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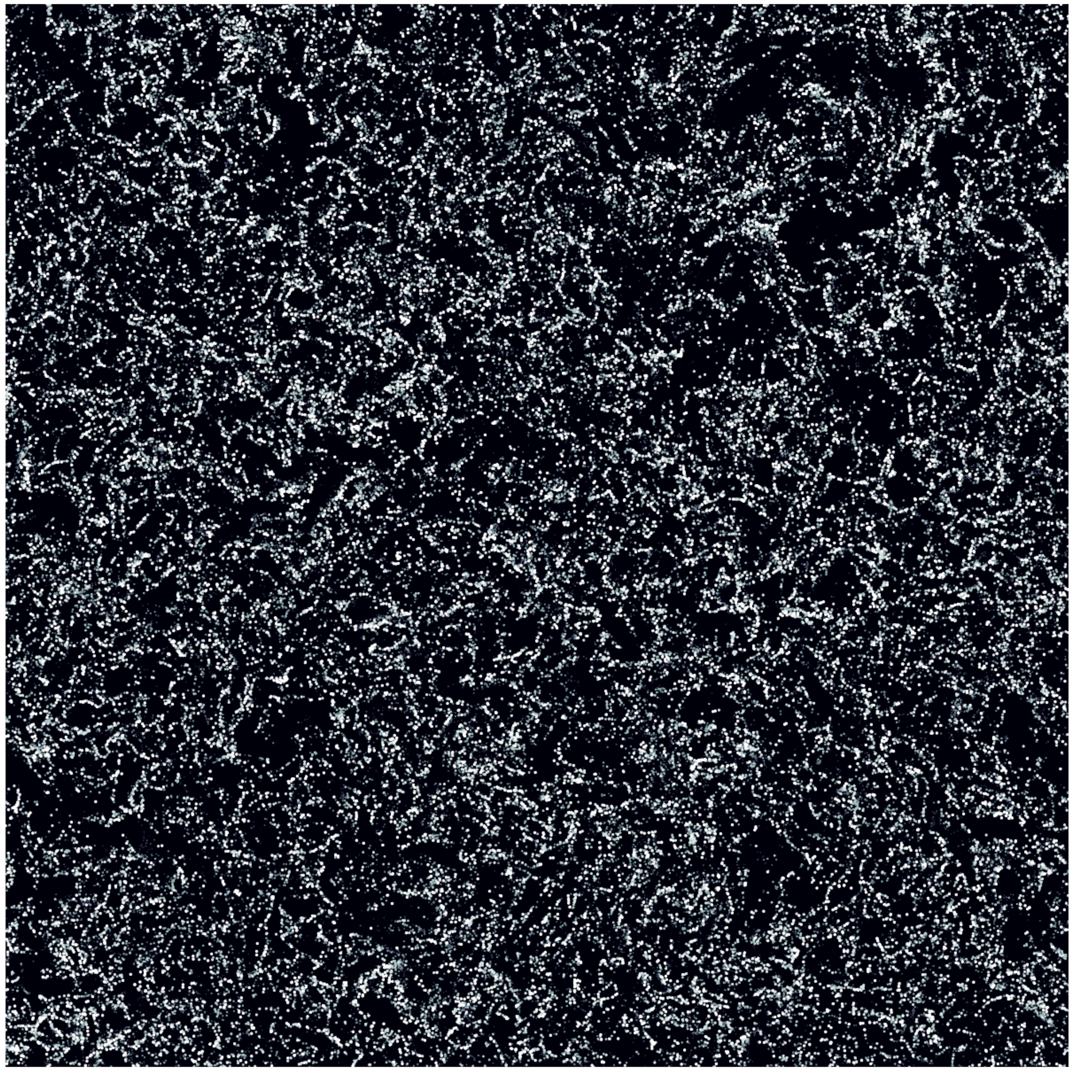
Turbophoresis of heavy inertial particles in statistically homogeneous flow



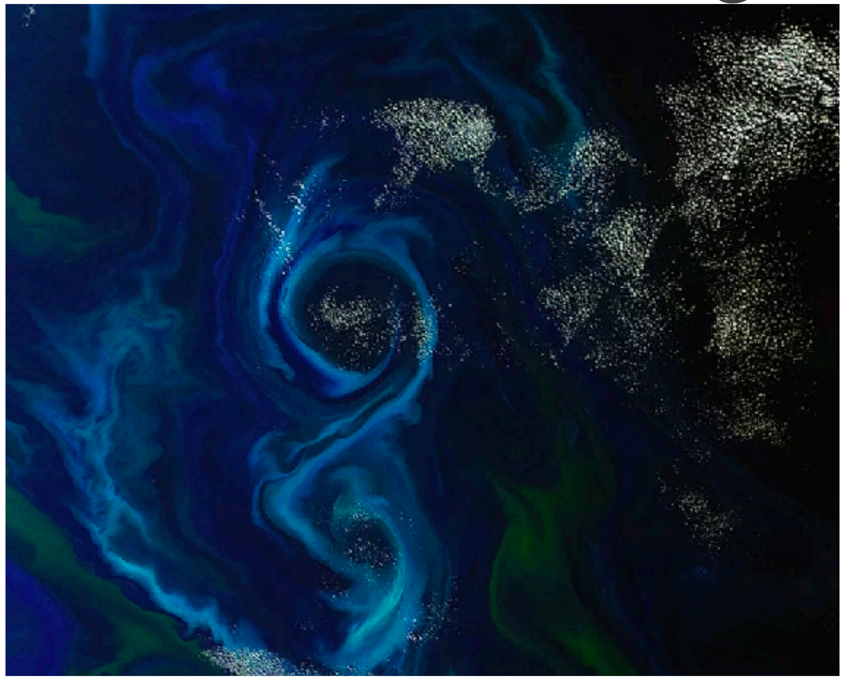
joint work with ***Robin Vallée*** *(Cemef, Mines Paris & Schneider Electric)*

Unifying perspectives on geophysical and environmental multiphase flows – KITP, October 2022

Disperse particles in turbulence

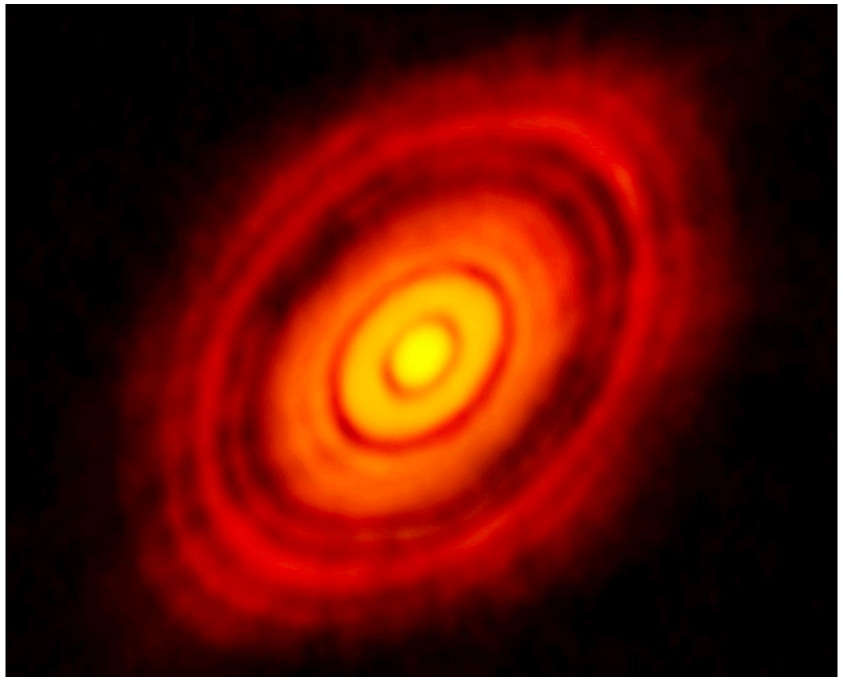
Sediments

Ocean biomixing



Warm clouds

Planet formation



How to predict concentrations in the inertial range of turbulence?

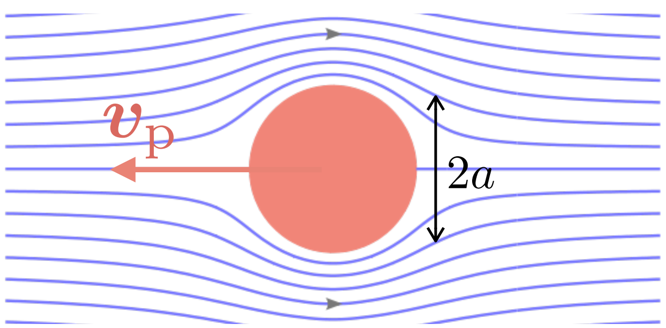
Heavy inertial particles

- Incompressible turbulence

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = -\frac{1}{\rho_\mathrm{f}}\nabla p + \nu\,\nabla^2\boldsymbol{u} + \boldsymbol{f}_\mathrm{ext} \quad \text{with} \quad \nabla\cdot\boldsymbol{u} = 0$$

- **Particles**: small, rigid, heavy, dilute with moderate slip

$a \ll \ell_\mathrm{f} \approx \eta \qquad \rho_\mathrm{p} \gg \rho_\mathrm{f} \qquad Re_\mathrm{p} = \dfrac{|\boldsymbol{v}_\mathrm{p} - \boldsymbol{u}|\,a}{\nu} \ll 1$



$$\frac{\mathrm{d}\boldsymbol{v}_\mathrm{p}}{\mathrm{d}t} = -\frac{1}{\tau}\left[\boldsymbol{v}_\mathrm{p} - \boldsymbol{u}(\boldsymbol{x}_\mathrm{p}, t)\right]$$

Stokes drag, no settling

Response time $\quad \tau = \dfrac{2\,\rho_\mathrm{p}\,a^2}{9\,\rho_\mathrm{f}\,\nu}$

- **Dimensionless parameters:**

Fluid inertia $\quad Re = \dfrac{U\,L}{\nu} = \left(\dfrac{L}{\eta}\right)^{3/4}$

Particle inertia $\quad St = \dfrac{\tau}{\tau_\eta} = \dfrac{2}{9}\dfrac{\rho_\mathrm{p}}{\rho_\mathrm{f}}\left(\dfrac{a}{\eta}\right)^2$

Dissipative scales

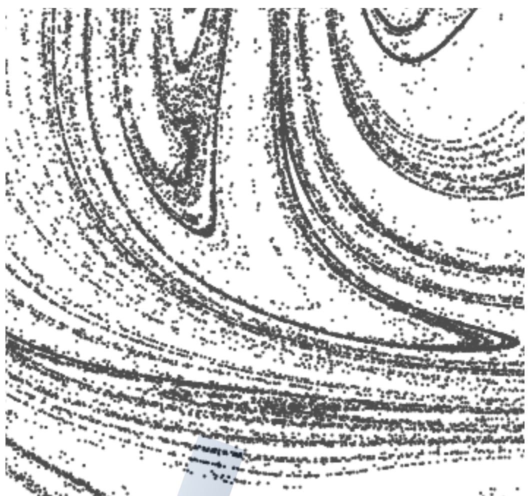
$$\eta = \nu^{3/4}/\varepsilon^{1/4}$$
$$\tau_\eta = \nu^{1/2}/\varepsilon^{1/2}$$

Volume fraction $\Phi_\mathrm{v} \ll 1$ no interactions, no feedback

Particle clustering

$\ell \ll \eta$

fractal distribution at
dissipative scales

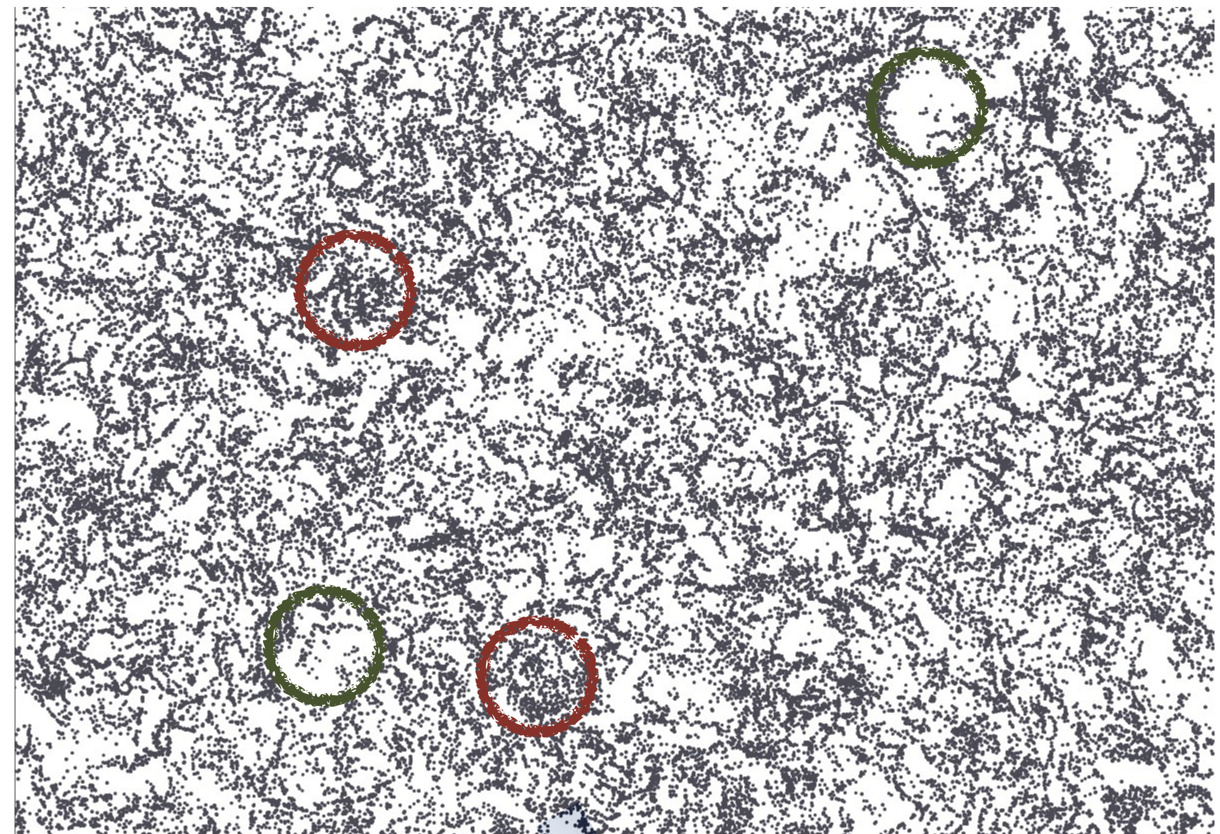


dissipative dynamics:
attractor, SRB measure

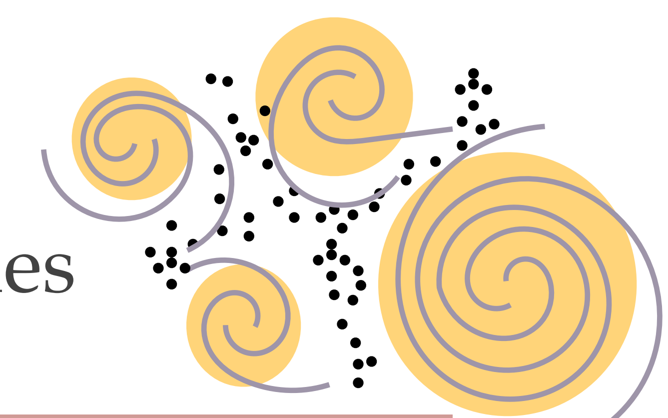
stretching & folding in phase space
+ projection onto configurations

