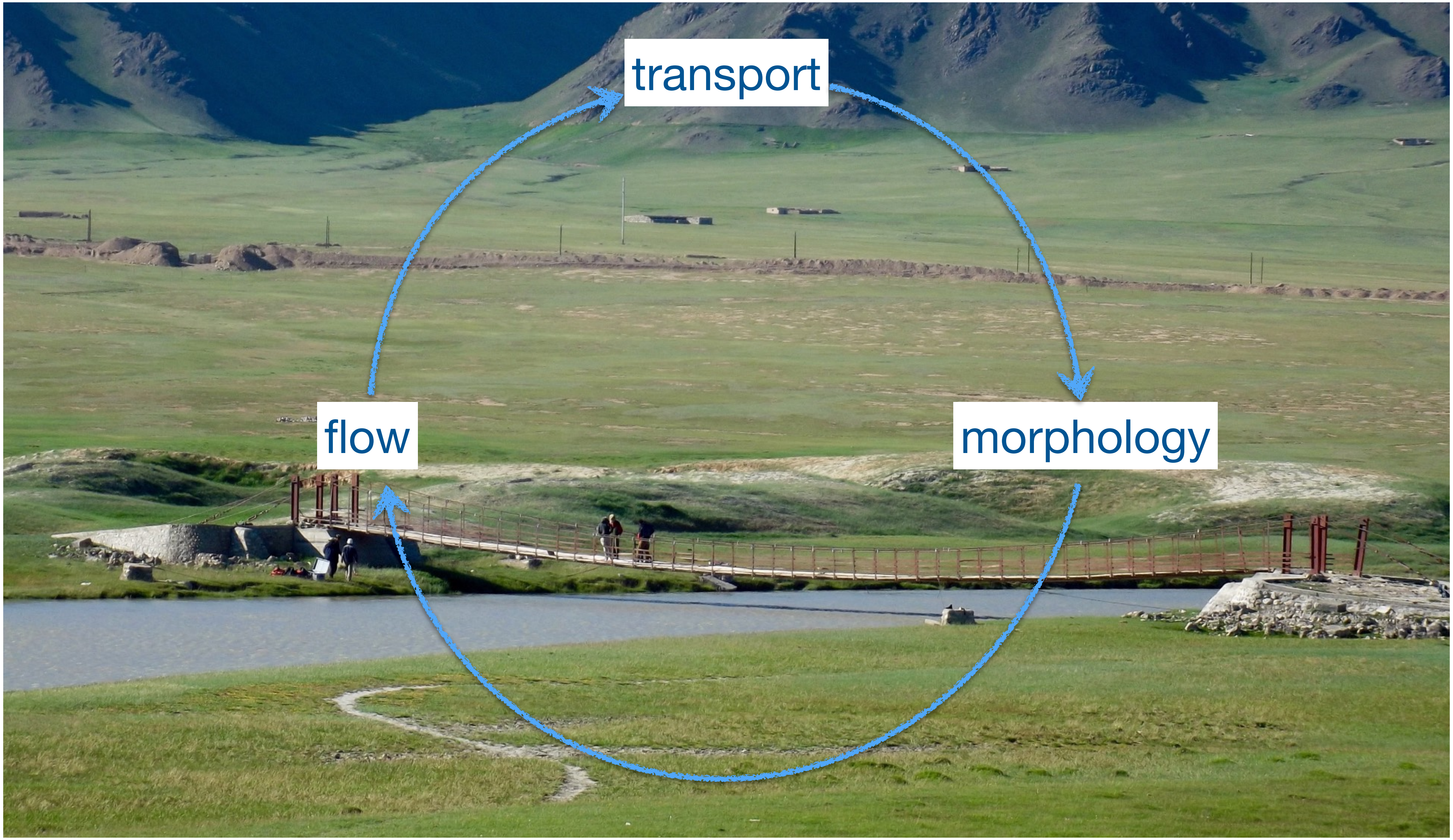
Equilibrium shape and size of alluvial rivers

E. Lajeunesse,
A. Abramian, O. Devauchelle, F. Métivier, P. Popović, G. Seizilles

Geological Fluid Dynamics Team,
Institut de Physique du Globe de Paris– Université Paris Cité

Alluvial river



transport

morphology

flow

Kaidu river, Tian-Shan, China

Bedforms

Ripples, Urumqi river
(chinese Tian-Shan)

Alternate bars, Ornain, Bar le Duc

Channel shape

Kaidu river, Tian-Shan, China

What selects the shape and the size of an alluvial channel?

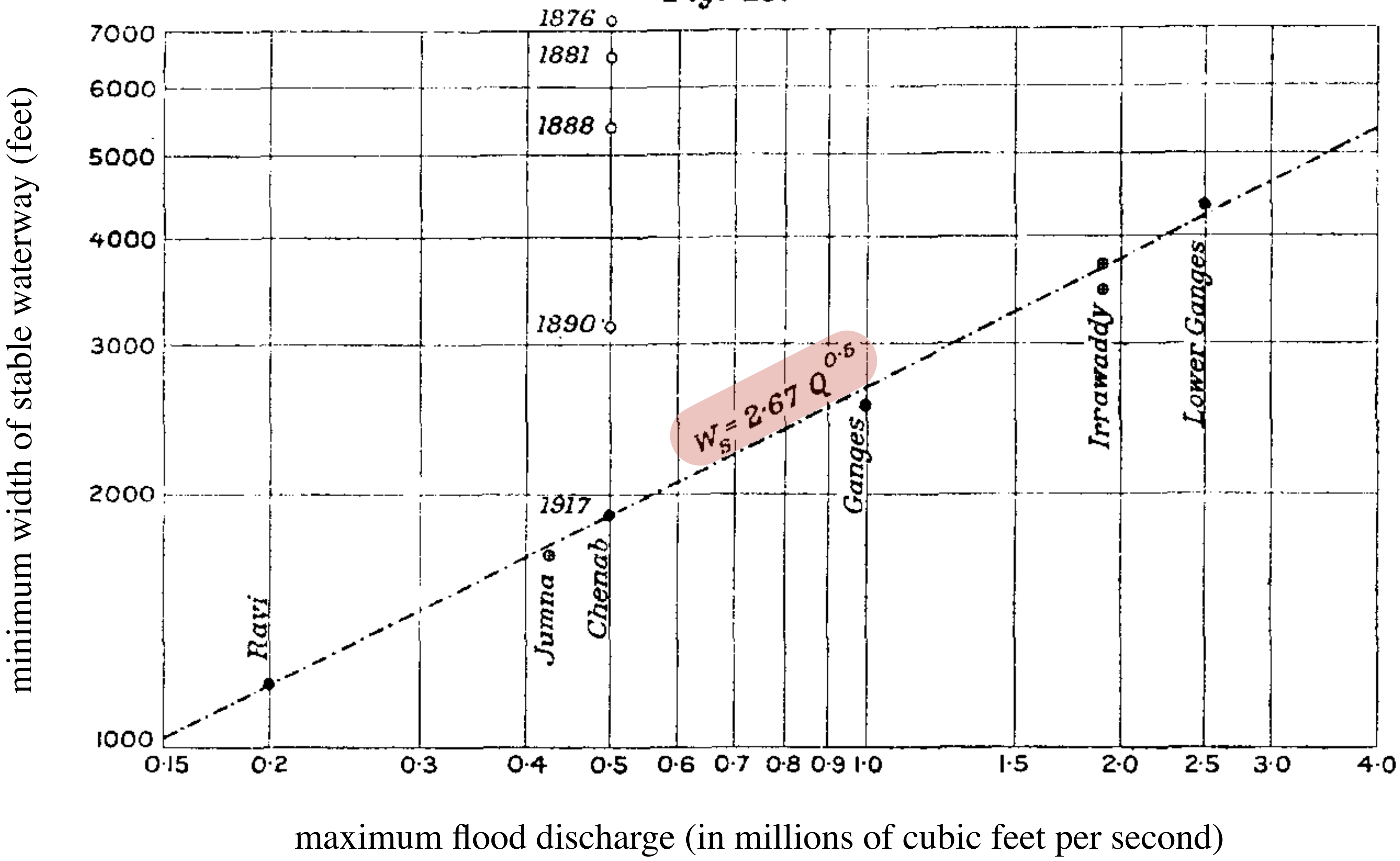
Lacey’s law

“Stable Channels in Alluvium.”

