Gauge-Invariant Couplings of Noncommutative Branes to Ramond-Ramond Backgrounds

Sunil Mukhi

Tata Institute of Fundamental Research, Mumbai

1. **Introduction**
2. **Noncommutative RR Couplings (in DBI Approximation)**
3. **Gauge Invariant Operators on Noncommutative D-Branes**
4. **Gauge Invariant RR Couplings and the Seiberg-Witten Map**
5. **Transverse Scalars**
6. **Non-BPS Branes and Tachyons**
7. **Conclusions**

Based on:

S.M. and N.V. Suryanarayana, hep-th/0009101

S.M. and N.V. Suryanarayana, hep-th/0104045

*The results of Sec.***4** *were independently obtained in:*

Y. Okawa and H. Ooguri, hep-th/0104036

H. Liu and J. Michelson, hep-th/yymmnnn

1. Introduction

- The couplings of BPS D-branes to Ramond-Ramond fields hold the key to their stability and dynamics.

- A collection of N BPS Dp-branes couples to all RR forms, via the formula:

$$\mu_p \int \mathrm{Tr}\left(P[e^{i\, \mathrm{i}_\Phi \mathrm{i}_\Phi} \sum_n C^{(n)}\, e^B]\, e^F\right)$$

  Here, i_Φ is an inner product between the transverse matrix-valued scalar and a form index: $\mathrm{i}_\Phi C \sim \Phi^i C_{i\ldots}$.

- We will ignore gravitational couplings localised on the brane.

- In general, the above formula has derivative corrections. However, there are some cases where the correction is known to be absent or can be absorbed into a field redefinition.

- Turn on a constant B field. We would like to know what is the analogue of the above formula in noncommutative variables.

- This is interesting for various reasons:

(i) Much insight was gained by asking the same question for the DBI part of the brane action.

(ii) The Chern-Simons action tells us about interesting physical effects such as anomaly inflow, bound states, the dielectric effect etc. How are these effects desribed in noncommutative variables?

(iii) The investigation reveals interesting mathematical identities in noncommutative field theory.

- We will address this problem in two parts: constant fields (analogue of the DBI approximation), and spatially varying fields.

- Generalisations to unstable non-BPS branes will also be discussed.

- The starting point is matrix theory. We will make heavy use of the ideas in:
  J. Liu, hep-th/0011125
  S. Das and S. Trivedi, hep-th/0011131

2. Noncommutative RR Couplings (in DBI Approximation)

- Let us start with N D-instantons, whose RR couplings are:

$$\mu_{-1}\,\mathrm{Tr}\left(e^{i\;\mathrm{i}_\Phi \mathrm{i}_\Phi}\sum_n C^{(n)}\right)$$

Here we keep the 0-form part.

- For infinite N, we can build up a Euclidean 9-brane by choosing the classical configuration

$$\Phi^i = x^i, \qquad [x^i, x^j] = i\,\theta^{ij}$$

This configuration carries the charge of all p-branes if θ^{ij} has maximal rank. Thus it is a D9-brane with induced lower-brane charges.

- Parametrising the fluctuations by

$$X^i = x^i + \theta^{ij}\hat{A}_j(x)$$

it is easy to check that

$$[X^i, X^j] = i\,(\theta^{ij} - \theta^{ik}\hat{F}_{kl}\theta^{lj}) \equiv i\,Q^{ij}$$

where

$$\hat{F}_{ij} \equiv \partial_i\hat{A}_j - \partial_j\hat{A}_i - i\,[\hat{A}_i, \hat{A}_j]_*$$

- Here, $*$ denotes the Moyal product:

$$f(x) * g(x) \equiv f(x)\, e^{\frac{i}{2}\overleftarrow{\partial_i}\theta^{ij}\overrightarrow{\partial_j}}\, g(x)$$

