebras we get

$$(\mathcal{L}_X + d\xi) e^{i\omega} = 0 \Rightarrow \mathcal{L}_X i\omega + d\xi = 0$$
$$\Rightarrow \mathcal{L}_X \omega = 0$$
$$d\xi = 0$$

i.e. we get again $C^\infty(M)/\mathbb{R}$.

If (M, Ω) - compact complex CY (assume also Kahler). We get

$$(\mathcal{L}_X + d\xi)\Omega = 0 \Rightarrow \mathcal{L}_X \Omega = 0 \quad d\xi \wedge \Omega = 0$$
$$\Rightarrow X = 0$$

$d\xi$ is of type $(1,1)$
$\Rightarrow$ by $\partial\bar{\partial}$ lemma

$$d\xi = \partial\bar{\partial} f$$

Hence we get again $C^\infty(M)/\mathbb{R}$.

<u>Caution</u>: This gives Lie algebras which are isomorphic as vector spaces but have different Lie brackets:
Poisson bracket in the symplectic case and trivial bracket in the CY case

Question (Greg Moore): Can one incorporate quantum corrections on the symplectic side in order to make the Lie brackets match?

If M^{4k} is hyperkähler we have 3 symplectic forms

$$\omega_1, \omega_2, \omega_3$$

ω_1 - Kahler

$\omega_2 + i\omega_3$ - holo symplectic

Now B-field ↘ ↙ symplectic

$$e^{\omega_2 + i\omega_3}$$ - generalized complex structure

The spinor

$$\varphi = t^k e^{\frac{\omega_2 + i\omega_3}{t}}$$

defines a B-field transform of a symplectic manifold

Now $\varphi \xrightarrow[t \to 0]{} (\omega_2 + i\omega_3)^k = \Omega$

↑ pure spinor defining an ordinary CY structure.

Thus: Generalized complex structures allow us in some cases to interpolate between complex and symplectic manifolds.

As we mentioned before: If $\varphi = \varphi_0 + \varphi_2 + \cdots$ is a pure spinor, then $\varphi_0 \neq 0$ and

$$\varphi = \varphi_0 e^{B + i\omega}$$

for some B, ω.

The GCY condition is simply that $d\varphi = 0 \Rightarrow \varphi_0 = \text{const}$.

However for a general generalized complex manifold this is not the case and we can see a type change for our pure spinor from point to point on the manifold.

Type change for pure spinors - even case

$$L \subset \Lambda^{\cdot} T^* \otimes \mathbb{C} \longrightarrow \Lambda^0 T^* \otimes \mathbb{C}$$
$$\varphi \longrightarrow \varphi_0$$

φ_0 defines a section of L^* which vanishes where the type changes.

In particular if $c_1(L) \neq 0 \Rightarrow$ type change must occur

<u>Examples</u>: (1) $M = \mathbb{R}^4 = \mathbb{C}^2$

$$\varphi = z_1 + dz_1 \wedge dz_2 = z_1 \exp\left(\frac{dz_1}{z_1} \wedge dz_2\right)$$

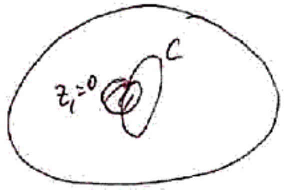
When $z_1 \neq 0 \Rightarrow \varphi$ = B-field transform of a symplectic manifold

when $z_1 = 0 \Rightarrow \varphi = dz_1 \wedge dz_2$ = complex manifold

Note that this is a generalized complex manifold. Indeed

- φ - pure $\Rightarrow E_\varphi$ closed under the Courant bracket if $d(f\varphi) = 0$ for some function.
- $\langle \varphi, \bar\varphi \rangle = \langle z_1 + dz_1 dz_2, \bar z_1 + d\bar z_1 d\bar z_2 \rangle$
  $= dz_1 dz_2 d\bar z_1 d\bar z_2 \neq 0$
- $d(z_1 \varphi) = 0 \Rightarrow E_\varphi$ - integrable

(2) If M is a complex surface with β a holomorphic section of $K^{-1} \Rightarrow \beta^{-1}$ is a holomorphic 2-form outside $C = \mathrm{zero}(\beta)$.



$z_1 + z_1 \beta^{-1}$

Topological conditions: Assume that we have a generalized almost complex manifold: M together with

- $E \subset (T \oplus T^*) \otimes \mathbb{C}$
- E - isotropic
- $E \cap \bar{E} = 0$.

