07-31-03

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Lecture 4

Last time we looked at: $X = \mathbb{C}$, $W = z^n$

Consider the brane

$$\mathcal{O}_X \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{O}_X \qquad \begin{array}{l} F = z \\ G = z^{n-1} \end{array}$$

We saw that

$$\mathrm{End}\left(\mathcal{O}_X \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{O}_X \right) = \langle 1, \theta \mid \theta^2 = 0 \rangle$$

for all $n > 2$.

We got the algebra $\mathrm{End}\left(\mathcal{O}_X \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{O}_X \right)$ as the cohomology of a dg-algebra with differential

$$D = [Q_{boundary}, \cdot\,]$$

Here $Q_{boundary} = \begin{pmatrix} 0 & z^{n-1} \\ z & 0 \end{pmatrix}$

In particular we have an A_∞-structure on $\mathrm{End}\left(\mathcal{O}_X \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{O}_X \right)$ which carries additional physical information.

Consider $n = 3$. In this case $m_2(\theta, \theta) = \theta^2 = 0$

What is $m_3(\theta, \theta, \theta) = ?$

In general if we have a dg-algebra and the A_∞-structure on the cohomology is such that

$$m_2 = m_3 = \dots = m_k = 0$$

$\Rightarrow$ m_{k+1} is ~~ ~~ unambiguously defined.

In particular if $m_2 = 0$ $\Rightarrow$

$$m_3([a],[b],[c]) = [\alpha c - (-1)^a a\beta]$$

where

$$\alpha: \quad ab = \partial\alpha \qquad ([a]\cdot[b] = 0)$$

$$\beta: \quad bc = \partial\beta \qquad ([b]\cdot[c] = 0)$$

In our case: $\quad \theta = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

$$\theta^2 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = D\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

Hence $\quad m_3(\theta,\theta,\theta) = \left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right] = -1.$

In general (for arbitrary n) we expect that

$$m_k(\theta,\dots,\theta) = 0 \qquad \text{for all } k < n$$

$$m_n(\theta,\dots,\theta) = -1$$

(This observation, as well as the preceding computation of m_3, is due to Kontsevich)

This suggests that the theory has a degree n superpotential in the open string moduli space (= moduli space of deformations of $\mathcal{O}_X \rightleftarrows \mathcal{O}_X$ in our original dg category)

Recall that a brane in the B-model was

$$E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1$$

with
$$F \circ G = W \cdot id_{E_1} - W_0 \cdot id_{E_1}$$
$$G \circ F = W \cdot id_{E_0} - W_0 \cdot id_{E_0}$$

where W_0 is a constant.

Now one notes:

- branes with different W_0's donot interact (no non-zero morphisms between them)
- the only non-trivial branes correspond to W_0 which is a critical value of W.

This was analyzed by Hori in some cases: Linear models of D-branes ---

In particular: Category of branes breaks into pieces = branes corresponding to the critical values W_0 of W.

Orlov made this mathematically precise:

If $W_0 \in \mathbb{C}$ - any value then the fiber has functions

$$\Gamma(\mathcal{O}(X_{W_0})) = \mathbb{C}[x_1, \ldots, x_n] / W(x) - W_0$$

Now consider

$$\mathcal{D}_{w_0} := D^b(X_{w_0}) / \mathrm{Perf}(X_{w_0})$$

$\Rightarrow$ if $w_0 \neq$ critical value $\Rightarrow$ X_{w_0} - smooth and hence $\mathcal{D}_{w_0} = 0$

~~if w_0 = critical value $\Rightarrow$~~ Orlov showed

$\mathcal{D}_{w_0} \cong$ category of branes for value w_0.

Hence
$$D(\text{B-branes}) = \coprod_{\substack{w_0 - \text{critical} \\ \text{for } W}} \mathcal{D}_{w_0}.$$

<u>Note</u>: One can show that for $X = \mathbb{C}$, $W = z^n$

then all branes are of the form

$$\oplus \left(\text{of} \quad \mathcal{O}_X \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{O}_X \qquad \begin{array}{l} F = z^k \\ G = z^{n-k} \\ k = 1, \dots, n-1 \end{array} \right)$$

This follows from Orlov's theorem and the fact that the singular fibers of W are 0-dimensional and have simple 'singular sheaves' e.g.

$$\mathbb{C}[x]/x^i \qquad i \le n,$$

Example: Let $X = \mathbb{C}^2$

$$W = x^n + xy^2 \quad \text{(D-type minimal model)}$$

what are the branes here?

