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Lecture 5

As we explained before: in 6 dimensions the correct group of automorphisms of a generalized complex structure (for the purposes of moduli) was

$$\mathrm{Diff}_0 M \ltimes \Omega^2_{exact}$$

This turns out to be ~~the~~ right group in general.

Remark: It is unrealistic to expect that Ω^2_{closed} will be an appropriate group to use for moduli constructions.

For instance if M^{4k} - hyper-Kähler then

$$(\omega_1 + i\omega_2)^{k+1} = 0$$

and if we act by $t^k \exp\left(\frac{\omega_1 + i\omega_2}{t}\right)$ we get a B-field transform to a complex structure.

$$t^k \exp\left(\frac{\omega_1 + i\omega_2}{t}\right) \xrightarrow[t \to 0]{} (\omega_1 + i\omega_2)^k$$

$\frac{\omega_1}{t}$ - B-field

$\frac{\omega_2}{t}$ - symplectic.

If we divide by all closed B-fields we will not get convergence:

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$\left[t^k \exp \frac{i\omega_2}{t}\right] \underset{t \to 0}{\to} \infty \Rightarrow$ if we quotient by Ω^2_{closed} we will get a non-Hausdorff space.

Let us study the Kuranishi theory of a generalized complex structure: $E \subset (T \oplus T^*) \otimes \mathbb{C}$

- E - max. isotropic
- $E \cap \bar{E} = 0$
- sections of E are closed under the Courant bracket.

The first obstacle of doing deformation theory here is the fact that the Courant bracket is not a Lie bracket.

If $A, B, C \in C^\infty(T \oplus T^*)$, then we have the Courant Jacobiator:

$$[[A,B],C] + [[B,C],A] + [[C,A],B] =$$

$$= -\frac{1}{3} d\left(([A,B],C) + ([B,C],A) + ([C,A],B)\right)$$

Since $E \subset (T \oplus T^*) \otimes \mathbb{C}$ - isotropic it follows that the Jacobiator $\equiv 0$ if $A, B, C \in E$.

Thus E together with its natural projection

$$E \xrightarrow{\pi} T \otimes \mathbb{C}$$

and the Courant bracket is a <u>Lie algebroid</u>.

Consider how

$$d : C^\infty(\Lambda^k E^*) \to C^\infty(\Lambda^{k+1} E^*)$$

given by

$$d\sigma(A_0, \dots, A_n) = \sum (-1)^i \pi(A_i)\, \sigma(A_0, \dots, \hat{A}_i, \dots, A_n)$$
$$+ \sum_{i<j} (-1)^{i+j} \sigma(\underbrace{[A_i, A_j]}_{\text{Courant bracket}}, A_0, \dots, \hat{A}_i, \dots, \hat{A}_j, \dots, A_n)$$

Since $[\,,\,]$ satisfies Jacobi on $E \Rightarrow d^2 = 0$

(This in fact works on any Lie algebroid)

Next recall that E had a complementary subbundle $\bar{E}$: $E \cap \bar{E} = 0$. Since $E \subset (T \oplus T^*) \otimes \mathbb{C}$ was isotropic $\Rightarrow E = E^\perp \Rightarrow (\,,\,)$ induces an isomorphism

$$\bar{E} \cong E^*$$

(This turns E into a Lie bialgebroid.)

Infinitesimal deformations of $E \subset (T \oplus T^*) \otimes \mathbb{C}$: these are given by the tangent space to the Grassmannian:

$$\mathrm{Hom}(E, (T \oplus T^*) \otimes \mathbb{C} / E) = \mathrm{Hom}(E, \bar{E})$$
$$= \mathrm{Hom}(E, E^*) = E^* \otimes E^*$$

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The infinitesimal deformations of E as an isotropic subspace of $(T \oplus T^*) \otimes \mathbb{C}$ are then in

$$\Lambda^2 E^* \subset E^* \otimes Z.$$

Also we would like to study not just infinitesimal deformations but infinitesimal deformations modulo our symmetries: Lie $(\mathrm{Diff}_0 \ltimes \Omega^2_{exact})$.

<u>Note</u>: We have a natural complex

$$\begin{array}{c} C^\infty(E^*) \xrightarrow{d} C^\infty(\Lambda^2 E^*) \xrightarrow{d} C^\infty(\Lambda^3 E^*) \\ \uparrow \\ (T \oplus T^*) \otimes \mathbb{C} \end{array}$$

Here $(T \oplus T^*) \otimes \mathbb{C} \xrightarrow{pr} \bar{E} \cong E^*$ is the natural map

$$X + \xi \mapsto X + d\xi$$

$\uparrow$ Lie Diff_0 $= C^\infty(T)$; $\nwarrow$ $\Omega^1 / d\Omega^0 = \Omega^2_{exact}$

Proposition This is the deformation complex for generalized complex structures.