$\ell \gg \eta$

clusters and **voids** in the inertial range



Ejection from eddies

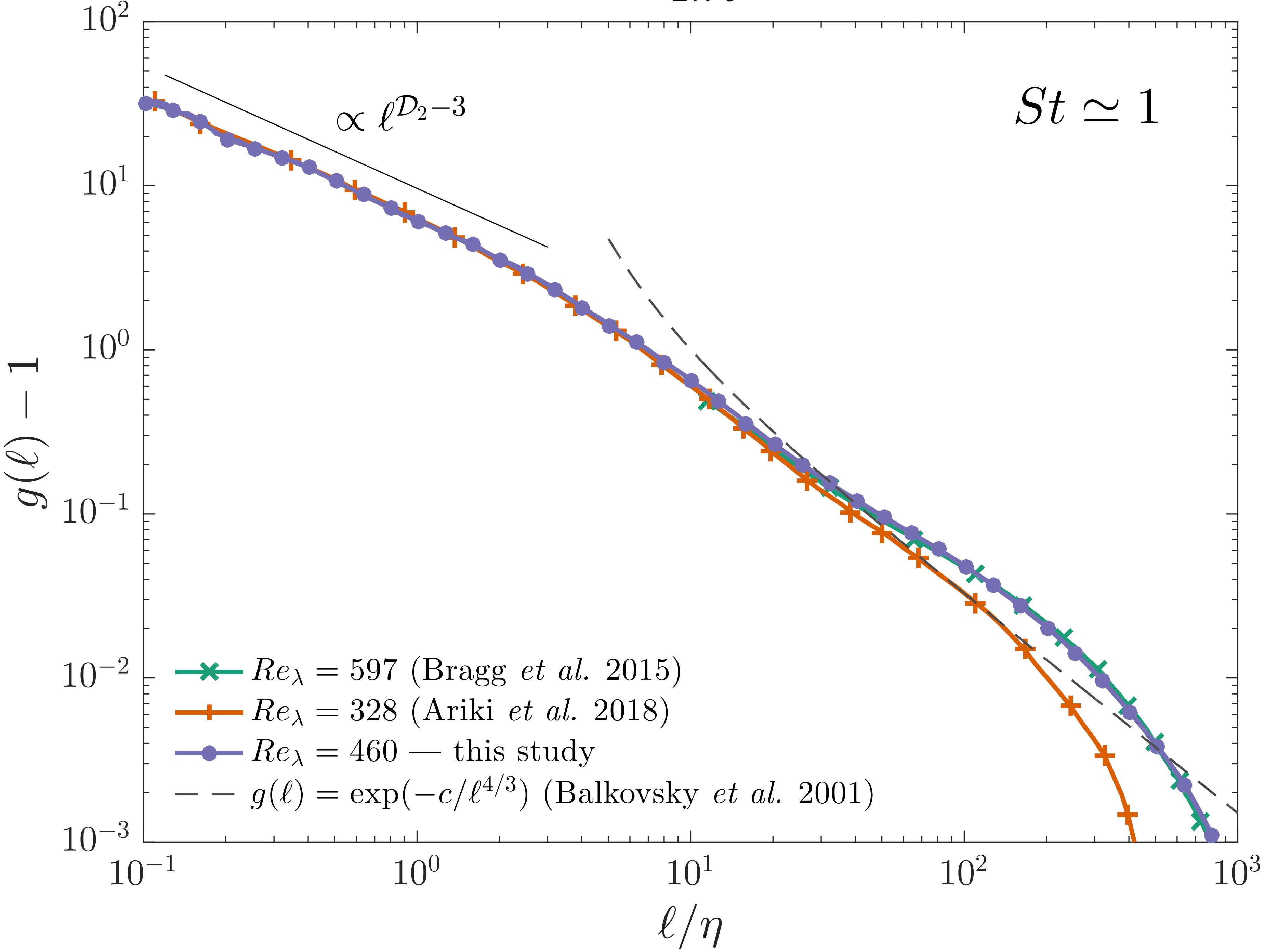


How to model and quantify?

Radial distribution function

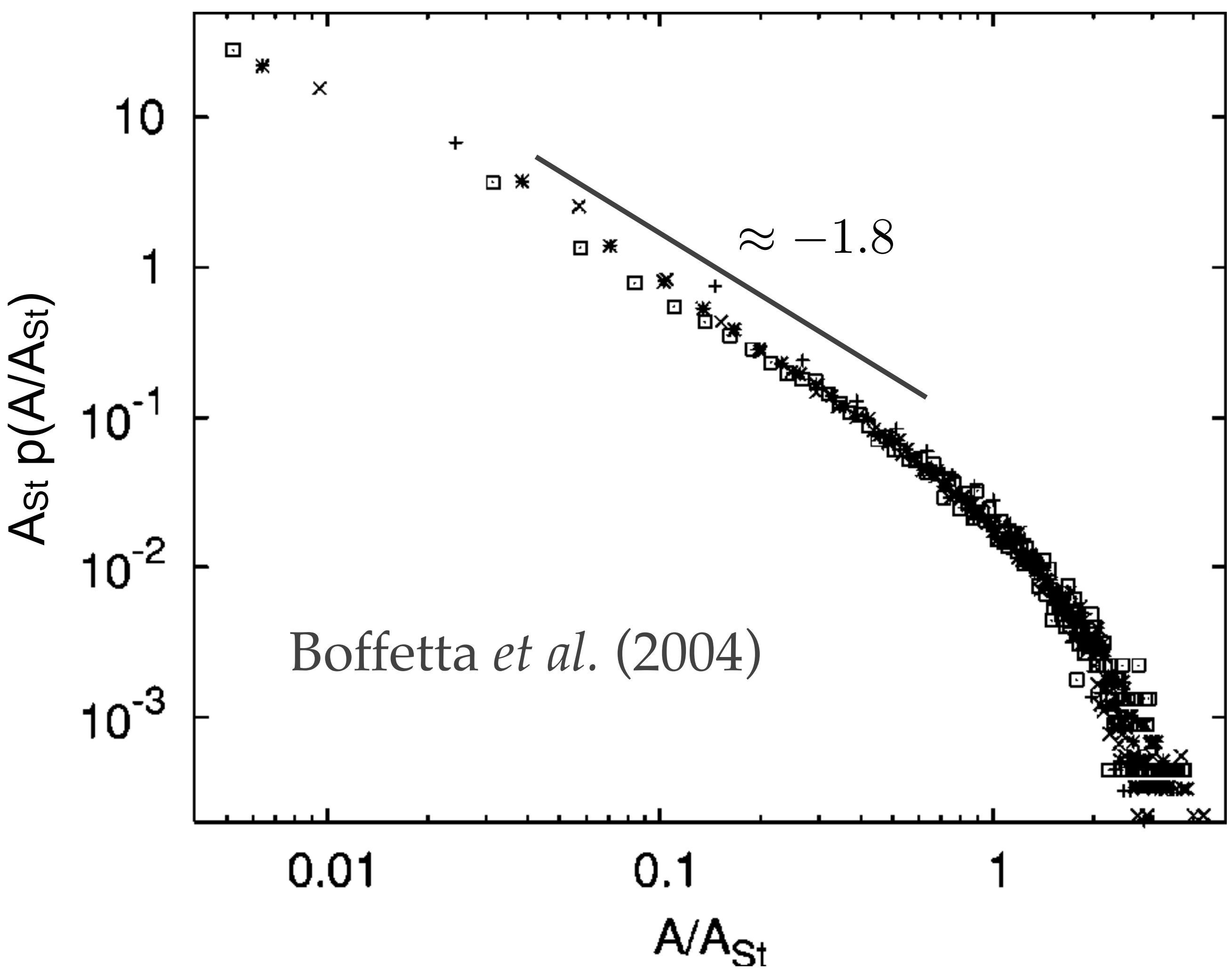
a.k.a. pair-correlation function, measures the excess in the distribution of distances between particle pairs

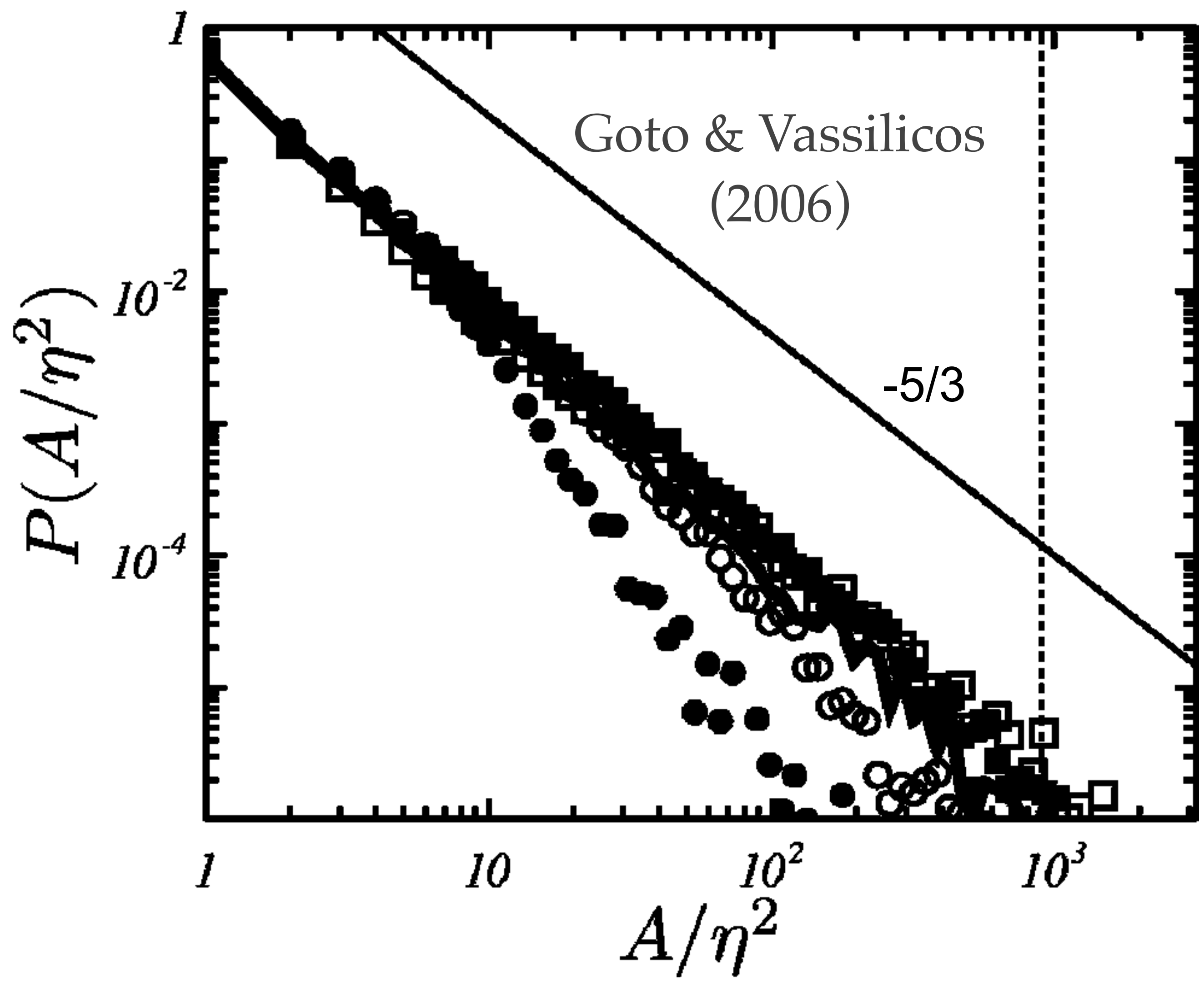
$$g(\ell) = \frac{p_2(\ell)}{4\pi\ell^2} \quad (= 1 \text{ if uniform})$$



