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Lecture 3

X - complex manifold

$W: X \to \mathbb{C}$ - holomorphic function

LG model on (X, W) : today $X = \mathbb{C}^n$

We would like to find the right boundary conditions.

Try Neumann boundary conditions on $\phi : \Sigma \to X$:

$$\begin{cases} \partial_n \phi_{|\partial\Sigma} = 0 \\ \psi_+ - \psi_{-\,|\partial\Sigma} = 0 \end{cases}$$

- fermionic boundary condition

$$\delta S_{bulk} = \int_{\partial\Sigma} \frac{i}{2} \bar{\epsilon} \left(\psi^{\bar{i}} \partial_{\bar{i}} \bar{W} + \psi^{i} \partial_{i} W \right) + \left(\text{hermitian conjugate}\right)$$

So if $\partial_i W \neq 0$ Neuman boundary conditions brake $N=2$ susy.

N. Warner, Hori-Iqbal-Vafa, ... : add Dirichlet boundary conditions in some directions in order to restore susy.

2.

This approach does not always work.
Instead - try to restore susy by adding a boundary term to the action.

Consider

$$S_{boundary} = \frac{1}{2} \int d\tau\, d\theta \left(\bar{\Gamma} D_\theta \Gamma + \Gamma T + \bar{T} \cdot \bar{\Gamma} \right)$$

where:

- $\Gamma(\tau, \theta)$ is some new superfield living on the boundary

  (If we have a brane-antibrane configuration $E_0, E_1 \Rightarrow$
  $\Gamma \in Hom(E_0, E_1)$)

$$\Gamma(\tau, \theta) = \eta(\tau) + i\theta\, l(\tau)$$

$$\eta(\tau),\ l(\tau) \in Hom(E_0, E_1)$$

(will assume E_0, E_1 - unitary line bundles)

- $T = T(\Phi^I(\tau, \theta)) \in Hom(E_0, E_1)$

  tachyon field

$$\Phi^I(\tau, \theta) = \phi^I + \theta \psi^I.$$

3.

In terms of η, $\mathcal{L}$, T we have

$$S_{boundary} = \frac{1}{2} \int_{\partial\Sigma} i \bar{\eta} \partial_0 \eta + \frac{i}{2} \left(\left(\psi^I \partial_I T\right) \eta + \text{hermitian conj.} \right) - \frac{1}{2} T \bar{T}.$$

We need to check that η, $\mathcal{L}$, T can be chosen so that the $N=2$ susy variation

$$\delta S_{bulk} + \delta S_{boundary}$$

is zero.

Analyzing this we see that we must have

$$\delta \eta = \epsilon s_1 + \bar{\epsilon} s_2 \qquad s_1 = i F^{\dagger}, \quad s_2 = -i G$$

$$T = F + G^{\dagger}$$

with $F \in Hom(E_0, E_1)$
$G \in Hom(E_1, E_0)$

and

$$\delta S_{boundary} = \frac{1}{2} \int \bar{\epsilon} \left(\psi^{\bar{i}} \partial_{\bar{i}} (F^{\dagger} G^{\dagger}) - \psi^{i} \partial_i (F G) \right) + (\text{hermitian conjugate})$$

and one gets

$$FG = GF = W + \text{const}.$$

Remark: • With work one can check that this is the right condition even when E_0, E_1 have higher rank.

• This is worked out in papers by Kapustin-Li

To understand what the categories of branes are we need to look at the boundary piece of the BRST operator

$$Q_{\text{boundary}} = -iF\eta + iG\bar{\eta}$$

where $\{\eta, \bar{\eta}\} = 1$.