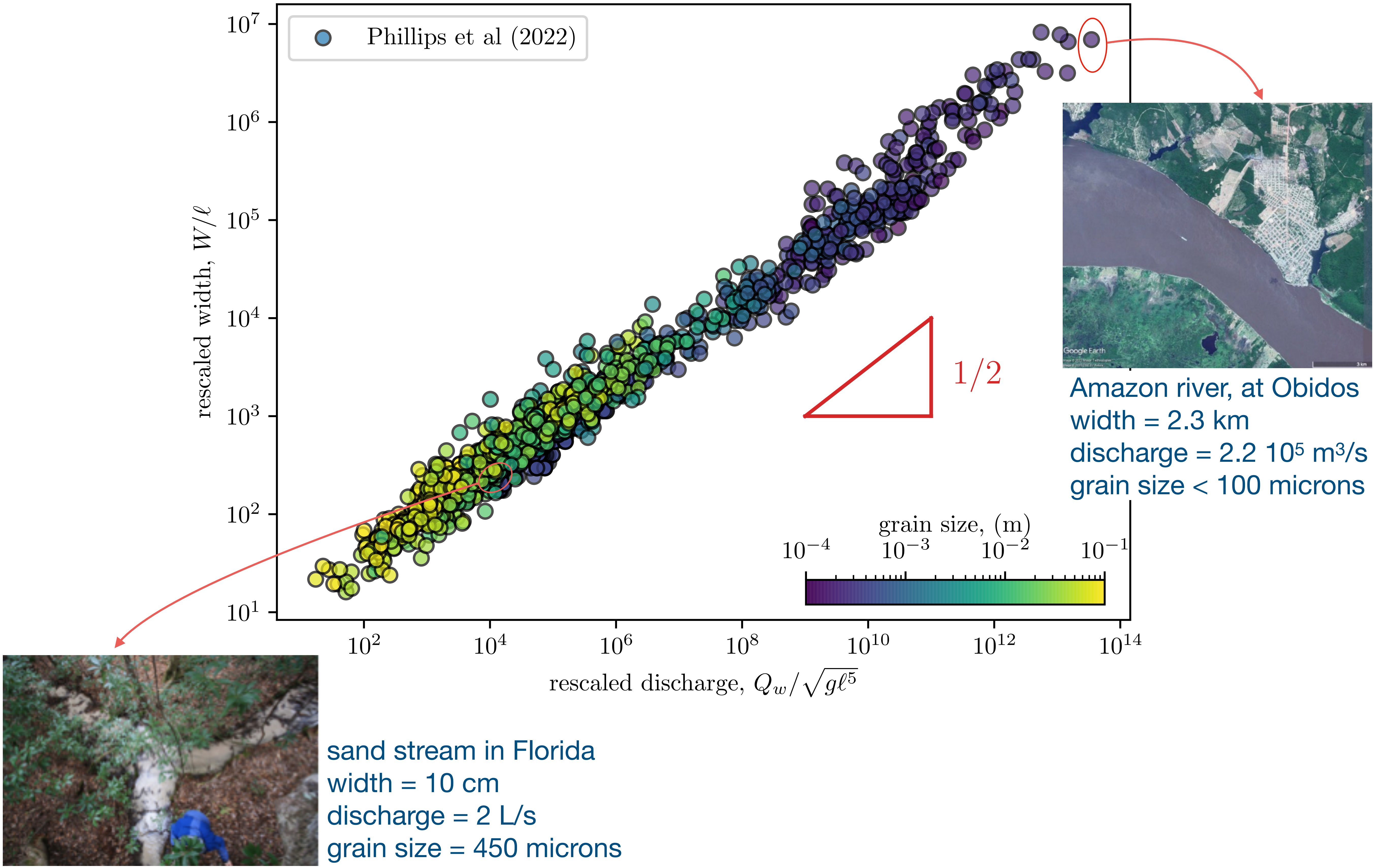
By Gerald Lacey, B.Sc., Assoc. M. Inst. C.E.

Minutes of the Proceedings of the Institution of Civil Engineers, Vol. 229 (1930)

Fig. 13.



Lacey's law

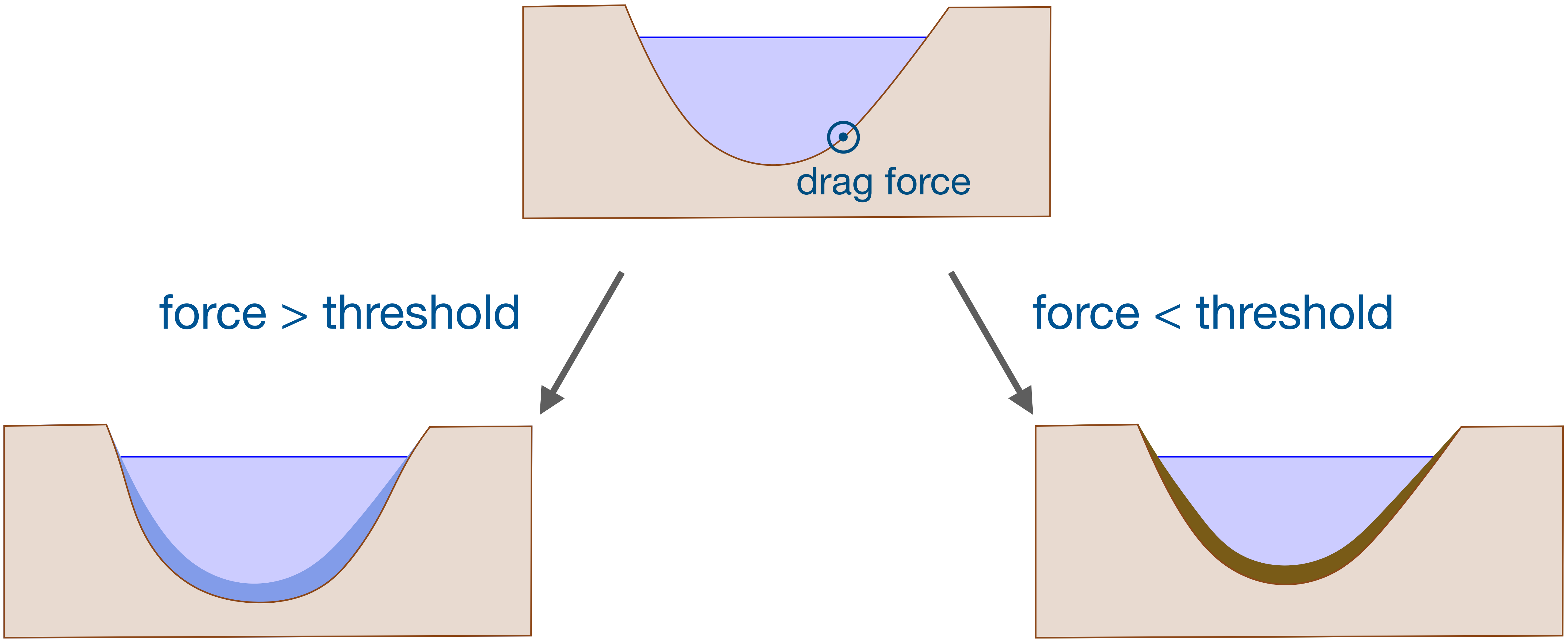


Amazon river, at Obidos
width = 2.3 km
discharge = 2.2 10^5 m^3/s
grain size < 100 microns

sand stream in Florida
width = 10 cm
discharge = 2 L/s
grain size = 450 microns

Threshold theory

Glover and Florey [1951] & Henderson [1961]



drag force

force > threshold

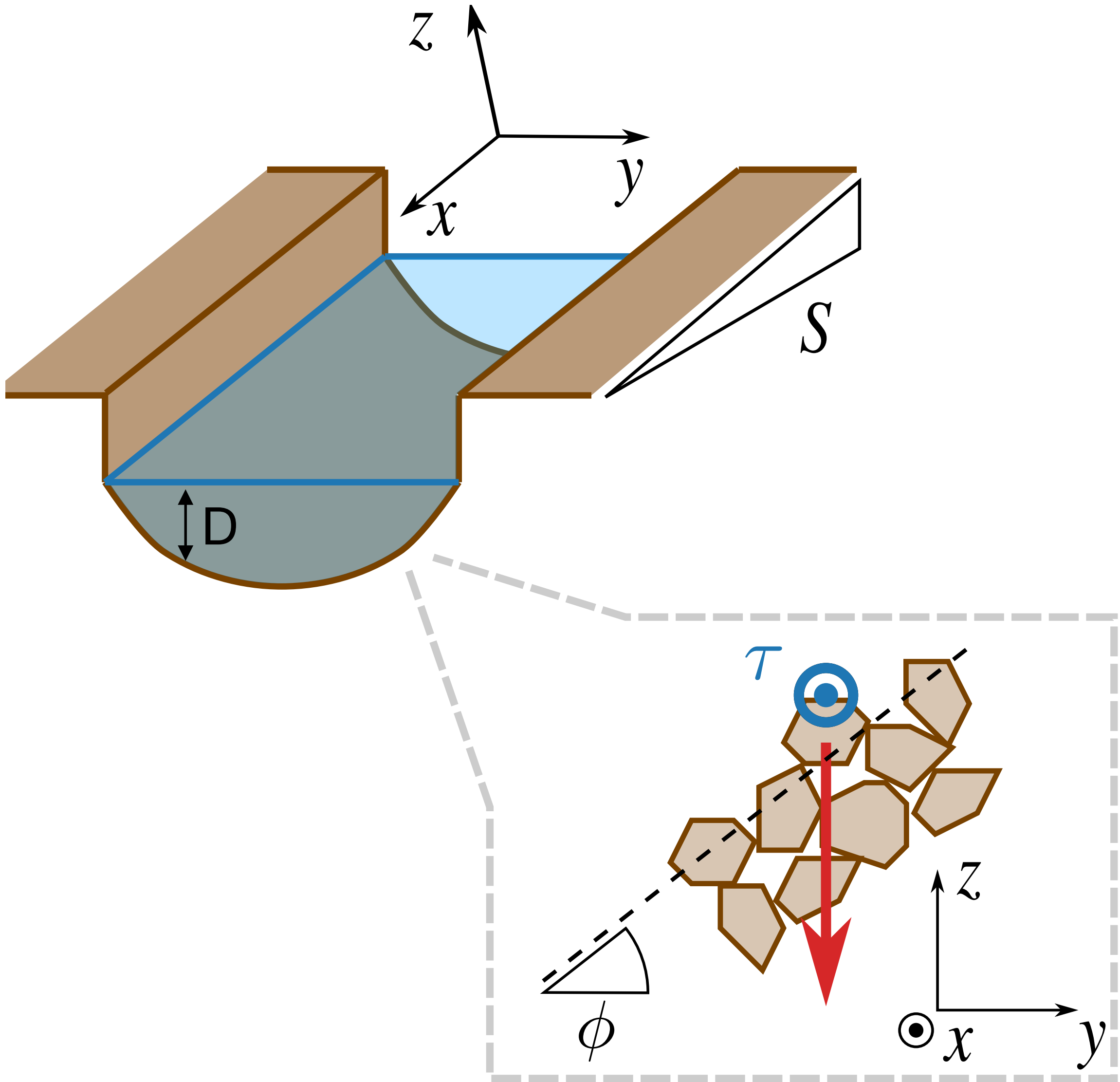
force < threshold

erosion

aggradation

→ river builds its bed at the threshold of entrainment.

