UNIVERSITY OF GOTHENBURG

Statistical models for alignment of spheroids in turbulence

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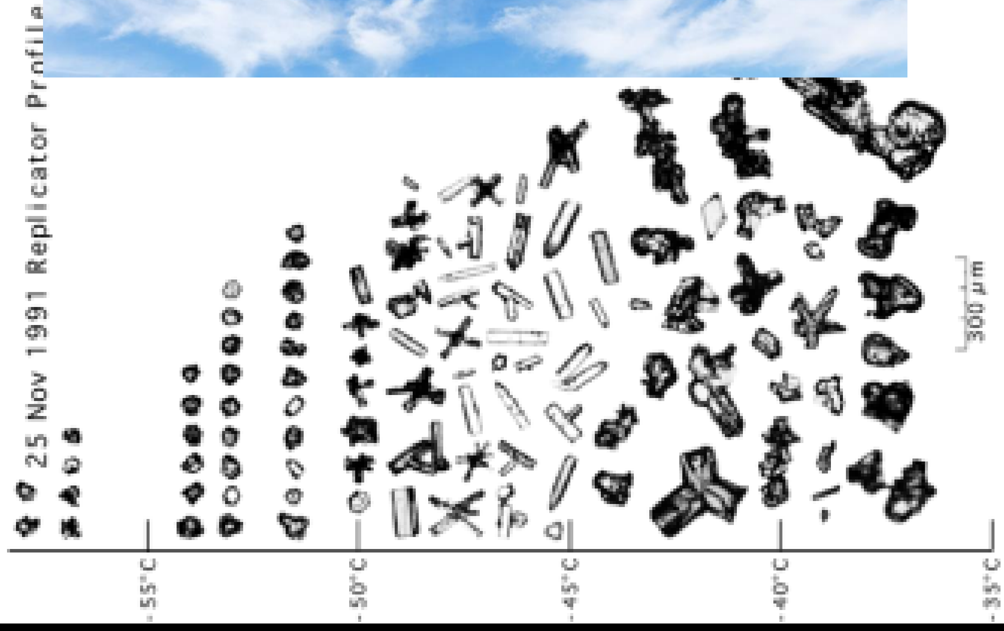
SANTA BARBARA OCT 27, 2022

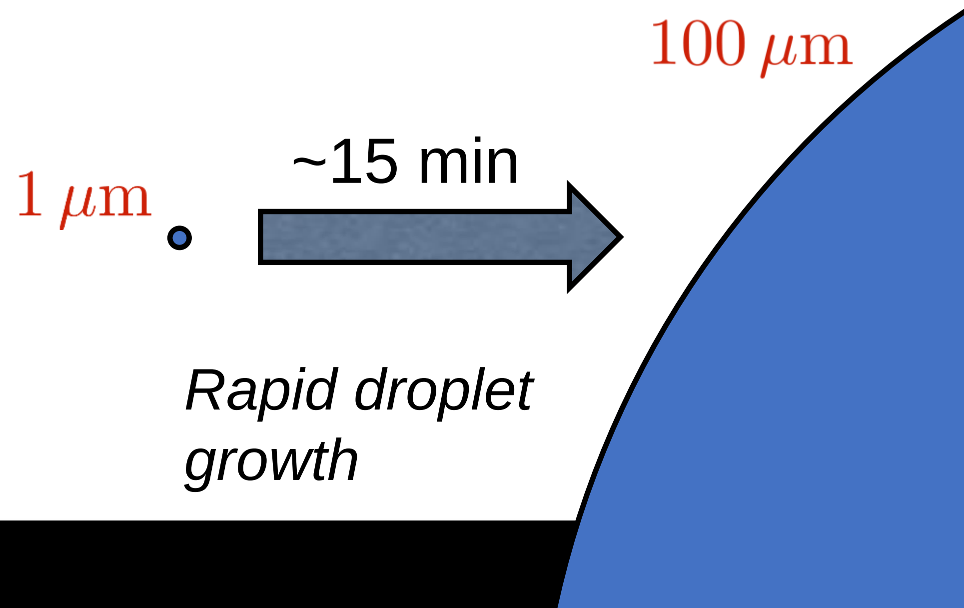
Small, dilute particles in turbulence

- Droplets and ice crystals in clouds
  Pruppacher and Klett, (Springer 1997)

Cumulus clouds

High altitude Cirrus clouds





Small, dilute particles in turbulence

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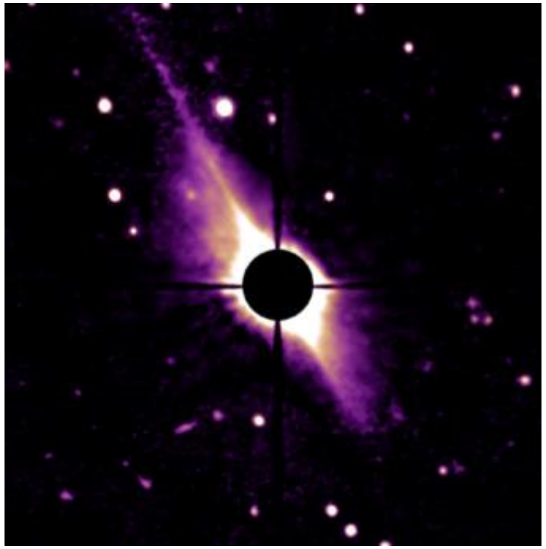
  Pruppacher and Klett, (Springer 1997)

- Dust grains in accretion disks
  Praburam and Goree, Astrophys. J. **441** (1995)



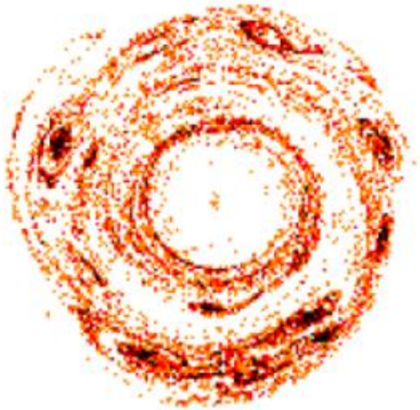


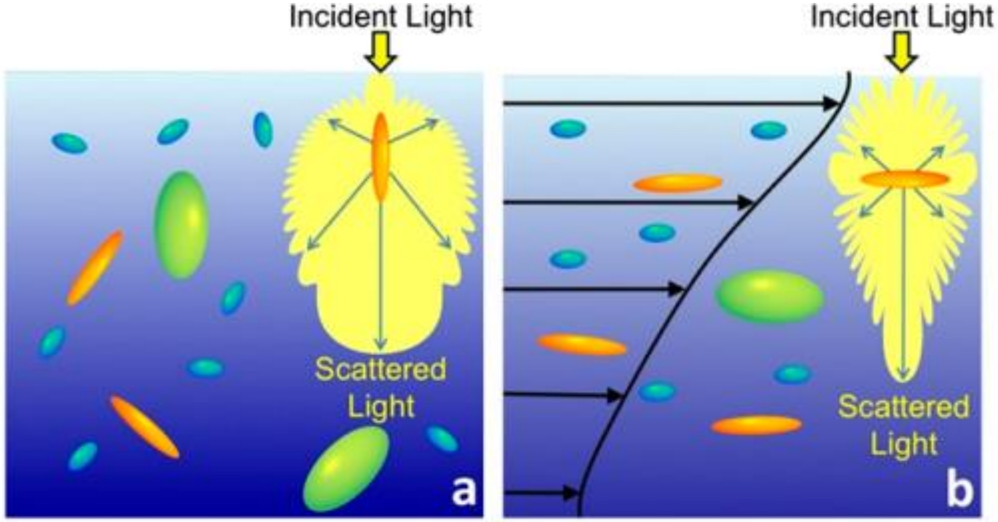
Photo and simulation of accretions disks

Small, dilute particles in turbulence

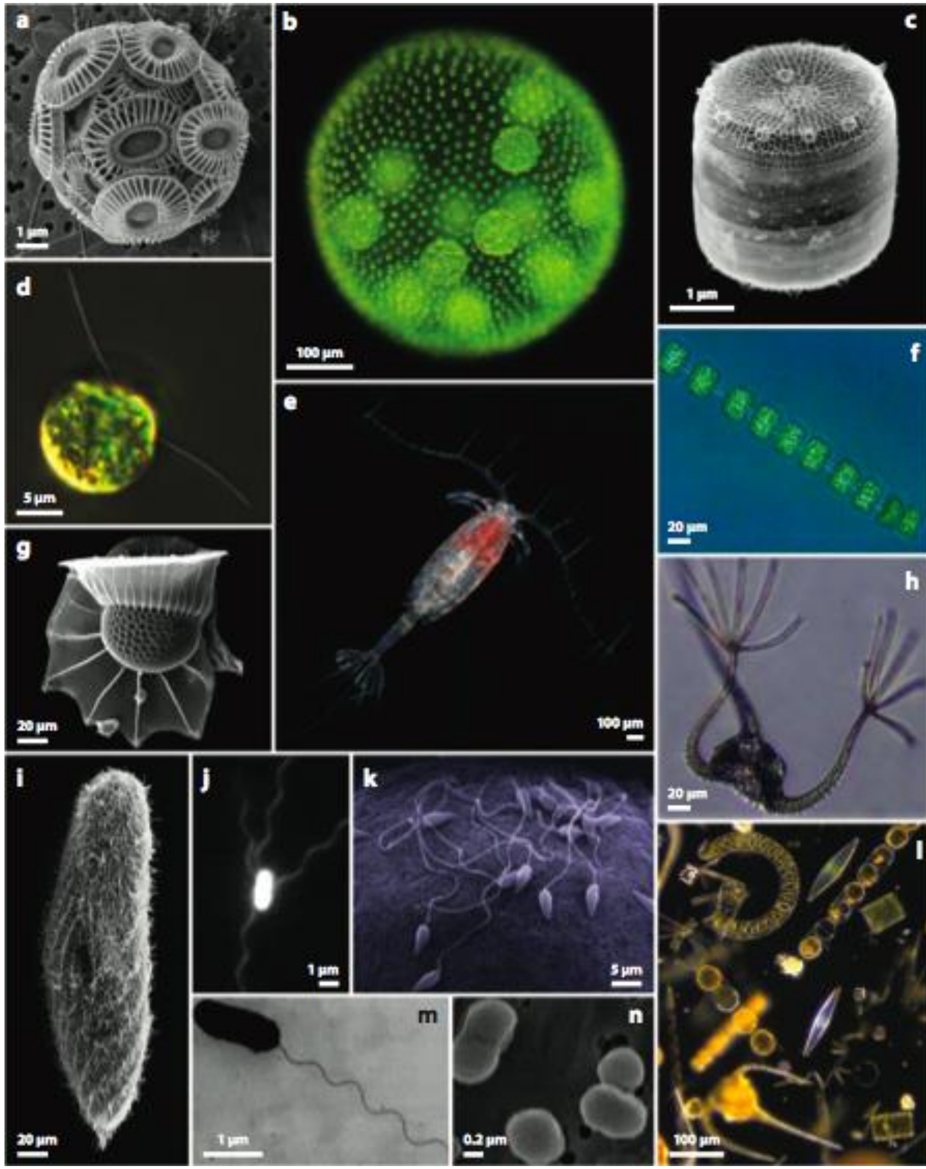
- Droplets and ice crystals in clouds
  Pruppacher and Klett, (Springer 1997)

- Dust grains in accretion disks
  Praburam and Goree, Astrophys. J. **441** (1995)

- Microswimmers and bacteriae
  Guasto et al, Annu. Rev. Fluid Mech. **44** (2012)



Increased backscattering in shear flow
Marcos et al, PNAS. **108** (2011)



KG & Mehlig, Adv. Phys. **65** (2016)

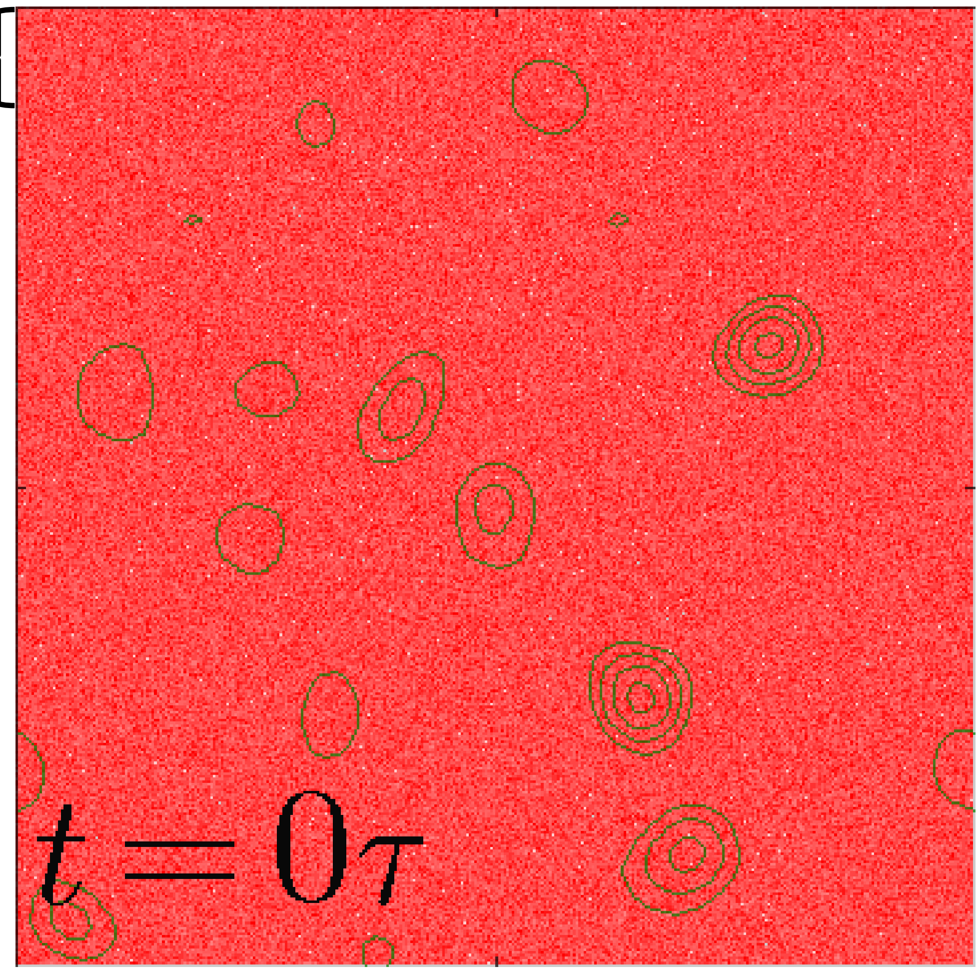
Statistical model

- Stationary incompressible random velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$
- No preferred direction or position in either space or time
- Single scale flow

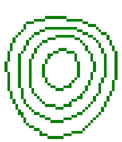
Length scale ℓ

Eulerian time scale τ

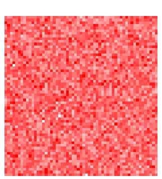
Advected time scale $\tau_{\mathrm{K}} \propto \dfrac{\ell}{u_{\mathrm{rms}}}$



- Kubo number $\mathrm{Ku} \propto \dfrac{\tau}{\tau_{\mathrm{K}}}$



Region of high vorticity

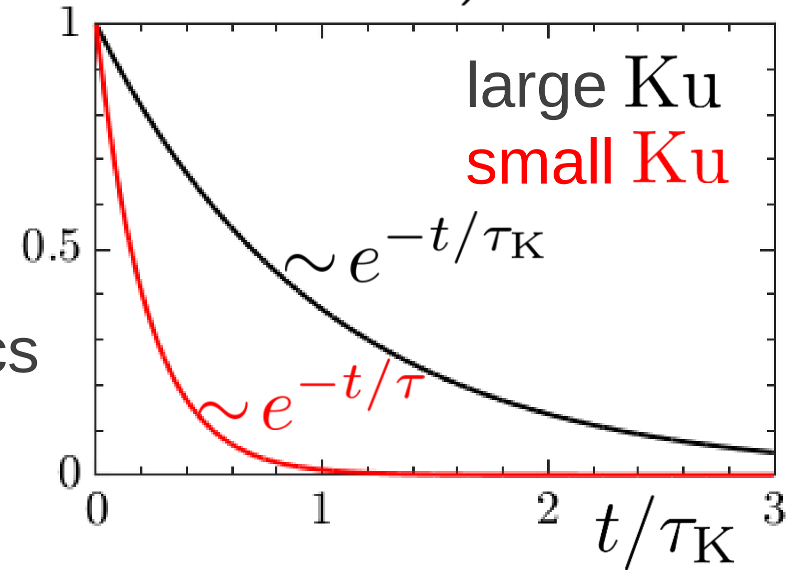


Particle probability density

Advantages of statistical models

- **Quick to simulate**

$$\mathrm{Tr}\left(\frac{\partial u_i}{\partial r_j}(\boldsymbol{x}_t, t)\frac{\partial u_i}{\partial r_j}(\boldsymbol{x}_0, 0)\right)\tau_{\mathrm{K}}^2$$