It is convenient to write

$$\eta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \bar{\eta} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and so

$$Q_{\text{boundary}} = \begin{pmatrix} 0 & iG \\ -iF & 0 \end{pmatrix}$$

In the bulk we have:

5.

$$Q_{bulk} = \int d\sigma \left(g_{i\bar{i}} (\psi_+ + \psi_-)^{\bar{i}} \partial_0 \phi^i + (\psi_+ - \psi_-)^{\bar{i}} \partial_1 \phi^i \right.$$
$$\left. + \frac{i}{2} (\psi_+ - \psi_-)^i \partial_i W \right)$$

We have

$$Q_{bulk}^2 = -i \, W_{|\partial\Sigma}$$

$$Q_{boundary}^2 = i \, W_{|\partial\Sigma}.$$

and so $(Q_{bulk} + Q_{boundary})^2 = 0$

<u>Note</u>: To see that Q_{bulk} anti-commutes with $Q_{boundary}$ one notices that in the zero-mode approximation $Q_{boundary}$ = holomorphic expression

$$Q_{bulk} = \bar{\partial}.$$

To work out the cohomology of $Q_{bulk} + Q_{boundary}$ we will work in the zero-mode approximation. This should be enough for the topological B-theory. We have:

$$\delta\phi^i = 0 \qquad \delta\phi^{\bar{i}} = -\bar{\epsilon}\, \psi^{\bar{i}}$$
$$\delta\psi^i = -2i\bar{\epsilon}\, \dot{\phi}^i, \qquad \delta\psi^{\bar{i}} = 0$$

$$\mathcal{O}_f = \psi^{\bar{i}_1} \cdots \psi^{\bar{i}_k} f_{\bar{i}_1 \cdots \bar{i}_k}(\phi)$$

$$\{ Q_{bulk}, \mathcal{O}_f \} = \mathcal{O}_{\bar{\partial} f}$$

For the boundary BRST operator:

$$f \in \mathrm{Hom}(E_0 \oplus E_1, E_0 \oplus E_1)$$

$\Rightarrow$ f - matrix and

$$\{ Q_{boundary}, \mathcal{O}_f \} = \mathcal{O}_{[Q_{boundary}, f]}$$

↑ matrix commutator

Hence our states are described by (in the zero-mode approximation)

$$f \in \bigoplus_p \mathcal{A}^{0,p}(X) \otimes \mathrm{End}(E_0 \oplus E_1)$$

with

$$Qf = \bar{\partial} f + [Q_{boundary}, f]$$

<u>Note</u>: $[\bar{\partial}, Q_{boundary}] = 0$

and $Q^2_{boundary} \in \mathrm{Center}(E_0 \oplus E_1) \otimes \cdots$

$\Rightarrow$ $[Q^2_{boundary}, f] = 0 \quad \forall f.$

We can simplify things further since we assumed $X = \mathbb{C}^n$.

If $f = f_0 + f_1 + \cdots + f_r$

$$f_i \in \Omega^{0,i} \otimes End(E_0 \oplus E_1)$$

$$\bar\partial f + [Q_{boundary}, f] = 0 \Rightarrow f_r = \bar\partial \beta_{r-1}$$

Redefine f as

$$f' = f - (\bar\partial + [Q_{boundary}, \cdot\,])\beta$$

$\Rightarrow f \sim f'$ - cohomologous and

$$f' = f_0' + \cdots + f_{r-1}'$$

Continuing this way we can replace f by a 0-form cohomologous to f.

Thus we may assume $f = f_0$

$$\bar\partial f_0 = 0$$
$$[Q_{boundary}, f_0] = 0.$$

$$f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$A \in End(E_0)$, $D \in End(E_1)$ - holomorphic endomorphisms

$B \in Hom(E_0, E_1)$, $C \in Hom(E_1, E_0)$ - holomorphic homomorphisms

Remark: It is natural to consider A, D as even and B, C as odd since

$$f = \alpha_0 + \eta\alpha_1 + \bar{\eta}\alpha_2 + \eta\bar{\eta}\alpha_3$$

Now the differential Q becomes

$$f \longmapsto \underbrace{[Q_{\text{boundary}}, f]}_{\text{supercommutator}}$$

$$\| $$

$$\begin{pmatrix} 0 & G \\ F & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}\begin{pmatrix} 0 & G \\ F & 0 \end{pmatrix}$$