Note: If we have a general X and $W = W_1 + W_2$ and a brane with differential

$$D = \begin{pmatrix} 0 & G \\ F & 0 \end{pmatrix}$$

we can try to break D into D_1, D_2 s.t.

$$D_1^2 = W_1$$
$$D_2^2 = W_2$$

For this write

$$D = \eta_1 D_1 \otimes 1 + 1 \otimes D_2 \eta_2$$

$$\eta_1^2 = 1 \quad \eta_2^2 = 1 \qquad \eta_1\eta_2 + \eta_2\eta_1 = 0$$

$\Rightarrow \quad D^2 = W.$

$\rightarrow$ can apply this to $W_1 = x^n$ on $\mathbb{C}$
$W_2 = xy^2$ on $\mathbb{C}^2$

in order to compute the branes

Recall: Vafa's computation of the one point function in closed string theory:

$$f \in \mathbb{C}[\vec{x}]/\partial_i W \quad \rightsquigarrow \quad \langle f \rangle = \operatorname{res} \frac{f}{\partial_1 W \cdots \partial_n W}$$

where

$$\operatorname{res} \frac{f}{\partial_1 W \cdots \partial_n W} = \int_\Gamma \frac{f \, dx_1 \wedge \cdots \wedge dx_n}{\partial_1 W \cdots \partial_n W}$$

Γ - Lagrangian cycle

$$|\partial_i W| = \varepsilon_i$$

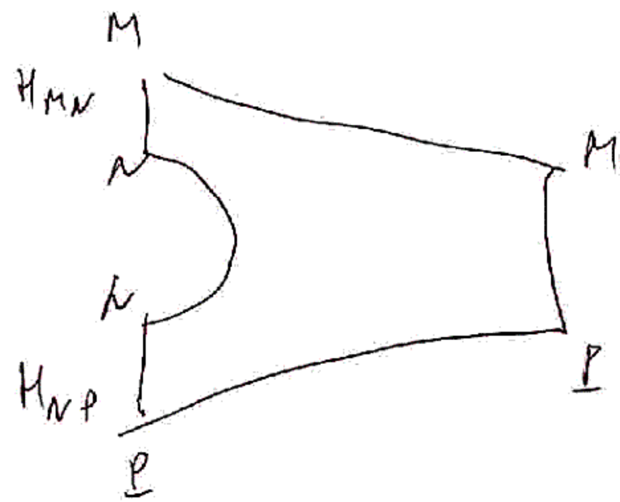
ε_i - small.

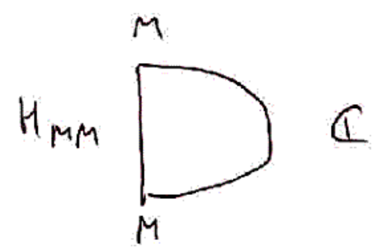
We get a bilinear form

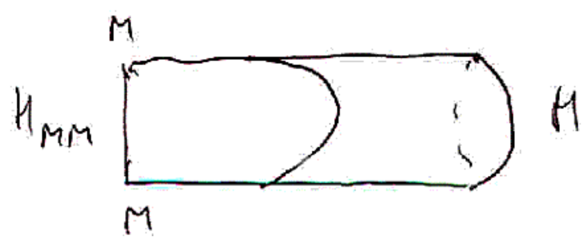
$$\langle f, g \rangle := \langle fg \rangle$$

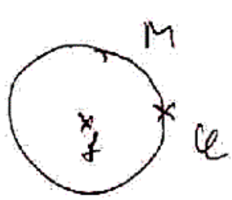
<u>Note</u>: It is non-trivial that this form is non-degenerate.

We would like now to do the same for open strings:









$$D = \begin{pmatrix} 0 & G \\ F & 0 \end{pmatrix}$$

$$D = z_i \Gamma_i + \cdots$$

$$\{\Gamma_i, \Gamma_j\} = Q_{ij}$$ – Clifford algebra relation

$$\Rightarrow \quad \langle f \ell \rangle = \operatorname{res} \frac{f \, \mathrm{Str}\left((\partial D)^n \ell\right)}{\partial_1 W \, \partial_2 W \cdots \partial_n W}$$

8.

Here $(\partial D)^n \sim dx_1 \cdots dx_n$

Since this expectation value was defined in terms of the path integral it will be good if we can independently check that the formula makes sense and has the right properties:

- $\operatorname{res} \dfrac{f \operatorname{str}((\partial D)^n \varphi)}{\partial_1 W \cdots \partial_n W}$ is well defined

Indeed if $\varphi = [D, \alpha] \Rightarrow$ ok

Since

$$D^2 = W$$

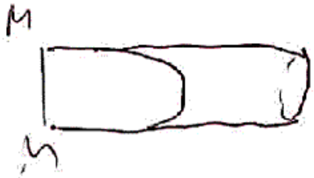
$$dD \cdot D + D \cdot dD = dW$$

- $\langle f \varphi_1 \varphi_2 \rangle \overset{?}{=} \pm \langle f \varphi_2 \varphi_1 \rangle$

$$[D, \varphi_1] = 0$$

$$[dD, \varphi_1] + [D, d\varphi_1] = 0$$

We can use the residue formula to read off the boundary bulk maps:

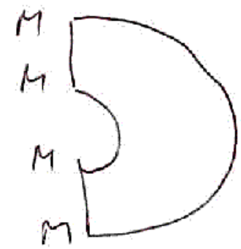
H_{MM} 

$H \rightsquigarrow \varphi \mapsto \operatorname{str}((\partial D)^n \varphi)$

Again we will have the consistency conditions for all this data that need to be checked mathematically:

- $$\langle \varphi_1, \varphi_2 \rangle = \operatorname{res} \frac{\operatorname{str}\left((\partial D)^n \varphi_1 \varphi_2 \right)}{\partial_1 W \cdots \partial_n W}$$

  Should be non-degenerate (this is just the pairing in the endomorphism algebra):



- Cardy condition

If $\alpha \in H_{MM}$, $\beta \in H_{NN}$ $\Rightarrow$ ~~Should~~ get $f_{\alpha\beta} : H_{MN} \to H_{MN}$, and the cardy condition expresses $\operatorname{str} f_{\alpha\beta}$ in terms of inner product of bulk states corresponding to α, β.

For example, In the case $\alpha = 1$, $\beta = 1$ then should get

$$\chi(H_{MN}) = \operatorname{res} \frac{\operatorname{str}\left((\partial D_M)^n \right) \operatorname{str}\left((\partial D_N)^n \right)}{\partial_1 W \cdots \partial_n W}$$

This can be checked in examples.

We would like to understand how the closed strings can be identified with the Hochschild cohomology of the category of branes in this case.

Start with a brane $E_0 \underset{G}{\overset{F}{\rightleftarrows}} E_1$

$$F \circ G = G \circ F = W$$

<u>Recall</u>: a dga (A, d): $d(a_1 a_2) = da_1 \cdot a_2 \pm a_1 da_2$

$$d^2 = 0$$

a module over (A, d): M, $D_M : M \to M$

s.t. $D_M^2 = 0$

$$D_M(am) = da \cdot m \pm a D_M m$$

Similarly we have: cdg algebra

$$(A,\ d : A \to A,\ B) + \text{Leibnitz}$$

$$d^2 = [B, \cdot]$$

cdg module

$$(M, D_M) + \text{Leibnitz}$$

$$D_M^2 = B \cdot$$

<u>Note</u>: A cdg algebra is not a CDG module over itself in general.

Now consider $(A = \mathbb{C}[x_1, \dots, x_n],\ d = 0,\ W)$ as a cdg algebra

$\Rightarrow$ the branes are just the derived category of cdg modules over this cdg algebra.

The HH of D^b (cdg mod $-(A, 0, W)$) is simply the HH of $(A, 0, W)$

Explicitly
$$(A, 0, W) \otimes (A, 0, W)^{op}$$
$$\parallel$$
$$(A, 0, W) \otimes (A, 0, -W)$$
$$\parallel$$
$$(A \otimes A, 0, W \otimes 1 - 1 \otimes W)$$

$$\Rightarrow HH^{\bullet}(A, 0, W) := \mathrm{Ext}^{\bullet}_{(A,0,W) \otimes (A,0,W)^{op}}\left((A,0,W),(A,0,W)\right)$$

<u>Note</u>: $(A, 0, W)$ is now a cdg module over $(A, 0, W) \otimes (A, 0, W)^{op}$ since
$$d^2 a = Wa - aW = 0.$$

To compute this HH look e.g. at
$$X = \mathbb{C}$$
$$W = W(x)$$

Then can look at $\mathbb{C}^2$ (x, y)
$$W(x, y) = W(x) - W(y)$$

$$D = \begin{pmatrix} 0 & \frac{W(x)-W(y)}{x-y} \\ x-y & \end{pmatrix}$$

this deformation of the Koszul resolution of the diagonal in $\mathbb{C}^2$ has been ~~people~~ found by Lev Rozansky.

If now we look at a module

$\Rightarrow$ have

$$\begin{pmatrix} \alpha(x,y) & 0 \\ 0 & \alpha(x,y) \end{pmatrix}$$

$\alpha \in \mathbb{C}[x,y]$ and

$$\alpha(x,y) \sim \alpha(x,y) + * (x-y) + * \frac{W(x)-W(y)}{x-y}$$

$\Rightarrow$ get the Jacobi ring of $W(x)$.

~~(this computation was explained to me by~~ is

Now we can recover Vafa's formula

$$\mathrm{res}\ \frac{\mathrm{str}\left((\partial D)^2 \varphi\right)}{\partial_x W\, \partial_y W} = \mathrm{res}\ \frac{\alpha(x,x)}{\partial_x W}$$