- **Tunable flow properties**

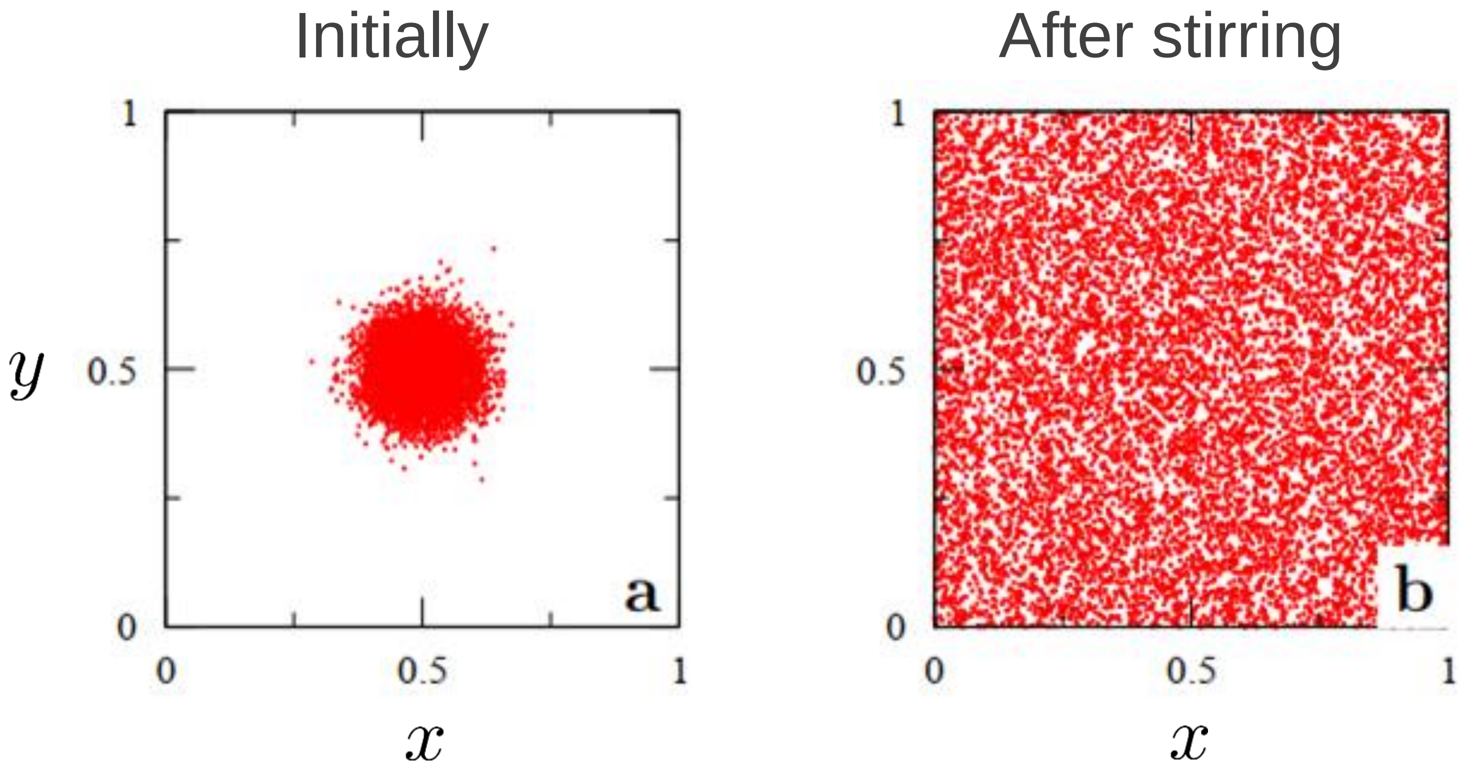
$$\mathrm{Ku} \propto \frac{\tau}{\tau_{\mathrm{K}}}$$

Allows to distinguish effects due to particle dynamics contra fluid dynamics

- **All flow statistics known**
  => Allows for analytical solutions
  - Fokker-Planck description
    KG et al., New J. Phys. **10** (2008); Wilkinson et al., Europhys. Lett. **89** (2010)
  - Trajectory expansion ('Kubo expansion')
    KG and Mehlig, Europhys. Lett. E **96** (2011); Adv. Phys. (2016)
  - Matched asymptotics
    KG and Mehlig, Phys. Rev. E **84** (2011); J. Turbulence **15** (2014)

Mixing by random stirring

- Computer simulation of 10^4 tracer particles (red) in two-dimensional random flow (periodic boundary conditions)



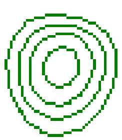
‘Unmixing’ of slightly inertial particles

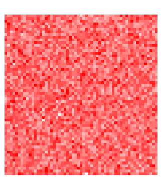
- Non-interacting, non-colliding particles (red) suspended in a random flow
- Stokes’ dynamics

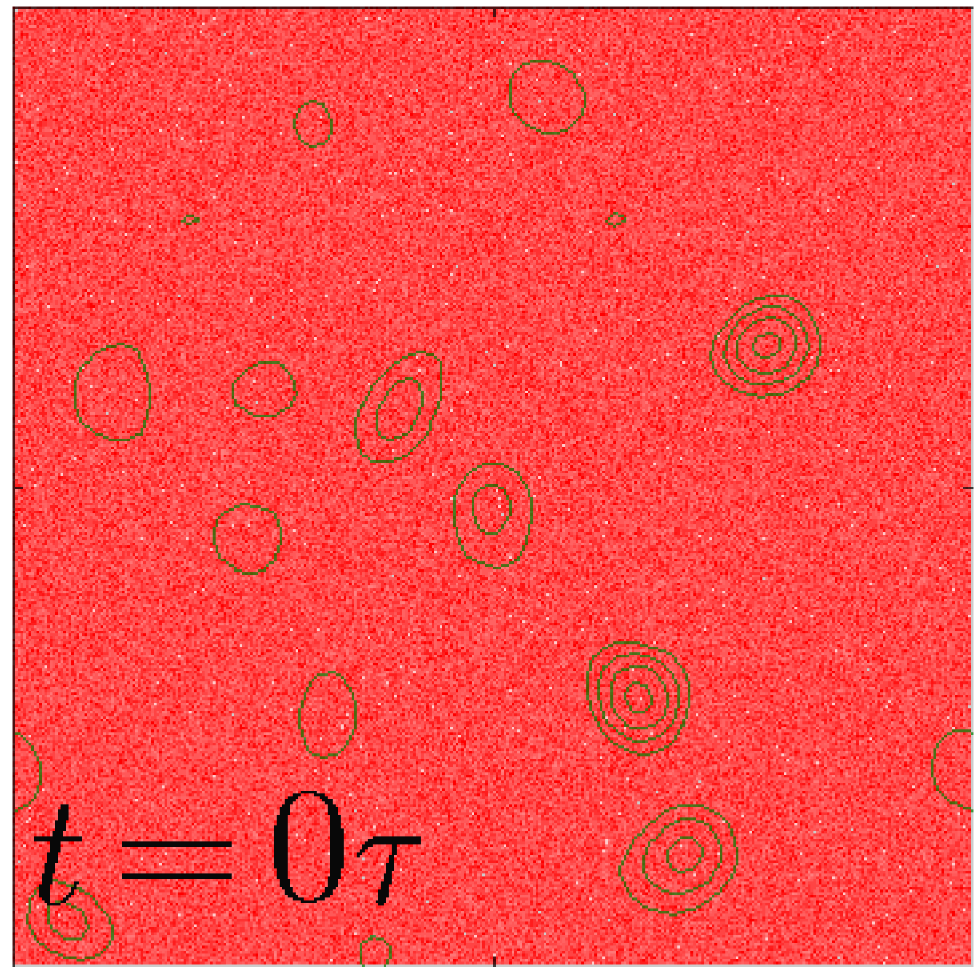
$$\ddot{\boldsymbol{r}} = \frac{1}{\tau_{\mathrm{p}}}(\boldsymbol{u}(\boldsymbol{r},t) - \dot{\boldsymbol{r}})$$

$$\mathrm{St} = \frac{\tau_{\mathrm{p}}}{\min(\tau,\tau_{\mathrm{K}})} = 0.05$$

$$\mathrm{Ku} \propto \frac{\tau}{\tau_{\mathrm{K}}} = 1$$

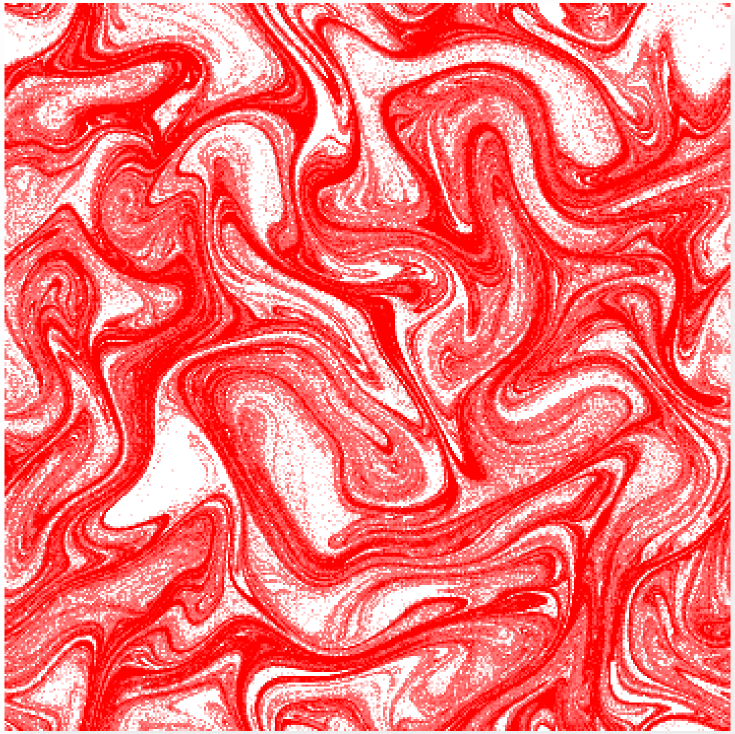

Region of high vorticity


Particle probability density

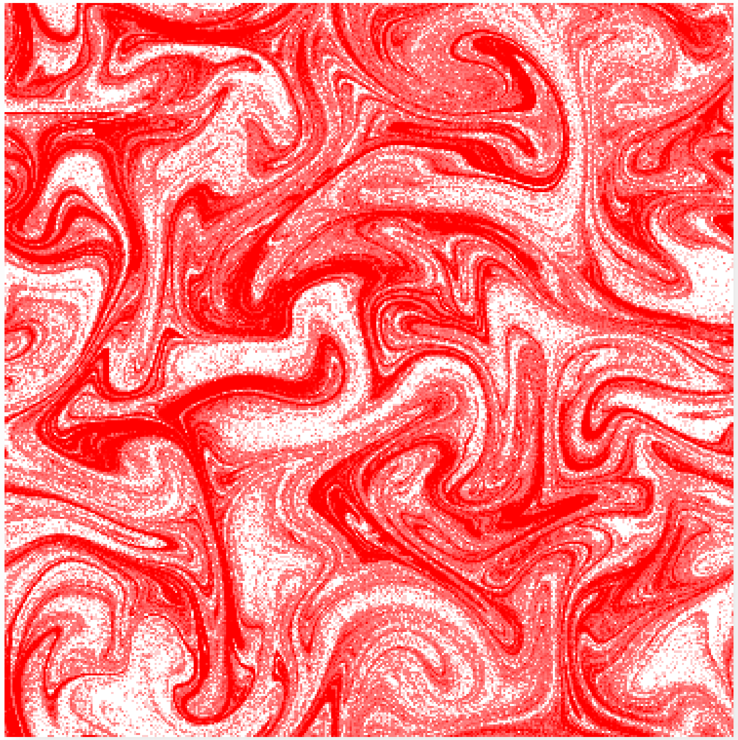


Comparison to compressible flows

A hint of what is going on...



Slightly inertial particles ($\mathrm{St} = 0.05$) in an incompressible random flow.



Tracer particles ($\mathrm{St} = 0$) in a compressible random flow.

Centrifuge mechanism

Maxey, J. Fluid Mech. **174**, 441, (1987)

- Inertial particles are centrifuged out of vortices

- For slightly inertial particles ($\mathrm{St} \ll 1$)

$$\boldsymbol{v} = \boldsymbol{u} - \tau_{\mathrm{p}}\Big[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\Big]$$

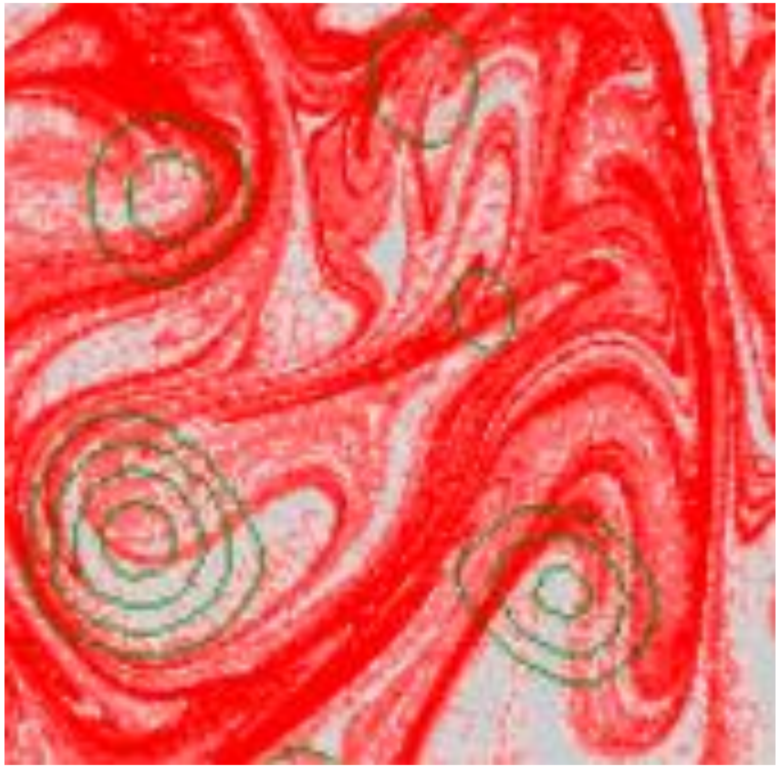
- Particles velocity field $\boldsymbol{v}$ is compressible

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = -\tau_{\mathrm{p}}\mathrm{Tr}\big[(\boldsymbol{\nabla}\boldsymbol{u})^2\big]$$

$$= -\tau_{\mathrm{p}}[\mathrm{Tr}(\mathbb{S}^2) - \mathrm{Tr}(\mathbb{O}^2)]$$

Strain Rotation

- Particles cluster due to long-lived flow structures



Particles avoid regions of high vorticity

Trajectory expansion

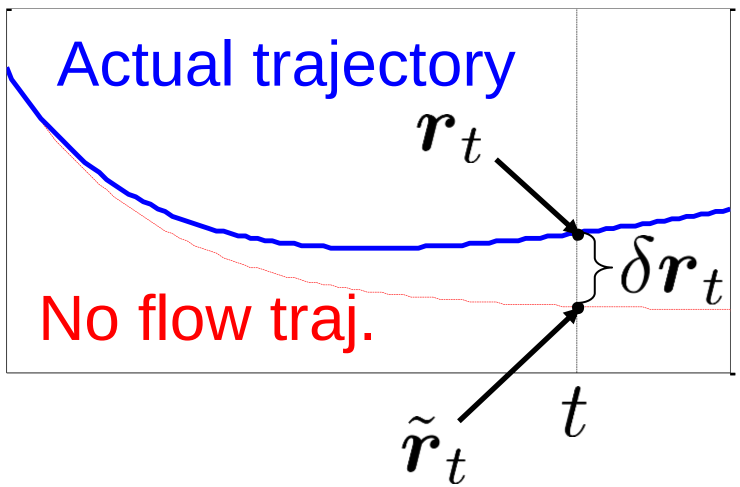
Dynamics in the absence of flow

$$\dot{\boldsymbol{r}} = \boldsymbol{v}$$
$$\dot{\boldsymbol{v}} = (\boldsymbol{u}(\boldsymbol{r}_t, t) - \boldsymbol{v})/\tau_\mathrm{p}$$

Deterministic solution

$$\tilde{\boldsymbol{r}}_t = \boldsymbol{r}_0 + \boldsymbol{v}_0\tau_\mathrm{p}(1 - e^{-t/\tau_\mathrm{p}})$$

with initial position $\boldsymbol{r}_0$ and velocity $\boldsymbol{v}_0$.