Examples:

- symplectic – de Rham complex for complex forms
- Complex – $\Lambda^\bullet(T \oplus \bar{T}^*)$, T = holomorphic tangent bundle

$$1 \xrightarrow{\bar\partial} \bar{T}^* \xrightarrow{\bar\partial} \Lambda^2\bar{T}^* \xrightarrow{\bar\partial}$$
$$T \xrightarrow{\bar\partial} T\otimes\bar{T}^* \xrightarrow{\bar\partial}$$
$$\Lambda^2 T \xrightarrow{\bar\partial}$$

Theorem (Gualtieri) There is a Kuranishi space which is smooth and has a tangent space $\cong H^2$ (complex).

$$d\sigma + \tfrac{1}{2}[\sigma,\sigma] = 0$$

"Schouten bracket"

Examples: Deformations of a complex manifold as a generalized complex manifold

$$H^2 = H^2(\mathcal{O}) \oplus H^1(T) \oplus H^0(\Lambda^2 T)$$

(0,2) part of a closed B-field

defos of complex structures

Poisson structures

Recall: On complex manifolds the holomorphic Poisson structures act on the complex structure to produce a generalized complex structure:

$$E = \left\{\frac{\partial}{\partial z_i}\right\} \oplus \{d\bar{z}_i\} \qquad \text{If } \eta = \sum \eta^{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

$$\bar{E} = \left\{\frac{\partial}{\partial \bar{z}_i}\right\} \oplus \{dz_j\} \qquad \text{then}$$

$$\bar{E} \mapsto \left\{ \frac{\partial}{\partial \bar{z}_i},\ dz_j + \eta^{jk}\frac{\partial}{\partial z_k} \right\}$$

sends us to another generalized complex structure

- If η^{-1} exists $\Rightarrow$ η^{-1} is a holomorphic symplectic form and η acting on E is the same as the η^{-1}-transform of E where η^{-1} is considered as a B-field.

- The obstruction space in this case is

$$H^3(\text{complex}) = \bigoplus_{p+q=3} H^p(\Lambda^q T).$$

K3 surfaces

If X - K3 surface, then

$$H^2(\text{complex}) = \underset{1}{H^{0,2}} \oplus \underset{20}{H^1(T)} \oplus \underset{1}{H^0(\Lambda^2 T)}$$

The obstruction space is $= 0$.

One can check that near a complex pt every generalized complex structure on a K3 is a generalized CY structure.

This can be made more explicit by taking

$$\varphi = \varphi_0 + \varphi_2 + \varphi_4$$

pure spinor - describing a generalized complex structure.

Now:

$$\left| \begin{array}{l} d\varphi = 0 \\ \langle \varphi, \varphi \rangle = 0 \qquad (\varphi_2^2 - 2\varphi_0\varphi_4 = 0) \\ \langle \varphi, \bar{\varphi} \rangle > 0 \end{array} \right.$$

are the conditions that φ defines a generalized complex structures.

Consider now the analogous locus in cohomology:

$$[\varphi] \in H^{\cdot}(M, \mathbb{C}) \ : \quad Q \subset \mathbb{P}(H^{\cdot}(M, \mathbb{C}))$$

22 dimensional quadric given by $\langle [\varphi], [\varphi] \rangle = 0$

+ take an open set of all

$$[\varphi] \in Q \; : \; \langle [\varphi], [\bar{\varphi}] \rangle > 0$$

This is the period space for generalized complex structures on M.

We have a period map

$$\varphi \longmapsto [\varphi].$$

The period space is open in

$$Q = \frac{SO(4,20)}{SO(2) \times SO(2,20)}$$

$\Rightarrow$ Can use as a local model for the moduli space.

How about the moduli space of generalized CY metrics?

We need two pure spinors φ_1, φ_2:

$$\langle \varphi_i, \varphi_i \rangle = 0 \qquad \langle \varphi_1, \varphi_2 \rangle = 0$$
$$\langle \varphi_i, \bar{\varphi}_i \rangle > 0 \qquad \langle \varphi_1, \bar{\varphi}_2 \rangle = 0$$
$$\langle \varphi_1, \bar{\varphi}_1 \rangle = \langle \varphi_2, \bar{\varphi}_2 \rangle$$

If we write

$$\varphi_i = x_i + \sqrt{-1}\, y_i$$

$\Rightarrow$

x_1, x_2, y_1, y_2 - orthogonal w.r.t. Mukai pairing + have unit length

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So the image of the moduli of generalized CY metrics on M in the period space is open in

$$\frac{SO(4,20)}{SO(2)\times SO(2)\times SO(20)}$$

Let us now look at the generalized CY metrics on a K3 M from the point of view of the generalized Kähler metrics.