Threshold theory



density of sediment
fluid density
grain size

- gravity force: $F_g \propto (\rho_s - \rho_f)\ g\ d_s^3$

shear stress

- drag force: $F_d \propto \tau\ d_s^2$

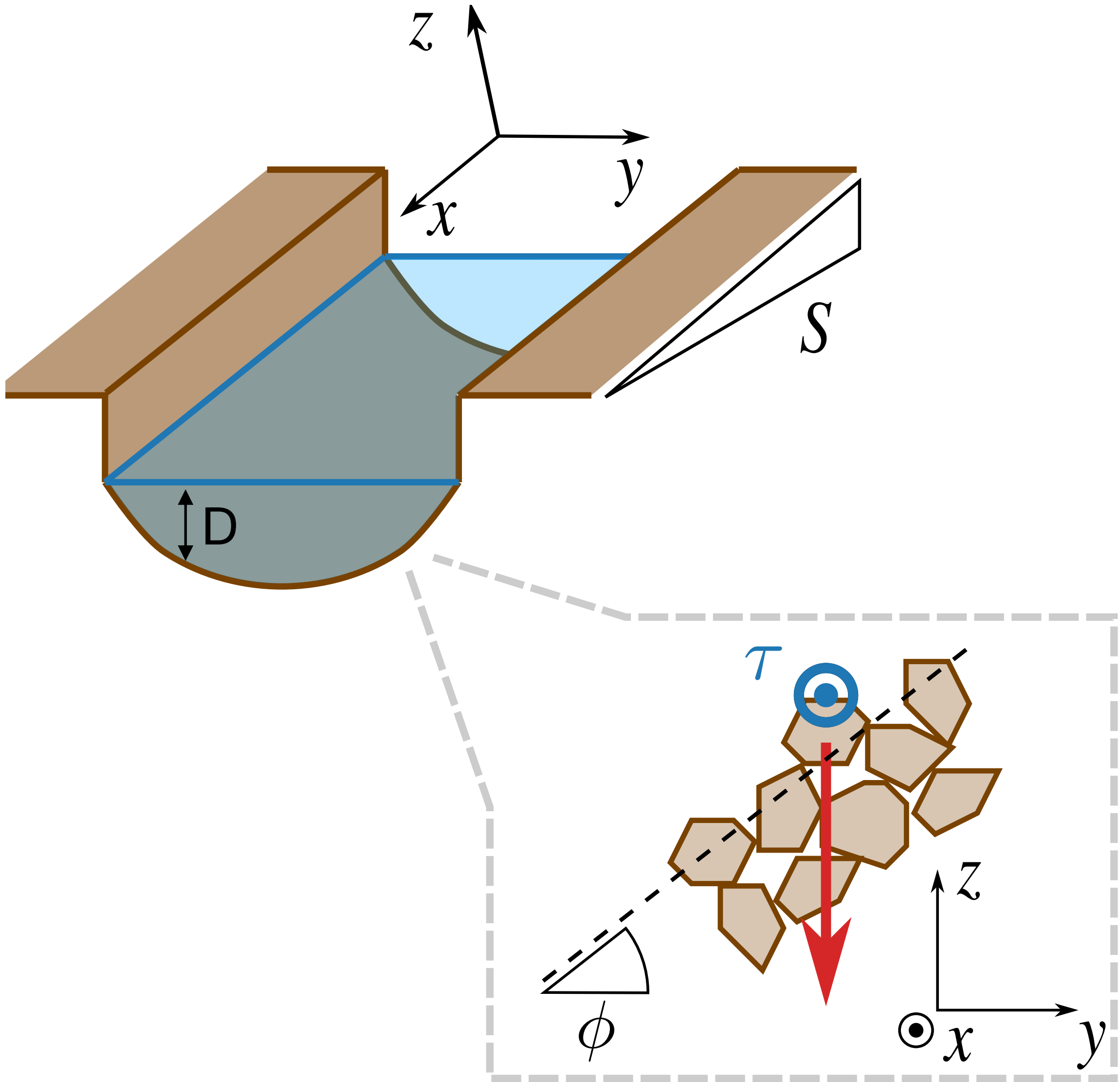
- shallow water approximation: $\tau = \rho g D S$

flow depth
slope

friction coefficient

Coulomb's law of friction → $$\frac{\text{tangential force}}{\text{normal force}} = \mu_t$$

Threshold theory



threshold Shields stress

grain size

$$L = \frac{\theta_t}{\mu_t}\,\frac{\rho_s - \rho_f}{\rho_f}\,d_s$$

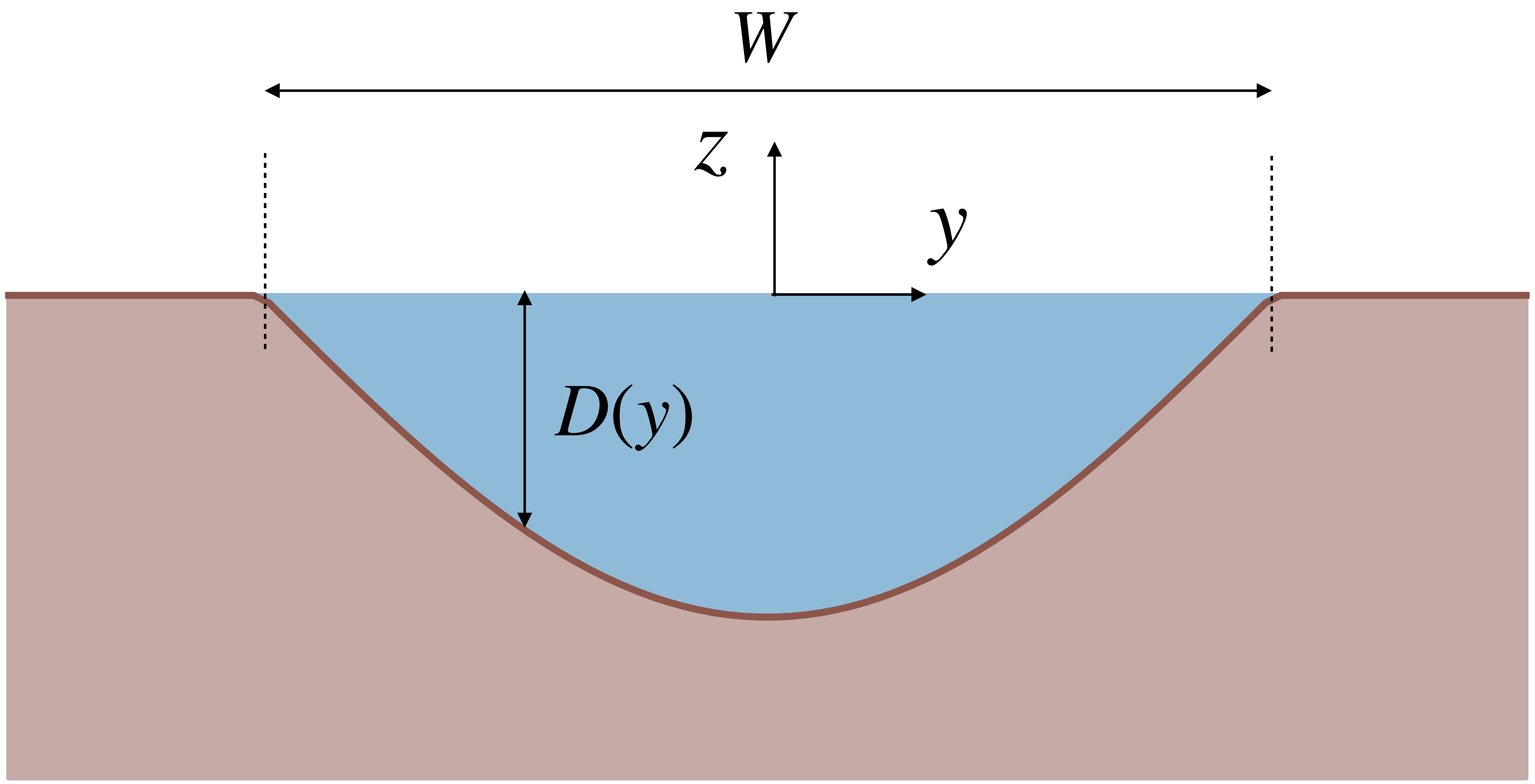
friction coef.

tangential/normal force

friction coef.

Coulomb's law of friction → $$\sqrt{\underbrace{\boxed{\left(\frac{SD}{L}\right)}^{2}}_{\text{drag force}} + \underbrace{\boxed{D'^{2}}}_{\text{gravity}}} = \mu_t$$

Threshold theory



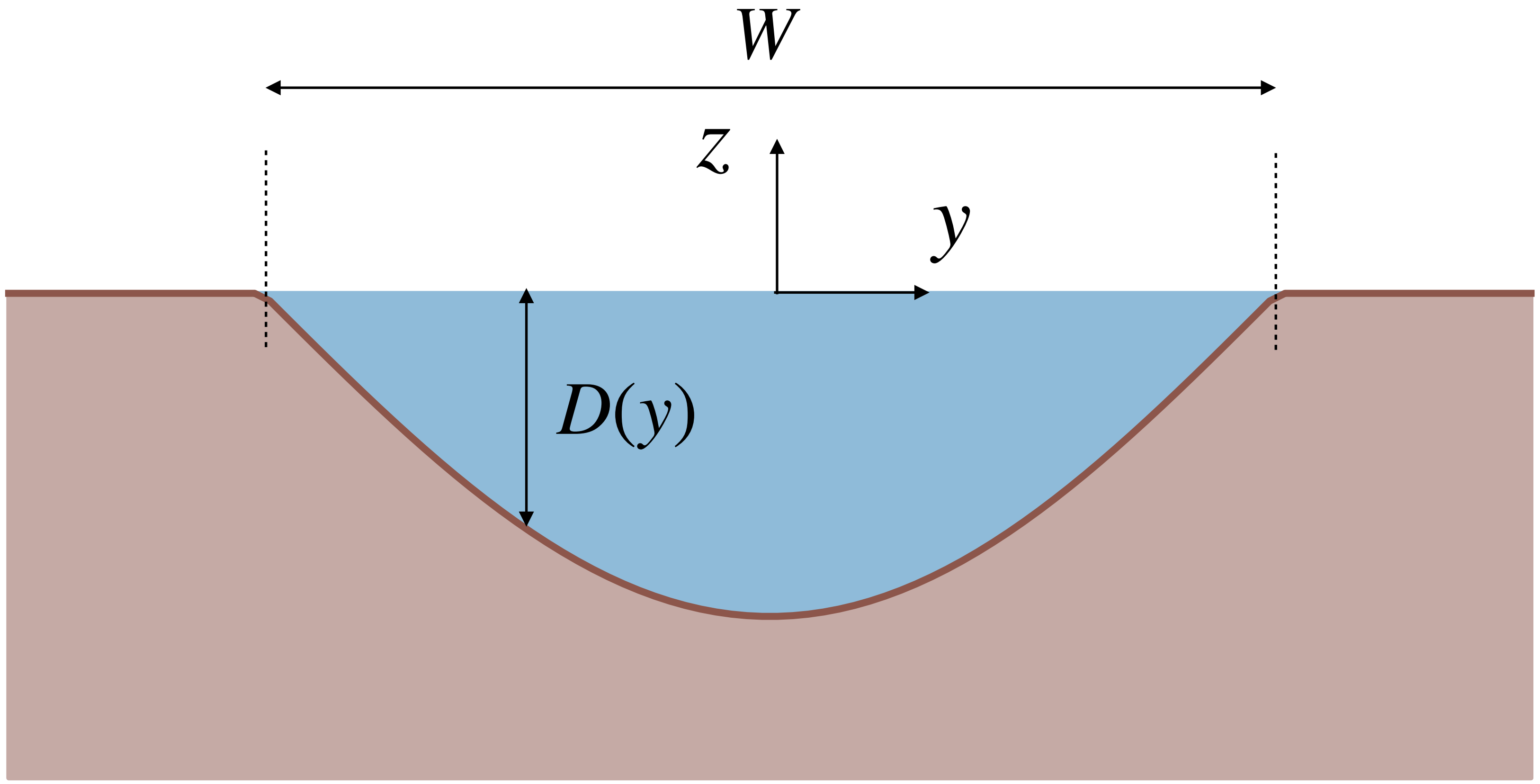
$$D = \overset{\text{friction coef.}}{\mu_t}\frac{L}{S}\cos\left(\frac{\overset{\text{slope}}{S}y}{\underset{\sim\ d_s}{L}}\right)$$