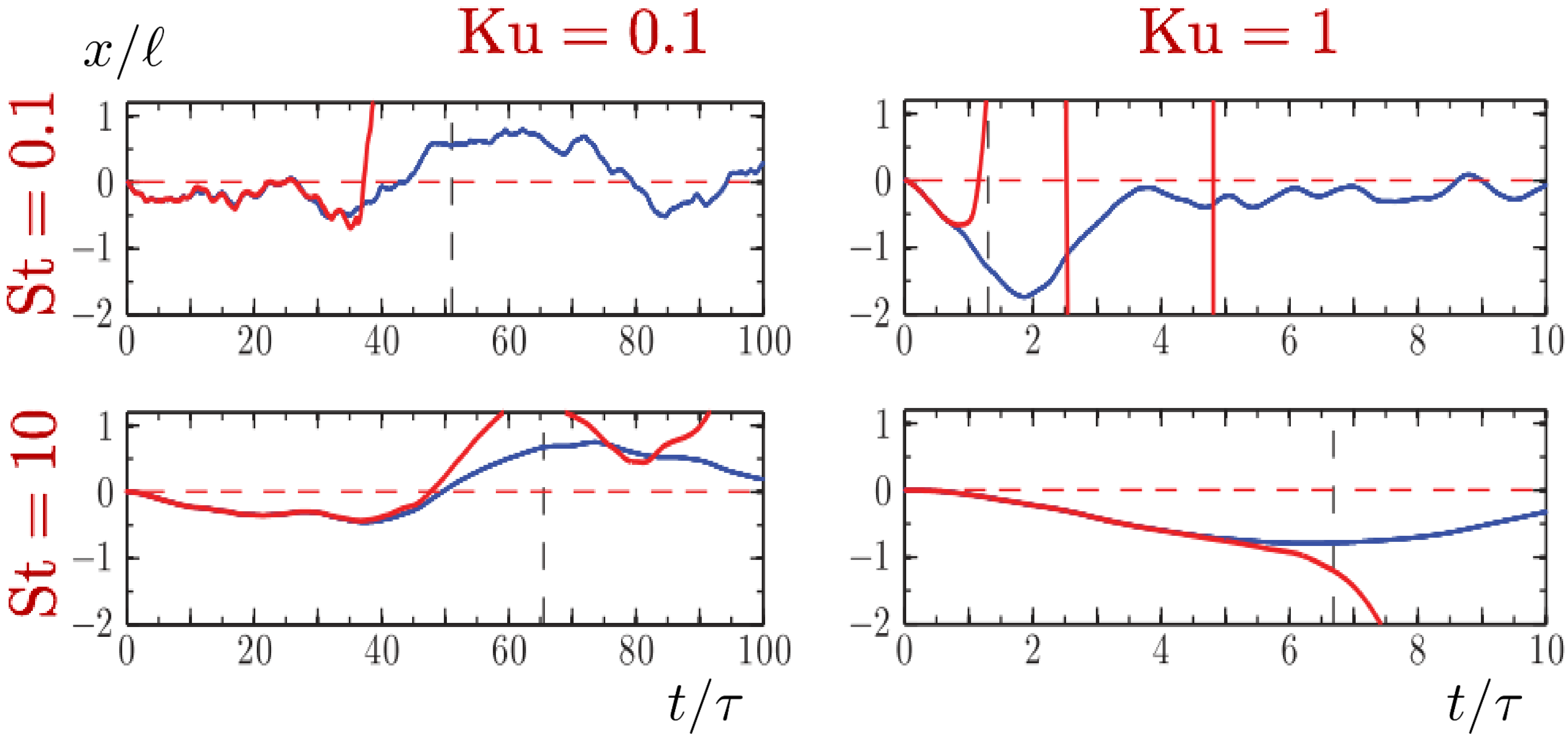
Trajectory with flow is $\boldsymbol{r}_t$

Expand flow in terms of small $\delta\boldsymbol{r}_t = \boldsymbol{r}_t - \tilde{\boldsymbol{r}}_t$

=> Can be evaluated using known flow statistics along $\tilde{\boldsymbol{r}}_t$

Trajectory approximation ($d = 1$)

Expansion works up to parameter-dependent time scale



Numerical trajectory

Analytical trajectory $O(\delta x_t^5)$ using $u(x_0, t)$,$\partial_x u(x_0, t)$ ...

DNS from Bec et al, Phys. Fluids **18** (2006)

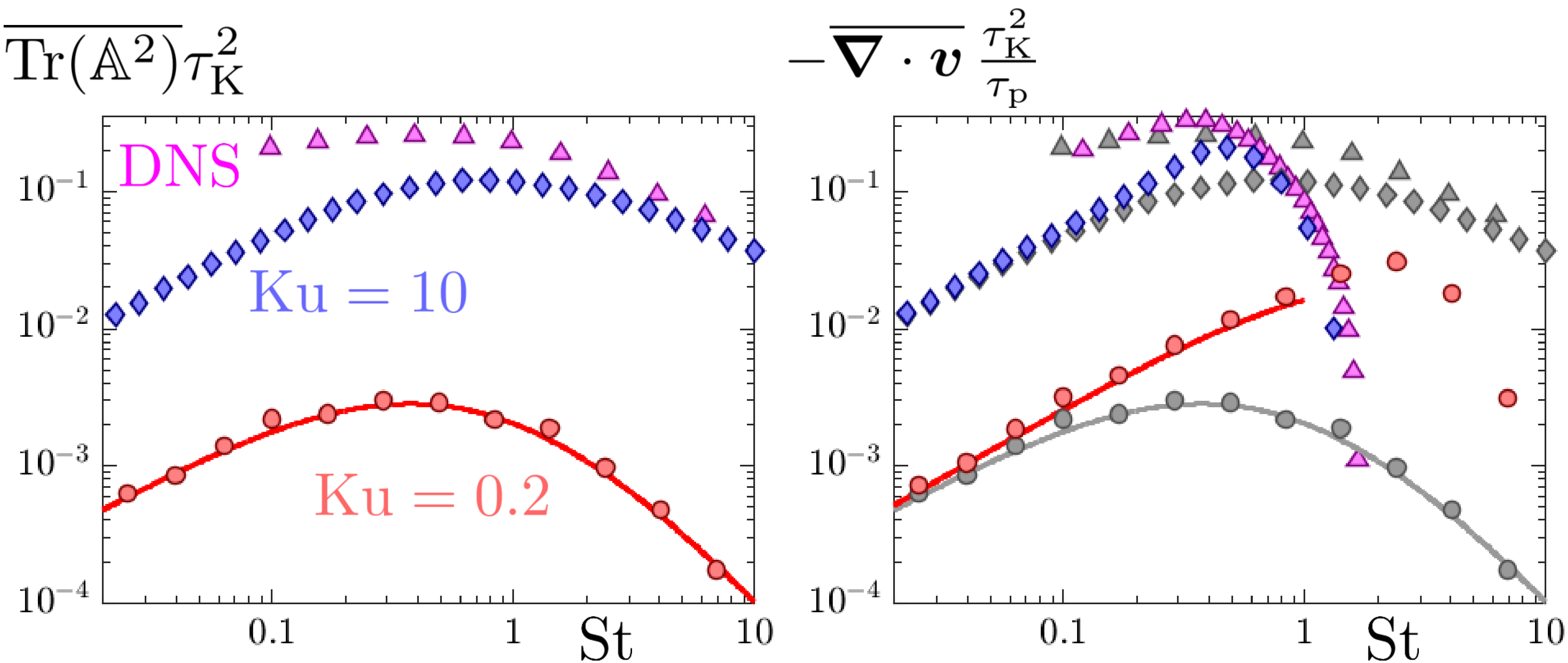
Preferential concentration

Centrifuge mechanism $\nabla \cdot \boldsymbol{v} = -\tau_{\rm p}\mathrm{Tr}(\mathbb{A}^2)$ with $\mathbb{A} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}$

Average along trajectories

$$\overline{\mathrm{Tr}(\mathbb{A}^2)}\tau_{\rm K}^2 = \tfrac{2}{3}\,\mathrm{Ku}^2\,\mathrm{St}\,\frac{1}{(1+\mathrm{St})^2(1+2\,\mathrm{St})} + \ldots$$

$$-\overline{\nabla \cdot \boldsymbol{v}}\,\frac{\tau_{\rm K}^2}{\tau_{\rm p}} = \tfrac{2}{3}\,\mathrm{Ku}^2\,\mathrm{St}\,\frac{1+3\,\mathrm{St}+\mathrm{St}^2}{(1+\mathrm{St})^3} + \ldots$$



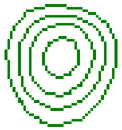
‘Unmixing’ of very inertial particles

- Non-interacting, non-colliding particles (red) suspended in a random flow
- Stokes’ dynamics

$$\ddot{\boldsymbol{r}} = \frac{1}{\tau_{\mathrm{p}}}(\boldsymbol{u}(\boldsymbol{r}, t) - \dot{\boldsymbol{r}})$$

$$\mathrm{St} = \frac{\tau_{\mathrm{p}}}{\min(\tau, \tau_{\mathrm{K}})} = 10$$

$$\mathrm{Ku} \propto \frac{\tau}{\tau_{\mathrm{K}}} = 0.1$$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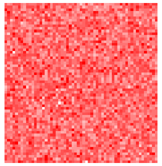
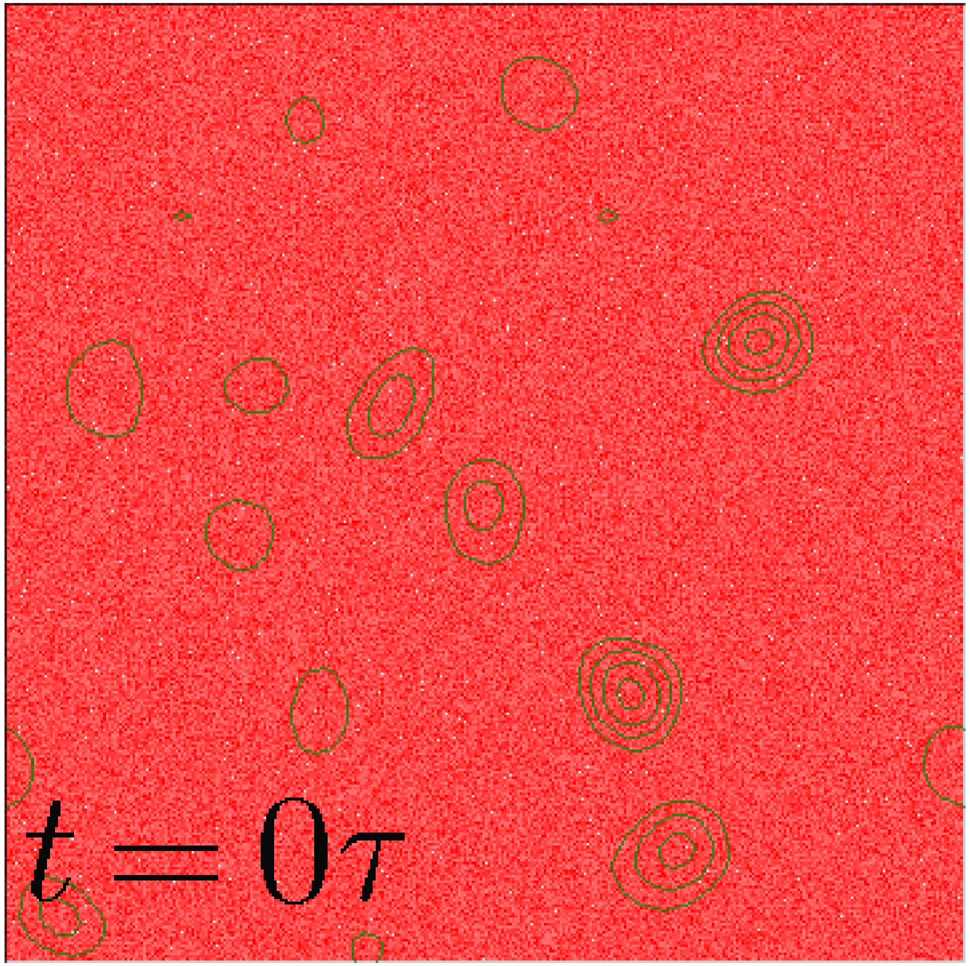
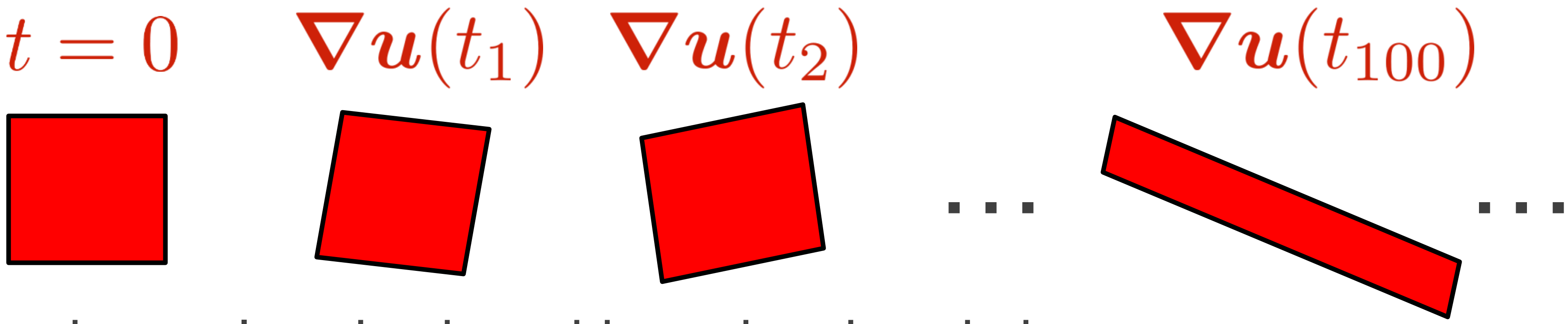

Particle probability density



Multiplicative amplification

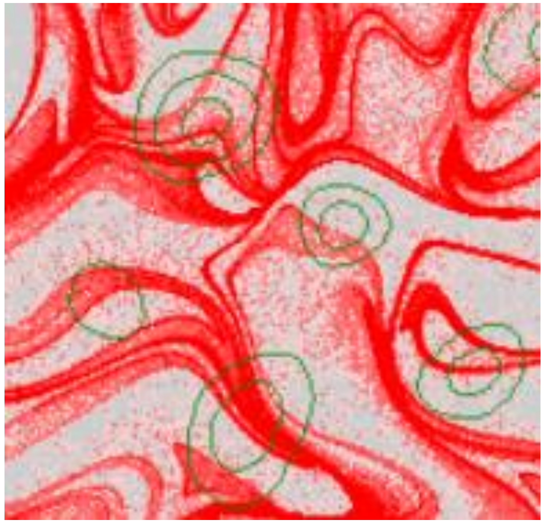
- Effect from history of many small independent fluid deformations



- Can be analyzed using white-noise description

Wilkinson et al., Phys. Fluids **19** (2007)

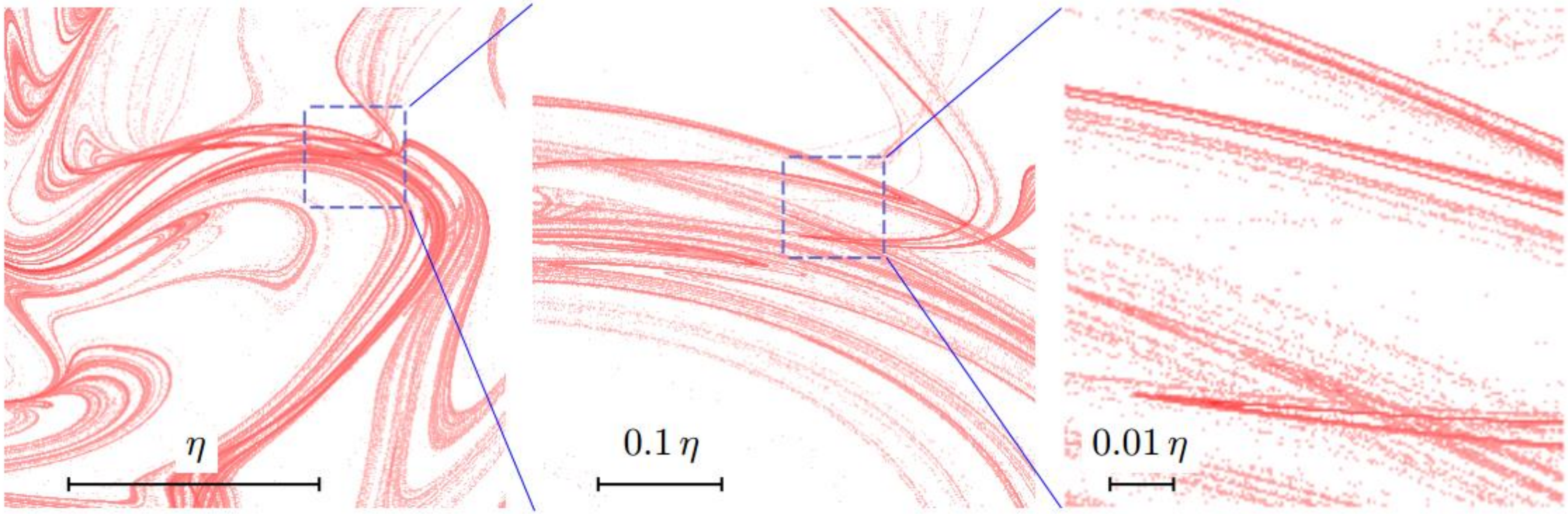
- Clustering uncorrelated to instantaneous flow structures



