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Lecture 3

Want to motivate the definition of a generalized complex manifolds. It is a particular case of a more general set of geometric structures on $T \oplus T^*$. Such geometric structures arise naturally in geometry:

For instance – if $(M, \Omega \in H^0(\Omega^3_M))$ is a 3-dim CY manifold, then one can look at

$[\mathrm{Re}\,\Omega] \in H^3(M, \mathbb{R})$

local modulus of the CY complex structure in the special coordinates on the moduli space.

Natural question: How do we reconstruct the complex structure on M starting with a real 3-form only?

To understand this we need to study ~~the~~ the geometry attached to a real 3-form on a 6-dimensional real space.

Start with linear algebra:

$$V \cong \mathbb{R}^6$$

$GL(6, \mathbb{R})$ acts on $\Lambda^3 V^*$ $(\dim \Lambda^3 V^* = 20)$

$\dim GL(6, \mathbb{R}) = 36$ and $GL(6, \mathbb{R})$ acts with an open orbit on $\Lambda^3 V^*$

$\dim(\text{Stabilizer}) = 36 - 20 = 16$

Among the different open orbits of $GL(6,\mathbb{R})$ on $\Lambda^3 V^*$ there is a unique one which has stabilizer $\subset SL(3,\mathbb{C})$.

Call this orbit $(\Lambda^3 V^*)_0$

Now every $\rho \in (\Lambda^3 V^*)_0$ defines:

- a complex structure I on V
- a complex form $\Omega \in \Lambda^3 (V, I)^*$ with $\operatorname{Re}\Omega = \rho$

In particular $\rho \mapsto \Omega \mapsto i\,\Omega \wedge \bar{\Omega} \in \Lambda^6 V^*$

<u>Conclusion:</u> As far as linear algebra is concerned there is nothing special about real parts of holomorphic 3-forms.

With this in mind: define ~~[illegible]~~

$$\phi : (\Lambda^3 V^*)_0 \longrightarrow \Lambda^6 V^* \quad \leftarrow GL(6,\mathbb{R}) \text{ equivariant}$$

s.t. $\qquad \lambda^3 \rho \longrightarrow \lambda^6 \phi(\rho)$

Then ϕ - homogeneous of degree 2 (uniquely determined by this property since $GL(6,\mathbb{R})$ acts transitively)

Then
$$D\phi_\rho : \Lambda^3 V^* \to \Lambda^6 V^*$$

and we can define $\hat{\rho} \in \Lambda^3 V^*$ by

$$\hat{\rho} \wedge \rho = D\phi_\rho(\rho) = 2\,\mathrm{Vol}(\rho)$$

$\hat{\rho}$ is a function of ρ (non-linear)

and $\Omega = \rho + i\hat{\rho}$.

Let M^6 be a compact manifold

$\rho \in \Omega^3_M$ (real 3-forms)

s.t. $\rho(m) \in (\Lambda^3 T^*)_0$

Define $\mathrm{Vol}(\rho) = \int_M \phi(\rho)$ — volume function associated with ρ

Given a cohomology class $a \in H^3(X, \mathbb{R})$
Consider the space of all $\rho \in \Omega^3_M$
for which

- $d\rho = 0$
- $[\rho] = a$
- $\rho(m) \in (\Lambda^3 T^*)_0$

Look for critical points of the functional Vol restricted to the space of such ρ's.

If $\dot{\rho} = d\xi$ - exact,
then

$$\delta \operatorname{vol}(\dot{\rho}) = \int D\phi(\dot{\rho}) = \int \hat{\rho} \wedge \dot{\rho}$$
$$= \int \hat{\rho} \wedge d\xi = \int d\hat{\rho} \wedge \xi$$

Hence ρ is a critical point $\iff d\hat{\rho} = 0$

$\iff d\Omega = 0$

$\Rightarrow$ integrability of the almost complex structure defined by ρ

This gives another approach to studying the moduli of complex CY 3-folds.