- Now in the exponential pre-factor we can use

$$\mathrm{i}_\Phi \mathrm{i}_\Phi \to \frac{i}{2}\mathrm{i}_Q$$

where $\mathrm{i}_Q C \sim Q^{ji} C_{ij\ldots}$. Then the D-instanton coupling is

$$\mu_{-1}\,\mathrm{Tr}\, e^{-\frac{1}{2}\mathrm{i}_Q} \sum_n C^{(n)}$$

- As Q^{ij} is a 10×10 antisymmetric matrix, we can define its Pfaffian by

$$\mathrm{Pf}Q \equiv \frac{1}{2^5 5!}\,\epsilon_{i_1\ldots i_{10}}\, Q^{i_1 i_2} \ldots Q^{i_9 i_{10}}$$

It is easy to see that

$$\mathrm{Tr}\, e^{-\frac{1}{2}\mathrm{i}_Q} \sum_n C^{(n)} = \mathrm{Tr}\;\mathrm{Pf}Q \sum_n C^{(n)}\, e^{Q^{-1}}$$

where

$$Q^{-1} \equiv \frac{1}{2} Q^{-1}_{ij}\, dx^i \wedge dx^j$$

and Q^{-1}_{ij} is the matrix inverse of Q_{ij}. On the RHS, the rule is to expand the exponential in wedge products, and take the 10-form part of the expression.

- Thus the RR couplings on the D9-brane are, for constant fields, simply given by:

$$\mu_{-1}\mathrm{Tr}\ \mathrm{Pf}Q\sum_n C^{(n)}\, e^{Q^{-1}} = \mu_9 \int_x \frac{\mathrm{Pf}Q}{\mathrm{Pf}\theta}\sum_n C^{(n)}\, e^{Q^{-1}}$$

where the right side follows from:

$$\mathrm{Tr}\ \to \int d^{p+1}x \frac{1}{(2\pi)^{\frac{p+1}{2}}\mathrm{Pf}\theta}$$

- To see the equivalence to the commutative couplings, note that

$$\begin{aligned} Q^{-1}_{ij} &= \left(\frac{1}{\theta - \theta\hat{F}\theta}\right)_{ij} \\ &= \left(\frac{1}{\theta}\right)_{ij} + \left(\hat{F}\frac{1}{1-\theta\hat{F}}\right)_{ij} \\ &= B_{ij} + F_{ij} \end{aligned}$$

where

$$B \equiv \frac{1}{\theta}, \qquad F = \hat{F}\frac{1}{1-\theta\hat{F}}$$

The first equation defines the B-field in the "background-independent" formulation of noncommutativity. The second is the Seiberg-Witten map for constant $U(1)$ field strengths.

- Thus the noncommutative CS action becomes

$$\mu_9 \int_x \frac{\mathrm{Pf}Q}{\mathrm{Pf}\theta} \sum_n C^{(n)}\, e^{B+F}$$

which is equal to the original commutative CS action upto the factor

$$\frac{\mathrm{Pf}Q}{\mathrm{Pf}\theta} = 1 + \text{total derivative terms}$$

for constant field strengths.

(Compare with

$$T_9 \int \frac{\mathrm{Pf}Q}{\mathrm{Pf}\theta} \sqrt{\det\left(g_{ij} + Q^{-1}_{ij}\right)}$$

for the DBI case.)

- For lower-dimensional BPS Dp-branes, the noncommutative CS term for constant fields is:

$$\mu_p \int_x \frac{\mathrm{Pf}Q}{\mathrm{Pf}\theta} P\left[e^{i(\mathrm{i}_\Phi * \mathrm{i}_\Phi)} \sum_n C^{(n)}\right] e^{Q^{-1}}$$

Here $Q \equiv \theta - \theta \hat{F} \theta$ is of rank $p+1$ rather than 10, and Φ^i are the $9-p$ transverse scalars. Note the $*$ product between these scalars.

3. Gauge Invariant Operators on Noncommutative D-Branes

- For spatially varying RR fields and gauge field strengths, the above couplings are not gauge invariant. We must introduce open Wilson lines to make gauge invariant couplings.