Small-scale fractal clustering

- Inertial particles cluster on self-similar structures, 'fractals'

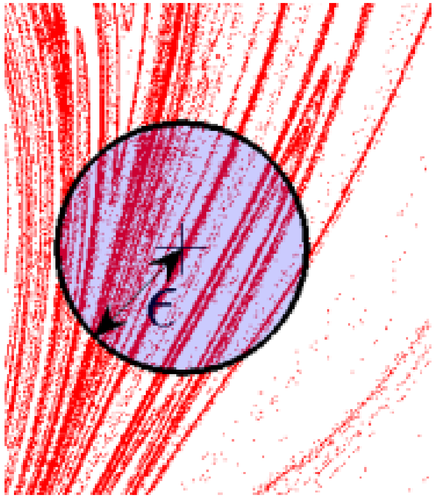
Sommerer & Ott, Science **259** , 334, (1993)



- Correlation dimension d_2

  Scaling of number of particles in sphere of radius ϵ

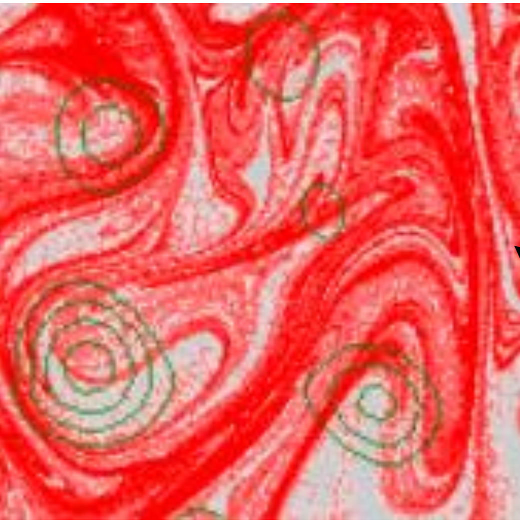
$$N \sim \epsilon^{d_2}$$



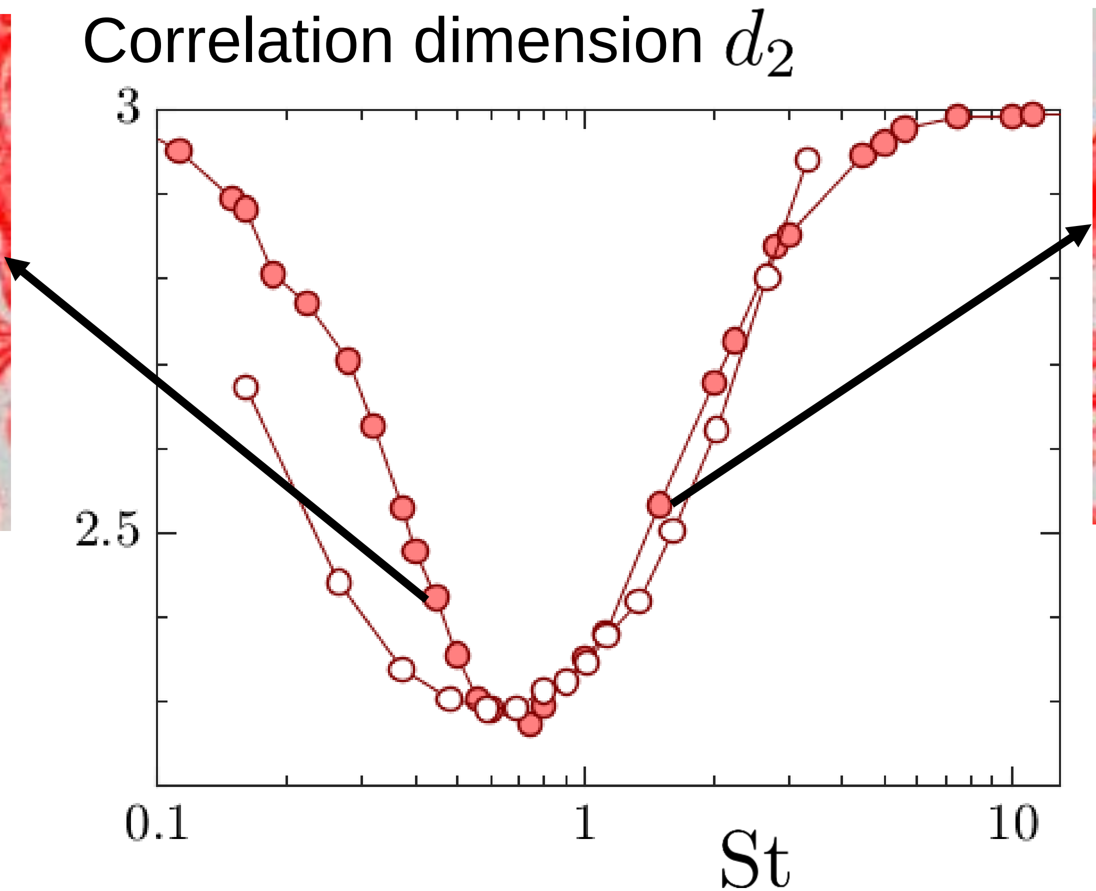
- Distribution of separations $P(R) \sim R^{d_2 - 1}$

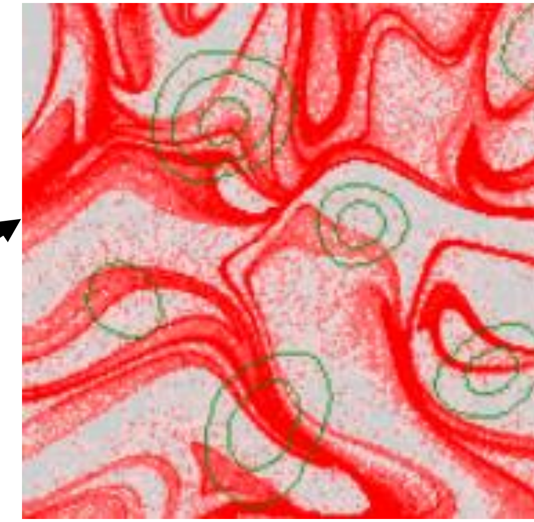
DNS from Bec et al, Phys. Rev. Let. **112** (2014)

Comparison to DNS



Preferential concentration





Preferential concentration + Multiplicative amplification

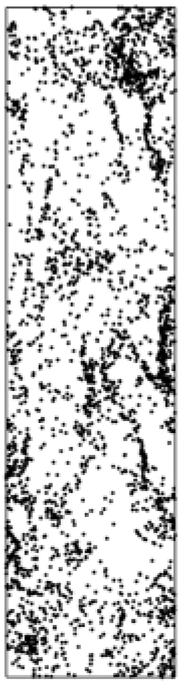
Model DNS

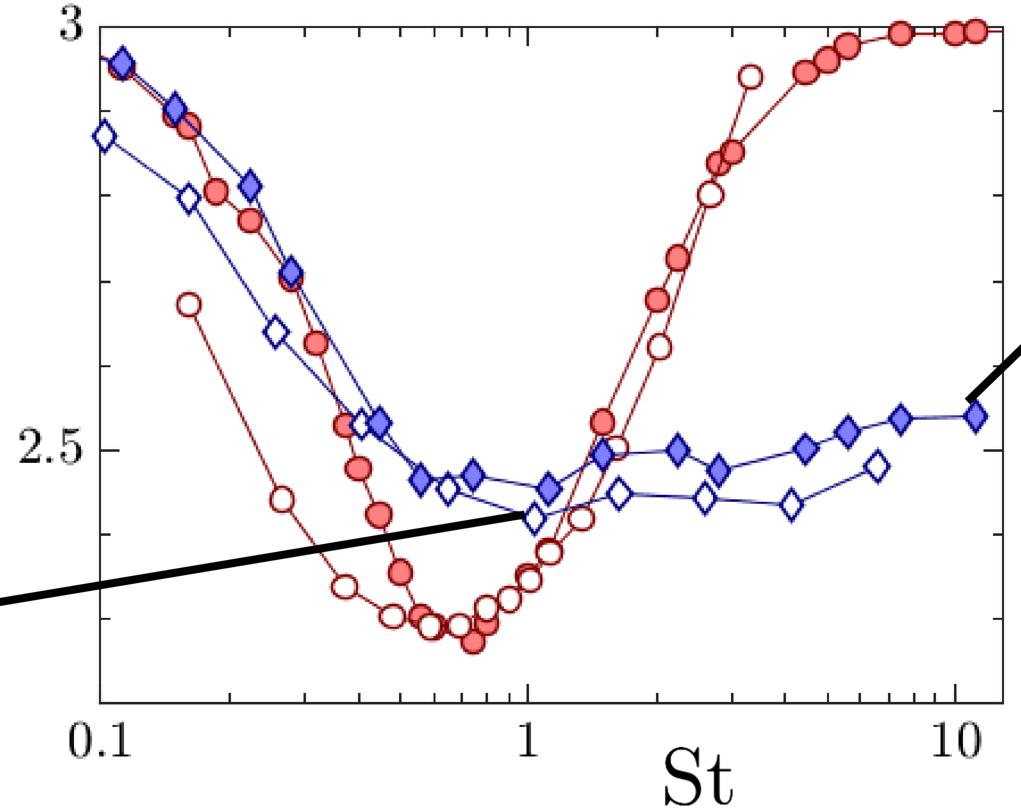
Comparison to DNS

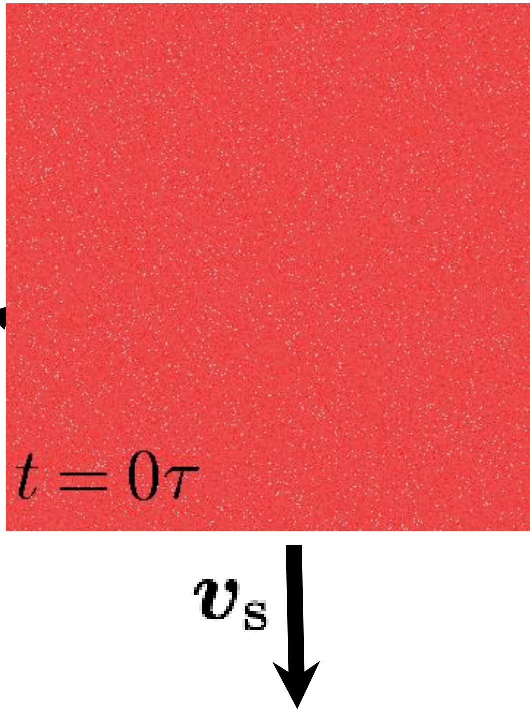
DNS from Bec et al, Phys. Rev. Let. **112** (2014)
KG, Vajedi & Mehlig, Phys. Rev. Lett. **112** (2014)

Correlation dimension d_2

Multiplicative amplification







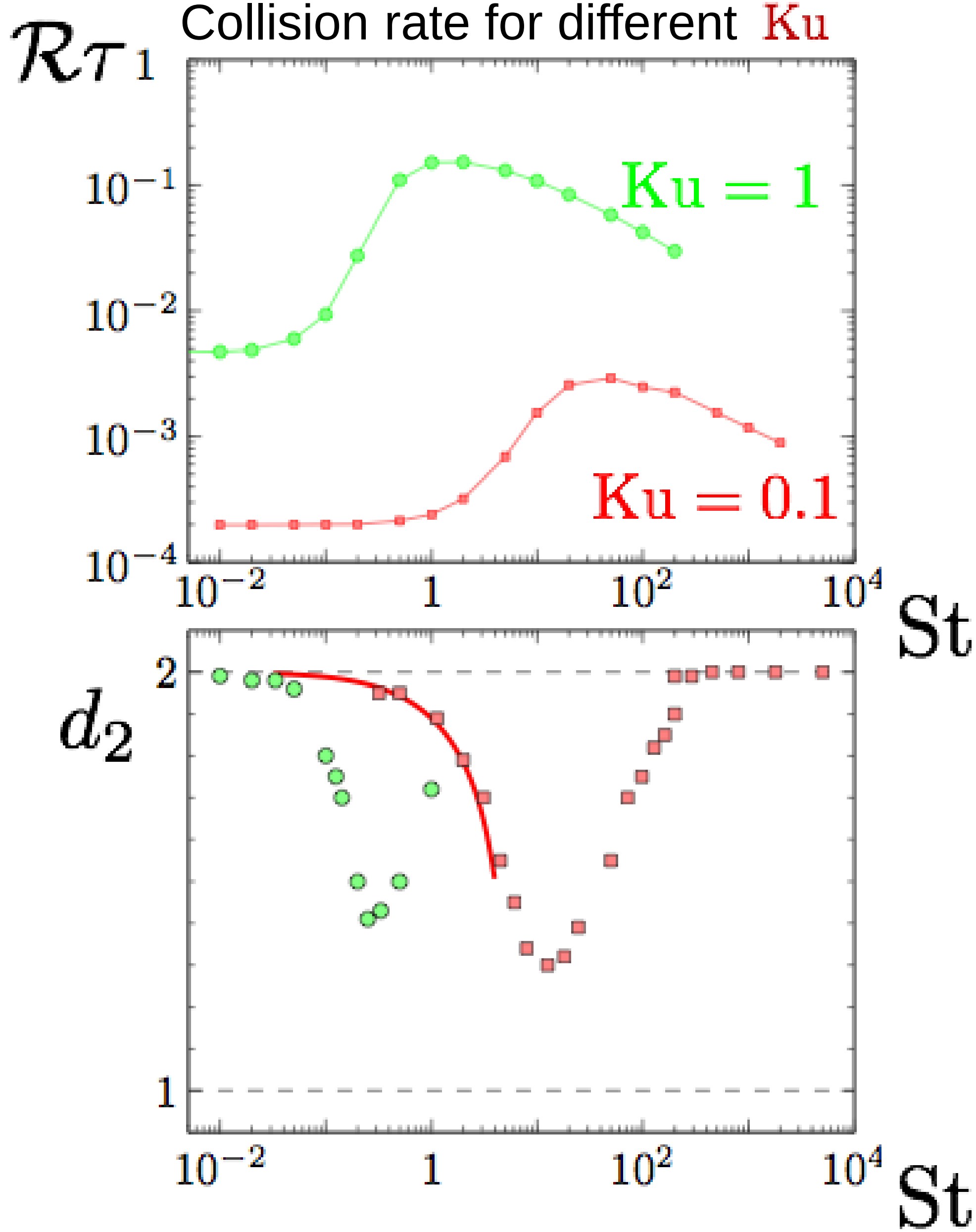
Froude number

$$\mathrm{Fr} = \frac{\eta_\mathrm{K}}{g\tau_\mathrm{K}^2}$$

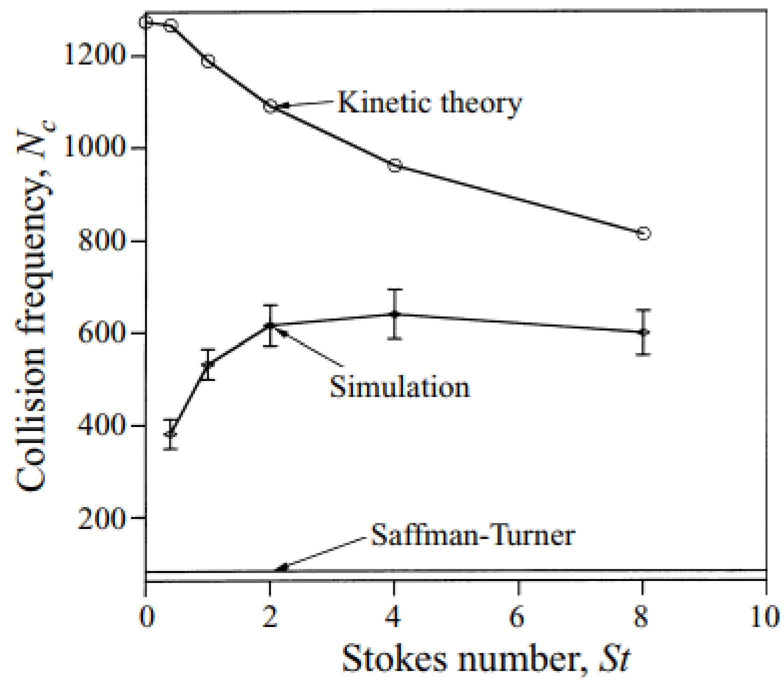
	Model	DNS
$\mathrm{Fr} = \infty$	● (filled circle)	○ (open circle)
$\mathrm{Fr} = 0.05$	◆ (filled diamond)	◇ (open diamond)