Glover and Florey [1951], Henderson [1961], Seizilles *et al.* [2013]

width and depth $\propto \dfrac{\overset{\text{grain size}}{d_s}}{\underset{\text{river slope} \sim 10^{-5}\text{-}10^{-2}}{S}}$

→ channel size ≫ grain size

Threshold theory



friction coef.

slope

$$D = \mu_t \frac{L}{S} \cos\left(\frac{Sy}{L}\right)$$

$\sim\ d_s$

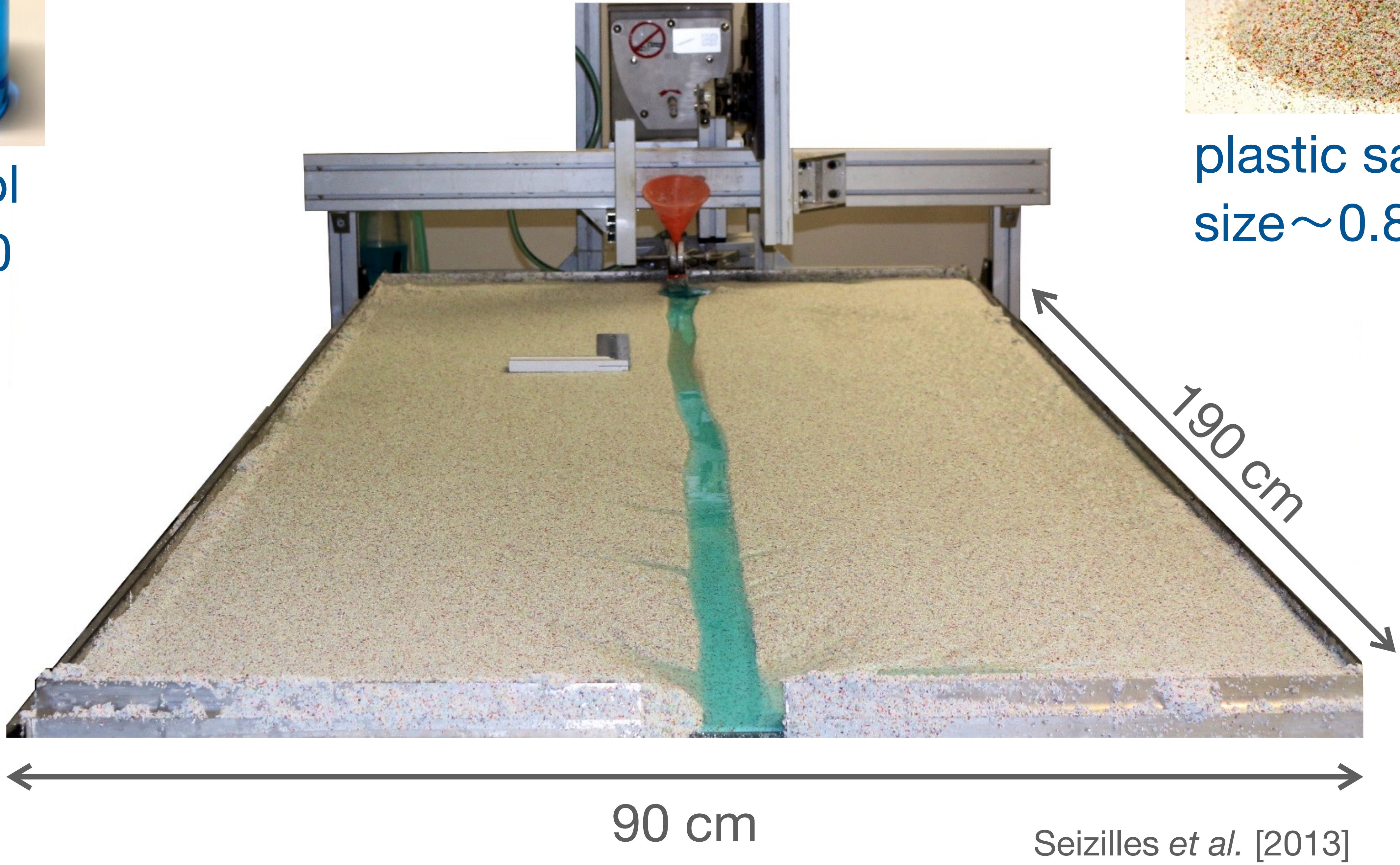
Glover and Florey [1951], Henderson [1961], Seizilles *et al.* [2013]

Shallow-water theory -> channel shape independent of the nature of the flow.

Laboratory laminar river

glycerol
Re∼10

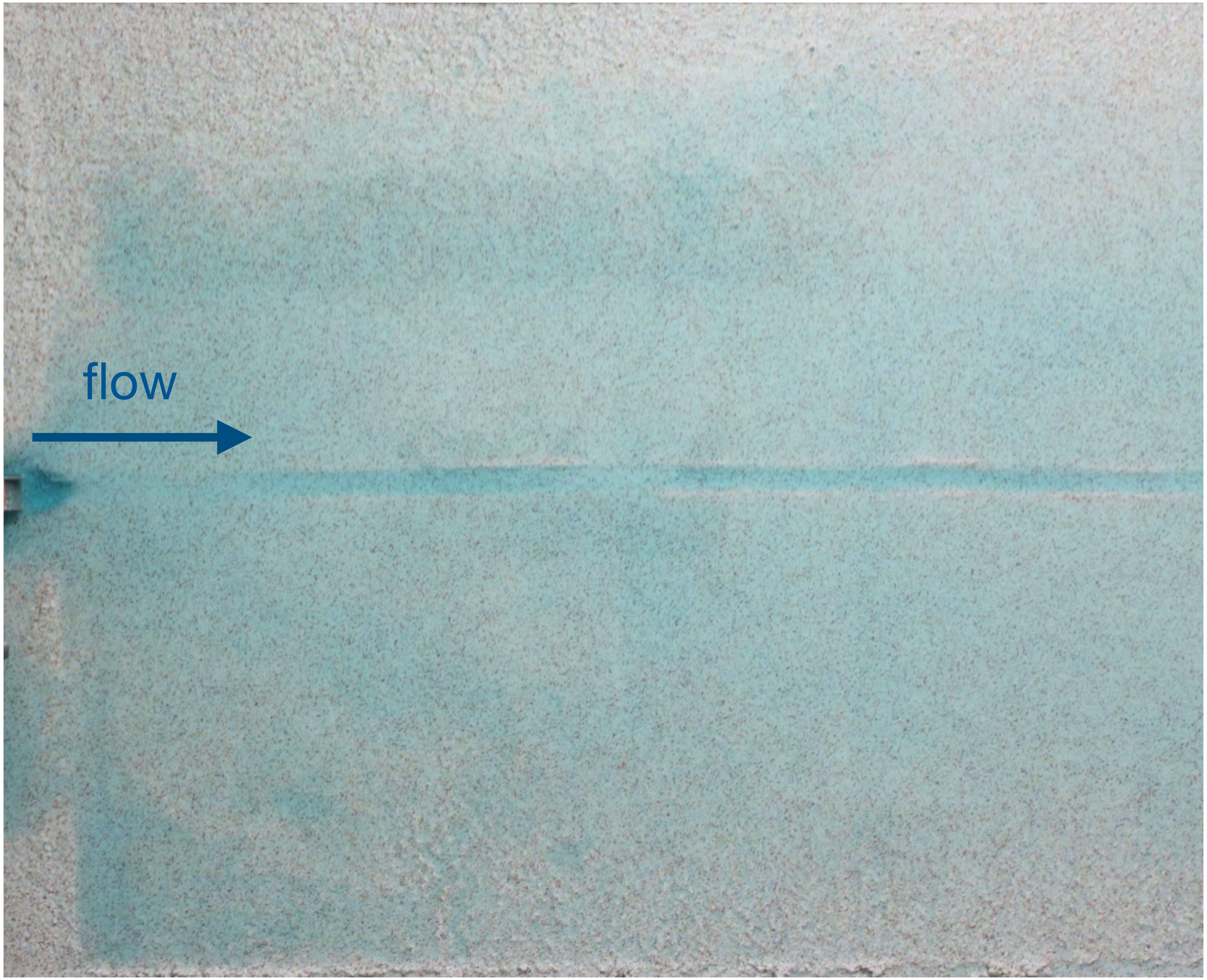
plastic sand
size∼0.83 mm



Seizilles *et al.* [2013]