- The open Wilson line for a noncommutative gauge theory is defined as follows:

$$W(x,C) \equiv P_* \exp\left(i \int_0^1 d\tau \frac{\partial \xi^i(\tau)}{\partial \tau} \hat{A}_i(x + \xi(\tau)) \right)$$

where $\xi^i(\tau) = \theta^{ij} k_j \tau$. Under noncommutative gauge transformations, it transforms as:

$$W(x,C) \to U(x^i)\, W(x,C)\, U^\dagger(x^i + \theta^{ij} k_j)$$

Since $U^\dagger(x^i + \theta^{ij} k_j) * e^{ik.x} = e^{ik.x} * U^\dagger(x^i)$, it follows that the Fourier transform:

$$\widetilde{W}(k,C) = \int \frac{d^{p+1}x}{(2\pi)^{p+1}}\, W(x,C) * e^{ik.x}$$

is gauge-invariant.

- Given local operators $\mathcal{O}_I(x)$ transforming in the adjoint, one can formally make a composite local operator

$$\prod_{I=1}^{n} \mathcal{O}_I(x)$$

Now, to this composite we can associate a gauge-invariant operator of momentum k^i, obtained by "smearing" the original operators along an open Wilson line:

$$\mathcal{Q}(k) = \int \frac{d^{p+1}x}{(2\pi)^{p+1}} \left(\prod_{I=1}^{n} \int_0^1 d\tau_I \right) \times$$
$$P_* \left[W(x,C) \prod_{I=1}^{n} \mathcal{O}_I(x+\xi(\tau_I)) \right] * e^{ik.x}$$

- As a result of the various τ-integrals, one finds that new kinds of $*$ products arise when expanding this. Indeed, we have

$$\mathcal{Q}(k) = \sum_{m=0}^{\infty} \int \frac{d^{p+1}x}{(2\pi)^{p+1}} \; \mathcal{Q}_m(x)\, e^{ik.x}$$

where

$$\mathcal{Q}_m(x) = \frac{1}{m!}(\theta\partial)^{i_1} \ldots (\theta\partial)^{i_m} \times$$
$$\langle \mathcal{O}_1(x), \ldots, \mathcal{O}_n(x), \hat{A}_{i_1}(x), \ldots, \hat{A}_{i_m}(x) \rangle_{*_{m+n}}$$

- We see that new commutative, nonassociative products called $*_n$ have appeared for every n. The simplest one is $*_2$, defined as:

$$\langle f(x), g(x)\rangle_{*_2} \equiv f(x)\frac{\sin(\frac{1}{2}\overleftarrow{\partial_i}\theta^{ij}\overrightarrow{\partial_j})}{\frac{1}{2}\overleftarrow{\partial_i}\theta^{ij}\overrightarrow{\partial_j}}g(x)$$

- These products are quite ubiquitous, because of identities like:

$$\int_0^1 d\tau(*_\tau + \mathrm{c.c}) = *_2$$

where $*_\tau$ is the Moyal product with parameter $\theta^{ij}\tau$.

- We will apply this procedure to find gauge-invariant RR couplings.

4. Gauge Invariant RR Couplings and the Seiberg-Witten Map

- We have seen that for constant fields on a p-brane, the highest-rank RR form $C^{(p+1)}$ couples to the operator

$$\frac{\mathrm{Pf}Q}{\mathrm{Pf}\theta} = \sqrt{\det(1-\theta\hat{F})}$$

It follows that the gauge-invariant coupling of this RR form is:

$$\mu_p\,\epsilon^{i_1\cdots i_{p+1}}\,\tilde{C}^{(p+1)}_{i_1\cdots i_{p+1}}(-k)\times$$
$$\int \frac{d^{p+1}x}{(2\pi)^{p+1}}\;L_*[\sqrt{\det(1-\theta\hat{F})}\;W(x,C)] * e^{ik.x}$$

where the symbol L_* stands for the smearing and path-ordering of everything inside the bracket.

- Now, however, this must agree precisely with the commutative coupling, not just up to total derivative terms. The commutative coupling is simply

$$\mu_p\,\epsilon^{i_1\cdots i_{p+1}}\,\tilde{C}^{(p+1)}_{i_1\cdots i_{p+1}}(-k)\;\delta^{(p+1)}(k)$$

- Thus we predict the identity:

$$\delta^{(p+1)}(k) = \int \frac{d^{p+1}x}{(2\pi)^{p+1}} L_*[\sqrt{\det(1-\theta\hat{F})}\, W(x,C)]*e^{ik.x}$$

- This is easily checked to lowest order in $\hat{A}$:

$$\sqrt{\det(1-\theta\hat{F})} \sim 1 - \frac{1}{2}\mathrm{tr}\,(\theta\hat{F})$$

  and

$$W(x,C) \sim 1 + \theta^{ij}\partial_j\hat{A}_i$$

  We have checked it to $\mathcal{O}(\hat{A}^3)$, and have also given a recursive proof to all orders for the special case where θ^{ij} has rank 2.