Distribution of voids – 2D

Simulation in two-dimensional inverse cascade

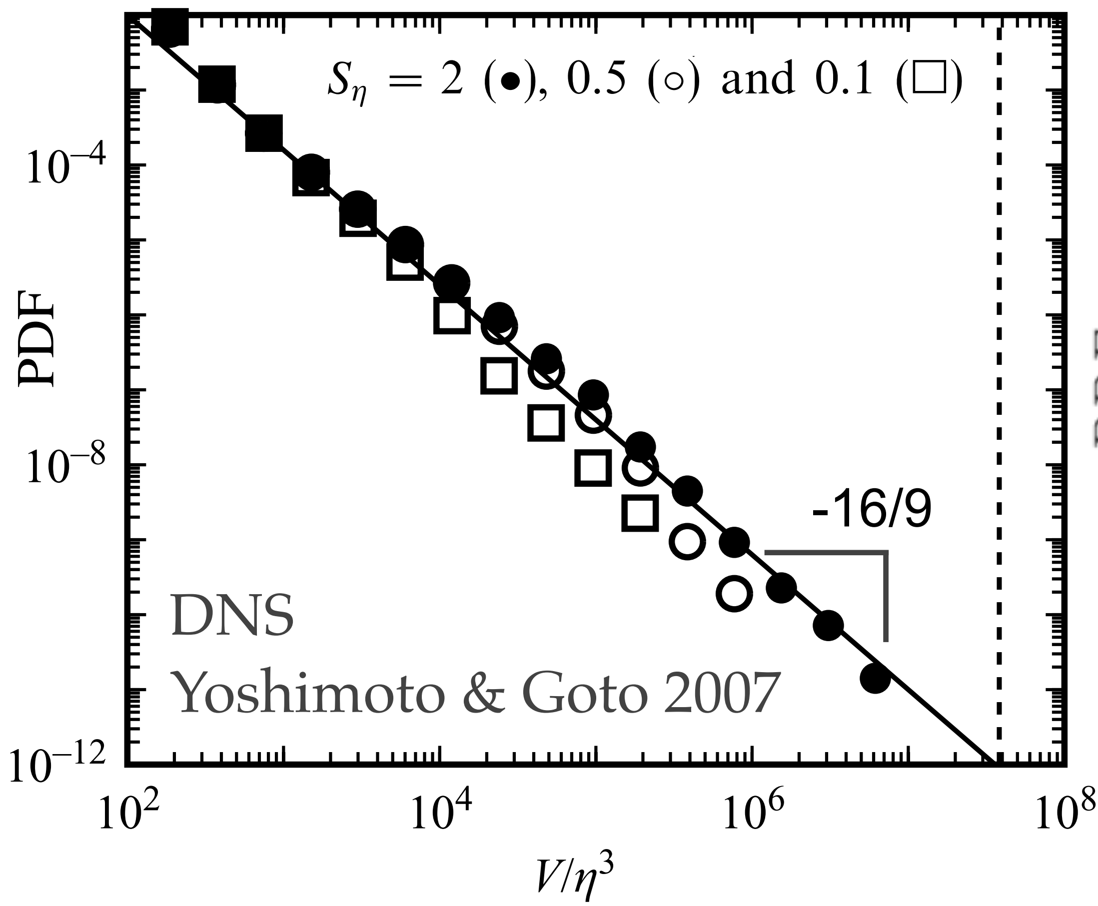


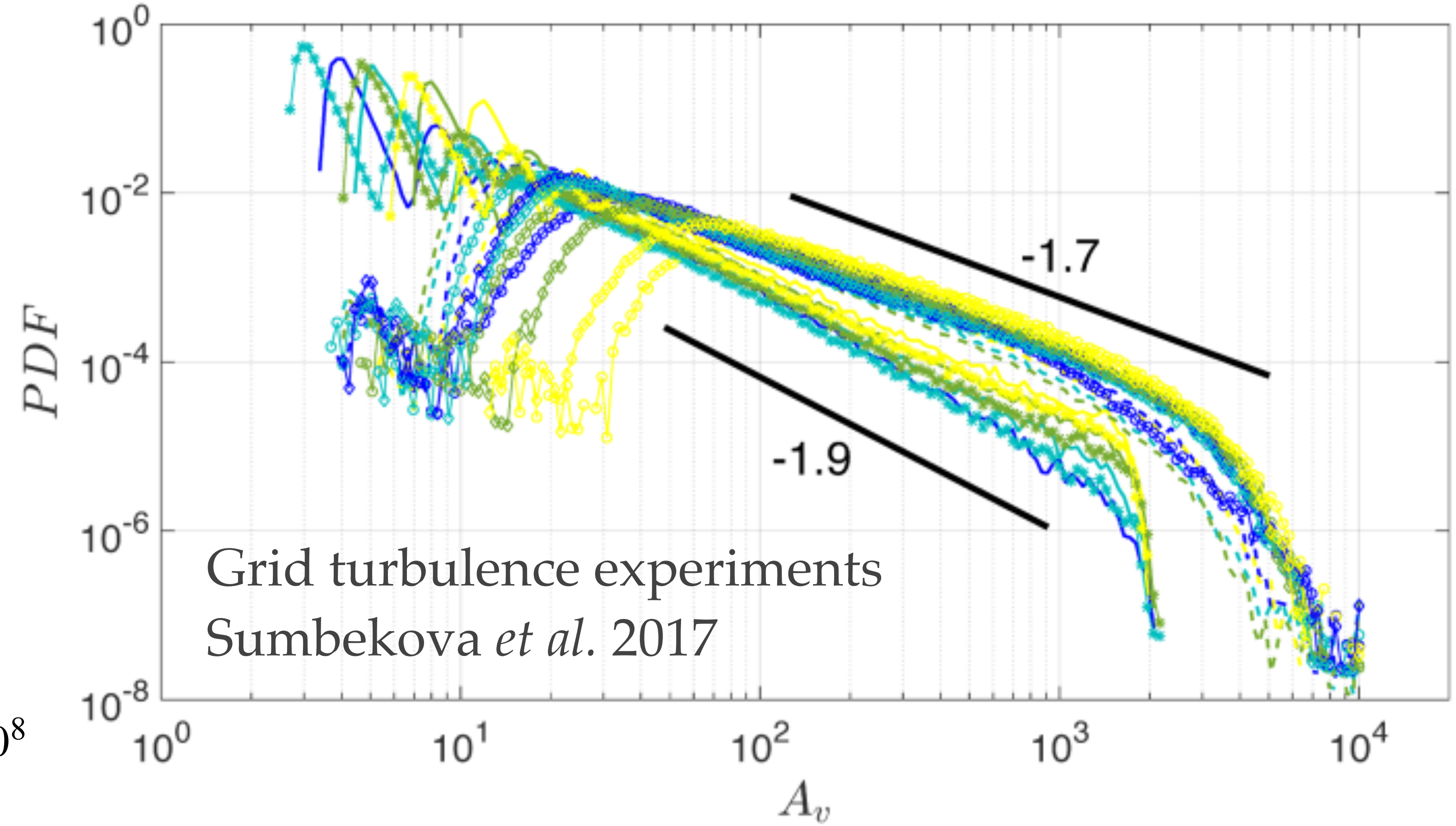


Goto & Vassilicos: Sweep-stick mechanism.
Particles follow the zero-acceleration of the flow

$p(A) \sim A^{-5/3}$ independent of Stokes

Distribution of voids – 3D





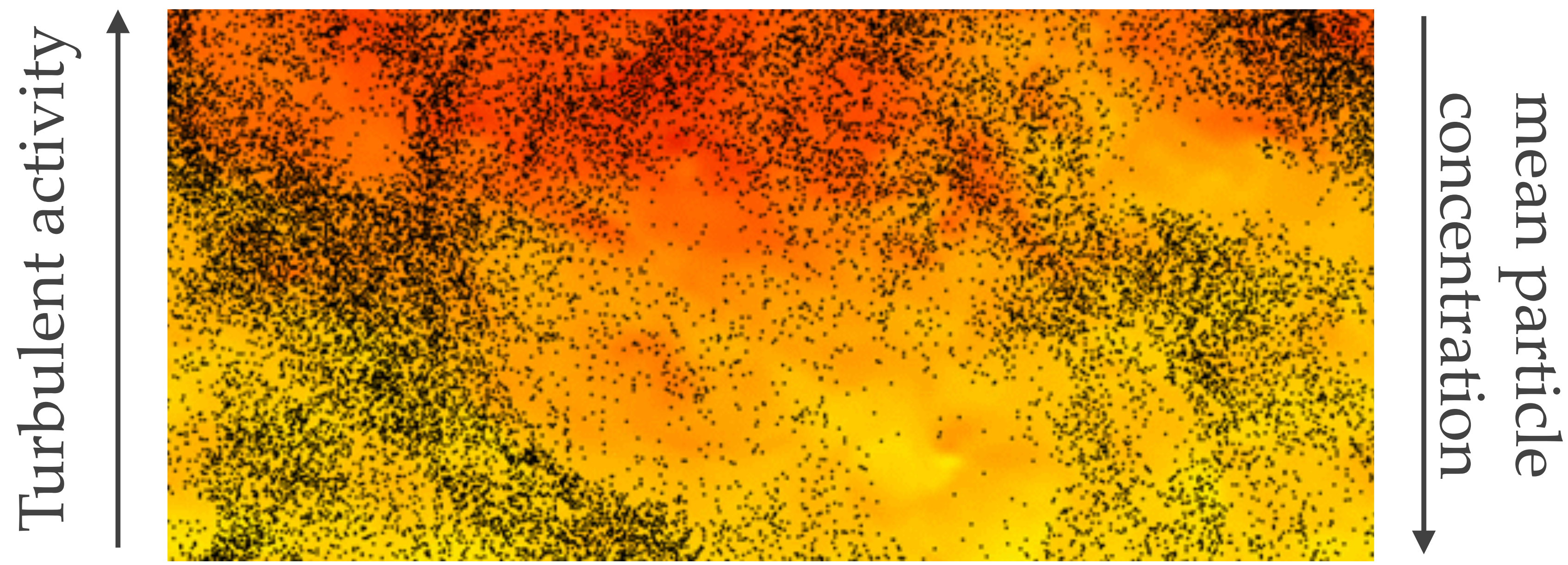
Difficult to assess…

Sweep-stick mechanism? $p(V) \sim V^{-16/9}$

Possible dependence on the Stokes number?

Another viewpoint: turbophoresis

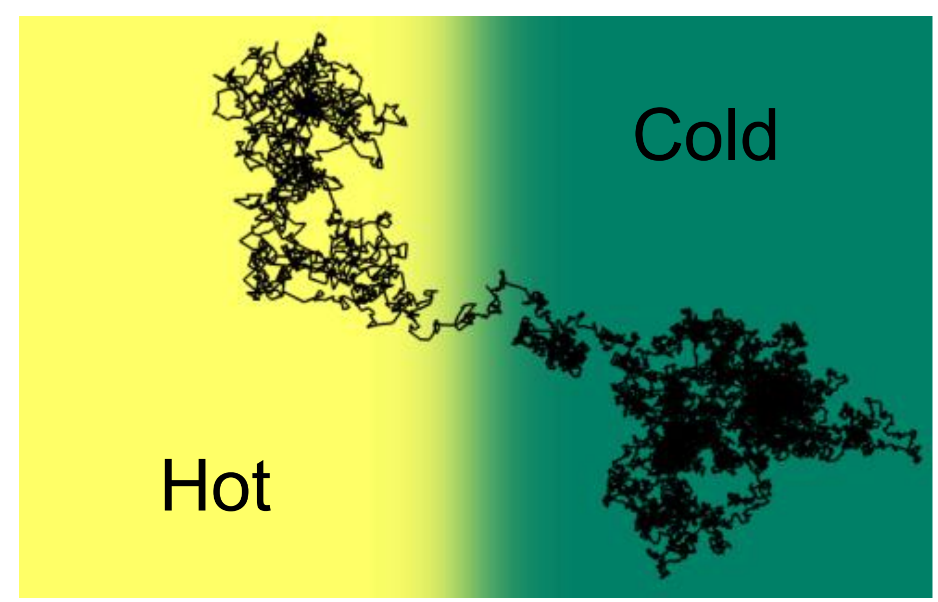
- In **inhomogeneous flow:** (Caporaloni et al. 1975, Reeks 1983)



(from De Lillo *et al*. 2016)

Effective diffusion equation for the average particle concentration

- Analogy with **thermophoresis**: diffusive particles spend more time in colder regions

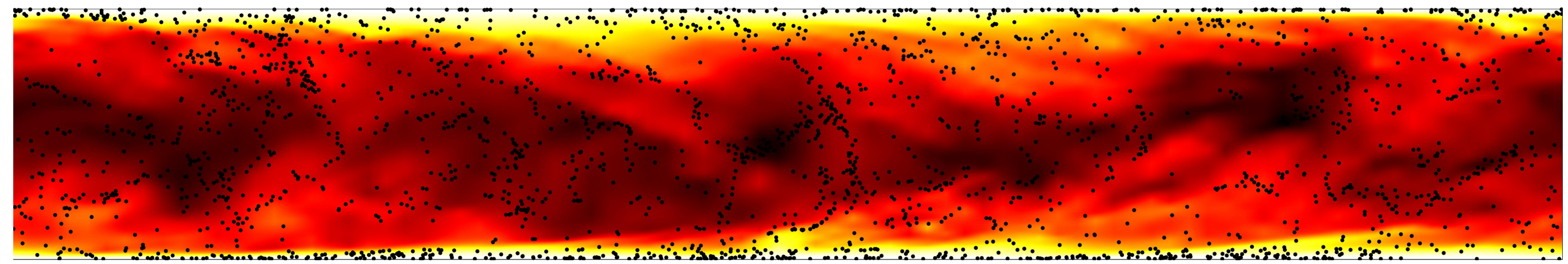


Inhomogeneous turbophoresis

- Turbulent boundary layers: channel flow

particle migrate
toward the walls

(Rouson & Eaton 2001,
Marchioli & Soldati 2002,
Costa *et al.* 2020)



ejection from high-kinetic-energy regions

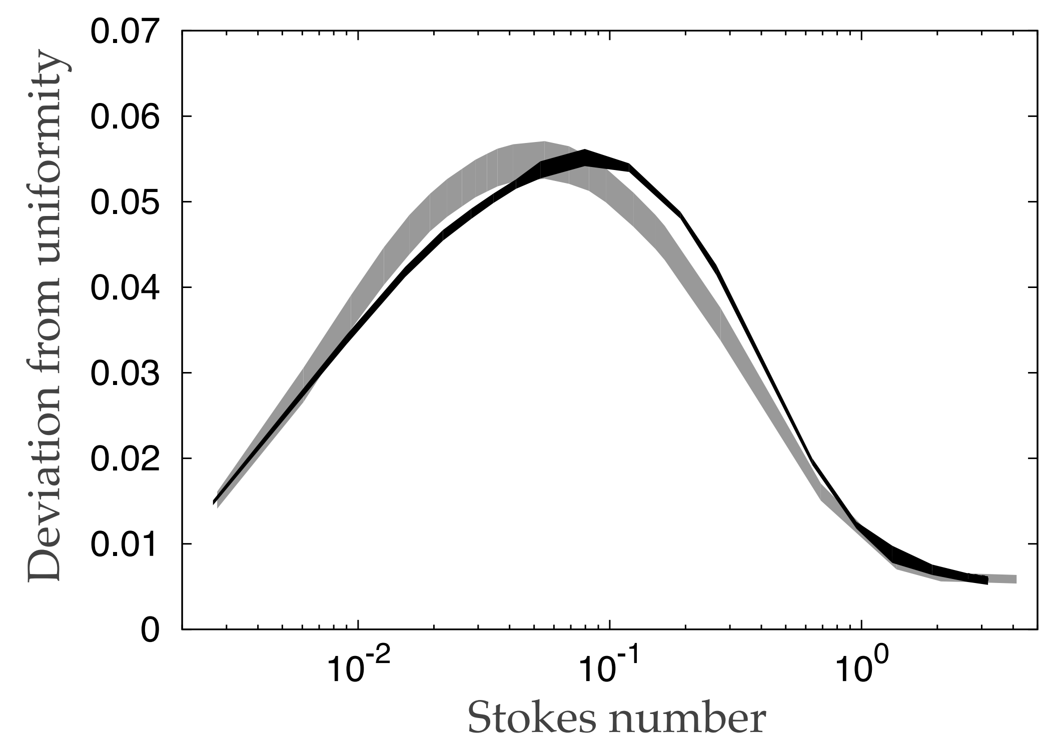
- Periodic flow with non-uniform forcing

Non-monotonic dependence
upon the particle response time

(De Lillo *et al.* 2016, Mitra *et al.* 2018)