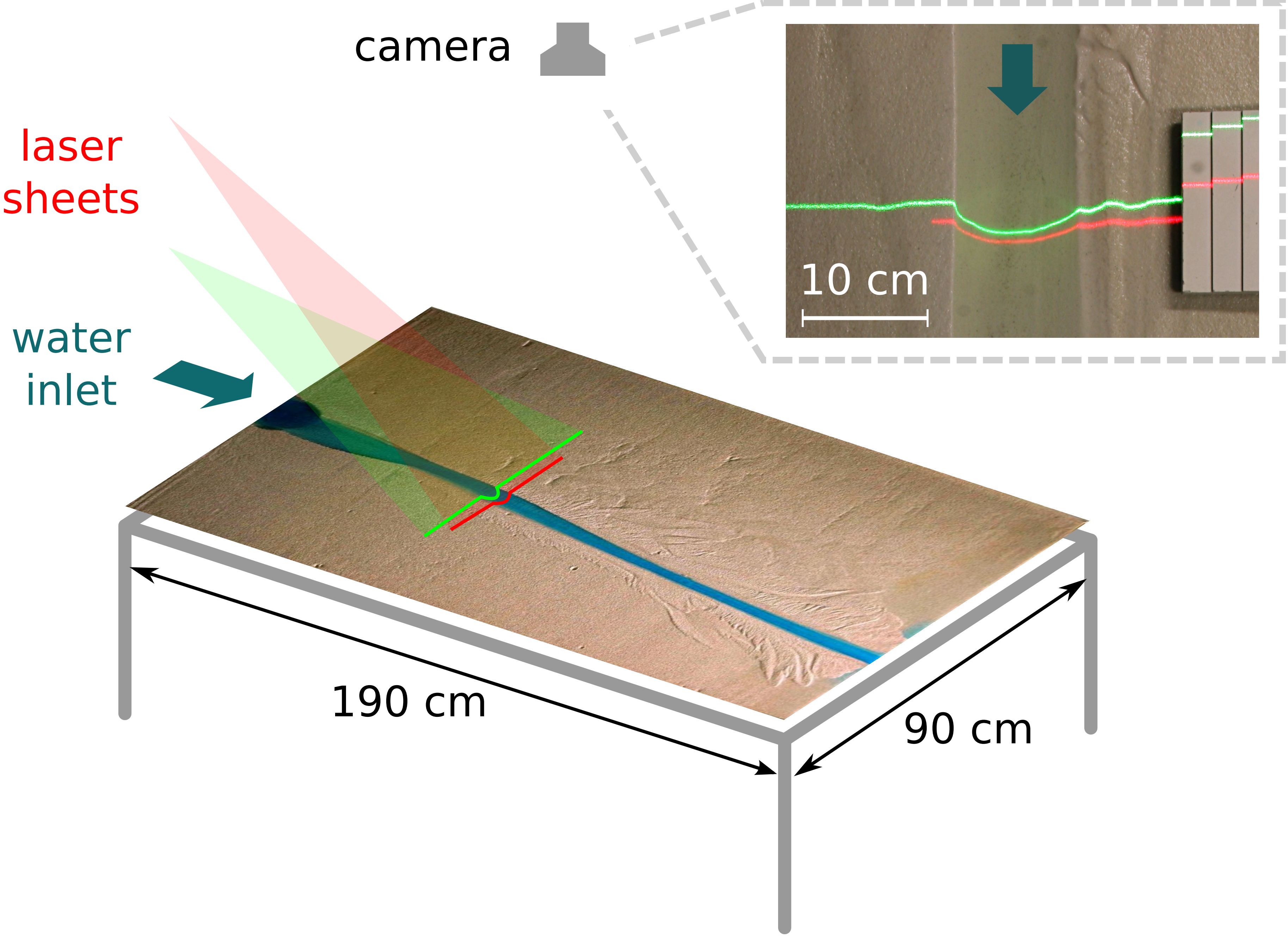
constant flow discharge

Laboratory laminar river

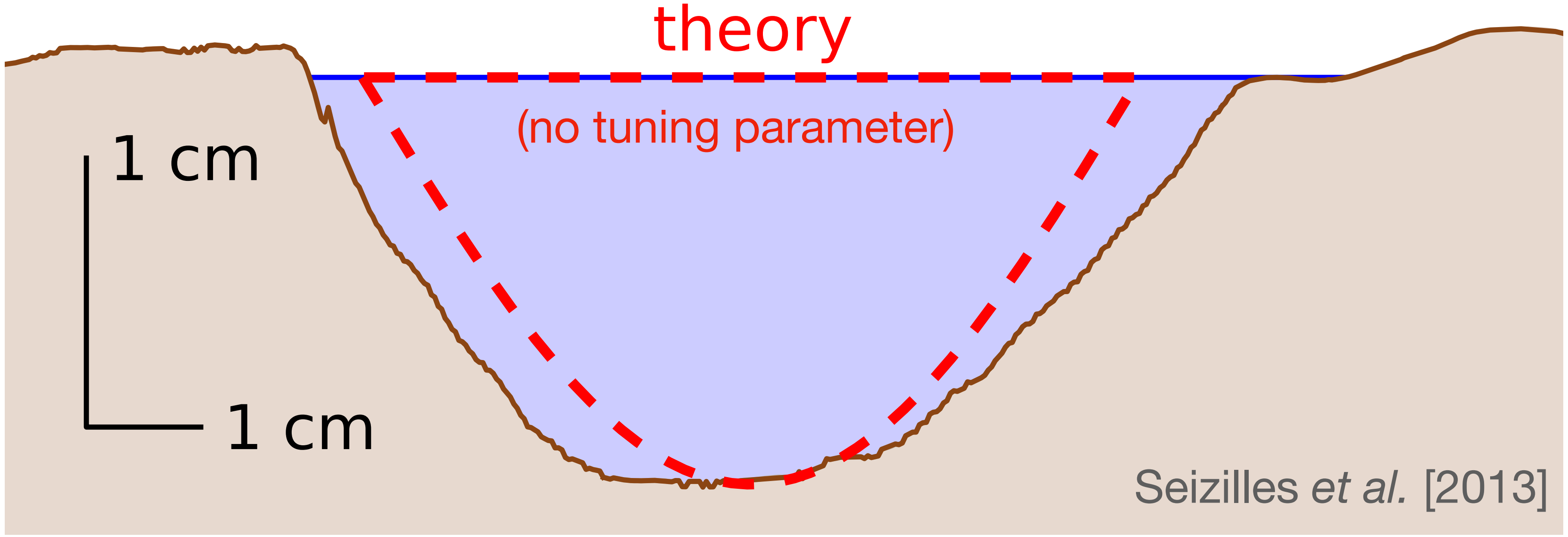


1 image every 5 minutes
duration ~ 20 hours

Laminar channel : shape

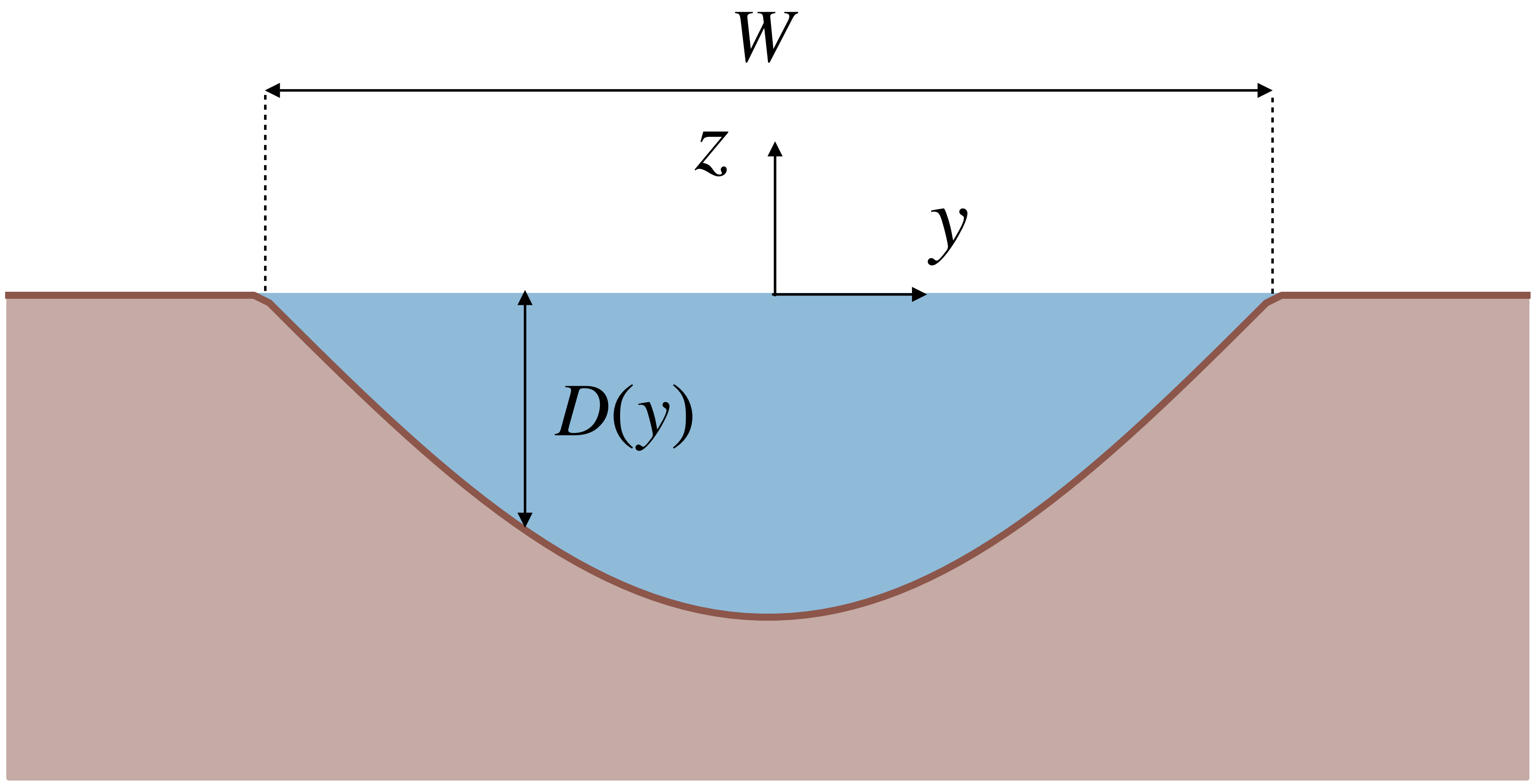


Laminar channel : shape



Seizilles *et al.* [2013]