Before we set it up consider a finite dimensional analogue:

Consider a function

$$\underset{\substack{\uparrow \\ \text{parameter}}}{f(x,t)} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$$

$$D_x f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$$

If Hessian of f in the x-directions is non-degenerate $\Rightarrow$ $D_x f$ is non-degenerate $\Rightarrow$

$\Rightarrow$ the implicit function theorem gives that if we have a critical point of $f(x, t_0)$ then we have a critical point ~~[illegible]~~ of $f(x,t)$ for all $t \in U \ni t_0$
neighborhood

We now want to apply this reasoning to Vol. However Vol is Diff(M) invariant
$\Rightarrow$ Vol is constant on Diff(M)-orbits

So we cannot expect the Hessian of Vol to be non-degenerate on the full cohomology class a. On the other hand we want to divide by Diff(M) so it suffices to have the Hessian be non-degenerate transverse to the orbit of Diff (~~[illegible]~~ Morse-Bott critical pt).

$\Rightarrow$ the usual Banach manifold techniques give that an open set in $H^3(M, \mathbb{R})$ (centered at a) is a local moduli space for the CY structure determined by a.

<u>Note</u>: If the $\partial\bar\partial$ lemma holds for the complex structure corresponding to a, then the Hessian of Vol is non-degenerate transverse to the Diff orbits

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More general setting: G - Lie group
V - representation space

Assume that G has an open orbit on V
Such a V is called a prehomogeneous vector space for G.

M. Sato & T. Kimura – complete classification
Nagoya Math J. 1977 of prehomogeneous pairs (V, G).

$GL(6, \mathbb{R})$ has an open orbit on $\Lambda^3 V^*$

$Spin(6,6) \times \mathbb{R}^*$ - Spin representation is 32 dimensional : $\Lambda^{ev}\mathbb{R}^6$.

$(Spin(6,6) \times \mathbb{R}^*, \Lambda^{ev}\mathbb{R}^6)$ is a prehomogeneous pair and the stabilizer has dim $1 + \frac{12 \cdot 11}{2} - 32 = 35$

There is a particular open orbit in $\Lambda^{ev}\mathbb{R}^6$ with stabilizer $SU(3,3)$: $(\Lambda^{ev}\mathbb{R}^6)_0$ - this orbit

<u>Proposition</u> If $\rho \in (\Lambda^{ev} V)_0 \Rightarrow \rho$ is the real part of a pure spinor φ with $\langle \varphi, \bar{\varphi} \rangle \neq 0$

$\uparrow$ Mukai pairing

<u>Note</u>: We have a similar statement for $\rho \in (\Lambda^{odd} V)_0$.

In particular $\rho \in (\Lambda^{ev/odd} V)_0$ gives $\varphi \Rightarrow$ get $i\langle \varphi, \bar{\varphi} \rangle \in \Lambda^6 V^*$ and

$$\phi(\rho) : (\Lambda^{ev/odd} V^*)_0 \longrightarrow \Lambda^6 V^*$$

The critical points of the associated volume functional on a given cohomology class are ρ's s.t.

$$d\rho = 0 \;, \; d\hat{\rho} = 0$$

$$\Updownarrow$$

$$d\varphi = 0$$

This was precisely the definition of a generalized CY structure.

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Let us now study the moduli problem!

Have ρ with $d\rho = 0$
$[\rho] = a$

The natural group acting on such ρ's is

$$\mathrm{Diff}_0(M) \ltimes \Omega^2_{exact}$$

where $d\xi \in \Omega^2_{exact}$ acts as

$$\rho \mapsto e^{d\xi}\rho$$

Note: $[e^{d\xi}\rho] = [\rho]$.

It turns out that there is a technical condition: "the dd^J lemma"

(When we have a complex manifold the dd^J-lemma is simply the $\partial\bar{\partial}$-lemma.