Effective diffusion

$$\kappa(x) \propto \text{temp} \propto \langle |V_{\mathrm{p},x}|^2 \rangle$$

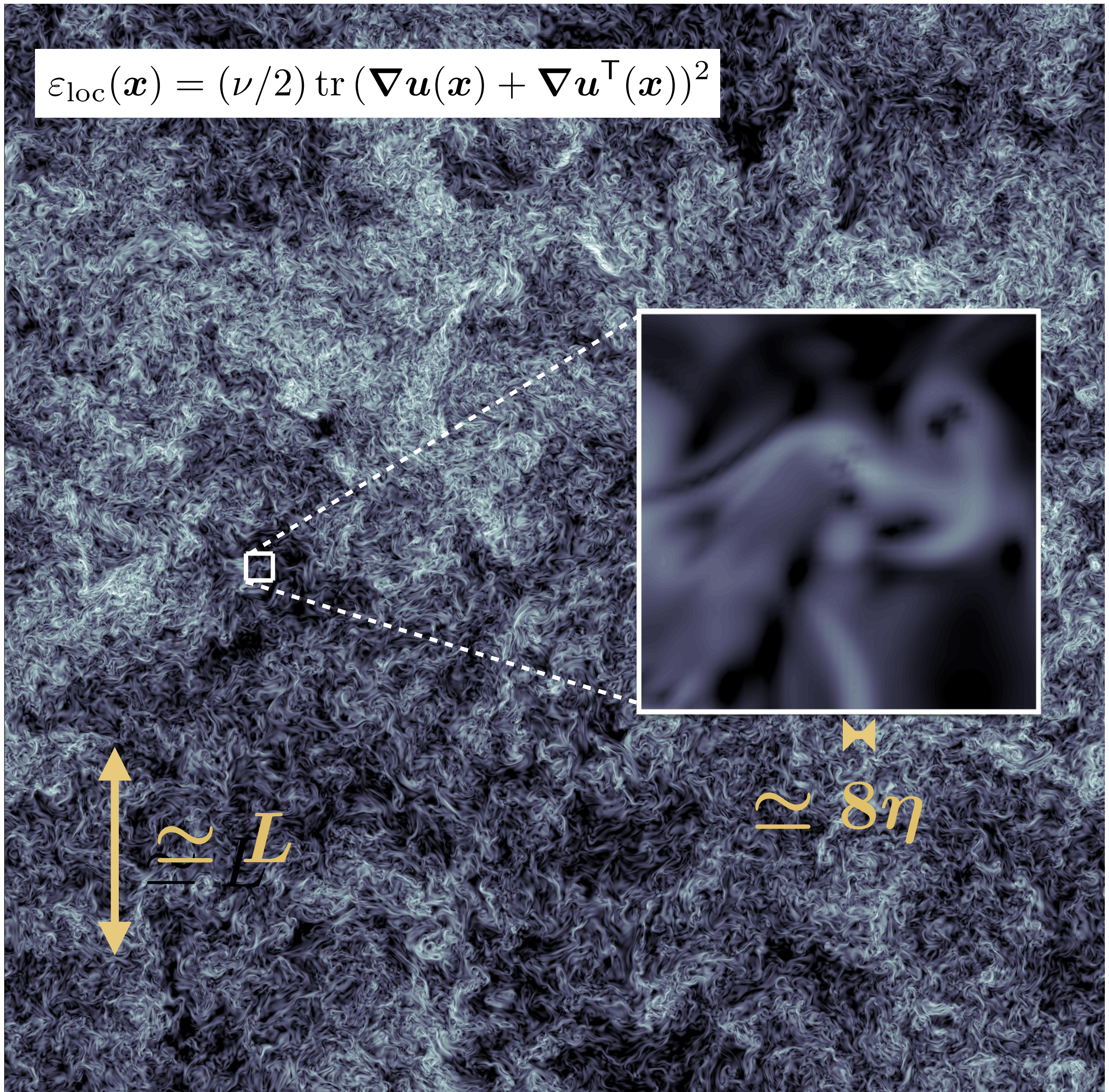


Do such considerations extend to statistically homogeneous flows?

Direct numerical simulations

- **Fluid + particles**
  Pseudo-spectral code LaTu

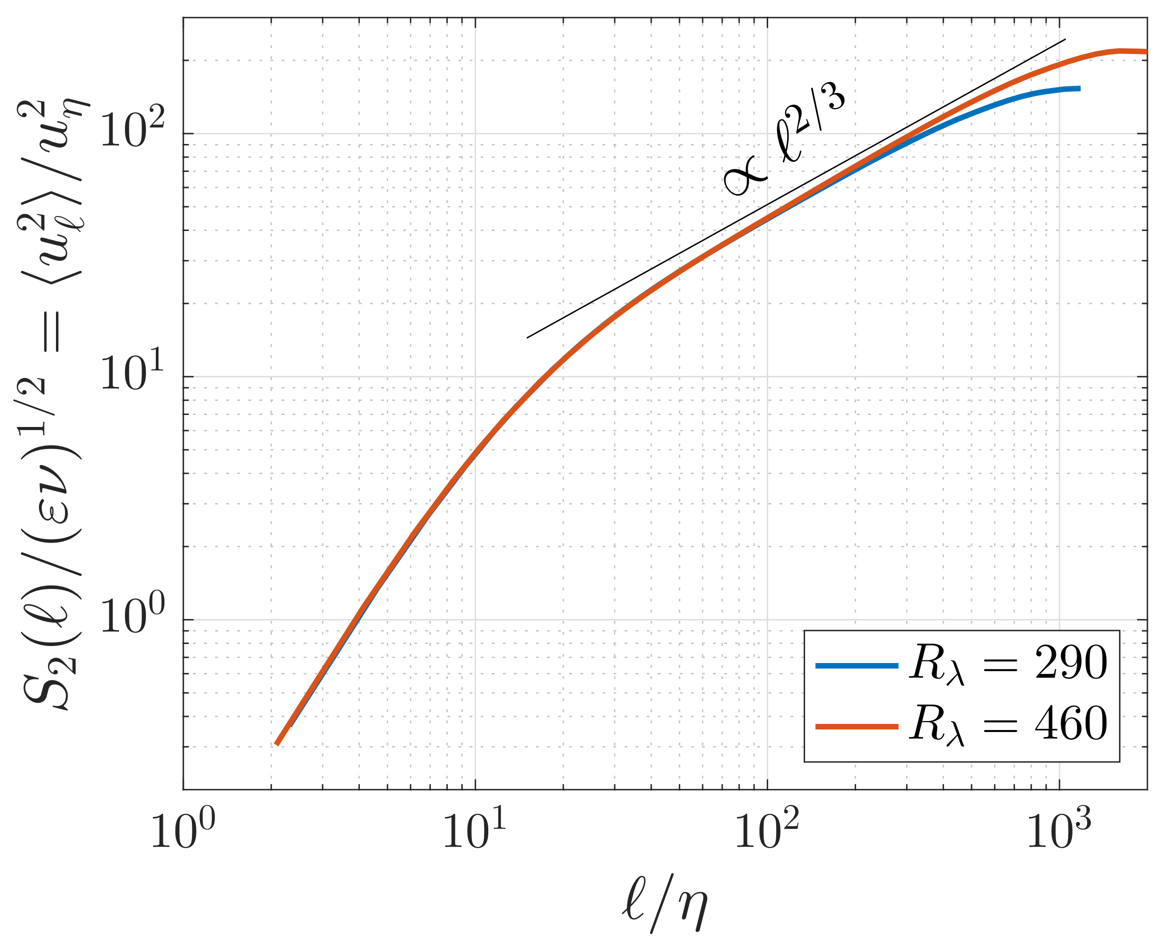
N^3	R_λ	N_p (per St)
1024^3	290	1.25×10^7
2048^3	460	10^8



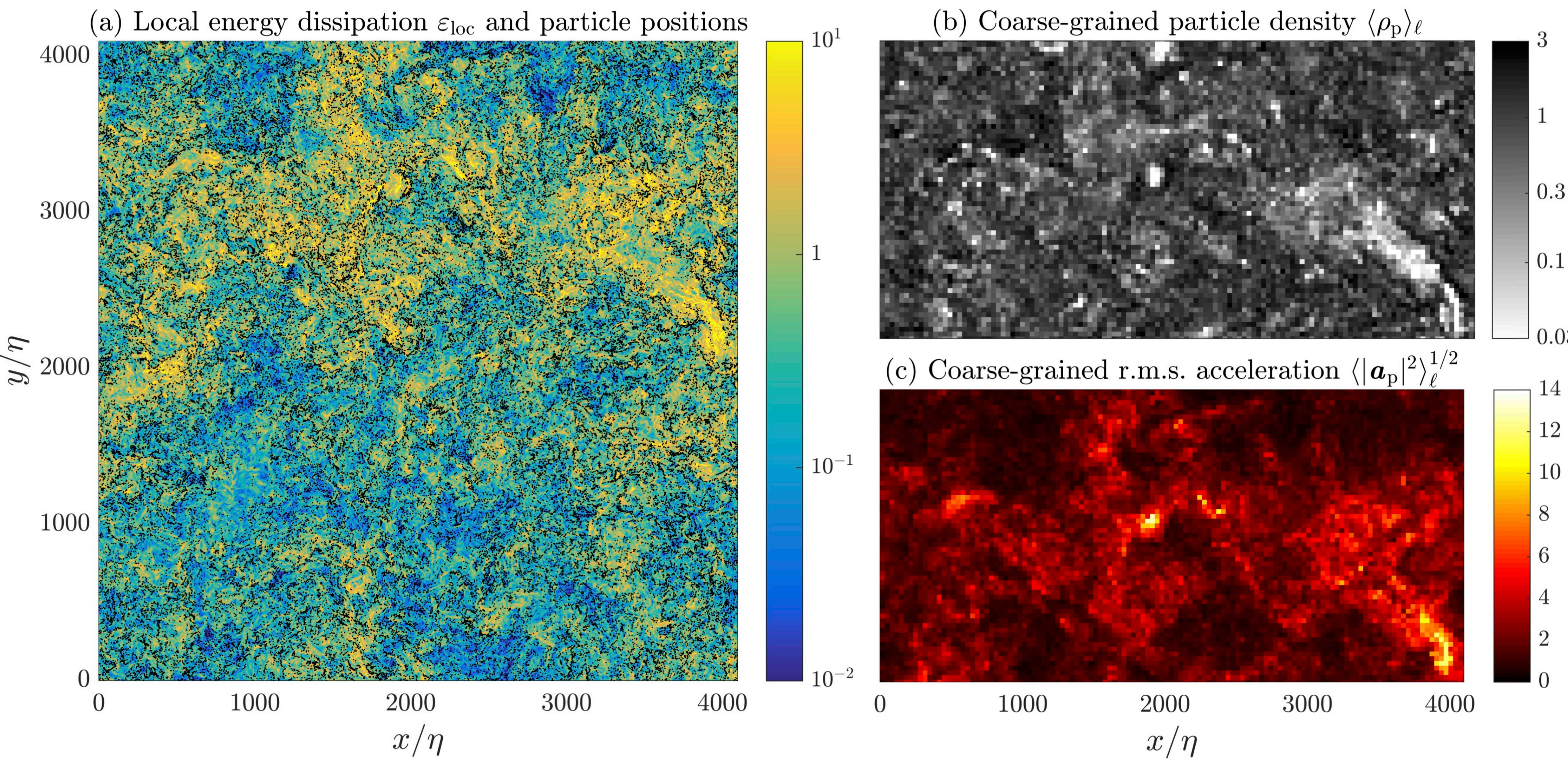
Turbulence similarity
(Kolmogorov 1941)

for $\eta \ll \ell \ll L$

$u_\ell = |u(x+\ell) - u(x)| \sim \varepsilon^{1/3}\,\ell^{1/3}$



Particle clusters at a glance



Clear correlations between particle positions, accelerations, and turbulent activity over inertial-range scales.

Diffusive process

- Importance of acceleration to measure deviations from the fluid

$$\boldsymbol{a}_\mathrm{p} = -\frac{1}{\tau}\left[\boldsymbol{v}_\mathrm{p} - \boldsymbol{u}(\boldsymbol{x}_\mathrm{p},t)\right] \quad \Rightarrow \quad \boldsymbol{v}_\mathrm{p} = \boldsymbol{u}(\boldsymbol{x}_\mathrm{p},t) - \tau\,\boldsymbol{a}_\mathrm{p}$$

- Acceleration is moreover:
  - ✓ correlated over short scales (in principle…)
  - ✓ very sensitive to flow activity

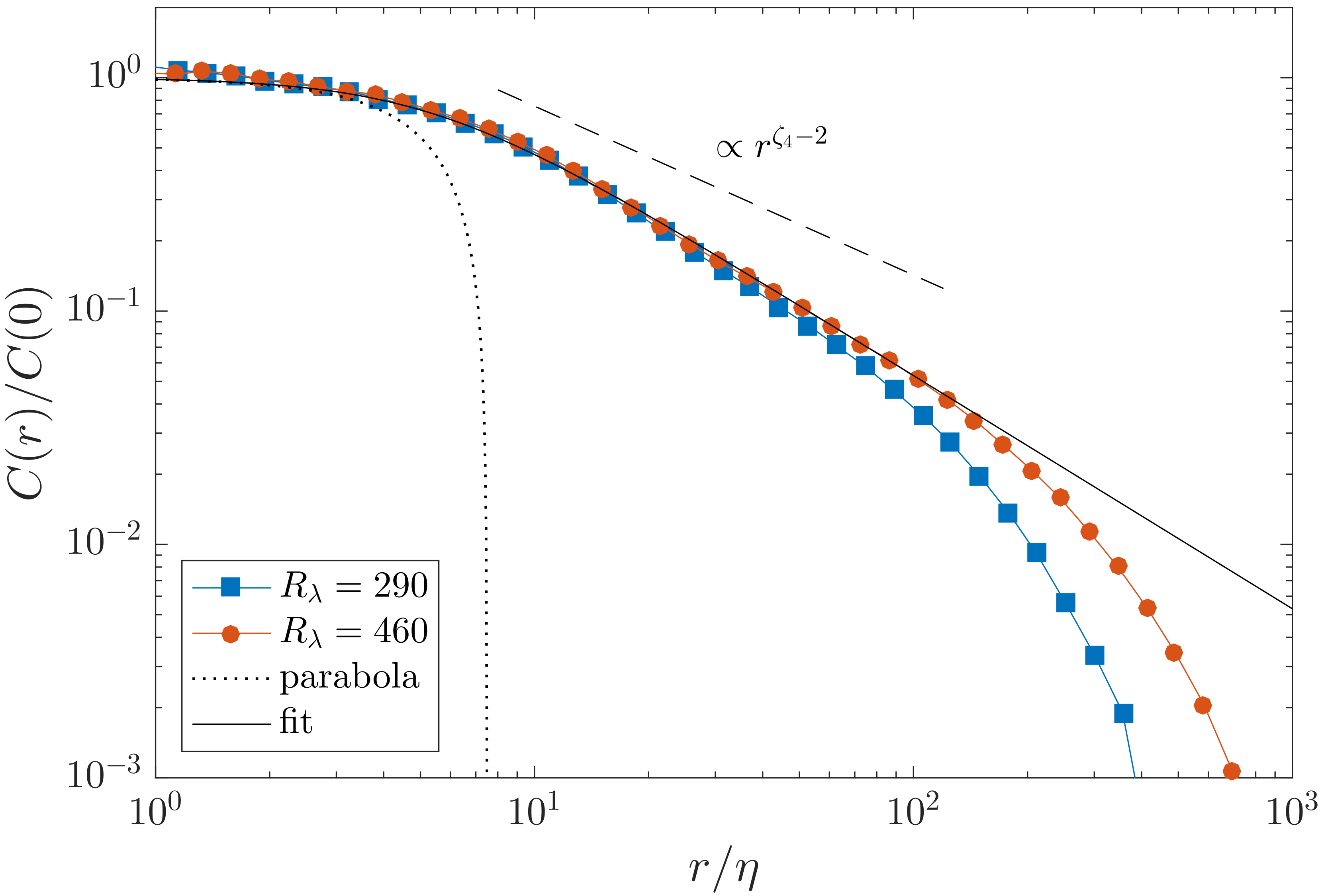
$$\boldsymbol{x}_\mathrm{p}(t+\delta t) - \boldsymbol{x}_\mathrm{p}(t) = \underbrace{\int_t^{t+\delta t} \boldsymbol{u}(\boldsymbol{x}_\mathrm{p}(s),s)\,\mathrm{d}s}_{\text{large-scale } \approx \text{ const.}} - \tau \underbrace{\int_t^{t+\delta t} \boldsymbol{a}_\mathrm{p}(s)\,\mathrm{d}s}_{\text{small-scale, Central-limit theorem?}}$$

$$\mathrm{d}\boldsymbol{x}_\mathrm{p}(t) \approx \left[\boldsymbol{u}(\boldsymbol{x}_\mathrm{p}(t),t) - \tau\,\overline{\boldsymbol{a}_\mathrm{p}}(t)\right]\mathrm{d}t + \boldsymbol{\sigma}(t)\,\mathrm{d}\boldsymbol{W}(t)$$

$$\boldsymbol{\sigma}^\mathsf{T}\boldsymbol{\sigma} = \tau^2 T_\mathrm{I}\left(\overline{\boldsymbol{a}_\mathrm{p}\otimes\boldsymbol{a}_\mathrm{p}} - \overline{\boldsymbol{a}_\mathrm{p}}\otimes\overline{\boldsymbol{a}_\mathrm{p}}\right)$$