Width vs discharge?



depth-averaged velocity

depth

discharge : $$Q_w = \int_{-W/2}^{+W/2} U \; D \; dy$$

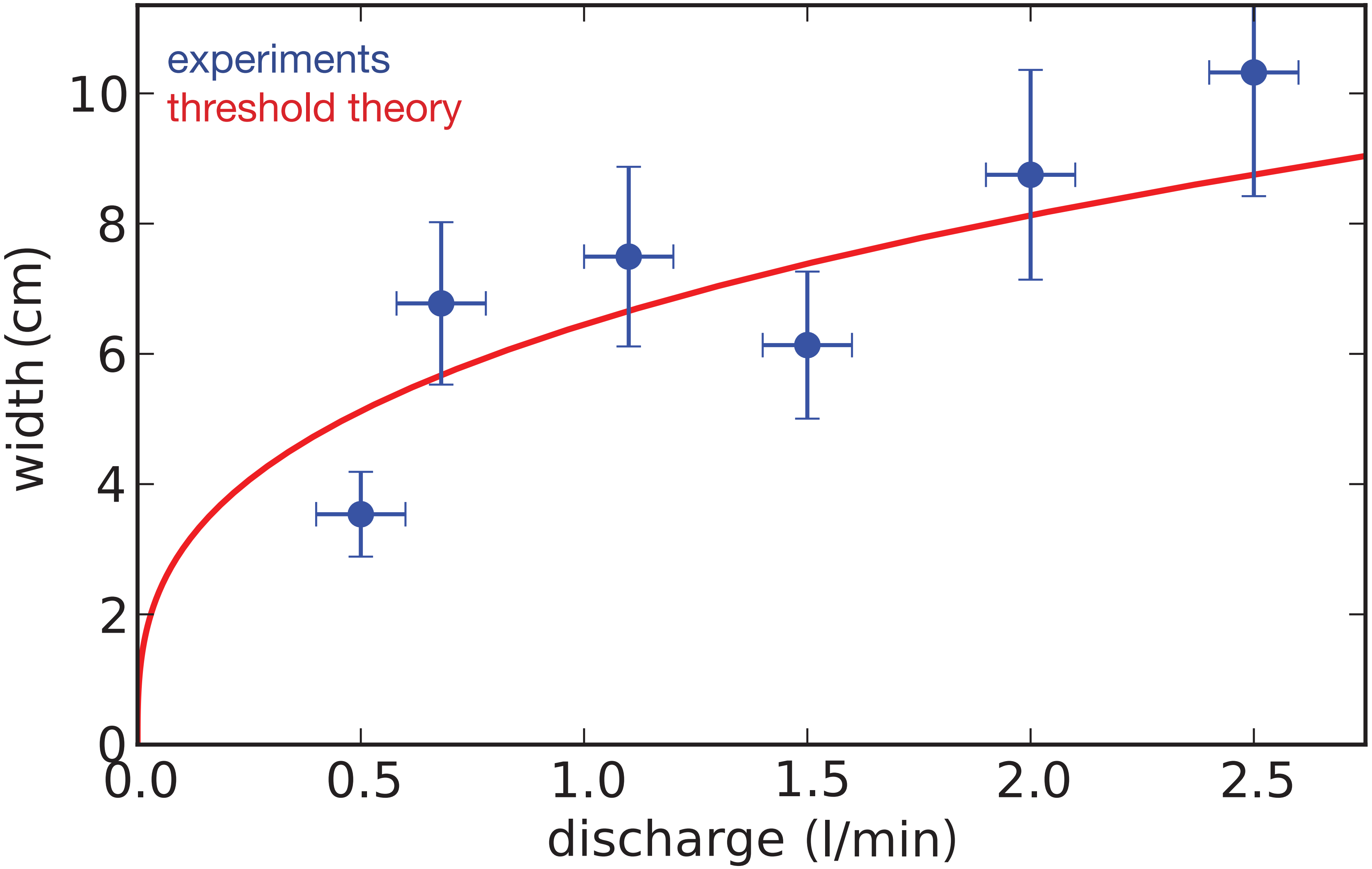
slope

lubrication theory : $$U = \frac{gSD^2}{3\nu}$$

viscosity

depth-averaged velocity

Laminar channel : width vs discharge



width

discharge

laminar flow

$$\frac{W}{\ell} = \pi \left(\frac{9}{4 \quad \theta_t \quad \mu_t^2} \right)^{1/3} \left(\frac{Q_w}{\ell^4 g/\nu} \right)^{\boxed{1/3}}$$

characteristic length : $\ell = \frac{\Delta\rho}{\rho} d_s$

threshold Shields stress

friction coef.

kinematic viscosity

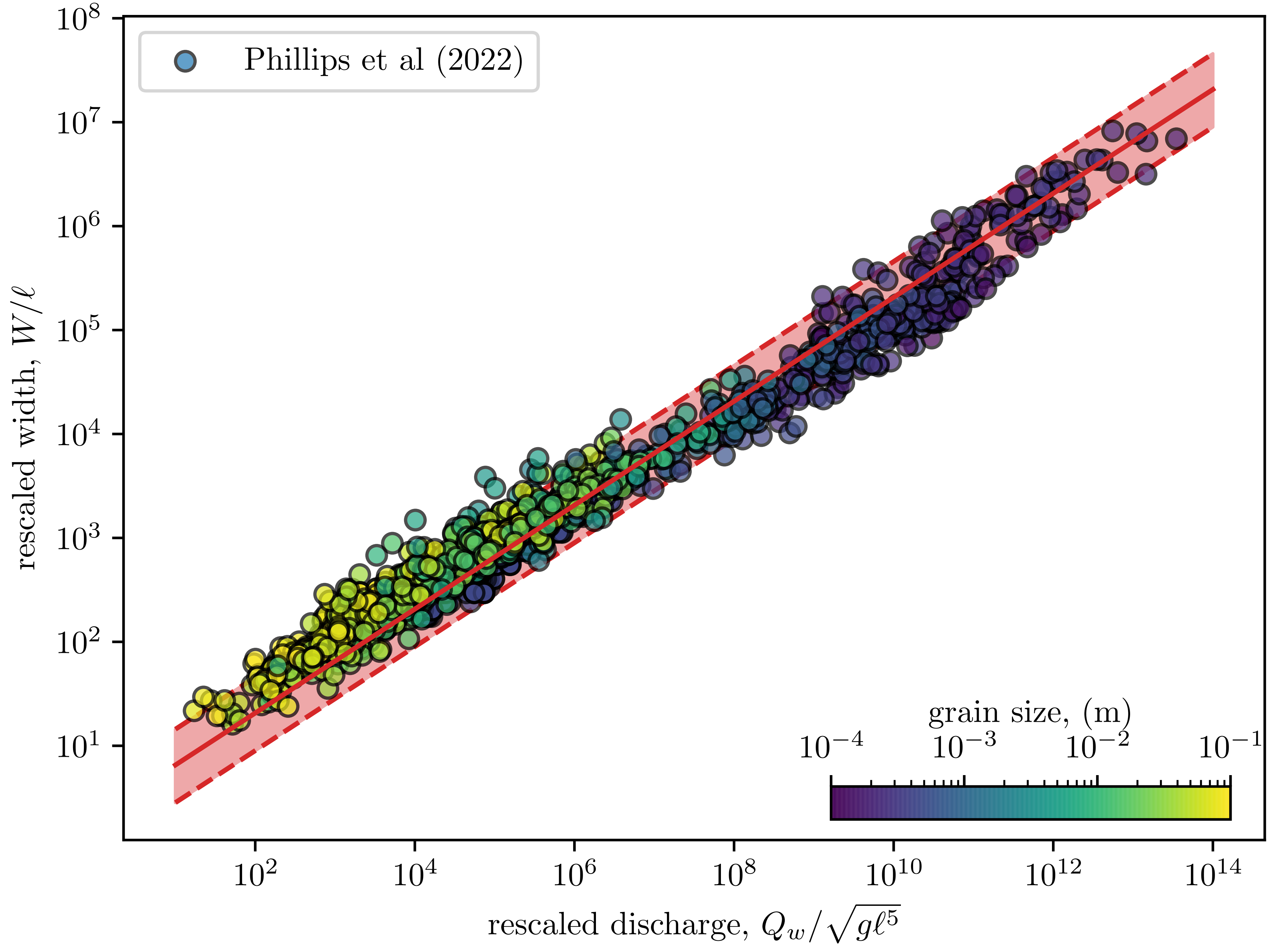
What of the field ?

river near Bayin-Buluk, chinese Tian-Shan

turbulent flow →

$$\underset{\text{depth-averaged velocity}}{U} = \left(\frac{g\overset{\text{slope}}{S}\overset{\text{depth}}{D}}{\underset{\text{turbulent friction coefficient}}{C_f}} \right)^{1/2}$$

What of the field ?