When we have a symplectic manifold the dd^J-lemma is equivalent to the hard Lefschetz property i.e. to

$$[\omega] : H^2(M) \xrightarrow{\sim} H^4(M) \;)$$

Also: Hessian is non-degenerate transverse to the Diff orbits if "dd^J lemma" holds.

In particular we get a local moduli space $\cong \underset{\Psi_a}{U} \subset H^{ev/odd}(M, \mathbb{R})$.

Let's go one dimension up: $V = \mathbb{R}^7$

$GL(7, \mathbb{R})$ acting on $\Lambda^3 V^*$ has again an open orbit.

$$\dim(\text{stabilizer}) = 49 - \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 14$$

There is an orbit $(\Lambda^3 V^*)_0$ with stabilizer G_2.

Define $\mathrm{vol}(\rho)$ = metric volume form

(G_2 preserves a metric $\Rightarrow$ ρ determines a metric (up to scale) on M)

$$\rho \longrightarrow \hat{\rho} = *_\rho \rho$$

($*_\rho$ – Hodge star for the metric determined by ρ)

Have $\phi(\rho) := \rho \wedge \hat{\rho}$ and again the critical pts are ρ's with $d\rho = d\hat{\rho} = 0$

This is precisely the condition for G_2-holonomy.

Similarly $Spin(7,7) \times \mathbb{R}^*$ has a spinor 64 dim representation with open orbits

$$\dim(\text{stabilizer}) = \frac{14 \cdot 13}{2} + 1 - 64$$
$$= 92 - 64 = 28$$

There is a special open orbit with stabilizer $G_2 \times G_2$ $\Rightarrow$ get

$\rho \in \Lambda^{ev} T^*$, $\hat{\rho}$ - odd form and

$$d\hat{\rho} = d\rho = 0$$

are again the equations of the critical points for the volume functional and one can study the moduli space : there are again B-fields acting non-trivially.

This is currently under investigation by Frederik Witt.

The $G_2 \times G_2$ generalized geometry is again a geometry on $T \oplus T^*$.

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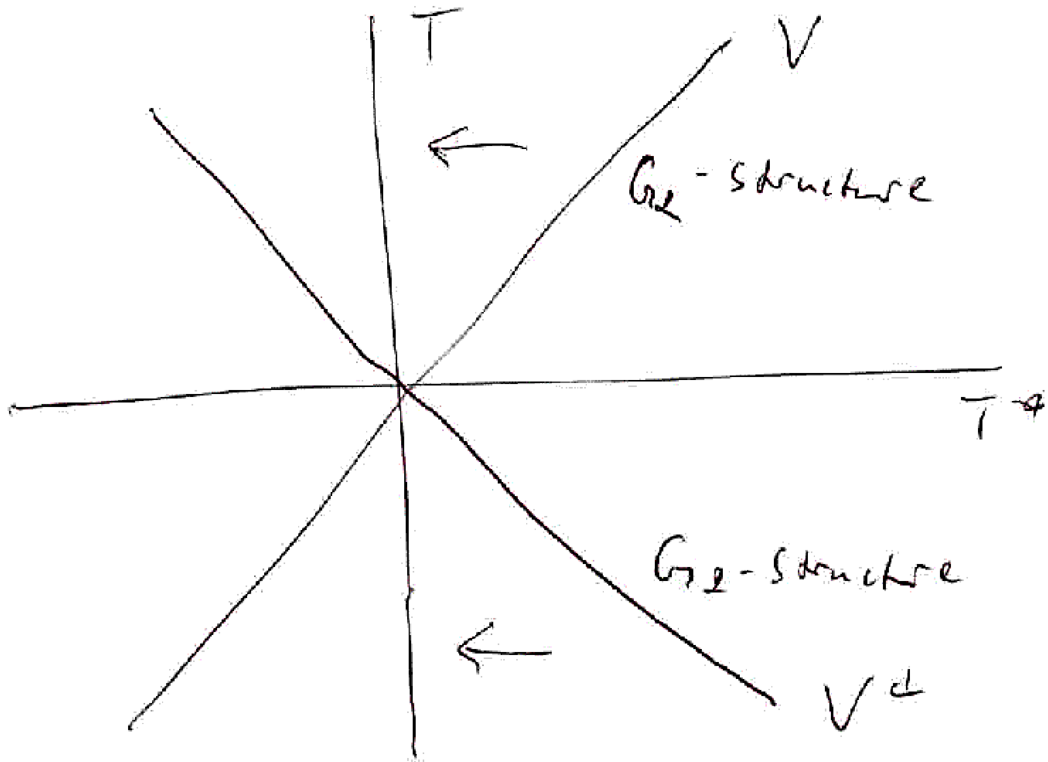