- Recalling, that $\mathsf{X}^i \equiv \mathsf{x}^i + \theta^{ij}\hat{A}_j$, it is easy to show that our identity takes the following neat form:

$$\mathrm{tr}\left(\mathrm{Pf}[\mathsf{x}^i,\mathsf{x}^j]\; e^{ik.\mathsf{x}}\right) = \mathrm{Str}\left(\mathrm{Pf}[\mathsf{X}^i,\mathsf{X}^j]\; e^{ik.\mathsf{X}}\right)$$

  where Str is the symmetrised trace.

- Next let us consider the coupling of the brane to the RR form $C^{(p-1)}$. Recall that for constant fields the coupling was:

$$\int \sqrt{\det(1-\theta\hat{F})}\, C^{(p-1)} \wedge Q^{-1}$$

With spatially varying fields the corresponding coupling must therefore be:

$$\mu_p\, \epsilon^{i_1\ldots i_{p+1}}\, \widetilde{C}^{(p-1)}_{i_1\ldots i_{p-1}}(-k)\times$$

$$\int \frac{d^{p+1}x}{(2\pi)^{p+1}} L_*\Big[\sqrt{\det(1-\theta\hat{F})}\,(Q^{-1})_{i_p i_{p+1}} W(x,C)\Big] * e^{ik.x}$$

- Compare this with the commutative coupling:

$$\mu_p\, \epsilon^{i_1\ldots i_{p+1}}\, \widetilde{C}^{(p-1)}_{i_1\ldots i_{p-1}}(-k) \int \frac{d^{p+1}x}{(2\pi)^{p+1}}\, (B+F)_{i_p i_{p+1}}(x)\, e^{ik.x}$$

Recall that we had

$$Q^{-1} = B + \hat{F}(1-\theta\hat{F})^{-1}$$

Using the previous identity, we can cancel off the terms proportional to B on both sides. That leaves a relation between commutative and non-commutative field strengths.

- Thus we obtain the new identity:

$$\tilde{F}_{ij}(k) = \int \frac{d^{p+1}x}{(2\pi)^{p+1}} \Big\{ L_*[\sqrt{\det(1-\theta\hat{F})}\,(\hat{F}(1-\theta\hat{F})^{-1})_{ij}\,W(x,C)] * e^{ik.x}\Big\}$$

  We recognise this as a closed-form expression for the $U(1)$ Seiberg-Witten map. It was conjectured recently by Liu (who extended the conjecture to $U(N)$).

- This comparison between commutative and non-commutative couplings makes sense despite the possibility of derivative corrections on both sides, because it is known that the derivative corrections can be absorbed into a field redefinition in this case. The SW map is anyway defined only up to a field redefinition.

5. Transverse Scalars

- So far we have largely ignored the dependence of the CS action on the transverse scalars. These make an appearance in the commutative CS action in several ways:

(i) The dependence of C:

$$C^{(n)}_{a_1 \ldots a_n} = C^{(n)}_{a_1 \ldots a_n}(\Phi) = e^{\Phi^i \partial_i} C^{(n)}_{a_1 \ldots a_n}(0)$$

(ii) The pullback of C:

$$P[C^{(n)}_{a_1 \ldots a_n}] = C^{(n)}_{a_1 \ldots a_n} + D_{a_1} \Phi^{i_1} C^{(n)}_{i_1 a_2 \ldots a_n} + \cdots$$

(iii) Commutator terms, involving $e^{i\, \mathrm{i}_\Phi \mathrm{i}_\Phi}$.

- This dependence on scalars gives rise to interesting generalisations of the previous results. I will focus on two of them.