The rate of collisions

Collision rate for different Ku



Collision velocities in DNS



Sundaram & Collins, J. Fluid Mech. **335** (1997)

Explanation:

‘Caustics’

Falkovich et. al, Nature **419**, 151, (2002)
Wilkinson et. al, Phys. Rev. Lett. **97** (2006)

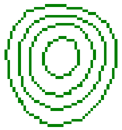
Collisions due to caustics

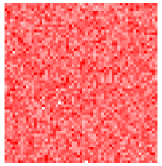
- Non-interacting, non-colliding particles (red) suspended in a random flow
- Stokes' dynamics

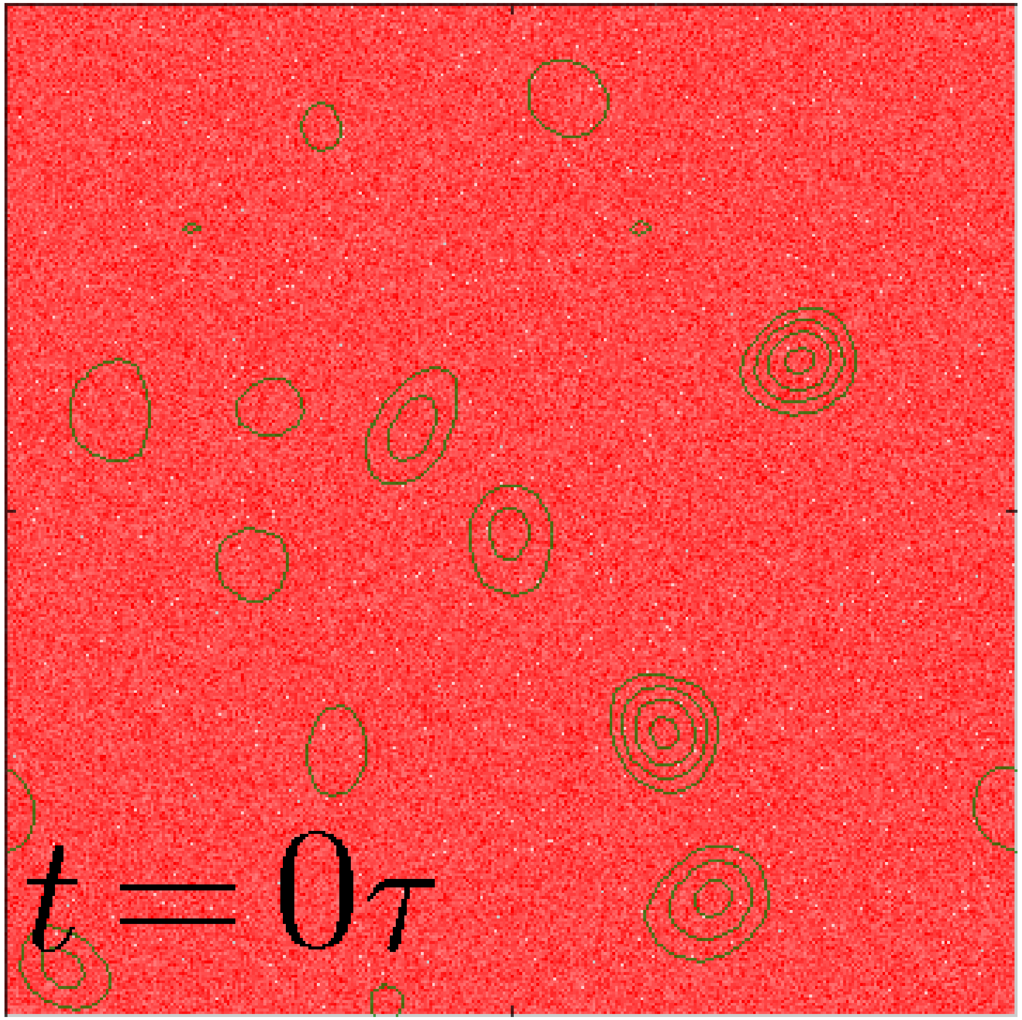
$$\ddot{\boldsymbol{r}} = \frac{1}{\tau_{\rm p}}(\boldsymbol{u}(\boldsymbol{r},t) - \dot{\boldsymbol{r}})$$

$$\mathrm{St} = \frac{\tau_{\rm p}}{\min(\tau,\tau_{\rm K})} = 5$$

$$\mathrm{Ku} \propto \frac{\tau}{\tau_{\rm K}} = 1$$

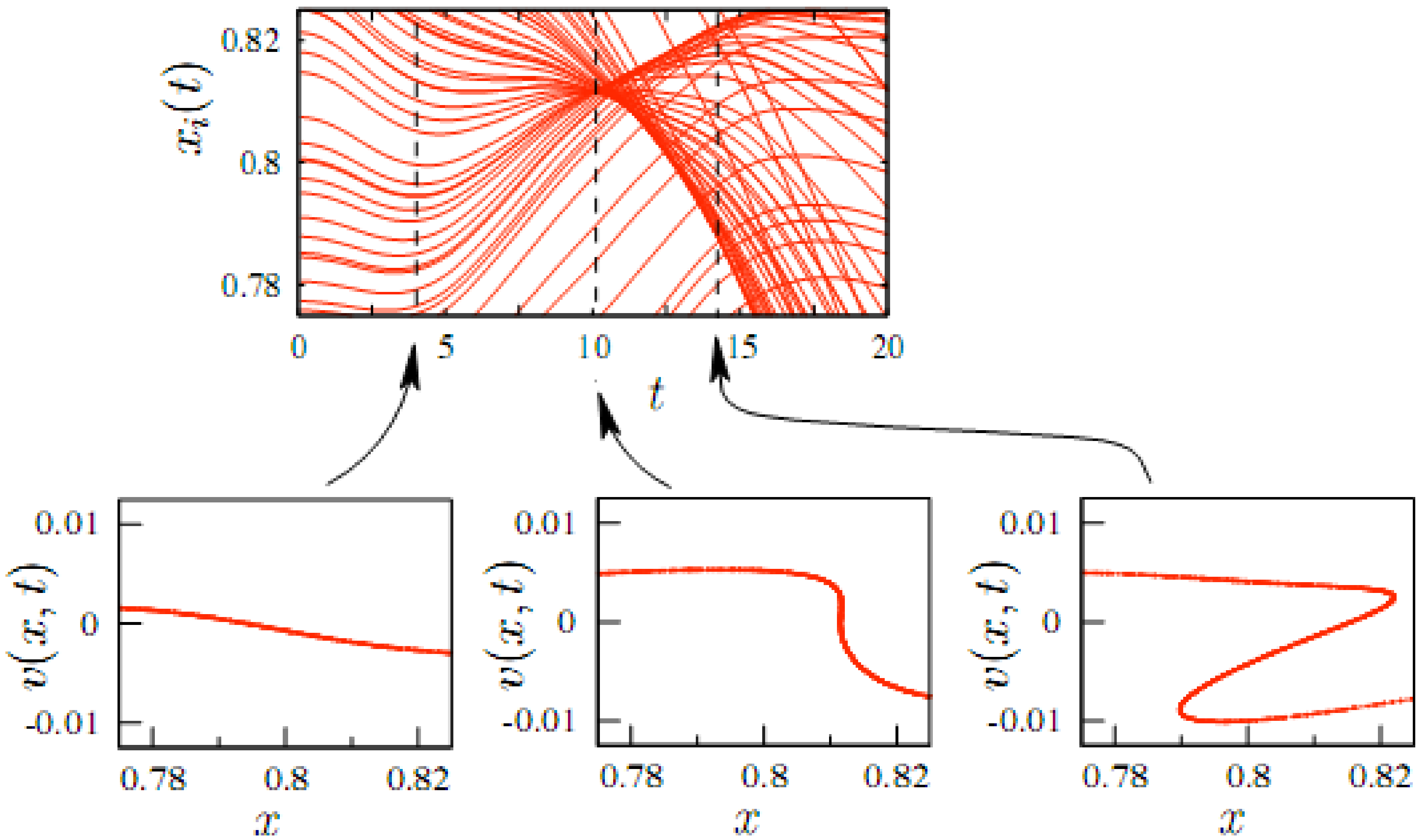

Region of high vorticity


Particle probability density



Formation of a caustic $(d = 1)$

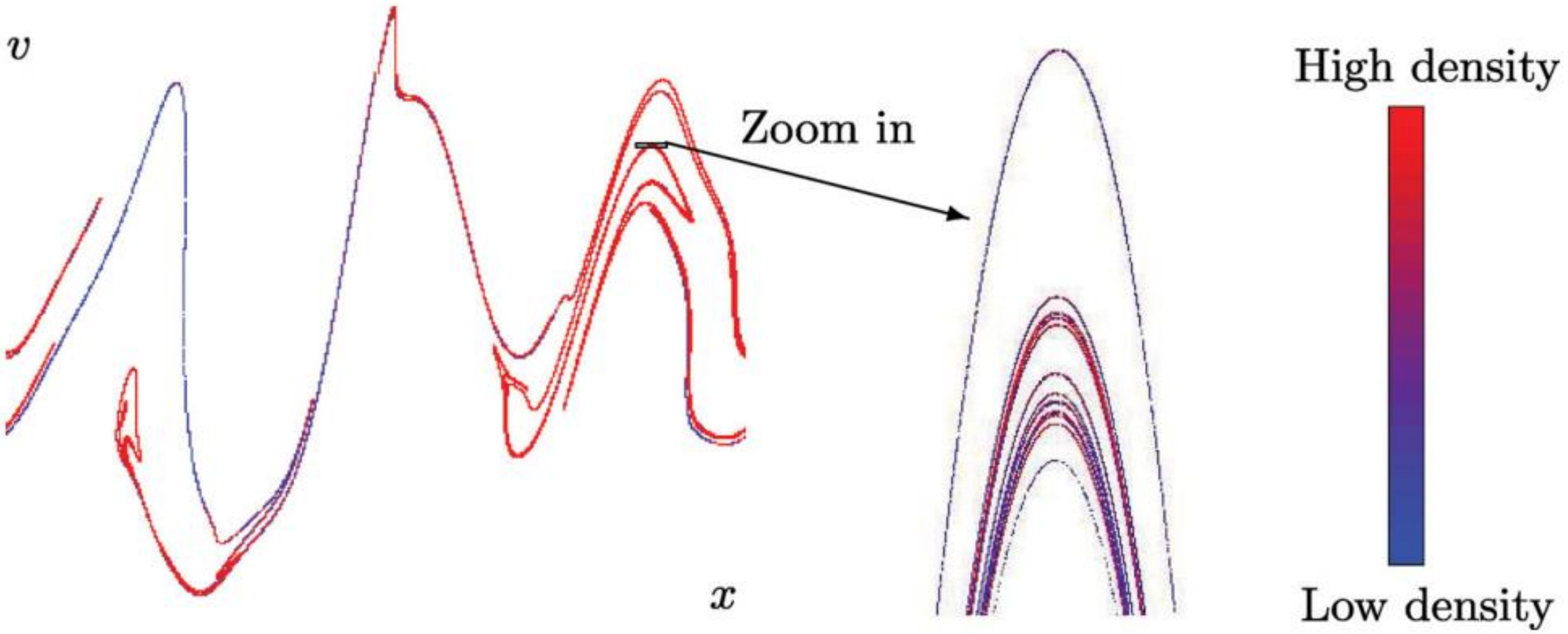
- Trajectories following $\ddot{x} = \frac{1}{\tau_\mathrm{p}}(\boldsymbol{u}(x,t) - \dot{x})$



KG & Mehlig, J. Turbulence **15** (2014)

Particle positions at large times ($d = 1$)

- Snapshot of positions following $\ddot{x} = \frac{1}{\tau_\mathrm{p}}(\boldsymbol{u}(x,t) - \dot{x})$



- Particles distribute on fractal in phase space with phase-space fractal dimension D_2. Here $D_2 \approx 0.24$.

KG & Mehlig, J. Turbulence **15** (2014)

Trajectories of separations ($d=1$)

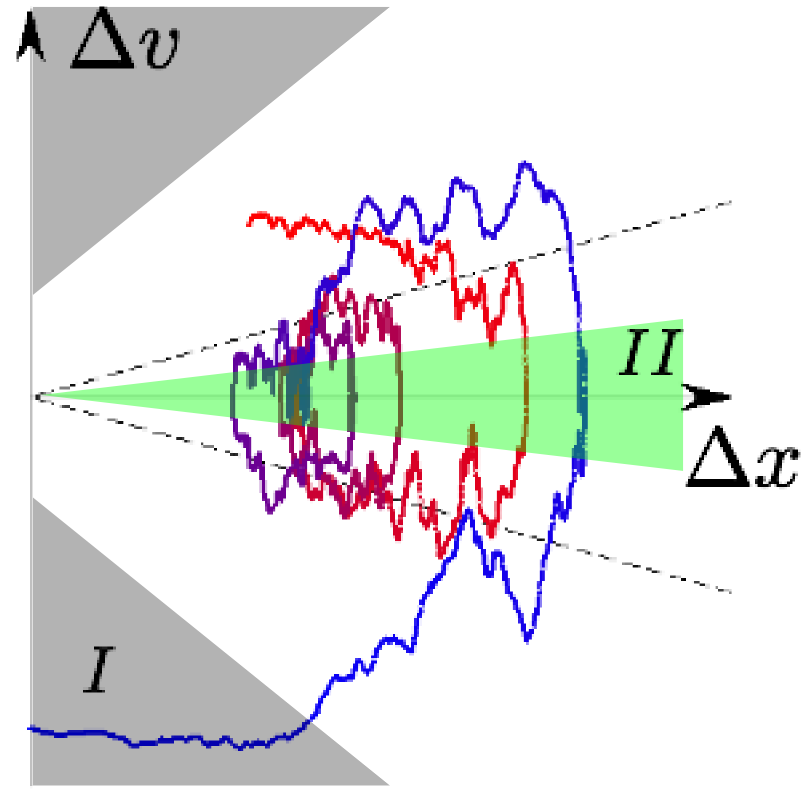
- Relative motion of two particles with separation Δx and relative velocity Δv.

- Case I ($|\Delta v| \gg |\Delta x|$)
  Dynamics $\Delta v \approx \mathrm{const.}$

$$\rho(\Delta v, \Delta x) \sim f_I(\Delta v)$$

- Case II ($|\Delta v| \ll |\Delta x|$)
  Dynamics $\Delta x \approx \mathrm{const.}$

$$\rho(\Delta v, \Delta x) \sim f_{II}(\Delta x)$$

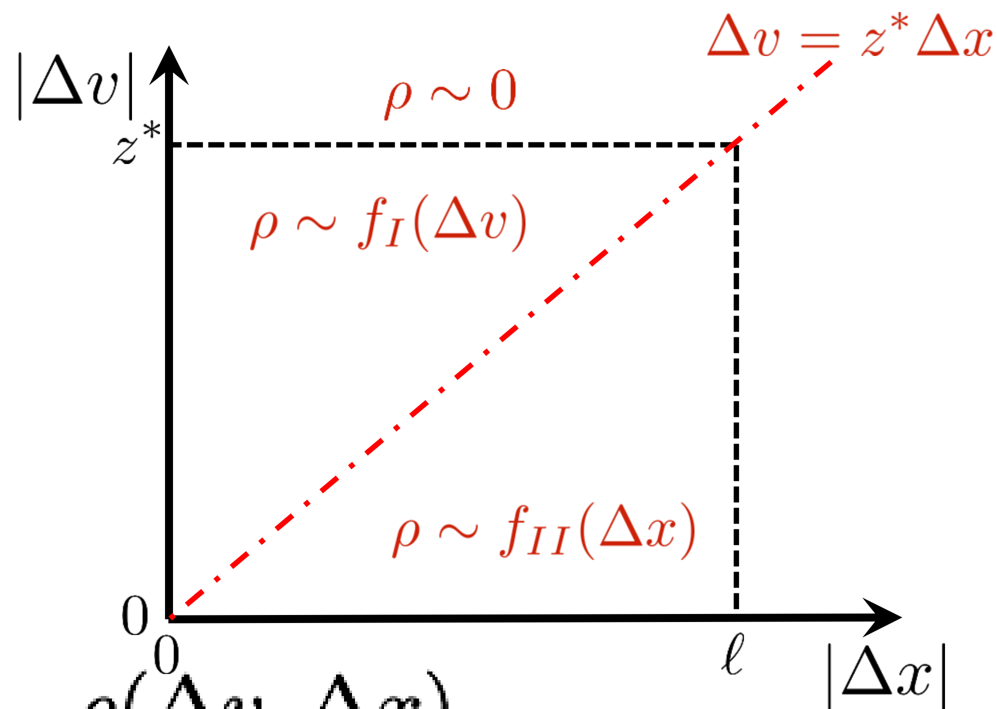


Example of relative trajectory between two droplets.