Metrics on $V, V^\perp$ induce opposite metrics on T by projection

$\Rightarrow$ get : • a metric g on T

• 2 (almost) G_2 structures on T

$\Leftrightarrow$ 2 non-zero spinors φ^+, φ^-.

• B-field

+ integrability condition

Note: $SV \otimes SV^\perp$ = spinors of $T \oplus T^* = \Lambda^{ev} T^*$

$$\rho = e^B \cdot (\varphi_+ \otimes \varphi_-)$$

Examples of $G_2 \times G_2$ structures :

(1) Assume M is a manifold with G_2 holonomy. Then take

$$\varphi_+ = \varphi_- = \Phi \text{ - the cov. constant 3-form defining the } G_2 \text{ structure}$$

Then

$$\rho = c \cdot (1 - *\Phi)$$

$$\hat{\rho} = c(-\Phi + \mathrm{vol})$$

(2) B-field transform of the previous example

$$\rho = e^B (1 - *\Phi)$$

for any closed B.

Expect : all B-field transforms of ~~[illegible]~~ a G_2 geometry (viewed as a $G_2 \times G_2$ geometry) fill up the moduli space of $G_2 \times G_2$ geometries near our G_2 geometry.

Note : This is exactly what happens for symplectic manifolds.

Integer invariant:

$$\begin{array}{l} U(S) \leftarrow 14 \text{ dimensional manifold} \\ \sigma \uparrow \downarrow \\ M^7 \end{array}$$

PD - Poincare dual.

$$[PD\,\sigma_+] \cup [PD\,\sigma_-] = \pm n \in \mathbb{Z}$$

n = # pts where φ_+ and φ_- coincide

Generalized complex manifolds in 6 dimensions

$$U(3,3) \sim U(3) \times U(3)$$ homotopy equivalence

$$SO(6)/U(3) = \mathbb{CP}^3 \Rightarrow \begin{array}{c} \mathbb{P}(S) \\ \downarrow \uparrow \sigma \\ M^6 \end{array}$$

again have an integer invariant

$$PD[\sigma_+] \cup PD[\sigma_-] \in \mathbb{Z}.$$

This invariant is related to $c_1(L)$

canonical bundle of the gen. Complex structure

If
$$c_1(L) = \ell$$
then have
$$n^{\pm} = \ell^3 - c_1^{\pm}\ell^2 + c_2^{\pm}\ell - c_3^{\pm}$$

where $c_i^{\pm}$ are the Chern classes of T for the two complex structures corresponding to the generalized complex structure on M

and $$n^+ - n^- = -\ell^2(c_1^+ - c_1^-) + \ell\left(\frac{c_1^{+2}}{2} - \frac{c_1^{-2}}{2}\right) = 0$$

Since $$\ell = \frac{c_1^+ + c_1^-}{2}$$

So $n^+ = n^- = n$ and

$$PD[\sigma_+] \cup PD[\sigma_-] = n.$$

When M = generalized CY $\Rightarrow \ell = 0$
and n = Euler characteristic.