NB: $\neq$ equilibrium Eulerian approaches, for which $\boldsymbol{a}_\mathrm{p} \approx \boldsymbol{a}_\mathrm{f}(\boldsymbol{x}_\mathrm{p},t)$ used to describe $St \ll 1$ (Maxey 1987, Ferry & Balachandar 2001)

Spatial correlations of acceleration



$$C(r) = \langle \boldsymbol{a}(\boldsymbol{r},t)\cdot\boldsymbol{a}(0,t)\rangle$$

Inertial-range expectation

Hill-Wilczak (1995), Xu *et al.* (2007)

$$C(r) \sim r^{\zeta_4 - 2} \qquad \zeta_4 \approx 1.27$$

Contributions from dissipative scales actually seem dominant:

$$C(r) \approx \frac{\langle |\boldsymbol{a}|^2\rangle}{(1+(c\,r/\eta)^2)^{1/2}} \sim r^{-1}$$

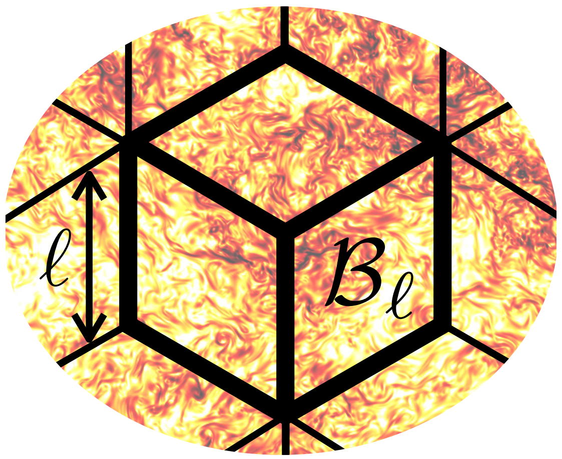
Reflect the intrinsic correlations of turbulent activity ⇒ Must be accounted for!

Refined self-similarity

❖ Coarse-grained dissipation

$$\varepsilon_\ell(\boldsymbol{x}) \equiv \frac{1}{|\mathcal{B}_\ell|}\int_{\mathcal{B}_\ell(\boldsymbol{x})} \varepsilon_{\rm loc}(\boldsymbol{x}')\, d^3x'$$

$$\langle \varepsilon_\ell \rangle = \varepsilon$$



Refined similarity (Kolmogorov 1962) $u_\ell = |u(x+\ell) - u(x)| \sim \varepsilon_\ell^{1/3}\,\ell^{1/3}$

accounts for instantaneous inhomogeneities in turbulent activity

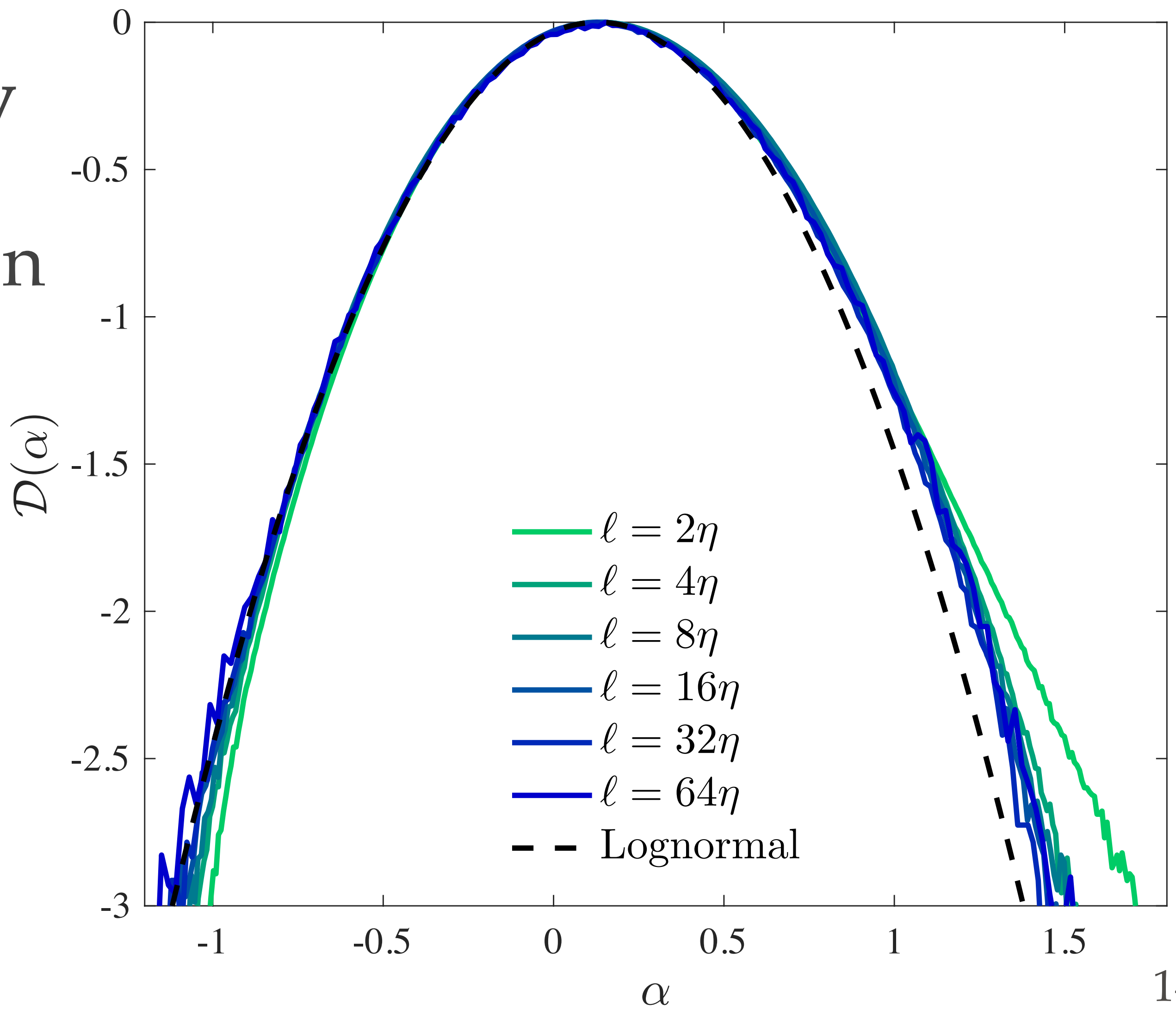
❖ "Multifractal" statistics of dissipation

$\varepsilon_\ell = \varepsilon\,(\ell/L)^\alpha$ with

$$p(\varepsilon_\ell)\,\mathrm{d}\varepsilon_\ell = (\ell/L)^{3-\mathcal{D}(\alpha)}\,\mathrm{d}\mu(\alpha)$$

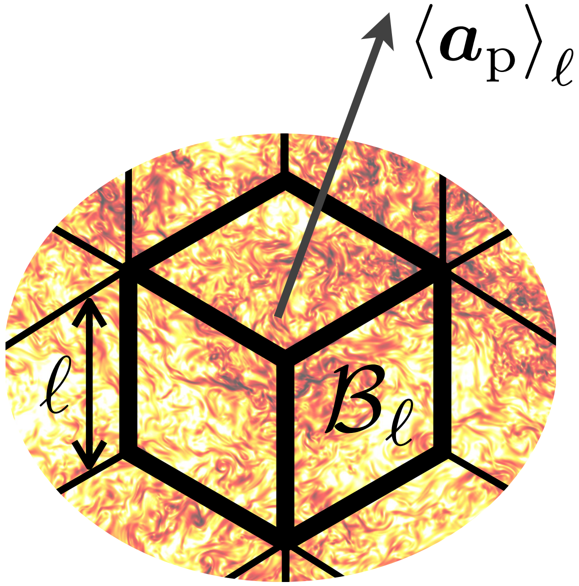
$$S_n(\ell) = \langle u_\ell^n \rangle \sim \ell^{\zeta_n}$$

$$\zeta_n = \frac{n}{3} + \inf_\alpha \left[\tfrac{1}{3}\alpha n + 3 - \mathcal{D}(\alpha)\right]$$



Coarse-grained dynamics

$\overline{(\cdot)} \equiv \langle \cdot \rangle_\ell$: average over a coarse-graining box of size ℓ



Residence time $\gg$ correlation time T_I

$\Rightarrow$ Box average $\approx$ time integral conditioned on local turbulent fluctuations

$$\boldsymbol{D}_\ell = \frac{1}{2}\boldsymbol{\sigma}_\ell^\mathsf{T}\boldsymbol{\sigma}_\ell = \frac{1}{2}\tau^2\, T_\mathrm{I}^{(\ell)}\left(\langle \boldsymbol{a}_\mathrm{p} \otimes \boldsymbol{a}_\mathrm{p} \rangle_\ell - \langle \boldsymbol{a}_\mathrm{p} \rangle_\ell \langle \boldsymbol{a}_\mathrm{p} \rangle_\ell\right)$$

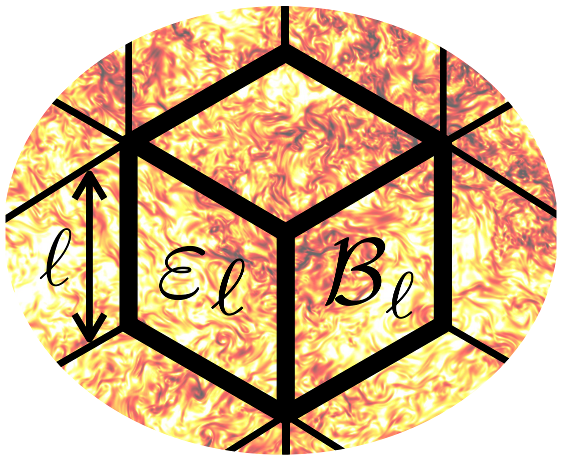
Evaluated in the box of reference $\Rightarrow$ depend upon $\boldsymbol{x}_\mathrm{p}$

$$\mathrm{d}\boldsymbol{x}_\mathrm{p}(t) \approx \left[\boldsymbol{u}(\boldsymbol{x}_\mathrm{p}(t),t) - \tau\,\langle \boldsymbol{a}_\mathrm{p} \rangle_\ell(\boldsymbol{x}_\mathrm{p}(t),t)\right]\mathrm{d}t + \boldsymbol{\sigma}_\ell(\boldsymbol{x}_\mathrm{p}(t),t)\,\mathrm{d}\boldsymbol{W}(t)$$

Itô, to be consistent with

$\langle \boldsymbol{v}_\mathrm{p} \rangle = \langle \boldsymbol{u}(\boldsymbol{x}_\mathrm{p}(t),t) \rangle$

Coarse-grained statistics



local turbulent fluctuations described by the coarse-grained dissipation rate

Refined-similarity hypothesis:

Particle statistics conditioned on ε_ℓ depend solely on:

- the local Reynolds number $Re_\ell = \varepsilon_\ell^{1/3}\ell^{4/3}/\nu$
- the local Stokes number $St_\ell = \tau\,\varepsilon_\ell^{1/2}/\nu^{1/2}$

$$T_{\mathrm{I}}^{(\ell)} = \nu^{1/2}\varepsilon_\ell^{-1/2}\Phi(Re_\ell, St_\ell)$$

$$\mathrm{Covar}_\ell(\boldsymbol{a}_{\mathrm{p}}) = \nu^{-1/2}\varepsilon_\ell^{3/2}\Psi(Re_\ell, St_\ell)\ \ldots$$

