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Lecture 2:

Physical motivation for the homological mirror conjecture Kontsevich proposed:

$$\begin{array}{ccc} \text{A-model} & & \text{B-model} \\ \downarrow & & \downarrow \\ DF(X) & & D^b(Coh(X)) \end{array}$$

To see how these categories naturally appear in physics - look at topological field theories with boundaries and try to make sense of the A and B twists of such theories.

How do objects in $D^b(Coh(X))$ appear in topological field theory?

Represents objects in $D^b(Coh(X))$ by complexes of vector bundles. Can we see these in B-twisted boundary TFT?

Simplest setup: Consider $\phi : \underset{\text{worldsheet}}{\Sigma} \to \underset{\sigma\text{-model}}{X}$ - field in the σ-model

the simplest boundary condition on ϕ:

(B-type) $\quad \partial_n \phi_{|\partial\Sigma} = 0 \quad$ (Neuman)

In modern language this means that we have taken a brane wrapping all of X which is just ~~to~~ the trivial line bundle + the trivial connection.

<u>Witten</u>: more generally can take
(E, ∇) on X : E - complex line bundle
∇ - connection s.t.
$\bar\partial = \nabla^{0,1}$: $\bar\partial^2 = 0$

Then we can modify the Neumann boundary condition on ϕ to take (E, ∇) into account:

$$\partial_n \phi = F \, \partial_\tau \phi + (\text{fermionic terms})$$

Here
$$F = -i \, \nabla^2$$

There are also <u>A-type boundary conditions</u>:

$Y \subset X$ Lagrangian submanifold
(E, ∇) - vector bundle with $\nabla^2 = 0$

and one can write Dirichlet boundary conditions at least classically.

<u>Note</u>: There are various quantum anomalies
E.g. K. Hori showed that the R-term anomaly is cancelled exactly when the Lagrangian Y is graded.

Going back to the B-model:

We can take more general branes if we look at

$$(Y, E, \nabla)$$

where

- $Y \subset X$ - complex submanifold
- $E \to Y$ - holomorphic vector bundle on Y
- $\bar\partial = \nabla^{0,1}$: $\bar\partial^2 = 0$

Fixing this data leads again to B-type boundary conditions on ϕ:

$$E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1 \qquad \begin{matrix} F \circ G = 0 \\ G \circ F \end{matrix}$$

brane-antibrane configuration
F, G - tachyonic modes.

The path $\int$ for the σ-model now looks like

bulk σ-model action

$$Z = \int \mathcal{D}\phi \, \mathcal{D}\psi \, e^{-S(X,\psi)} \cdot \operatorname{str} P\exp\left(\int d\theta \, d\tau \ a\right)$$

Here

$$\Phi^i = \phi^i + \theta \psi^i$$

$$a = \begin{pmatrix} A^1_\mu D_0 \phi^\mu & T^\dagger(\Phi) \\ T(\Phi) & A^2_\mu D^\mu \phi \end{pmatrix} \qquad T = F + G^\dagger$$

One checks that one can satisfy Susy
if $F \circ G = 0$
$G \circ F = 0$

and so the $\mathbb{Z}/2$ graded complexes of vector bundles

$$E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1 \qquad \text{with} \quad \begin{array}{l} F \circ G = 0 \\ G \circ F = 0. \end{array}$$

appear naturally in string theory.

<u>Note</u>: We can get such a $\mathbb{Z}/2$ graded creature from the derived category:
If

$$\to A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} \text{— —}$$

is a complex of vector bundles representing some object in $D^b(X)$
take

$$E_0 = \oplus A_{even} \qquad F = \Sigma\, d_{even}$$

$$E_1 = \oplus A_{odd} \qquad G = \Sigma\, d_{odd}$$

Note: It is not clear to me how to physically get from the $\mathbb{Z}/2$-graded complex to the complex $(A_\bullet, d)$.

How about morphisms:

$$M, N \text{ - objects in } D^b(X)$$

$$\begin{array}{c} \bullet\, M \\ \downarrow \\ N \end{array} \rightsquigarrow H_{MN}$$ space of strings stretching from M to N

If M, N - vector bundles

$$H_{MN} = \bigoplus_p H^p(X, M^* \otimes N)$$

comes naturally from the physics.

For more general objects it is not clear why

$$H_{MN} = \bigoplus_p \mathrm{Ext}^p(M, N)$$

defined physically in the B-model

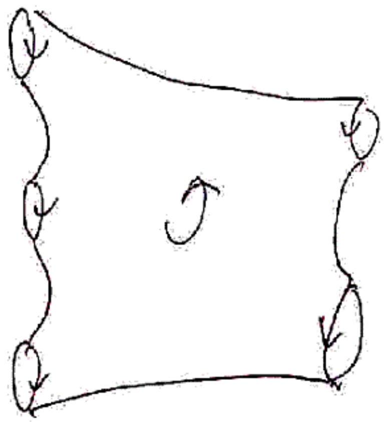
defined as a space of morphisms in $D^b(X)$.