width

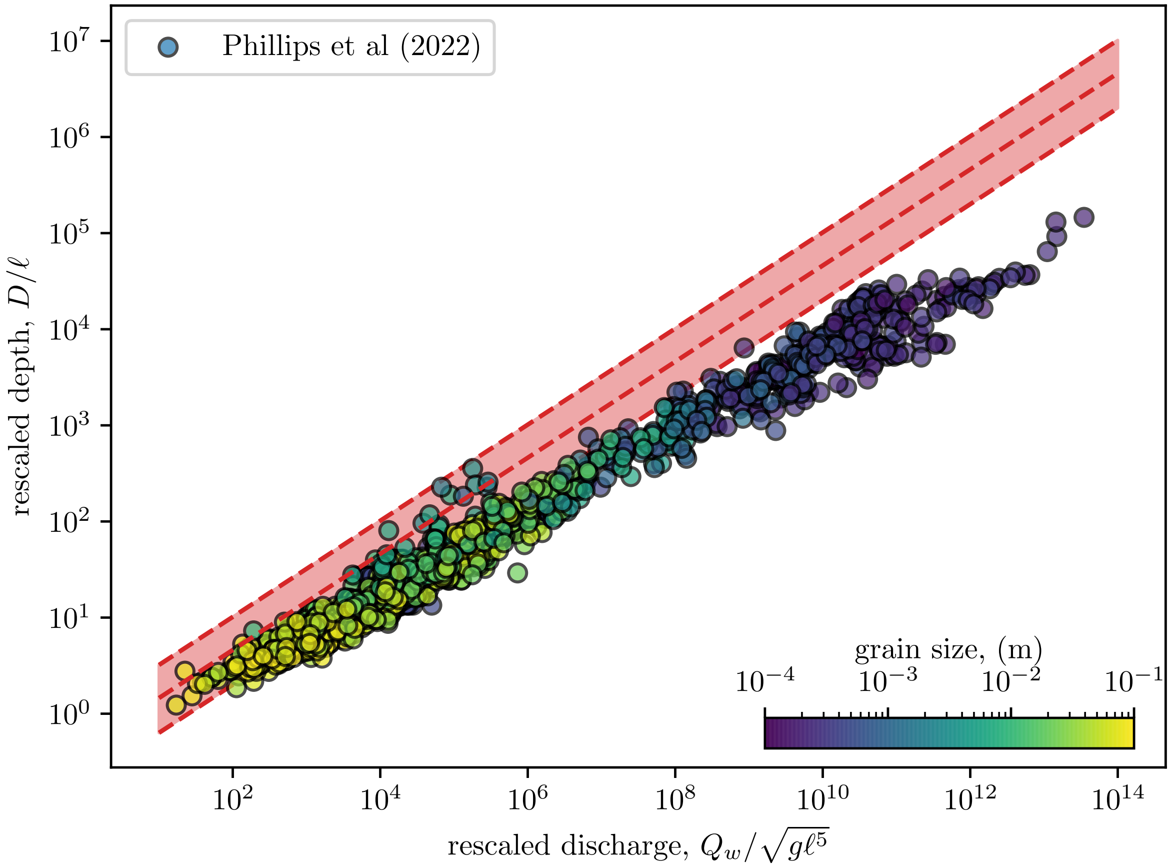
turbulent friction coef.

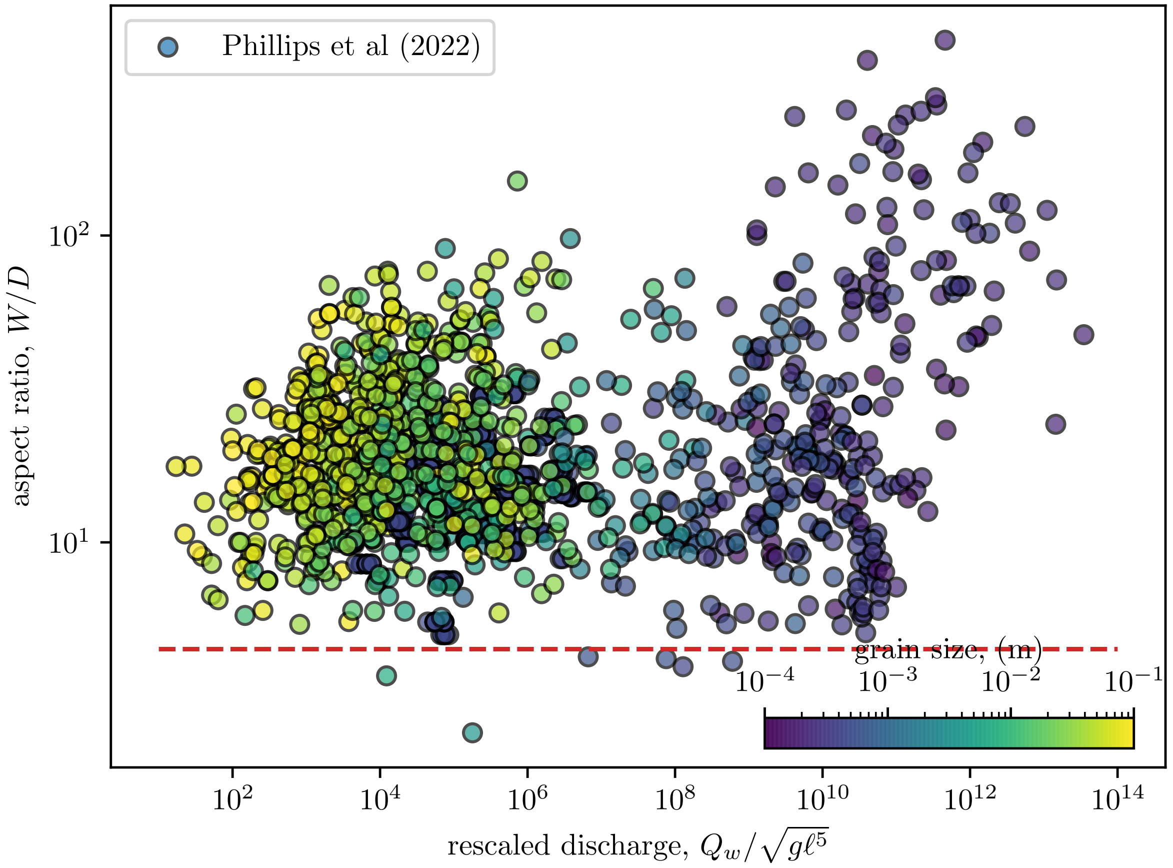
discharge

turbulent flow

$$\underset{\frac{\Delta\rho}{\rho}d_s}{\frac{W}{\ell}} = \frac{\pi}{\sqrt{I}}\left(\frac{C_f}{\theta_t\mu_t^2}\right)^{1/4}\left(Q_w/\sqrt{g\ell^5}\right)^{\boxed{1/2}}$$

Limits of the threshold theory





The threshold theory accounts for the width of rivers but not for their shape.

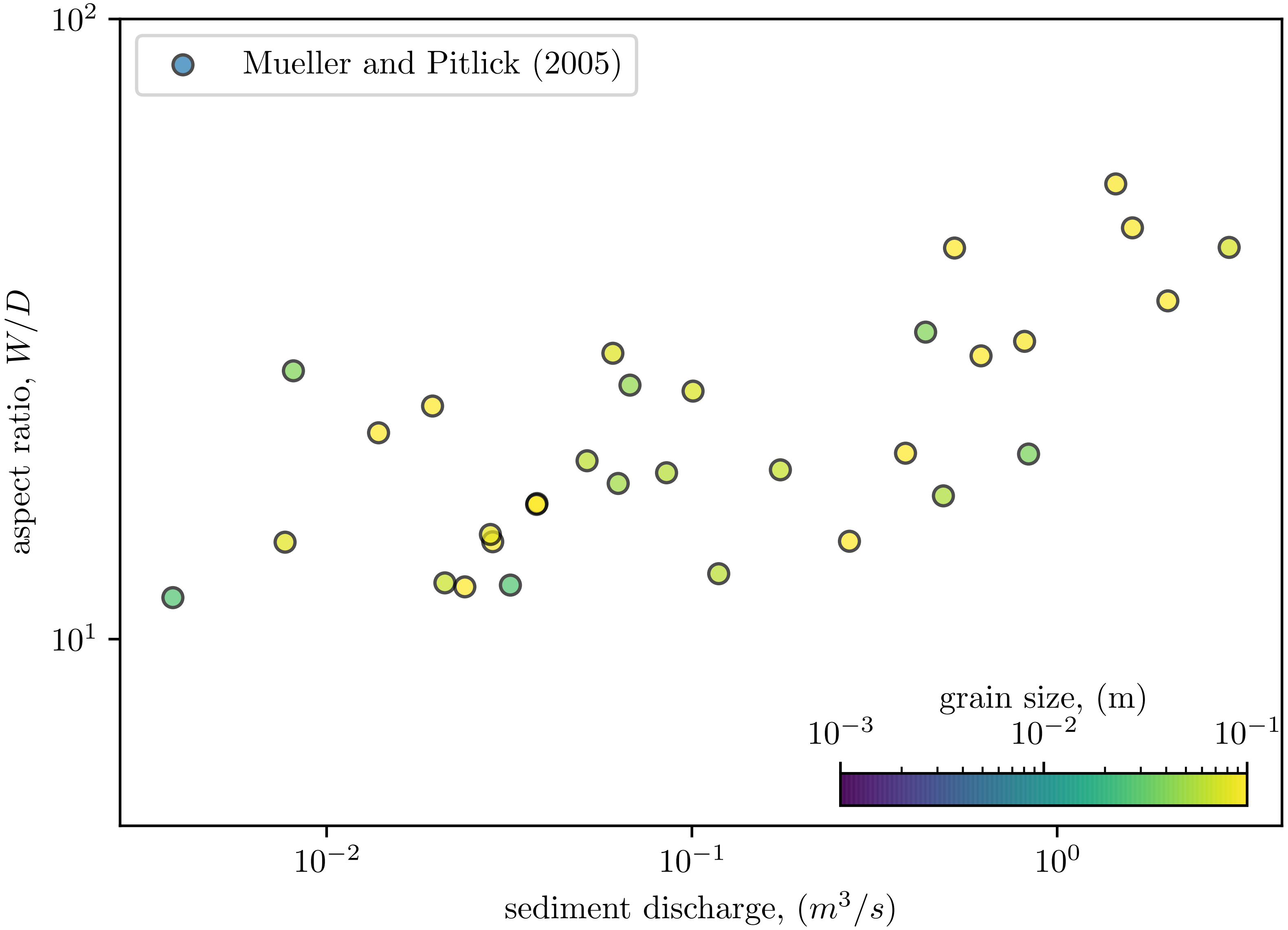
→ effect of sediment transport ?