- Consider first the dependence of C. In momentum space, the commutative coupling incorporating this is:

  $$\mu_p \, \epsilon^{i_1 \ldots i_{p+1}} \widetilde{C}^{p+1}_{i_1 \ldots i_{p+1}}(-k, q) \, \{ \delta^{(p+1)}(k) + i q_a \widetilde{\Phi}^a(k) + \cdots \}$$

  where q is the momentum transverse to the brane.

- In the noncommutative description, the dependence on q arises through a generalisation of the open Wilson line which now becomes:

  $$W'(x, C) = P_* \exp \left\{ i \int_0^1 d\tau \Big[\frac{\partial \xi^i}{\partial \tau} \hat{A}_i(x + \xi) + q_a \hat{\Phi}^a(x + \xi) \Big] \right\}$$

- Like the Wilson line at $q = 0$, this too has an expansion in terms of $*_n$ products:

  $$W'(x, C) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} \times$$
  $$(\theta\partial)^{i_1} \ldots (\theta\partial)^{i_k} \, (iq)_{a_{k+1}} \cdots (iq)_{a_m} \times$$
  $$\langle \hat{A}_{i_1}(x), \ldots, \hat{A}_{i_k}(x), \hat{\Phi}^{a_{k+1}}(x), \ldots, \hat{\Phi}^{a_m}(x) \rangle_{*_m}$$

- This gives a prescription to find the Seiberg-Witten map for the transverse scalars. Write down the noncommutative coupling of the RR field to this Wilson line, expand to linear order in q and compare with the commutative one.

- Explicitly, to third order in open-string fields, we find:

$$\begin{aligned}
\widetilde{\Phi}^a(k) = \hat{\Phi}^a &+ \theta^{ij}\langle \hat{A}_i, \partial_j \hat{\Phi}^a \rangle_{*_2} \\
&+ \frac{i}{2}\theta^{kl}\langle \hat{A}_l, [\hat{A}_k, \hat{\Phi}^a]_* \rangle_{*_2} \\
&+ \frac{1}{2}\theta^{ij}\theta^{kl}\langle \hat{A}_i, \partial_l \hat{A}_j, \partial_k \hat{\Phi}^a \rangle_{*_3} \\
&- \theta^{ij}\theta^{kl}\langle \hat{A}_i, \partial_j \hat{A}_l, \partial_k \hat{\Phi}^a \rangle_{*_3} \\
&+ \frac{1}{2}\theta^{ij}\theta^{kl}\langle \hat{A}_j, \hat{A}_l, \partial_i \partial_k \hat{\Phi}^a \rangle_{*_3} \\
&+ \cdots
\end{aligned}$$

To this order, this agrees with the SW map obtained from that of $\hat{A}$ by dimensional reduction.

- For the next application involving transverse scalars, we consider the dependence that comes from commutators. Consider the specific example of a (Euclidean) D-string coupling to the RR 4-form:

$$\mathrm{tr}\left(\frac{1}{2!2^2}\left(-i[\phi^{i_1},\phi^{i_2}]\right)\left(-i[\phi^{i_3},\phi^{i_4}]\right)C^{(4)}_{i_1 i_2 i_3 i_4}\right)$$

Now insert $\phi^1 = X^1, \phi^2 = X^2$ as we did earlier, making a D-string. The remaining ϕ^a are renamed $\hat{\Phi}^a$, representing transverse coordinates to the D-string.

- Thus we find that the operator coupling to $\frac{1}{2!2}\epsilon_{ij}\, C^{(4)}_{12ab}$ is the trace of:

$$\left(-i[X^i,X^j]\right)\left(-i[\hat{\Phi}^a,\hat{\Phi}^b]\right)-\left(-i[X^i,\hat{\Phi}^a]\right)\left(-i[X^j,\hat{\Phi}^b]\right)$$

Making the replacements

$$\begin{aligned}-i[X^1,X^2] &= Q^{12} = \theta^{12}(1+\theta^{12}\hat{F}_{12})\\ [X^i,\hat{\Phi}^a] &= i\theta^{ij}D_j\hat{\Phi}^a\end{aligned}$$

the operator turns into:

$$\theta^{12}\left\{(1+\theta^{12}\hat{F}_{12})(-i[\hat{\Phi}^a,\hat{\Phi}^b]) + \theta^{ij}D_j\hat{\Phi}^a D_i\hat{\Phi}^b\right\}$$