Recall that a generalized Kähler structure on M is given by a pair of complex structures $I_\pm$ giving say for I_+ a B-field automorphism:

$$A\left(\frac{\partial}{\partial z_i}\right) := \frac{\partial}{\partial z_i} + i\left(\frac{\partial}{\partial z_i}\right)(-\sqrt{-1}\,\omega + b)$$

Given a generalized Kähler structure on M we need to find φ_1, φ_2:

$$A\left(\frac{\partial}{\partial z_i}\right)\varphi_i = 0$$

But

$$A\left(\frac{\partial}{\partial z_i}\right)\varphi = 0 \iff i\left(\frac{\partial}{\partial z_i}\right)e^{i\omega - b}\varphi = 0$$

$$\Rightarrow e^{i\omega - b}\varphi \text{ is a } (0,0)+(0,2) \text{ form}$$

Hence

$$\varphi_1 = e^{-i\omega+b}(\alpha_1+\beta_1 v)$$

$$\varphi_2 = e^{-i\omega+b}(\alpha_2+\beta_2 v)$$

with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$

Impose algebraic conditions $\Rightarrow$ $\alpha_1, \beta_2 = 0$
or $\alpha_2, \beta_1 = 0$

So

$$\varphi_1 = \overset{\text{constant}}{c}\, e^{-i\omega+b}$$

$$\varphi_2 = e^{-i\omega+b}\underset{(0,2)\text{-form}}{\sigma}$$

So $d\varphi_1 = 0$ implies $d(-i\omega+b)=0$

$\Rightarrow d\omega + db = 0$

$\Rightarrow \nabla_+ = \nabla_- = \nabla$ - Levi-Civita

$\Rightarrow M$ - Kähler

Also $d\varphi_2 = 0$ implies $\bar\partial\sigma = 0 \Rightarrow \bar\sigma$ - holomorphic

Imposing $\langle\varphi_1, \bar\varphi_1\rangle = \langle\varphi_2, \bar\varphi_2\rangle \Rightarrow |\sigma| = 1$

Thus σ must be covariantly constant.

Thus we get an ordinary CY metric on the K3 M + closed B-field

So the moduli space now becomes

$$\frac{SO(4,20)}{SO(4)\times SO(20)} \sim \text{hk metrics} + B \text{ fields}$$

(modulo $\mathrm{Diff}_0 M \ltimes \Omega^2_{exact}$)

$\sim$ moduli of hk metrics on M

$\sim$ moduli of $N=(4,4)$ SCFT.

How will all this look for higher dimensional hk manifolds?

Look at a primitive hk manifold

$$\begin{array}{ccccc} H^{0,2} & \oplus & H^1(T) & \oplus & H^0(\Lambda^2 T) \\ 1 & & \overset{\shortparallel}{h^{1,1}} & & 1 \end{array}$$

Now Poisson structures are invertible
$\leadsto$ acting by those = taking a B-field transform by a holomorphic symplectic form.

Now one gets that near a complex structure on a hk M we can get all generalized complex structures on M as limits of B-field transforms of complex structures.

$\Rightarrow$ get a moduli space contained in

$$\mathbb{P}(H^{ev}(M, \mathbb{C})).$$

Verbitsky/Bogomolov: If $H = H^2 \subset H^{ev}(M^{4k}, \mathbb{C})$ then the subalgebra generated by H is

$$\mathrm{Sym}\, H \Big/ \begin{pmatrix} \text{ideal generated by } \sigma^{k+1} = 0 \\ \text{for all } \sigma : Q(\sigma) = 0 \end{pmatrix}$$

Here Q is the Beauville-Bogomolov form on H.

Periods lie in closure of $c\exp(B+i\omega)$ and one expects that there may be a bigger group acting on H and containing $O(Q)$ so that

$$\begin{pmatrix} \text{closure} \\ \text{of } c\cdot\exp(B+i\omega) \end{pmatrix} \subset \left(\mathrm{Sym}\, H \Big/ \begin{matrix} \sigma : Q(\sigma) = 0 \\ \sigma^{k+1} = 0 \end{matrix} \right)$$

|| pure spinors

orbit of the bigger group