KG & Mehlig, J. Turbulence **15** (2014)

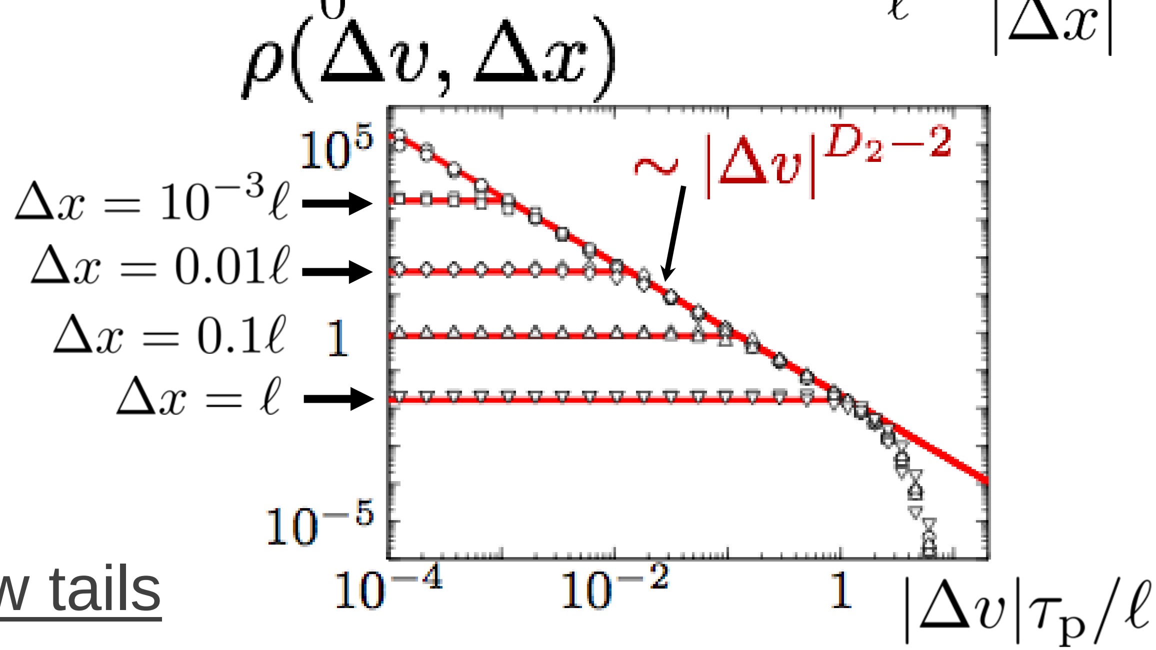
Distribution of relative velocities ($d = 1$)

- Match asymptotes along line $\Delta v = z^* \Delta x$



- Scaling $P(w) \sim w^{D_2 - 1}$ with phase-space correlation dimension D_2 gives

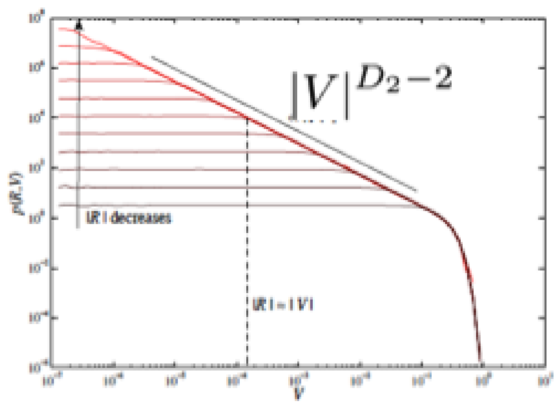
$$f_I(\Delta v) \sim |\Delta v|^{D_2 - 2}$$
$$f_{II}(\Delta x) \sim |\Delta x|^{D_2 - 2}$$



- Distribution has power-law tails

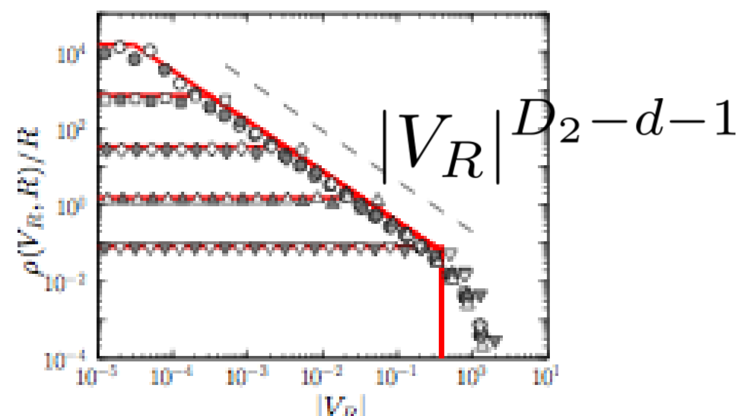
Universality

- Straightforward to generalize to higher dimension
- Asymptotic form does not depend on the driving flow



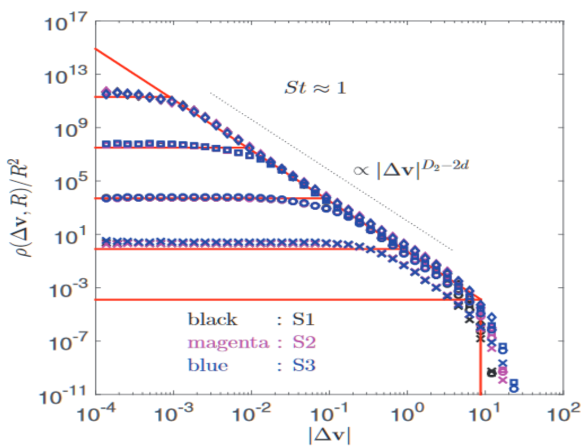
Smooth ‘Kraichnan flow’ ($d = 1$)

Cencini, Talk: MP0806_CG3.pdf (2009)



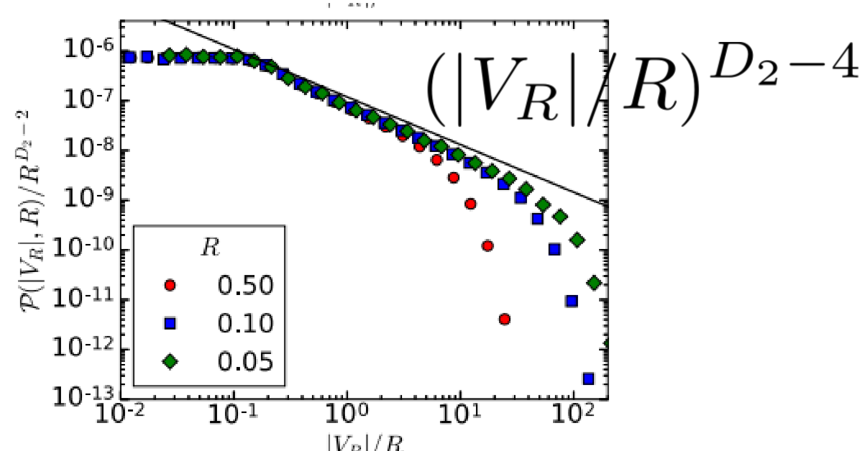
Distribution of radial velocities

KG & Mehlig J. turbulence **15** (2014)



DNS at different Reynolds numbers

Perrin & Jonker, Phys. Rev. E **92** (2015)



DNS at different Stokesnumbers

Bhatnagar et al, Phys. Rev. E **97** (2018)

Implications

- Moments of radial velocities in 3D

$$m_p(R) = \int_{-\infty}^{\infty} \mathrm{d}v_r |v_r|^p \rho(R, v_r)$$

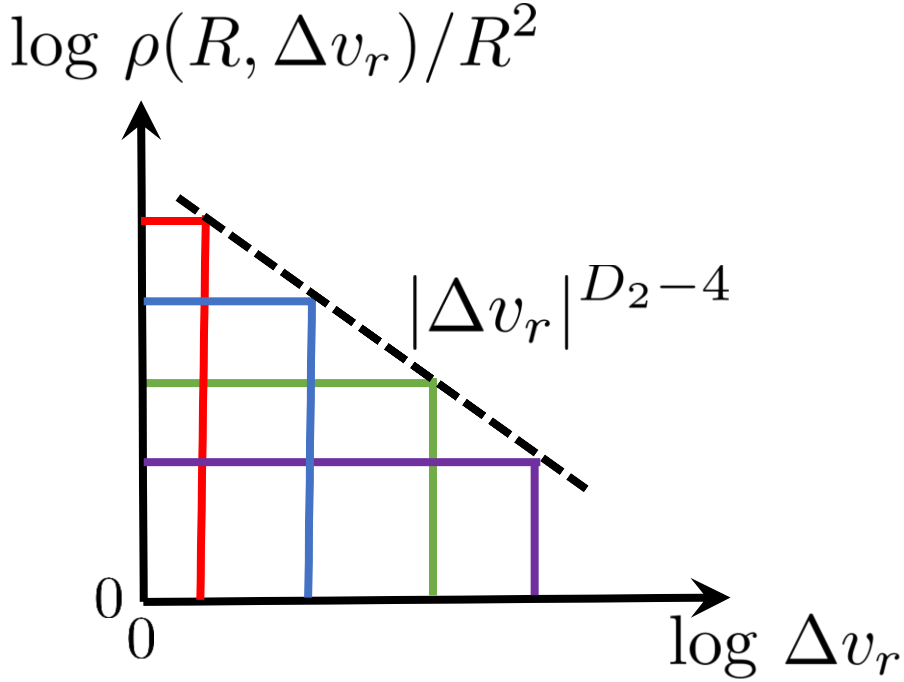
- Smooth dynamics $\langle \Delta v_r^2 \rangle \sim R^2$
  Distribution cut off at scale $\sim R$

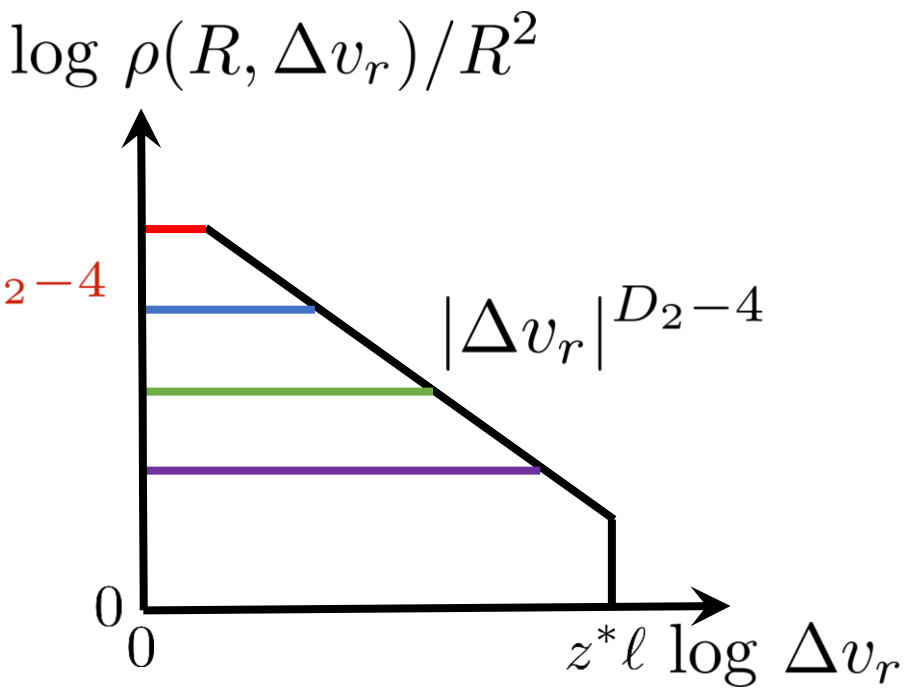
$$m_p(R) \sim \int_0^R \mathrm{d}v_r |v_r|^p R^2 R^{D_2-4}$$
$$\sim R^{p+D_2-1}$$

- Contribution from tails is scale free

$$m_p(R) \sim \int^{z^*\ell} \mathrm{d}v_r |v_r|^p R^2 |\Delta v_r|^{D_2-4}$$
$$\sim R^2$$

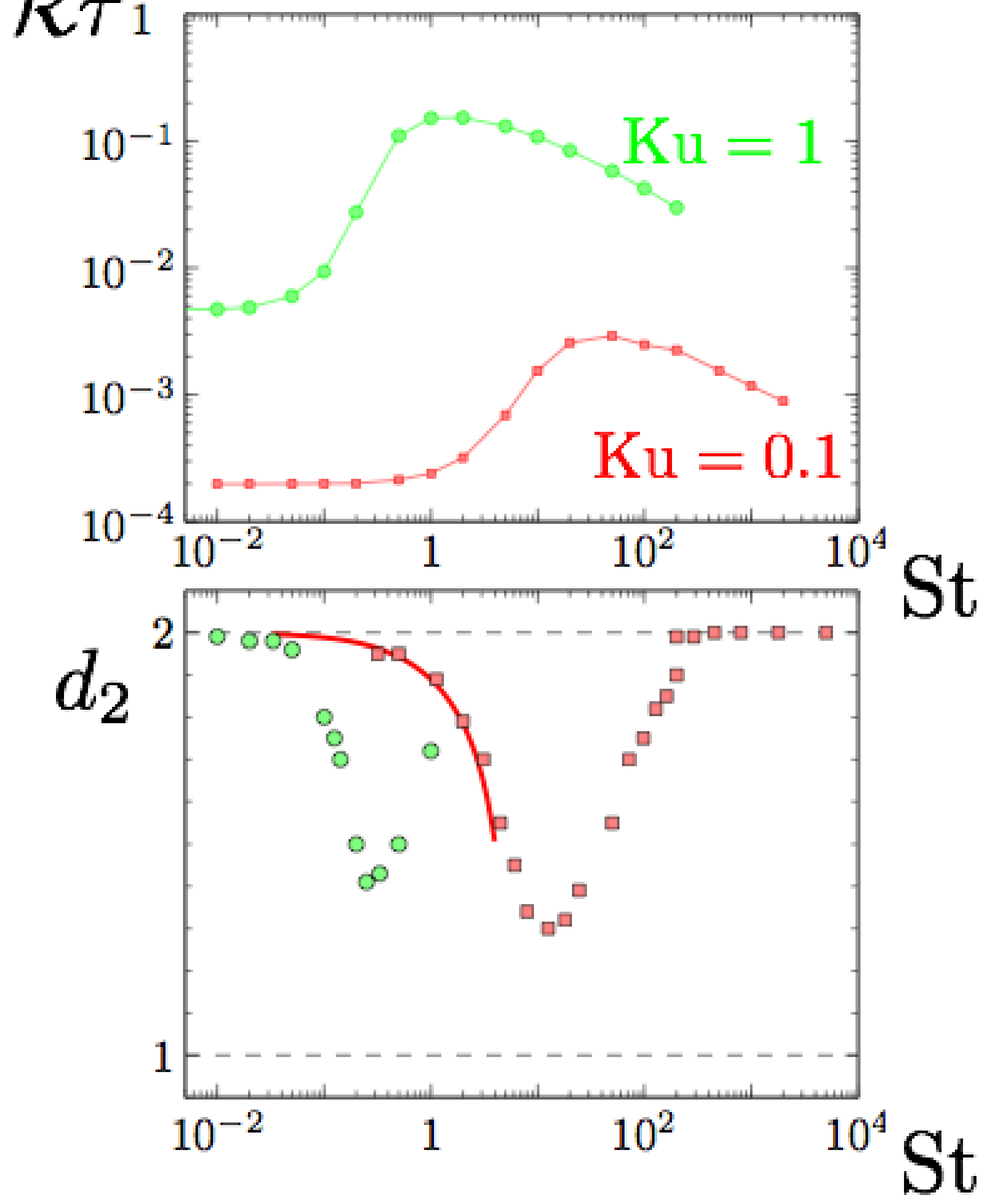
- Collision rate $m_1(2a)$ dominated by tail contribution if caustics common