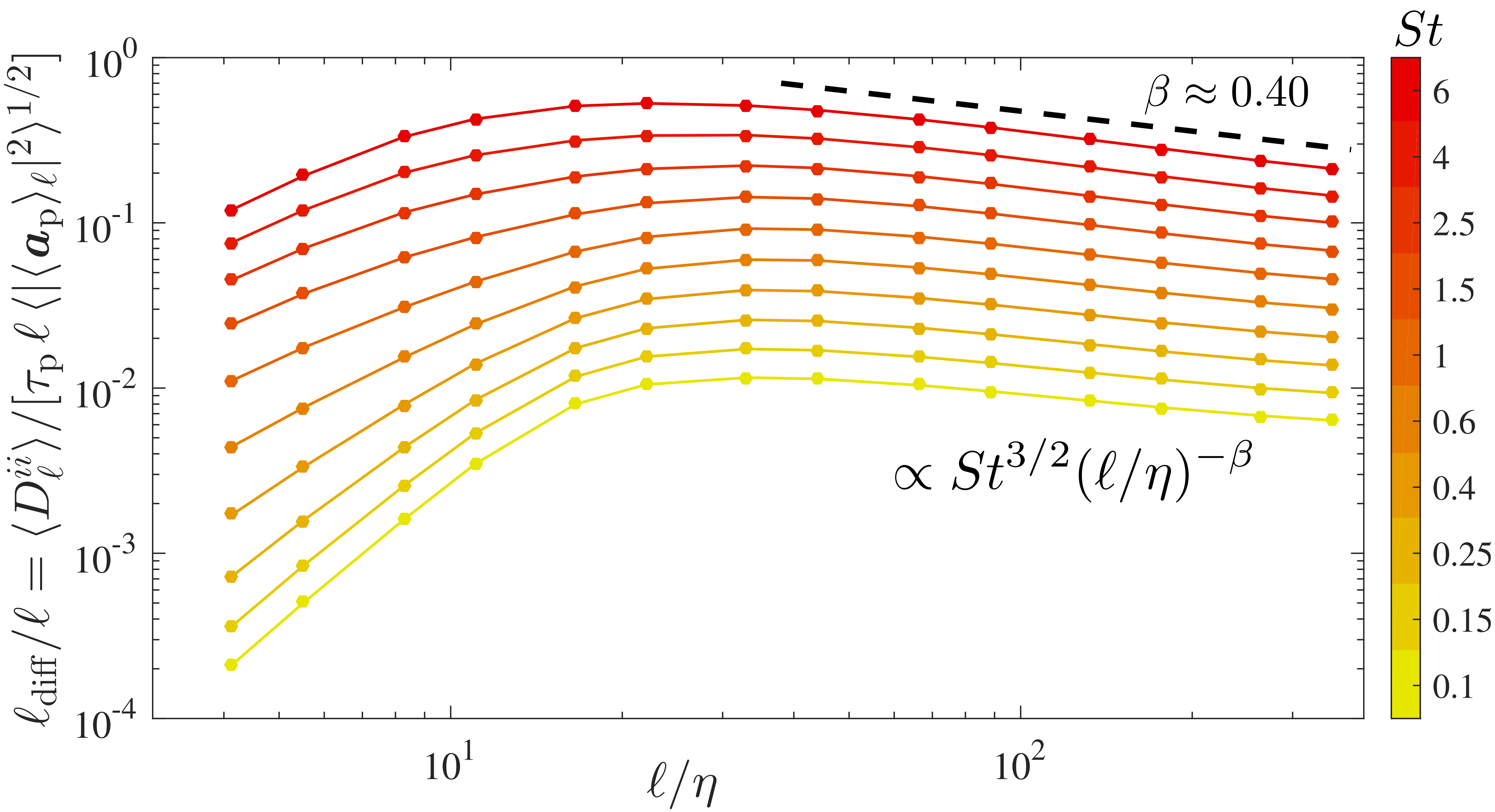
+ lognormal fluctuations

$\Rightarrow$ gives predictions on the statistics of $\langle \boldsymbol{a}_{\mathrm{p}}\rangle_\ell(\boldsymbol{x},t)$ and $\boldsymbol{\sigma}_\ell(\boldsymbol{x},t)$

Effect of diffusion

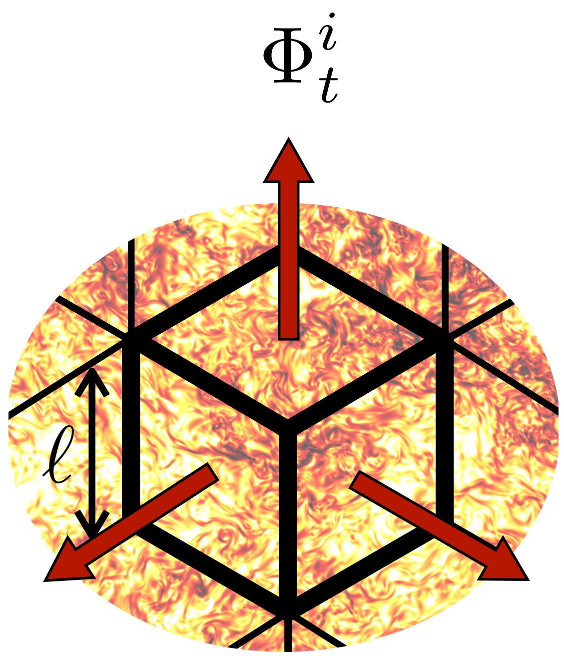
$$\mathrm{d}\boldsymbol{x}_\mathrm{p}(t) \approx [\boldsymbol{u}(\boldsymbol{x}_\mathrm{p}(t),t) - \tau\,\langle \boldsymbol{a}_\mathrm{p}\rangle_\ell(\boldsymbol{x}_\mathrm{p}(t),t)]\,\mathrm{d}t + \boldsymbol{\sigma}_\ell(\boldsymbol{x}_\mathrm{p}(t),t)\,\mathrm{d}\boldsymbol{W}(t)$$

dominates at scales $\gg \ell_\mathrm{diff}$ (Batchelor scale)



Drift prevails at moderate Stokes numbers and inertial-range scales

Ejection process



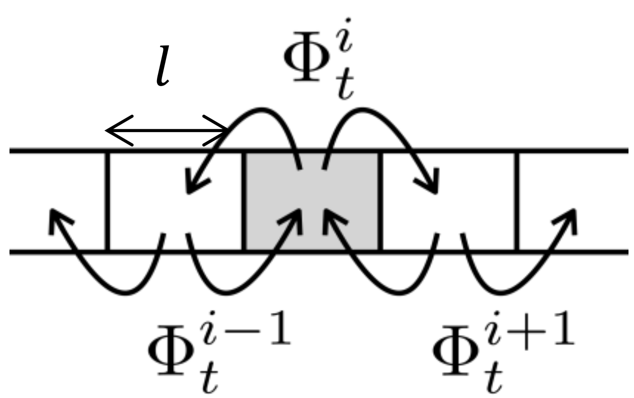
$\langle \rho_\mathrm{p} \rangle_\ell$ coarse-grained particle density

$$\boldsymbol{v}_p^\mathrm{eff}(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t) - \tau \, \langle \boldsymbol{a}_\mathrm{p} \rangle_\ell(\boldsymbol{x},t)$$

depends on past

Particle fluxes = transport by the fluid velocity + ejection due to inertia

Outgoing flux from the cell *i*: $\Phi_t^i \approx \int_{\partial\mathcal{B}_\ell} \tau \, \langle \rho_\mathrm{p} \boldsymbol{a}_\mathrm{p} \rangle \cdot \mathrm{d}\boldsymbol{S} \propto \tau \ell^2 \, \langle \rho_\mathrm{p} \rangle_\ell \, | \langle \boldsymbol{a}_\mathrm{p} \rangle_\ell |$



$$\frac{\mathrm{d}m_i}{\mathrm{d}t} = (1/2)\Phi_t^{i+1} - \Phi_t^i + (1/2)\Phi_t^{i-1}$$

$$\partial_t \, \langle \rho_\mathrm{p} \rangle_\ell + \boldsymbol{u} \cdot \nabla \, \langle \rho_\mathrm{p} \rangle_\ell \approx \nabla^2 (\kappa_\ell \, \langle \rho_\mathrm{p} \rangle_\ell)$$

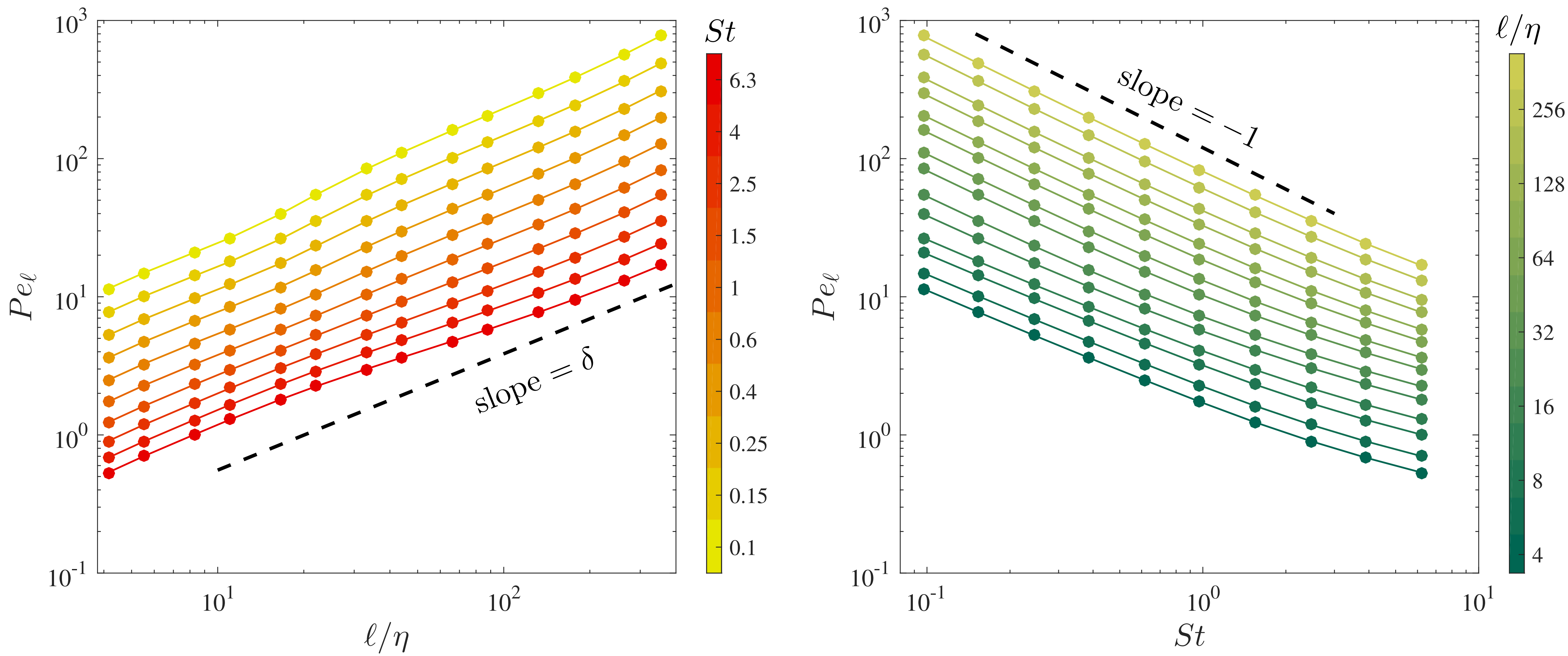
$$\kappa_\ell(\boldsymbol{x},t) \propto \tau \ell \, | \langle \boldsymbol{a}_\mathrm{p} \rangle_\ell |$$

Scale-dependent Peclet number

$$\partial_t \langle \rho_\mathrm{p} \rangle_\ell + \boldsymbol{u} \cdot \nabla \langle \rho_\mathrm{p} \rangle_\ell \approx \nabla^2 (\kappa_\ell \langle \rho_\mathrm{p} \rangle_\ell)$$

$$Pe_\ell = \frac{\delta_\ell u}{\ell\, \kappa_\ell} = \frac{\text{diffusive time}}{\text{advective time}}$$

$$Pe_\ell \propto \ell^\delta / \tau \text{ with } \delta \approx 0.9$$



Far inertial-range distribution depends solely on Pe_ℓ

Radial distribution function

Second-order moment of density: $g(\ell) = \langle \langle \rho_\mathrm{p} \rangle_\ell^2 \rangle / \rho_0^2 \qquad \rho_0 = \langle \langle \rho_\mathrm{p} \rangle_\ell \rangle$

$\langle \rho_\mathrm{p} \rangle_\ell = \rho_0 + \delta\rho \qquad \delta\rho \ll \rho_0 \;\Rightarrow\; \partial_t \delta\rho + \boldsymbol{u} \cdot \nabla \delta\rho \approx \rho_0 \nabla^2 \kappa_\ell$

Two regimes:

$Pe_\ell \gg 1 \; (i.e.\; \ell/\eta \gg St^{1/\delta})$

$$\delta u_\ell \, \delta\rho \sim \rho_0 \kappa_\ell / \ell$$

$$\Rightarrow \langle \delta\rho^2 \rangle \sim Pe_\ell^{-2}$$

$$g(\ell) - 1 \propto St^2 \, (\ell/\eta)^{-2\delta}$$

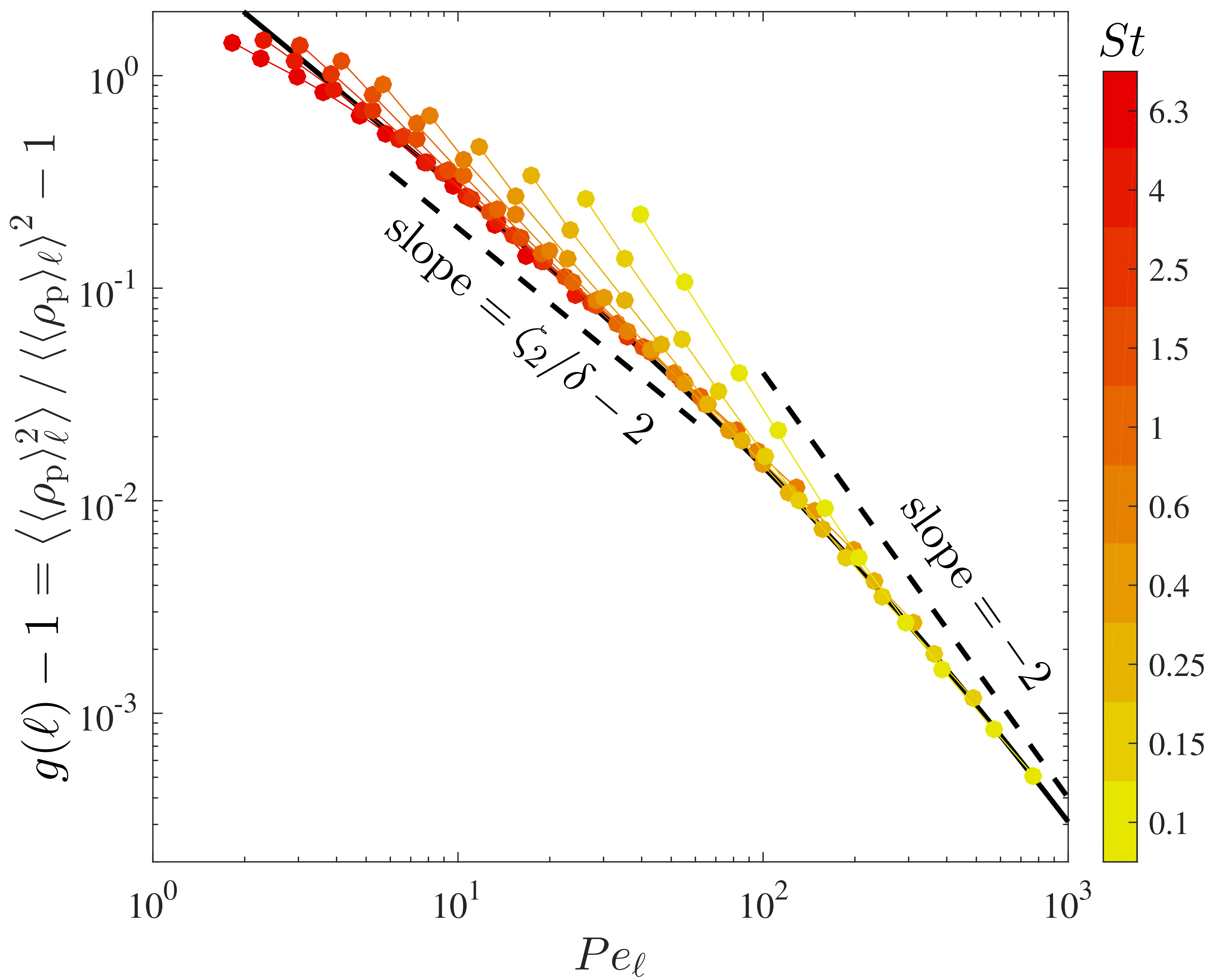
$Pe_\ell \lesssim 1 \; (i.e.\; \ell/\eta \lesssim St^{1/\delta})$