In other words we have an almost complex structure

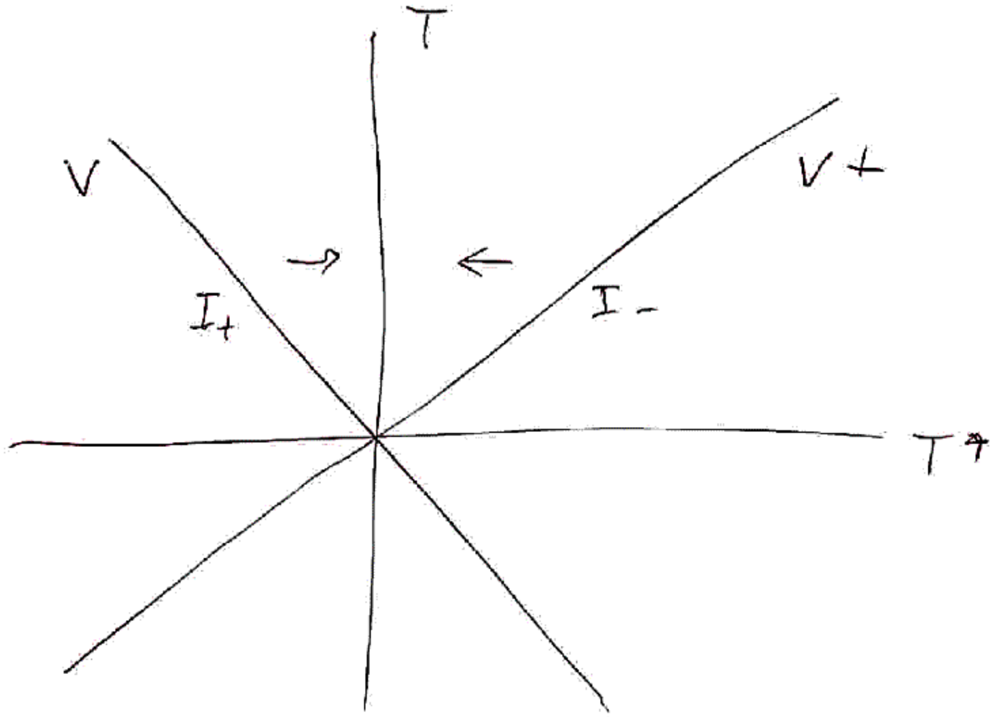
$$J : T \oplus T^* \to T \oplus T^*$$

and J is compatible with the indefinite metric on $T \oplus T^*$. In other words M + (generalized almost complex structure) is the same as

M + (reduction of the structure group $O(2m, 2m)$ of M to $U(m, m)$)

$$\text{Now} \quad U(m,m) \underset{\substack{\text{homotopy}\\ \text{equivalent}}}{\sim} U(m) \times U(m)$$

This gives the following picture:



where $V \oplus V^\perp = T \oplus T^*$

with

metric $|_V$ = positive definite

metric $|_{V^\perp}$ = negative definite

Also projections $V \to T$

$V^\perp \to T$

are isomorphisms and we get two almost complex structures (V, I_+)

$(V^\perp, I_-)$

In particular: if M - generalized almost complex $\Rightarrow$ T has two natural almost complex structures.

Thus: Spheres can not be generalized almost complex if dim > 2.

Now going back to I_+, I_-
note that if L - canonical bundle on M, then

$$-2c_1(L) = c_1^+ + c_1^-$$

<u>Note</u>: symplectic case: c_1^+, c_1^- - opposite
complex case: c_1^+, c_1^- - same

<u>Generalized complex submanifolds</u>

If $Y \subset M$ - submanifold

$$TY \subset TM_{|Y}$$

$$N^*Y = (TY)^{ann} \subset T^*M_{|Y}$$

$$\Rightarrow \quad TY \oplus N^*Y \subset T \oplus T^*_{|Y}$$

<u>Def</u>: A <u>generalized complex manifold</u> is a submanifold $Y \subset M$ in a generalized complex manifold, s.t. $TY \oplus N^*Y$ is preserved by J.

Examples: (1) M - complex

$$\mathcal{J} = \begin{pmatrix} I_T & 0 \\ 0 & -I_{T^*} \end{pmatrix}$$

$\Rightarrow$ $Y \subset M$ is a generalized complex submanifold $\Leftrightarrow$ Y is a complex manifold

(2) B-field transform of a complex manifold.

$$\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} I_T & 0 \\ 0 & -I_{T^*} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} I_T & 0 \\ -(BI + IB & -I_{T^*} \end{pmatrix}$$

$\Rightarrow$ $Y \subset M$ is a generalized complex manifold iff:

- $Y \subset M$ - complex submanifold
- $B|_Y$ is of type $(1,1)$

(because $B(Ix_1, x_2) + B(x_1, Ix_2) = 0$)

(3.) If M - symplectic, $Y \subset M$ - submanifold

$$J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

$\Rightarrow$ $\omega^{-1}\xi \in TY \Rightarrow Y$ - coisotropic

$\omega X \in (TY)^{ann} \Rightarrow Y$ - isotropic

$\Rightarrow$ Y - generalized complex submanifold

$\Leftrightarrow$ Y - Lagrangian

(4) B-field deformation of symplectic manifold

$$J \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} -\omega^{-1}(\xi + BX) \\ (\omega + B\omega^{-1}B)X + B\omega^{-1}\xi \end{pmatrix}$$

$\Rightarrow$ $Y \subset M$ - generalized complex $\Leftrightarrow$

- Y - coisotropic
- B-field is basic i.e. is a pull back from the leaf space of the null folliation on Y corresponding to ω.
- $(B + i\omega)$ is the $(2,0)$ form of a complex structure on the leaf space

In other words: Y is equipped with a transverse holomorphic symplectic structure for the rank foliation of ω.

<u>Remark</u>: • This is almost the same as Kapustin - Orlov generalized A-branes only there one requires

$$B = F_\nabla$$

for some connection ∇ on a line bundle on Y.

• Points are not always generalized complex submanifolds.
In the example with the Poisson structure on a surface

$\Rightarrow$ points on C are generalized complex submanifolds but points not on C are not generalized complex submanifolds.