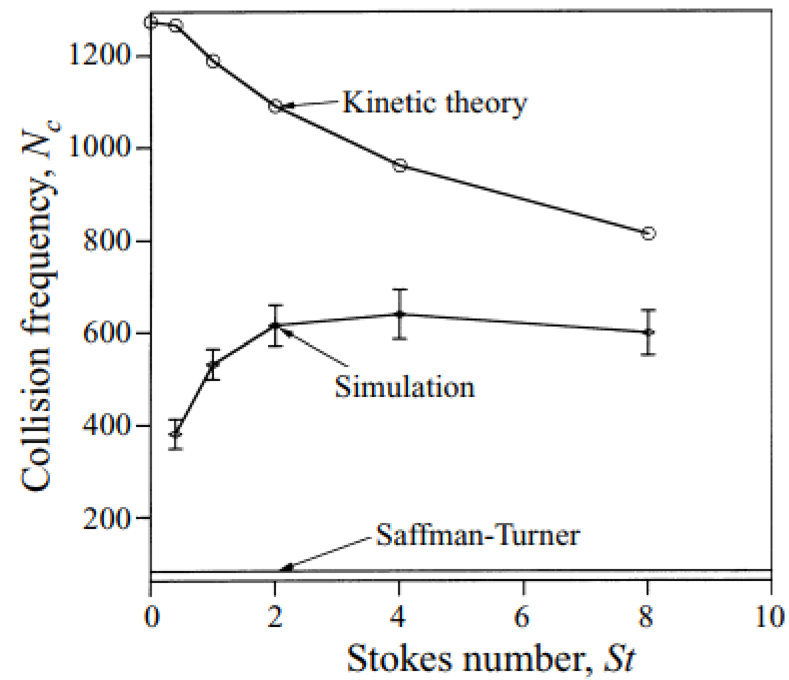


The rate of collisions

Collision rate for different Ku



Collision velocities in DNS



Sundaram & Collins, J. Fluid Mech. **335** (1997)

Explanation:

‘Caustics’

Falkovich et. al, Nature **419**, 151, (2002)
Wilkinson et. al, Phys. Rev. Lett. **97** (2006)

Summary, statistical model

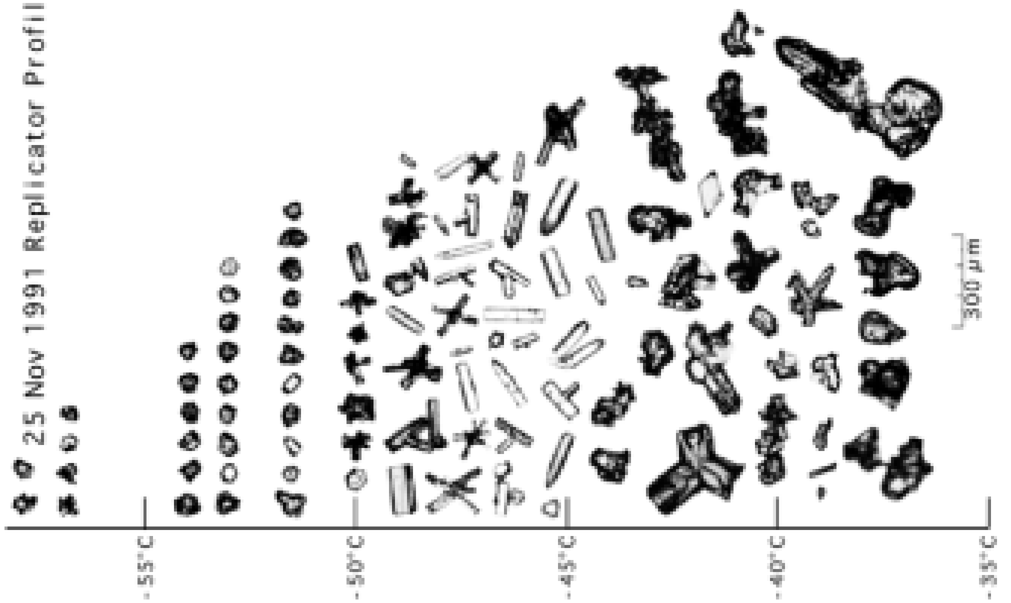
- Minimalistic model of turbulence.
- Qualitative agreement with DNS in most cases, sometimes quantitative agreement.
- Trajectory expansion allows analytical solution for limit of small Kubo number.
- Allowed to identify new clustering mechanism: multiplicative amplification
- Allowed to identify power-law tails in distribution of relative velocities. These have huge impact on collisions between particles of similar size.

Orientation of non-spherical particles

- Cloud ice crystals

  Pruppacher and Klett, (Springer 1997)

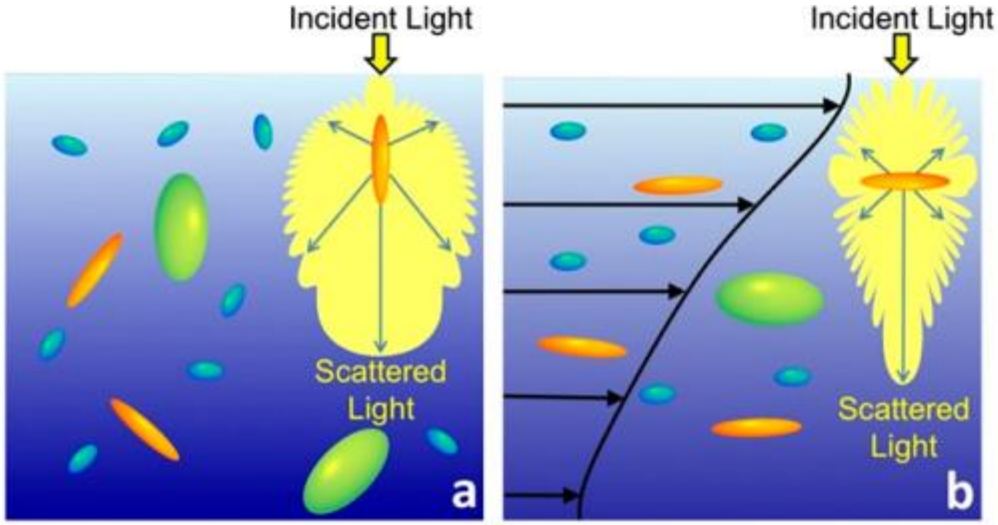
High altitude Cirrus clouds



Orientation of non-spherical particles

- Cloud ice crystals
  Pruppacher and Klett, (Springer 1997)

- Non-swimming plankton
  Marcos et al, PNAS. **108** (2011)



Increased backscattering in shear flow

Orientation of non-spherical particles

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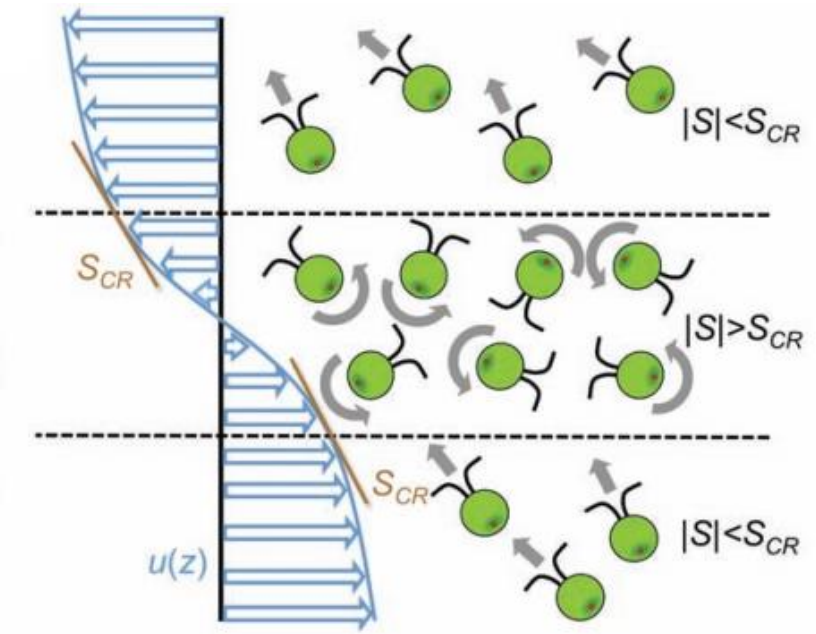
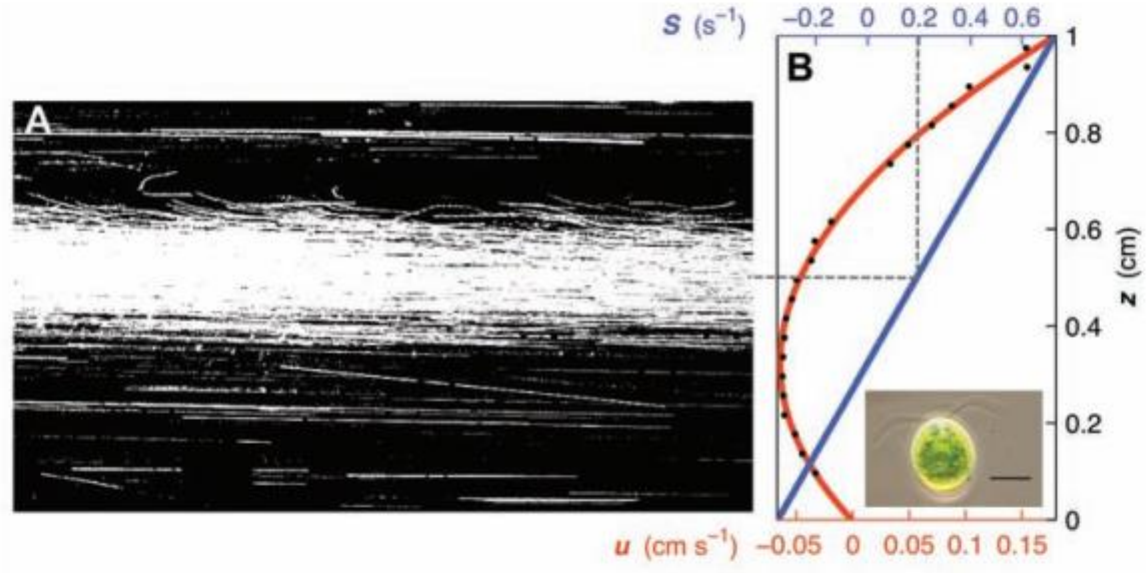
  Pruppacher and Klett, (Springer 1997)

- Non-swimming plankton
  Marcos et al, PNAS. **108** (2011)

- Swimming plankton
  Durham et al, Science **323** (2009)





Orientation of non-spherical particles

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  Marcos et al, PNAS. **108** (2011)

- Swimming plankton
  Durham et al, Science **323** (2009)

- Fibers in paper making
  Lundell et al, Annu. Rev. Fluid Mech. **43** (2011)

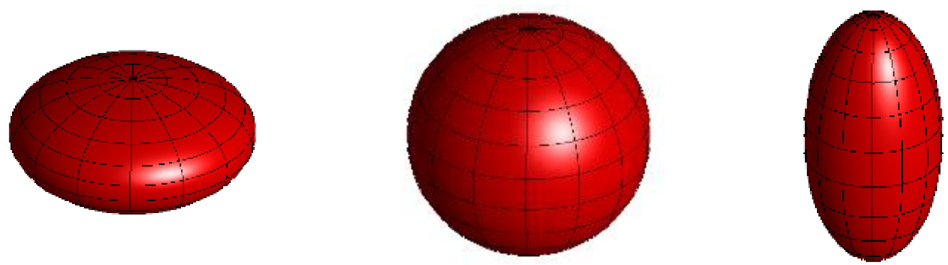
- Visualisation of turbulent flows
  A. Pumir and M.Wilkinson, NJP **13** (2011)

Happel & Brenner (1965); Jeffery, Proc. R. Soc. Lond. **102** (1922)

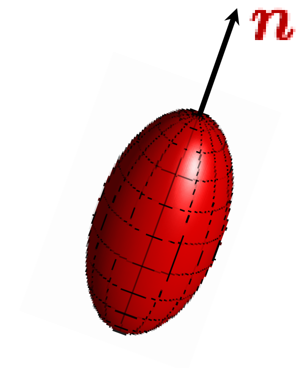
Dynamics of a small spheroid

Aspect ratio λ

$\lambda < 1$ $\lambda = 1$ $\lambda > 1$



Orientation $\boldsymbol{n}$



Force and torque in Stokes' approximation

$$\boldsymbol{F} = \mathcal{A}(\boldsymbol{u} - \dot{\boldsymbol{r}}) + \mathcal{B}(\boldsymbol{\Omega} - \boldsymbol{\omega}) + \mathcal{G} : \mathbb{S}$$

Axial symmetry

$$\boldsymbol{T} = \mathcal{B}^{\mathrm{T}}(\boldsymbol{u} - \dot{\boldsymbol{r}}) + \mathcal{C}(\boldsymbol{\Omega} - \boldsymbol{\omega}) + \mathcal{H} : \mathbb{S}$$

Mirror symmetry (x,y,z)

Neglect inertia $\boldsymbol{F} = 0 \Rightarrow \dot{\boldsymbol{r}} = \boldsymbol{u}$

$$\boldsymbol{T} = 0 \Rightarrow \boldsymbol{\omega} = \boldsymbol{\Omega} + \mathcal{C}^{-1}\mathcal{H} : \mathbb{S}$$

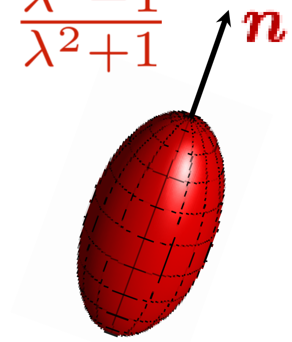
Jeffery eq. $\dot{\boldsymbol{n}} = \boldsymbol{\omega} \times \boldsymbol{n} = \mathbb{O}\boldsymbol{n} + \Lambda[\mathbb{S}\boldsymbol{n} - (\boldsymbol{n}^{\mathrm{T}}\mathbb{S}\boldsymbol{n})\boldsymbol{n}]$

Flow rotation $\mathbb{O}$, strain $\mathbb{S}$, shape $\Lambda = \frac{\lambda^2 - 1}{\lambda^2 + 1}$.

Bretherton, J. Fluid Mech. **14** (1962); Einarsson, PhD Thesis (2015)

Constant flow gradients

$\Lambda = \frac{\lambda^2-1}{\lambda^2+1}$



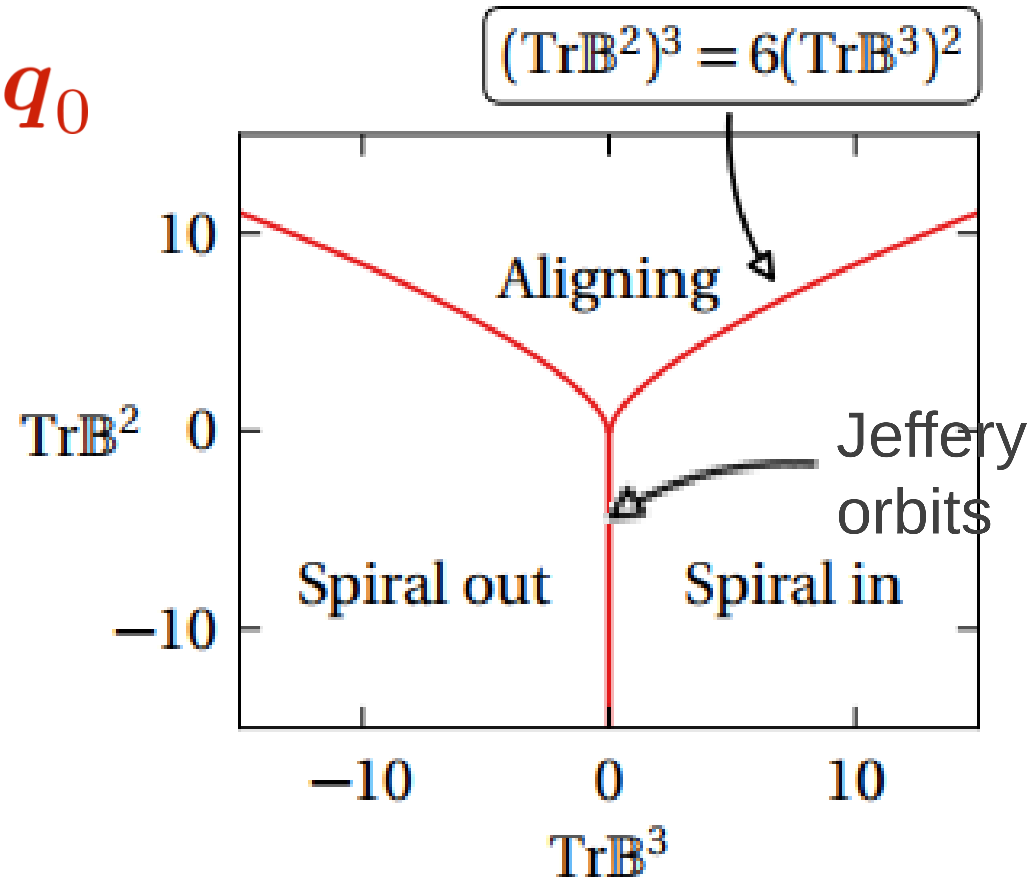
Jeffery eq. $\dot{\boldsymbol{n}} = \mathbb{O}\boldsymbol{n} + \Lambda[\mathbb{S}\boldsymbol{n} - (\boldsymbol{n}^{\mathrm{T}}\mathbb{S}\boldsymbol{n})\boldsymbol{n}]$