$$u_\mathrm{rms} \, \delta\rho \sim \rho_0 \kappa_\ell / \ell$$

$$\Rightarrow \langle \delta\rho^2 \rangle \sim Pe_\ell^{\zeta_2/\delta - 2}$$

$$g(\ell) - 1 \propto St^{2-\zeta_2/\delta} \, (\ell/\eta)^{\zeta_2 - 2\delta}$$

+ nontrivial dependences upon the Reynolds number



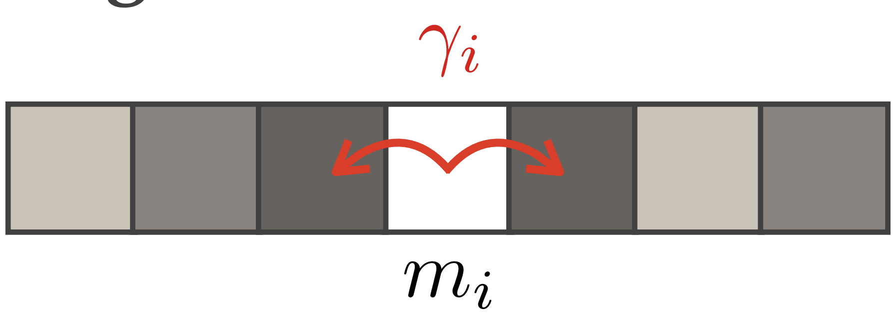
Other predictions from the model

Universal properties of ejection processes

Bec & Chétrite 2007
Krstulovic *et al.* 2012

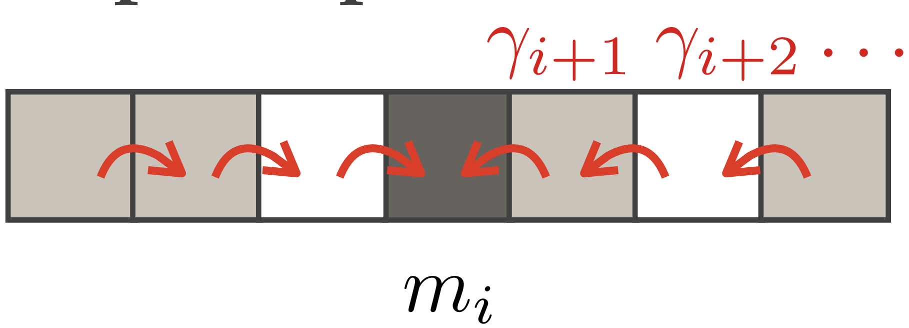
$$m_i(n+1) = \gamma_{i+1}(n)m_{i+1}(n) + [1-2\gamma_i(n)]m_i(n) + \gamma_{i-1}(n)m_{i-1}(n)$$

- **Algebraic** behaviour of the mass distribution at small values



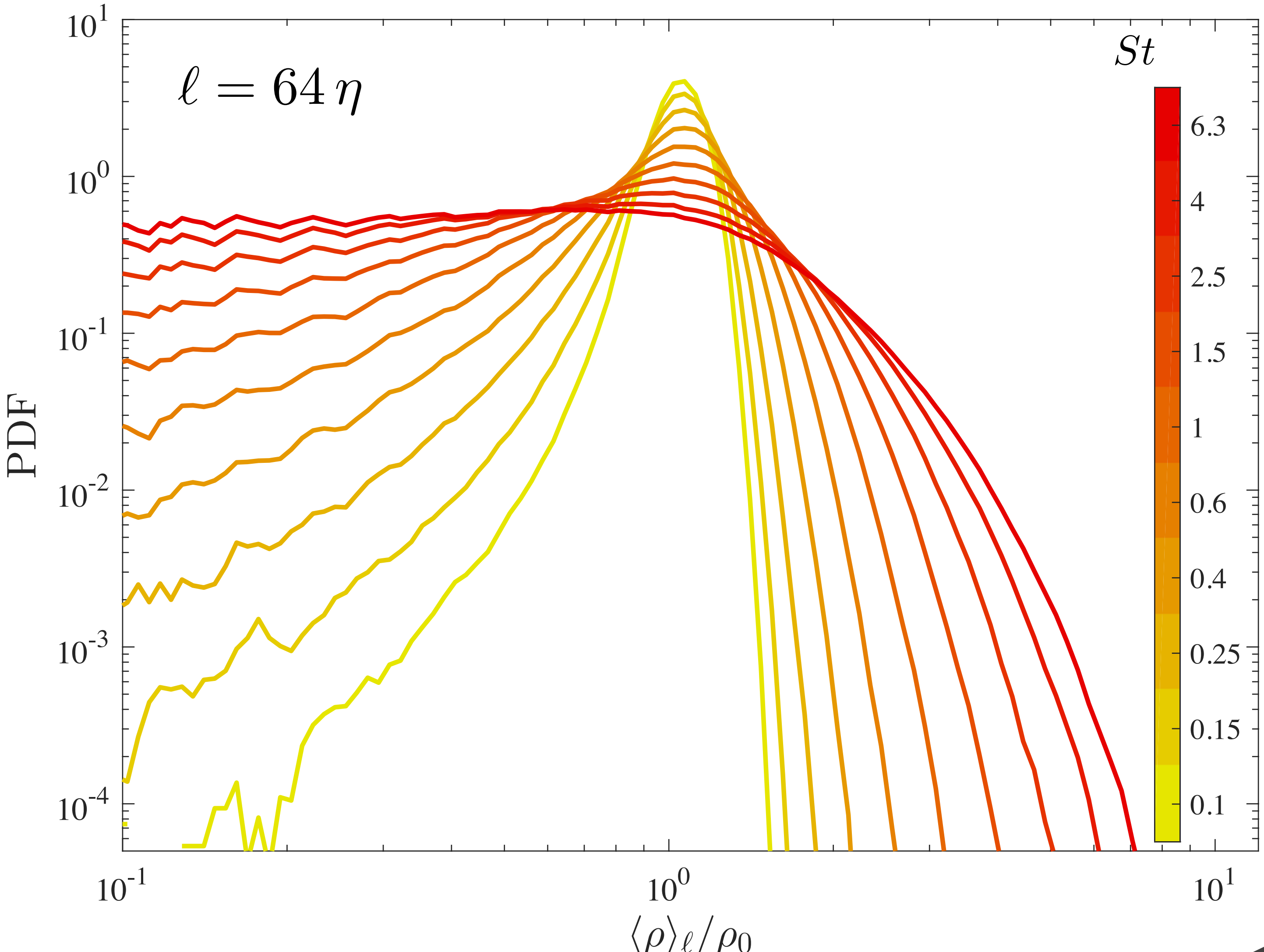
$$\Rightarrow \; p(m) \propto m^{\alpha}$$

- **Super-exponential** behaviour at large values



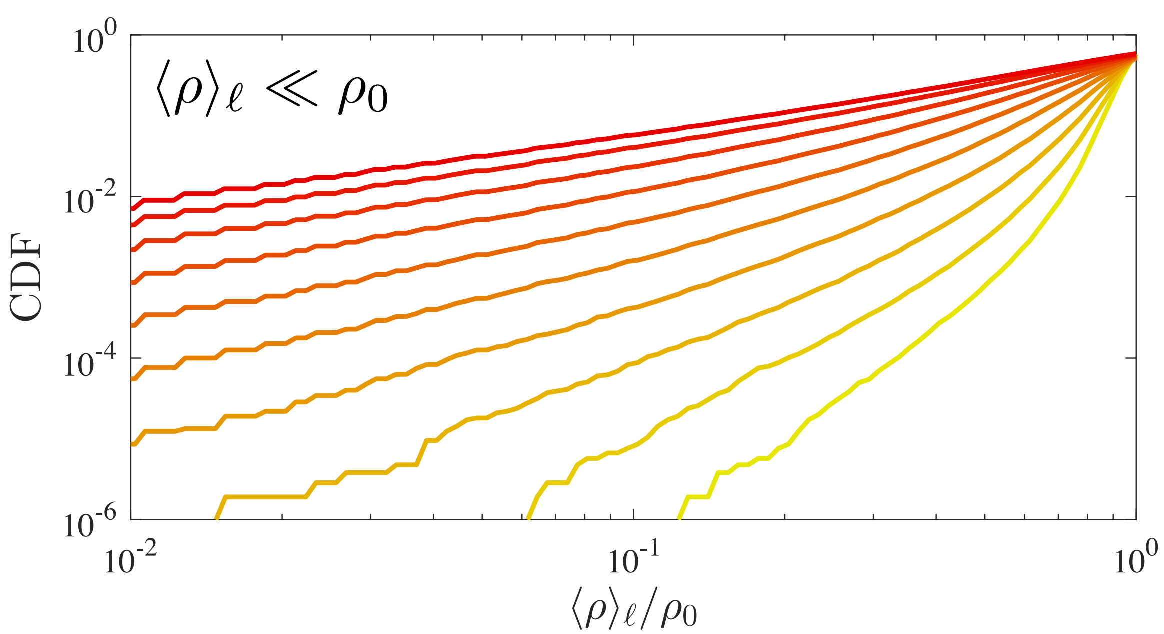
$$\Rightarrow \; p(m) \propto \exp(-c\,m\,\log m)$$