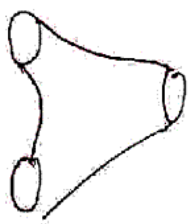
Several special cases for torsion sheaves were checked by Katz, Sharpe, Pantev, Caldararu.

TFT: (closed model à la Atiyah)

 $\mapsto H$

circle — Hilbert space

 $\mapsto H^{\otimes 3} \to H^{\otimes 2}$

 $\mapsto H \otimes H \to H$

 $\mapsto H \to \mathbb{C}$

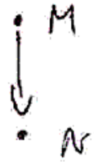
 $\mapsto \mathbb{C} \to H$

This is how we get the Frobenius algebra structure discussed last time.

What about boundary TFT?

Boundary TFT :

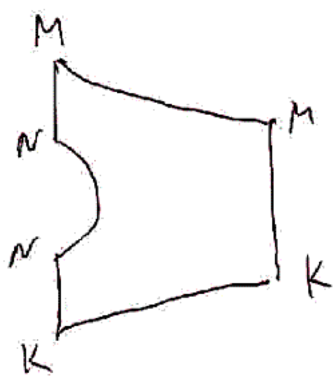
If we have branes M, N then



$\longmapsto H_{MN}$

$\longmapsto H$

$\Rightarrow$ have all the structure in the closed sector + extra:



$\longmapsto H_{MN} \otimes H_{NK} \to H_{MK}$

So one 'gets' a category with spaces of morphisms H_{MN}.

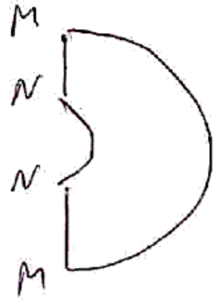
This is all evidence that the category of boundary TFT $= D^b(X)$. But there is no real argument for this statement.

The category of B-branes obeys some natural constraints (from the path integral derivation)

These can be posed and sometimes verified directly in the derived category. This provides further evidence for the claim

$$(\text{B-branes}) = D^b(X)$$

<u>For example:</u>

- 

gives a pairing

$$H_{MN} \otimes H_{NM} \to \mathbb{C}$$

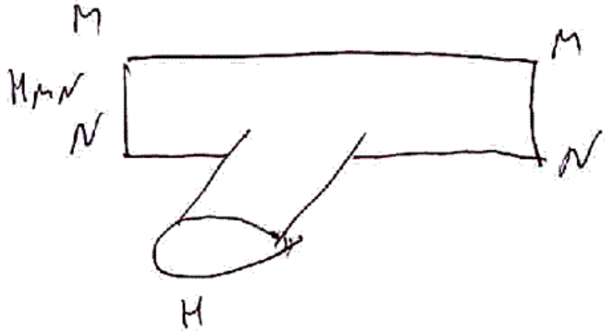
which can be shown to be non-degenerate.

In particular $H_{MN} \cong H_{NM}^*$ $\forall\, M, N$.

For a general X this does not hold in $D^b(X)$. But for a CY $\Rightarrow$

$$R\mathrm{Hom}(M,N) = R\mathrm{Hom}(N,M)$$

because of Serre duality

- 

$$H = \bigoplus_{p,q} H^p(\Lambda^q TX)$$

$$H_{MN} = \bigoplus_p \mathrm{Ext}^p(M,N)$$

and we should have an action

$$H \otimes H_{MN} \to H_{MN}$$

In particular we should have an action of

$$\bigoplus_{p,q} H^p(\Lambda^q TX) \quad \text{on} \quad \bigoplus_p \mathrm{Ext}^p(M,N)$$

Kontsevich in his 1994 ICM talk explained why there should be such an action:

identify $\bigoplus_{p,q} H^p(\Lambda^q TX)$ with $\bigoplus_k \mathrm{Ext}^k_{X\times X}(\Delta,\Delta)$

via the Gerstenhaber-Shack isomorphism

Use this to transplant the natural action of $\bigoplus_k \mathrm{Ext}^k_{X\times X}(\Delta,\Delta)$ on $\bigoplus_p \mathrm{Ext}^p(M,N)$

to a map

$$H \otimes H_{MN} \to H_{MN}.$$

<u>Note</u>: The action of $\bigoplus_k \mathrm{Ext}^k_{X\times X}(\Delta,\Delta)$ on $\bigoplus_p \mathrm{Ext}^p(M,N)$ is easy to see in the case when $X = \mathrm{Spec}\, A$ is affine:

$$\bigoplus_k \mathrm{Ext}^k_{X\times X}(\Delta,\Delta) = \bigoplus_k \mathrm{Ext}^k_{A\otimes A}(A,A) \ni h$$

$$h: A \to A[k]$$

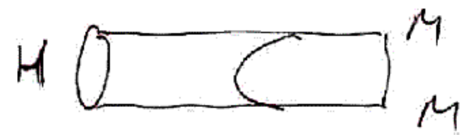
$\varphi : M \to N[\ell] \qquad \varphi \in \mathrm{Ext}^{\ell}_{A}(M,N)$