Equivalently $\dot{\boldsymbol{q}} = \mathbb{O}\boldsymbol{q} + \Lambda\mathbb{S}\boldsymbol{q}$ **with** $\boldsymbol{n} = \frac{\boldsymbol{q}}{|\boldsymbol{q}|}$

Solution $\boldsymbol{q}(t) = e^{[\mathbb{O}+\Lambda\mathbb{S}]t}\boldsymbol{q}_0$

Eigenvalue classification

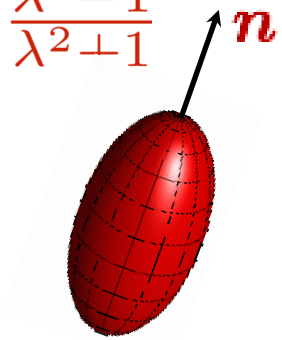
$$\mathbb{B} = \mathbb{O} + \Lambda\mathbb{S}$$



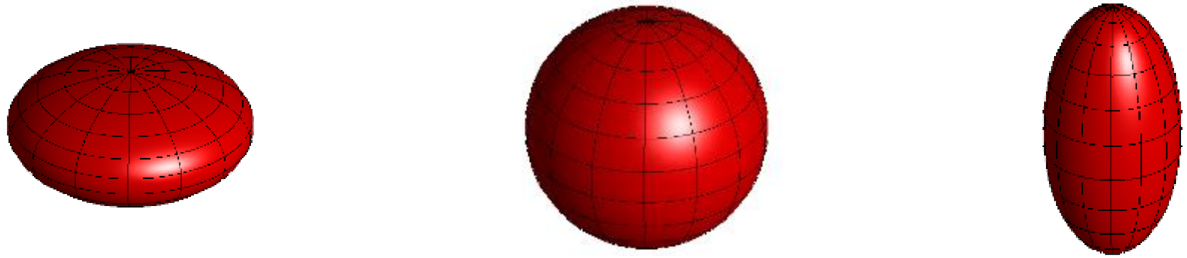
Parsa et al, Phys. Rev. Lett. **109** (2012); Byron et al, Phys. Fluids **27** (2015)

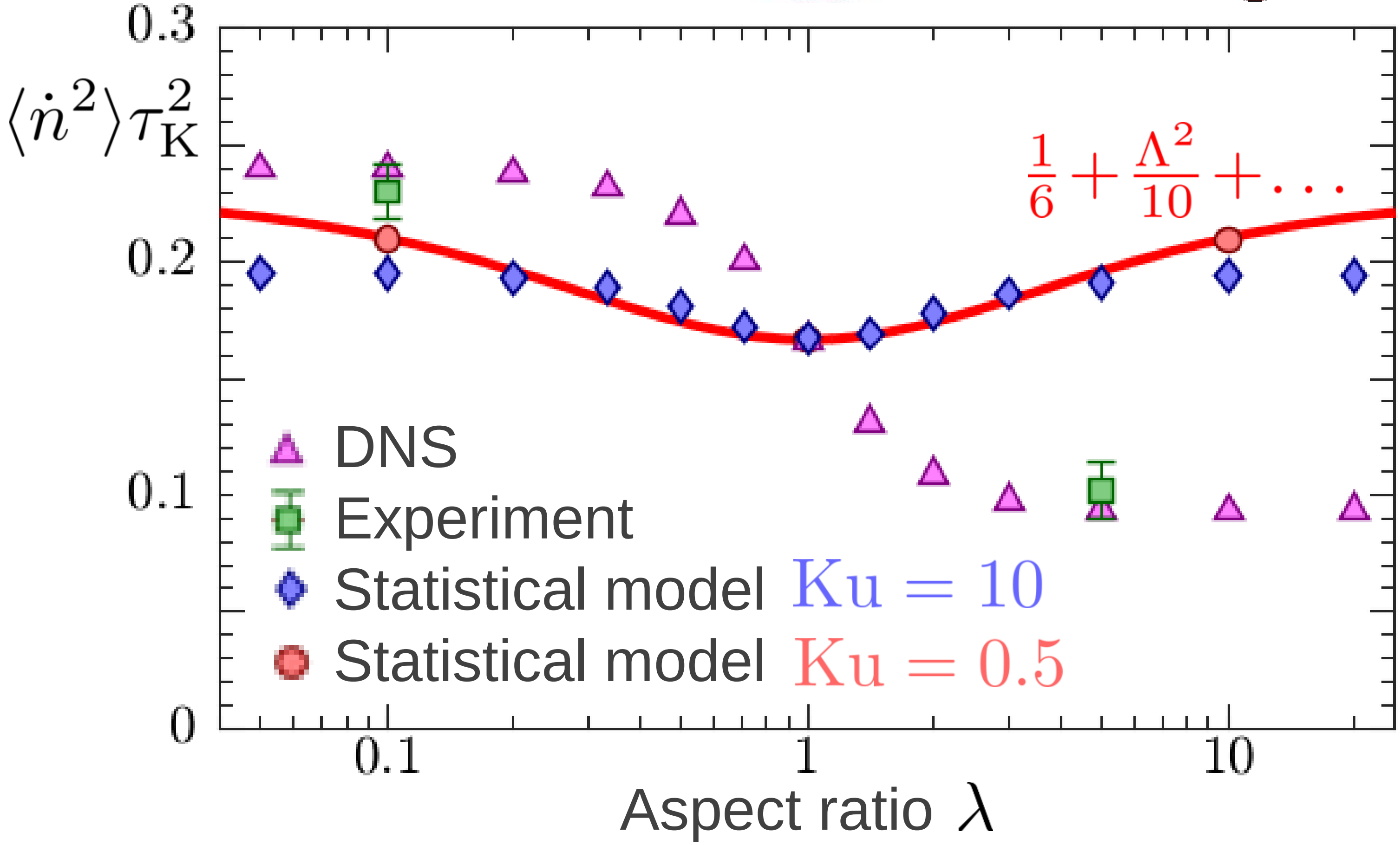
Tumbling rate in turbulence

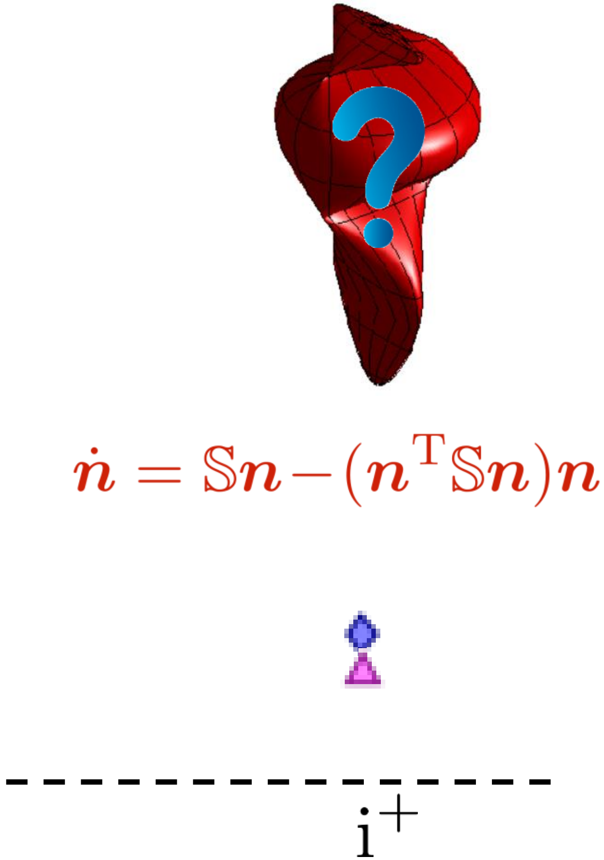
$\Lambda = \frac{\lambda^2-1}{\lambda^2+1}$



Dynamics $\dot{\boldsymbol{n}} = \mathbb{O}(t)\boldsymbol{n} + \Lambda[\mathbb{S}(t)\boldsymbol{n} - (\boldsymbol{n}^{\mathrm{T}}\mathbb{S}(t)\boldsymbol{n})\boldsymbol{n}]$



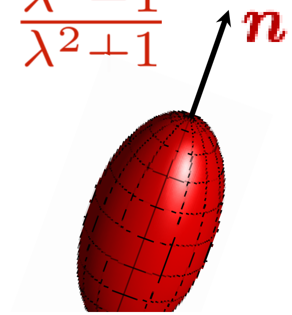




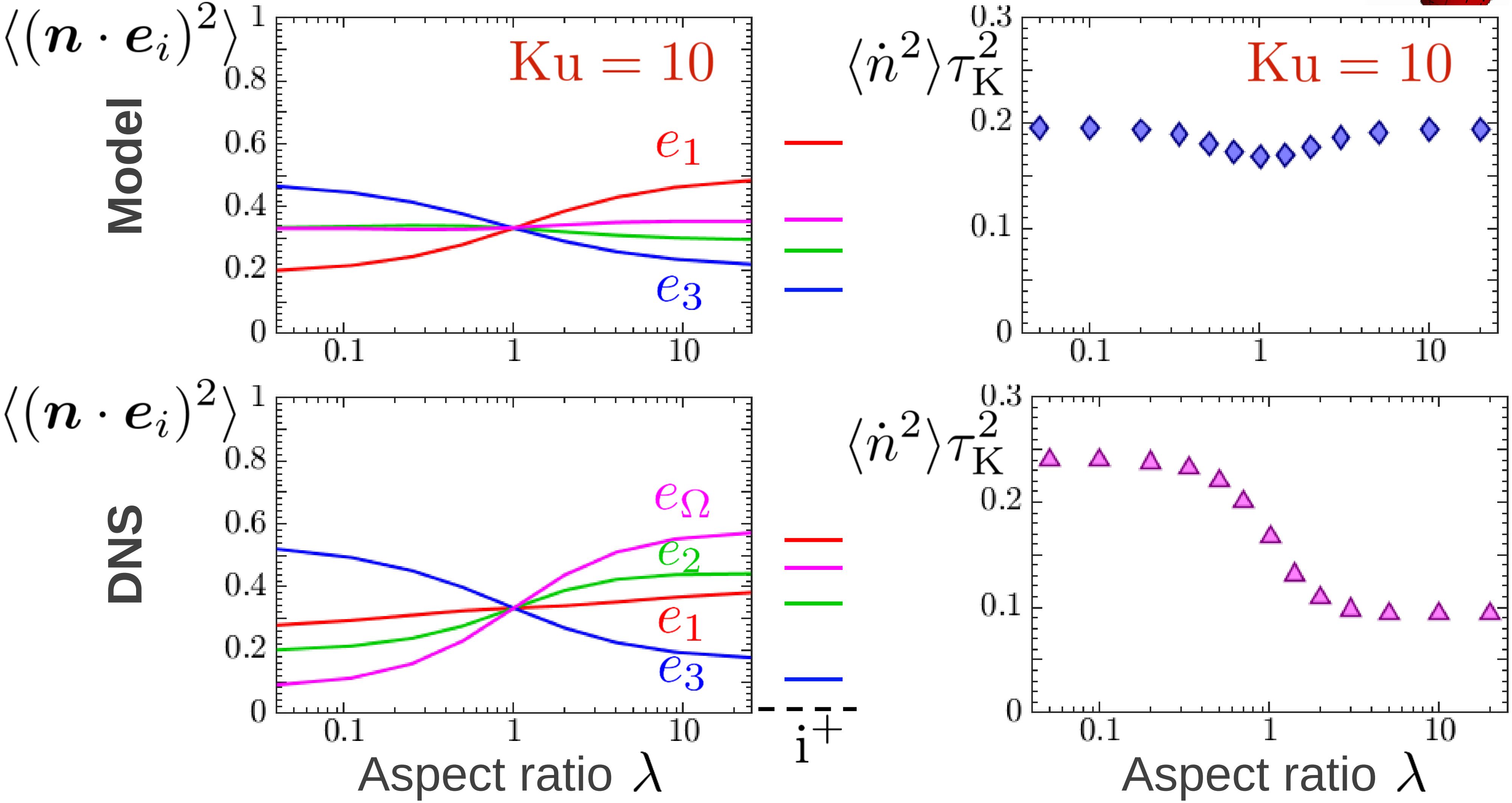
Pumir & Wilkinson, New Journal of Physics **13** (2011)

Preferential alignment

$$\Lambda = \frac{\lambda^2 - 1}{\lambda^2 + 1}$$



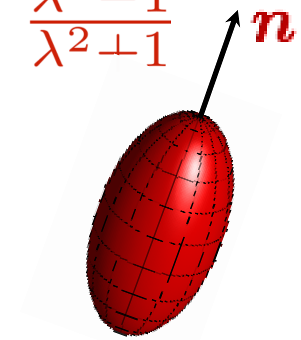
Dynamics $\dot{\boldsymbol{q}} = \mathbb{O}(t)\boldsymbol{q} + \Lambda \mathbb{S}(t)\boldsymbol{q}$; $\boldsymbol{n} = \dfrac{\boldsymbol{q}}{|\boldsymbol{q}|}$



KG, Einarsson & Mehlig, Phys. Rev. Lett. **112** (2014)

Statistical model explanation

$$\Lambda = \frac{\lambda^2-1}{\lambda^2+1}$$



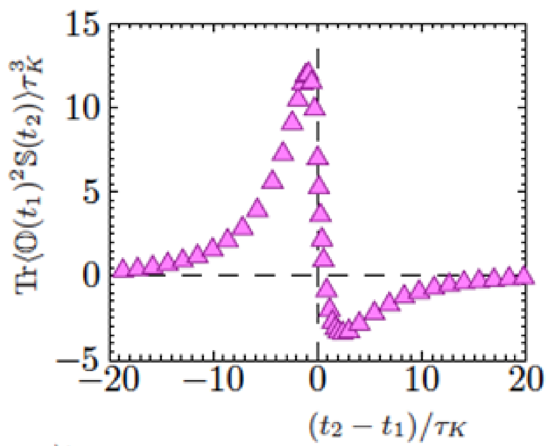
Trajectory expansion gives

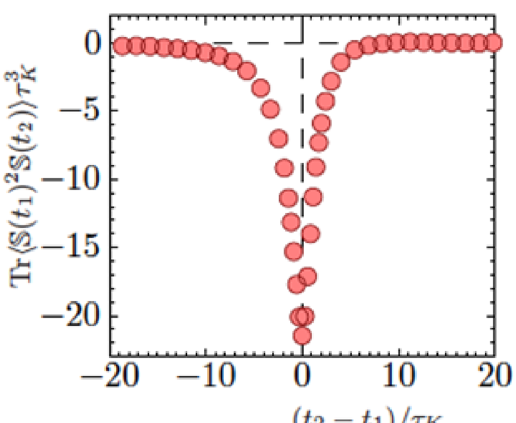
$$\langle \dot{\boldsymbol{n}}^2 \rangle = \frac{1}{15}\Big(5\mathrm{Tr}(\mathbb{O}_0^{\mathrm{T}}\mathbb{O}_0) + 3\Lambda^2\mathrm{Tr}(\mathbb{S}_0^{\mathrm{T}}\mathbb{S}_0)\Big) + \frac{2}{5}\Lambda\int_0^\infty \mathrm{d}t'\Big[-\mathrm{Tr}(\mathbb{O}_0^2\mathbb{S}_{-t}) + 2\Lambda\mathrm{Tr}(\mathbb{S}_0\mathbb{O}_0\mathbb{S}_{-t}) + \frac{3}{7}\Lambda^2\mathrm{Tr}(\mathbb{S}_0^2\mathbb{S}_{-t'})\Big]$$

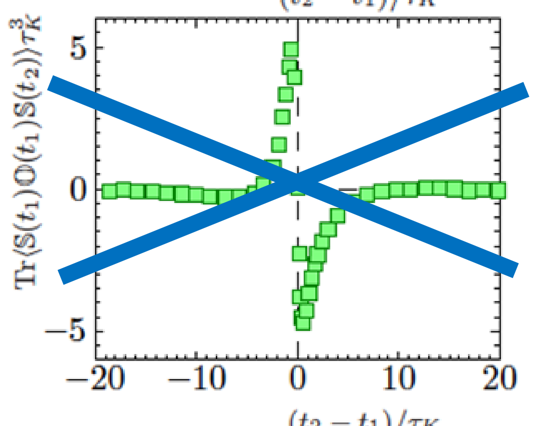
Dominant: $-\mathrm{Tr}(\mathbb{O}_0^2\mathbb{S}_{-t})$

Negligible: $2\Lambda\mathrm{Tr}(\mathbb{S}_0\mathbb{O}_0\mathbb{S}_{-t})$

DNS correlation functions







Comparison to DNS