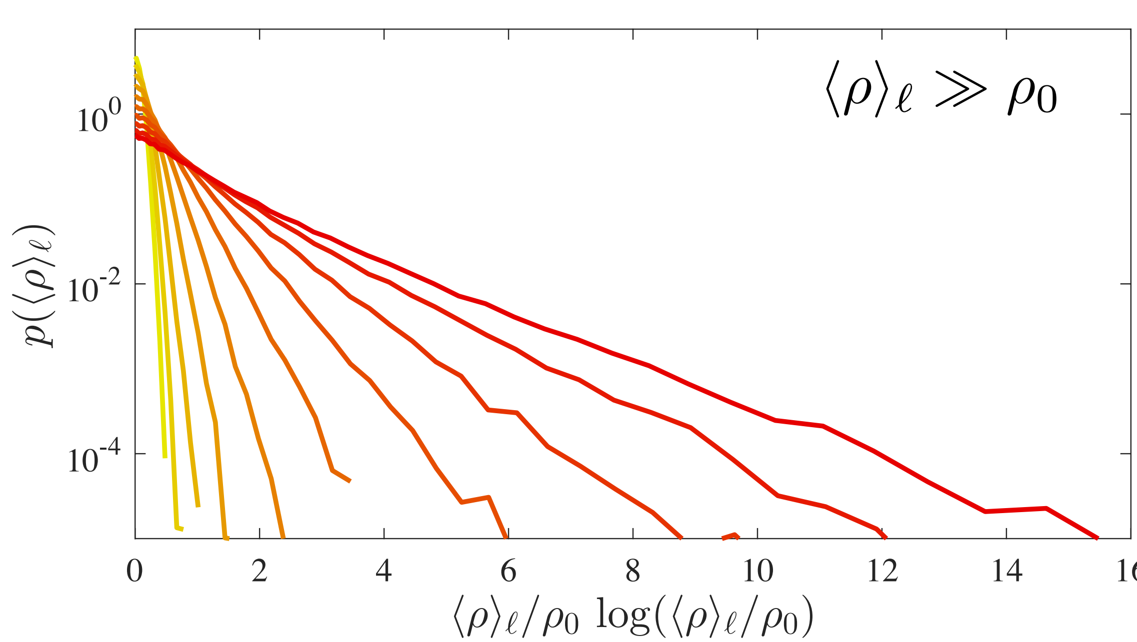
Coarse-grained distributions



Varying the Stokes number at a fixed scale

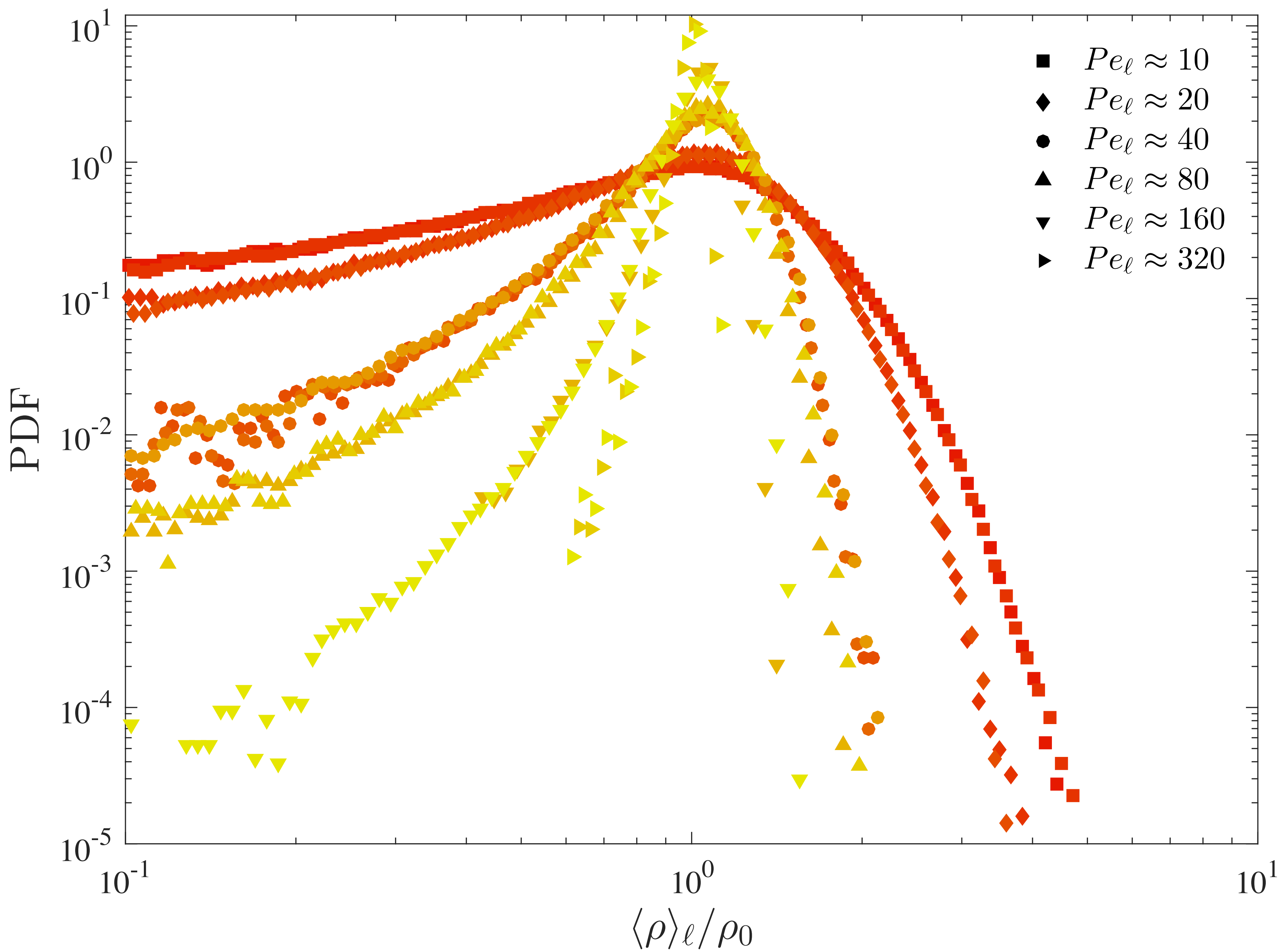
The two tails are well reproduced





Coarse-grained distributions

Varying both Stokes number and scale:
The distributions depend solely on Pe_ℓ

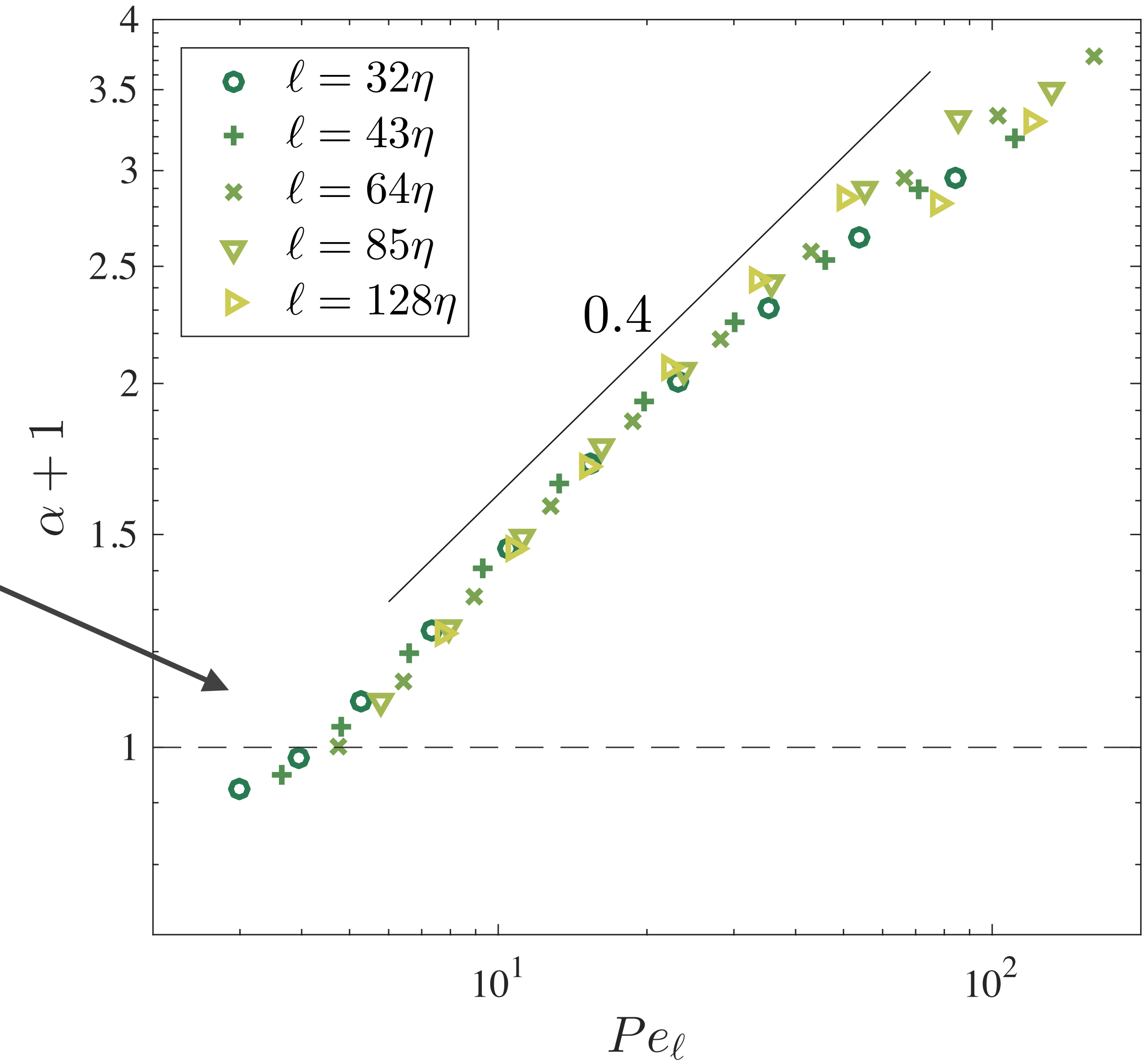


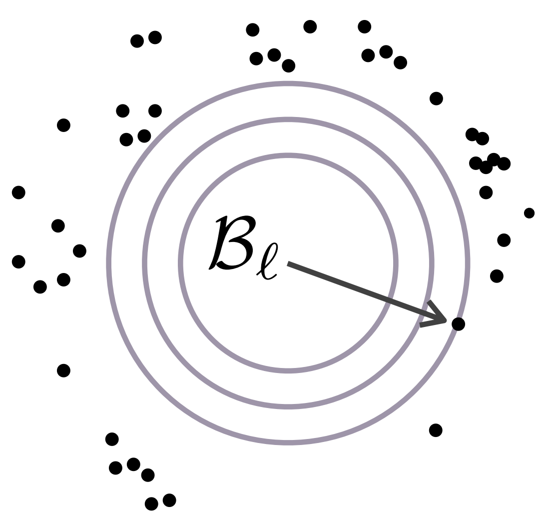
Connections with voids

$$p(\langle\rho\rangle_\ell) \propto \langle\rho\rangle_\ell^{\alpha(Pe_\ell)} \text{ when } \langle\rho\rangle_\ell \ll \rho_0$$

Transition?

$\alpha \equiv 0$ for $Pe_\ell < Pe_\star$





number density

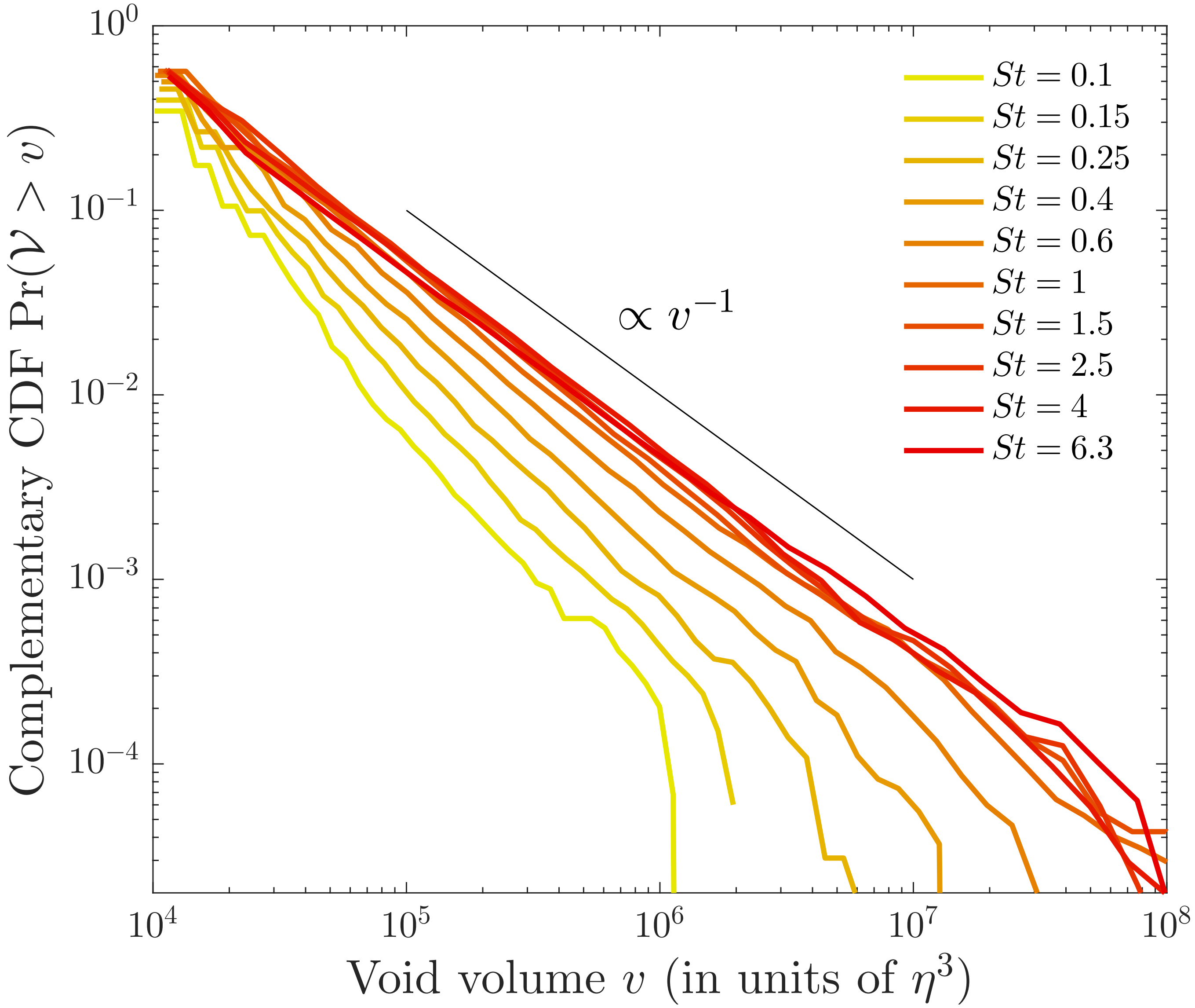
$$\langle\rho\rangle_\ell = \frac{\#(\mathcal{B}_\ell)}{\ell^3}$$

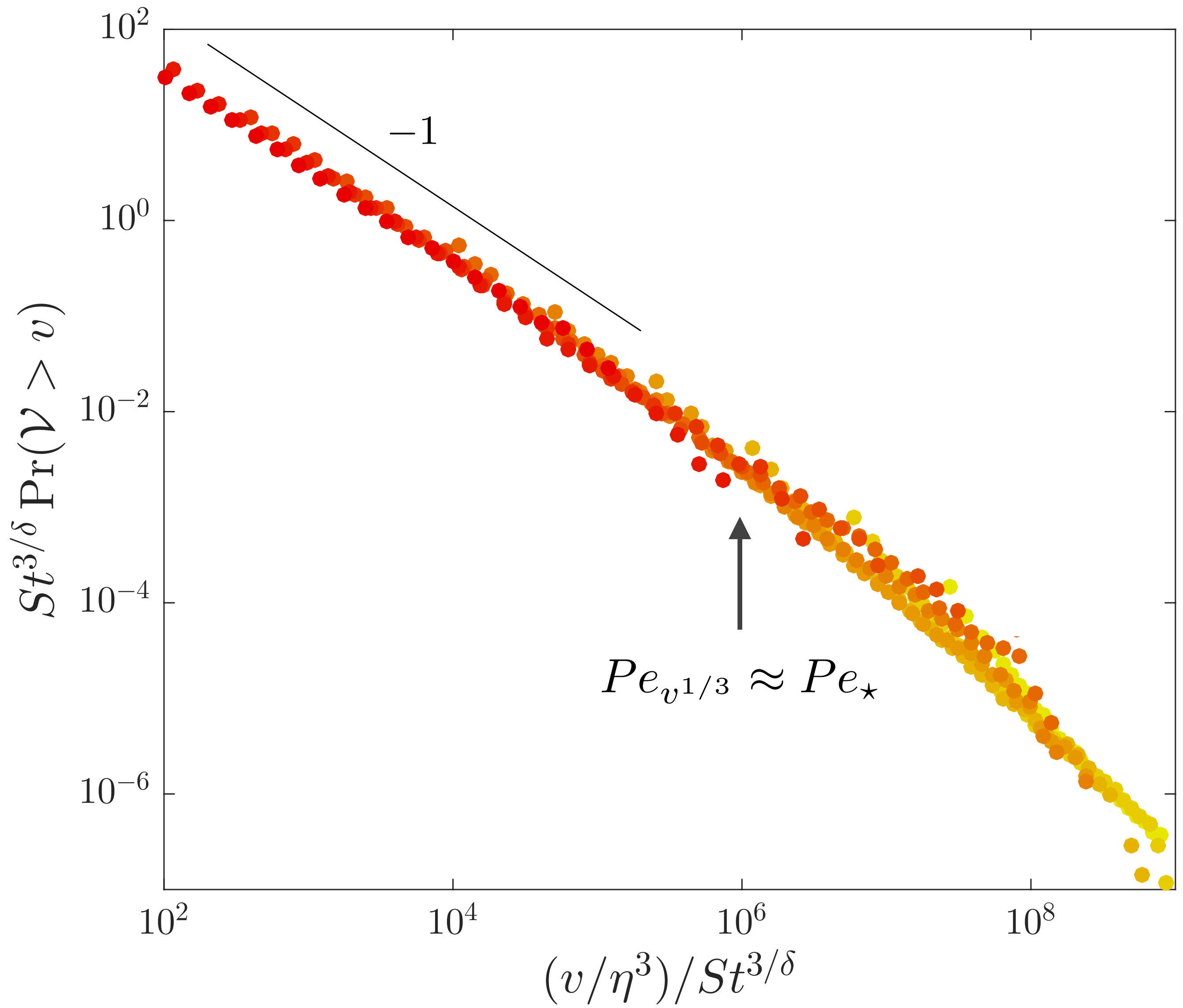
Void of volume $\mathcal{V}$ $\quad \Pr\left(\mathcal{V} > \ell^3\right) \simeq \Pr\left(\langle\rho\rangle_\ell < 1/\ell^3\right) \propto \ell^{-3(1+\alpha(Pe_\ell))}$

$$\Pr(\mathcal{V} > v) \propto \begin{cases} v^{-1} & \text{for } Pe_{v^{1/3}} < Pe_\star \\ \exp\left(-c\, v^\mu \log v\right) & \text{for } Pe_{v^{1/3}} > Pe_\star \end{cases}$$

Void distribution

Connected boxes of size ℓ that contain no particles





Summary / Conclusions

- Turbophoresis acts in statistically homogeneous flows because of their instantaneous non-uniformities
- Inertial-range particles dynamics can be described in terms of an effective diffusion equation with a space and time-dependent diffusivity determined by local turbulent activity
- The scaling properties of particle distributions can be inferred from such a model through a scale-dependent Peclet number. These include second-order moments of mass (RDF) and statistics of voids.