Hence the category of B-branes for (X, W) has hom spaces

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \begin{pmatrix} 0 & G \\ F & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}\begin{pmatrix} 0 & G \\ F & 0 \end{pmatrix} \right\}$$

More explicitly:

branes: $E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1$ $\quad F \circ G = W \cdot id_{E_1} + \text{const}$

$G \circ F = W \cdot id_{E_0} + \text{const}$

morphisms: $$\frac{\left\{ f = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : E_0 \oplus E_1 \to E_0' \oplus E_1' \right\} \quad [Q_{\text{boundary}}, f] = 0}{[Q_{\text{boundary}}, \cdot\,] - \text{exact}}$$

Remarks:

- this category was proposed by Kontsevich as a category of B-branes in LG models and we just got a physical derivation of this proposal
- If we keep the dg category rather than the cohomology category $\Rightarrow$ get more information. For instance higher products in the dg-category capture n-pt correlators in the string field theory.

Examples: $W = z^n$ on $X = \mathbb{C}$

$F = z^{n-k}$, $G = z^k$

$$\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \begin{array}{l} E_0 = \mathcal{O}_X \\ E_1 = \mathcal{O}_X \end{array}$$

$$D := [Q_{\text{boundary}}, \cdot\,]$$

$$D\varphi = \begin{pmatrix} Cz^k + Bz^{n-k} & (D-A)z^k \\ (A-D)z^{n-k} & Bz^{n-k} + Cz^k \end{pmatrix}$$

$$\ker D = \left\{ \begin{pmatrix} A & B \\ -Bz^{n-2k} & A \end{pmatrix} \right\}$$

modifying by D-exact terms gives

$$B \sim B + \# z^k$$
$$A \sim A + \# z^k$$

Hence

$$\mathrm{End}\left(E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1 \right) = \frac{\mathbb{C}[z]}{z^k} \oplus \frac{\mathbb{C}[z]}{z^k}$$

$$= \langle 1, a, \theta \mid a^k = 0, \ \theta^2 = -a^{n-2k} \rangle$$

where

$$a = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

$$\theta = \begin{pmatrix} 0 & 1 \\ -z^{n-2k} & 0 \end{pmatrix}$$

Interesting special cases:

- $n = 2, \ W = z^2, \ F = z, \ G = z$

$$\mathrm{End}\left(\mathcal{O}_z \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{O}_z \right) = \langle 1, \theta \mid \theta^2 = -1 \rangle$$

Clifford algebra

The appearance of a Clifford algebra here is a feature of any quadratic superpotential!

Let $X = \mathbb{C}^n$, $W = Q(x, x)$

Q - non-degenerate quadratic form

$$Q^2_{\text{boundary}} = W \quad \Rightarrow \quad Q_{\text{boundary}} = \Gamma_i z^i + \ldots$$

$$\{\Gamma_i, \Gamma_j\} = Q_{ij}$$

$\Rightarrow$ Clifford module

If W - Superpotential with non-degenerate critical pts on $\mathbb{C}^n$, then we get that the category of branes breaks into pieces labeled by the critical pts where each piece is the category of Clifford modules for the Clifford algebra corresponding to the ~~[illegible]~~ quadratic part of W at the corresponding critical pts.

__Note:__ Knörer periodicity: if

$$W' = W(x) + y_1^2 + y_2^2$$

then

$$D^b(\text{B-branes}_{W'}) \cong D^b(\text{B-branes}_W)$$

This was also recently proved by Orlov in the context of LG models

__Question:__ What happens if we add just one square?

__Proposal:__ Consider the category of pairs

$$\left(E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1 \ , \ \ \beta \in \mathrm{End}(E_0 \oplus E_1) \ \text{odd endomorphism s.t. } \beta^2 = 1 \right)$$

This is the category of ~~[illegible]~~ branes equipped with brane-antibrane symmetry

We expect:

$$D^b\left(\text{B-Branes}_W\right) \cong D^b\left(\text{B-branes}^{\,b/a\ \text{symmetry}}_{W+y^2}\right)$$

<u>Note:</u> This was checked for $W = z^4$, $W = \Sigma\,(\text{squares})$.