$\Rightarrow \quad M \otimes A \xrightarrow{h\circ\varphi} N \otimes A[k+\ell] \quad \Rightarrow$

$h \circ \varphi \in \mathrm{Ext}^{k+\ell}_{A}(M,N)$

- $\begin{array}{c} M \\ \downarrow \\ N \end{array} \mapsto H_{MN}$

$M = N$

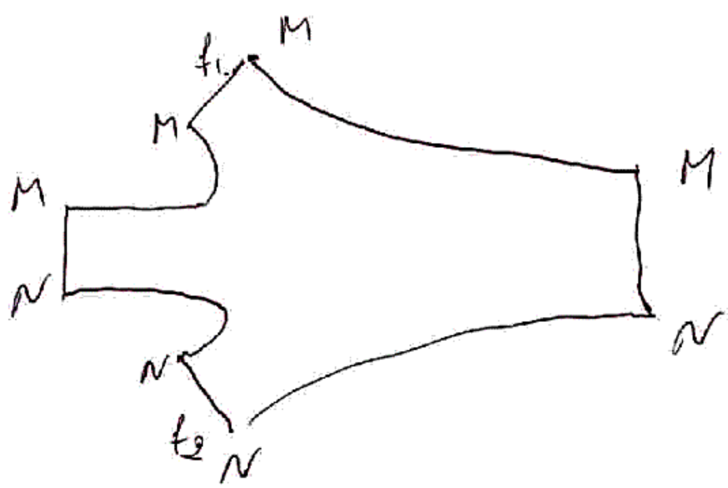


$0 \mapsto H$

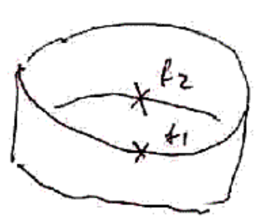
$\alpha^{*}_{M} : H \to H_{MM}$

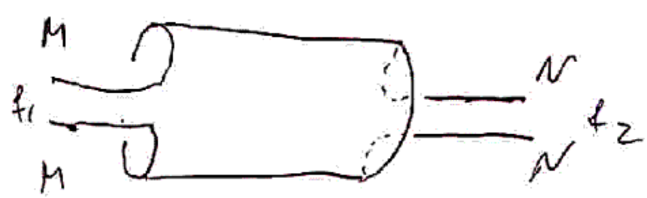
$\alpha_{M} : H_{MM} \to H$

$V = \bigoplus_{k} \mathrm{Ext}^{k}(M,N)$



$F(f_1, f_2) : V \to V$

$\mathrm{Str}\, F =$ 

$=$ 

$\shortparallel$

$\langle \alpha_M(f_1), \alpha_N(f_2) \rangle$

11.

This gives an identity in $D^b(X)$:

$$\mathrm{str}\, F = \langle \alpha_\mu(f_1), \alpha_\nu(f_2) \rangle$$

(B-model Cardy condition)

This is a non-trivial statement in the Derived category recently checked by Caldararu.

As seen from the previous discussion $\Rightarrow$ $D^b(X)$ provides a complicated mathematical description of the category and structure of B-branes.

There is a much simpler description (mathematically unexplored) in the context of Landau-Ginzburg models. Here we look at:

$X = \mathbb{C}^n$

$W : X \to \mathbb{C}$ - holomorphic function (Superpotential).

$N=2$ LG model on (X, W) $\leadsto$ B-model twist of topological theory

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$$\mathcal{O} \longmapsto H$$

$$H = \mathbb{C}[x_1, \ldots, x_n] / \langle \partial_{x_i} W \rangle$$

(Jacobi ring of W)

<u>D-branes</u> : (Hori-Iqbal-Vafa) need submanifolds in $\mathbb{C}^n$ on which W is constant

Now carrying out the previous analysis

$$A^{(1)}_\mu, \ A^{(2)}_\mu, \ T \xmapsto{\text{BRST}} \bar{\partial}^2_{1,2} = 0$$

$$T = F + G^\dagger$$

So one again get

$$E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1$$

E_i - holo vector bundles

but the susy conditions are

$$F \circ G = W \cdot 1 + \text{const}$$
$$G \circ F = W \cdot 1 + \text{const}$$

So one gets $D = \begin{pmatrix} 0 & G \\ F & 0 \end{pmatrix}$ with

$$D^2 = \begin{pmatrix} W \cdot 1 & 0 \\ 0 & W \cdot 1 \end{pmatrix}$$

So the B-branes in the LG model are given not by a complex of vector bundles but rather a twisted complex (twisted by the superpotential)

In particular the problem of finding the boundary conditions in TFT is captured in the problem of factorizing W as a product of matrix polynomials.

Expect analogues of all structures:

- Frobenius algebra structure
- "Serre duality"
- Cardy condition.