- Now we can smear this over the open Wilson line to find the operator coupling to $C_{12ab}(-k)$. However, a single D-string with a B-field does not couple to the RR 4-form! Hence we get the new identity:

$$\int \frac{d^{p+1}x}{(2\pi)^{p+1}} L_*\big[W'(x,C) \times \\ \big((1+\theta^{12}\hat{F}_{12})(-i[\hat{\Phi}^a,\hat{\Phi}^b]) + \theta^{ij} D_j\hat{\Phi}^a D_i\hat{\Phi}^b\big)\big] * e^{ik.x} = 0$$

- This can be thought of as an analogue of the earlier topological identity which involved only the gauge fields. In this case, the complicated expression at left has the same value at any $\hat{\Phi}$ that it has at $\hat{\Phi} = 0$. We have proved this recursively to all orders when $q_a = 0$.

- Note that in the DBI approximation, $\hat{\Phi}$ is linear in x and in this case,

$$-i[\hat{\Phi}^a,\hat{\Phi}^b] + \theta^{ij} D_j\hat{\Phi}^a D_i\hat{\Phi}^b = 0$$

  This only leaves the contribution proportional to $\hat{F}$, which is a total derivative.

- If we make N D-strings from matrices, the above identity will generalise, with the RHS being the known coupling of N D-strings to the RR 4-form.

6. Non-BPS Branes and Tachyons

- A single non-BPS Dp-brane in type II string theory has RR couplings like

$$\frac{\mu_{p-1}}{2T_0}\int dT \wedge \sum_n C^{(n)} e^{B+F}$$

where T is the tachyon and T_0 is its value at the minimum of its potential. There are nonlinear terms too, but we will ignore them.

- The first step, extension to the noncommutative case for constant fields, now requires that we find a replacement for dT. The obvious choice would be

$$\partial_i T \to D_i T = -i\theta^{-1}_{ij}[X^j, T]$$

but in fact this is incomplete. We have argued that the correct replacement is

$$\partial_i T \to \mathcal{D}_i T = -iQ^{-1}_{ij}[X^j, T]$$

which includes new couplings to the gauge field. This is background-independent in the sense of Seiberg-Witten.

- Thus, for example, the CS action for a noncommutative unstable D9-brane would be

$$\frac{\mu_8}{2T_0}\int \mathcal{D}T \wedge \sum_n C^{(n)} e^{Q^{-1}}$$

- We analysed this coupling from the point of view of brane decay via noncommutative solitons.

- One immediate result is that a noncommutative soliton, despite being a radial lump, does not carry RR charge even locally (because $[X^i_{cl}, T_{cl}] = 0$).

- Following the same procedure as described in the rest of this talk, we can write down the RR coupling of the non-BPS brane for spatially varying fields:

$$\frac{\mu_{p-1}}{2T_0}\,\epsilon^{i_1 i_2\cdots i_{p+1}}\,\tilde{C}^{(p)}_{i_2\cdots i_{p+1}}(-k)\int \frac{d^{p+1}x}{(2\pi)^{p+1}} \times$$
$$L_*\Big[\sqrt{\det(1-\theta\hat{F})}\,\mathcal{D}_{i_1}\hat{T}(x)\,W(x,C)\Big] * e^{ik.x}$$

- Comparison then leads to the SW map for the tachyon, which has been studied to low orders before.

7. Conclusions

- Noncommutative brane couplings to constant RR fields can be conveniently found using matrix theory. The couplings found previously by van Raamsdonk, Taylor, Myers are elegantly expressed in NC language.

- The generalisation to spatially varying fields also comes out naturally from matrix theory, with the introduction of open Wilson lines.

- Comparison of commutative and noncommutative RR couplings is a powerful source of identities including a closed form for the Seiberg-Witten map. Additional striking mathematical results may be encoded here.

- For the future: description-dependence and disk amplitudes (Liu-Michelson, Suryanarayana)...

- The justification for what we have done is much less strong for non-BPS branes. However, things do work out as expected, which is encouraging.