Rivers transport sediments

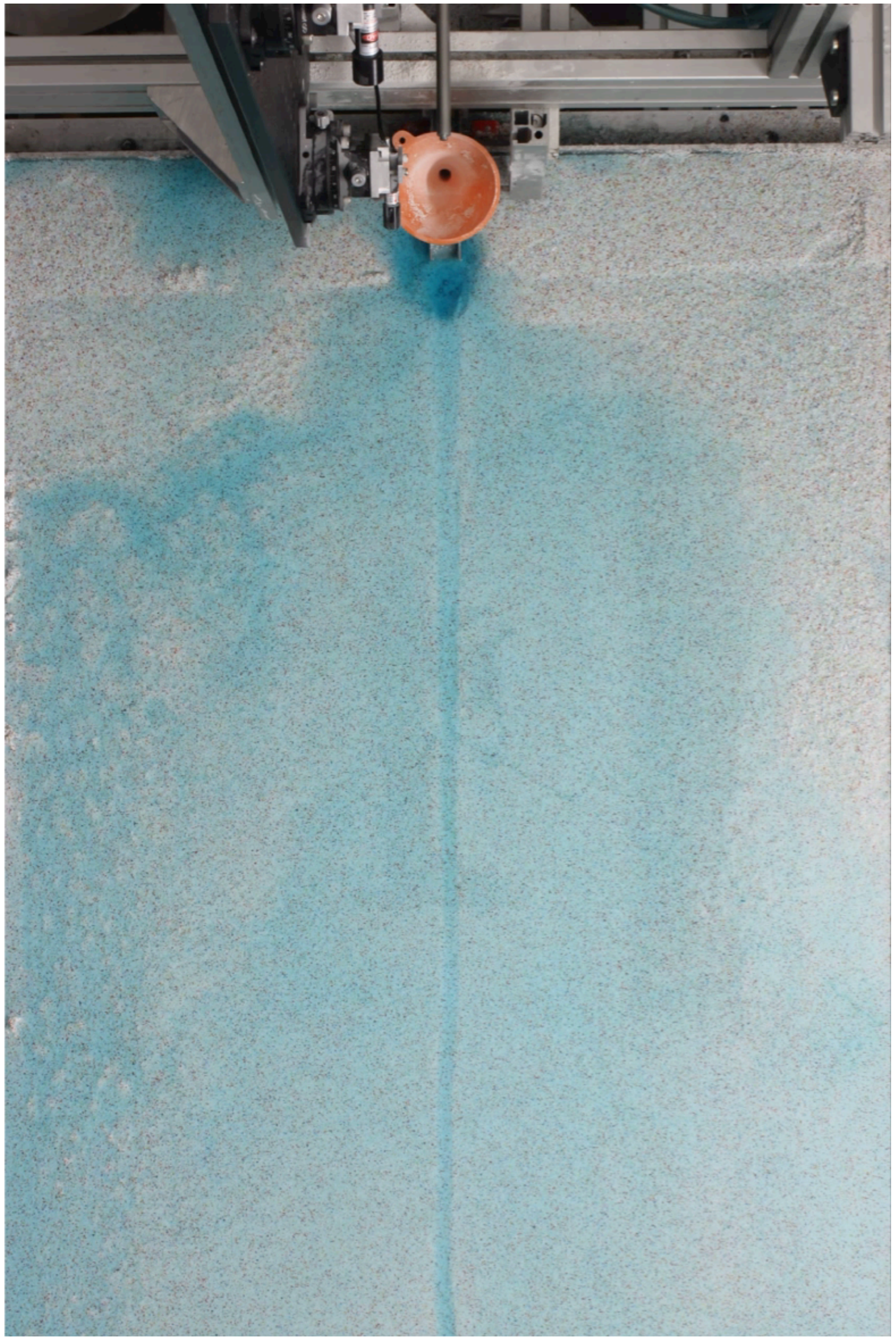
Sacramento river
© Ethan Mora

How does sediment transport influence the shape of rivers?

How does sediment transport influence the shape of rivers?

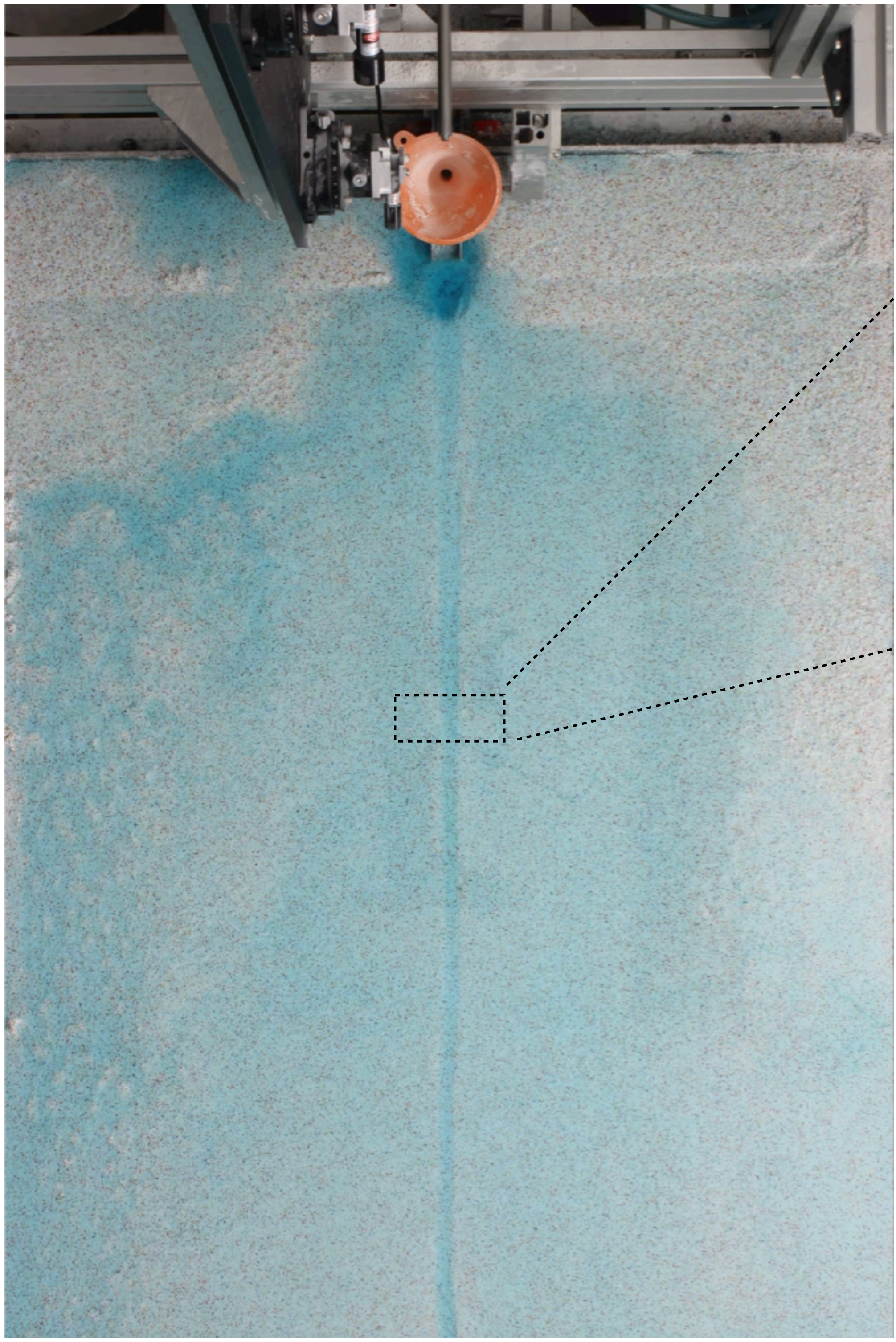


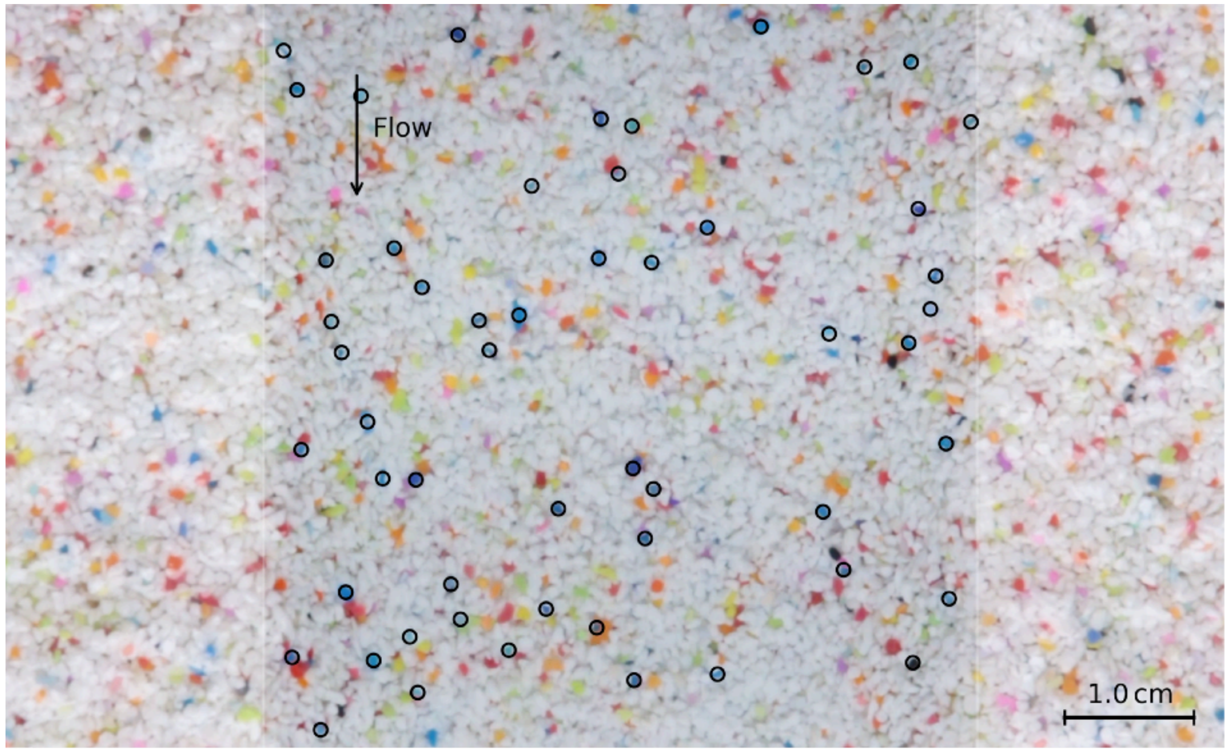
Active laboratory river

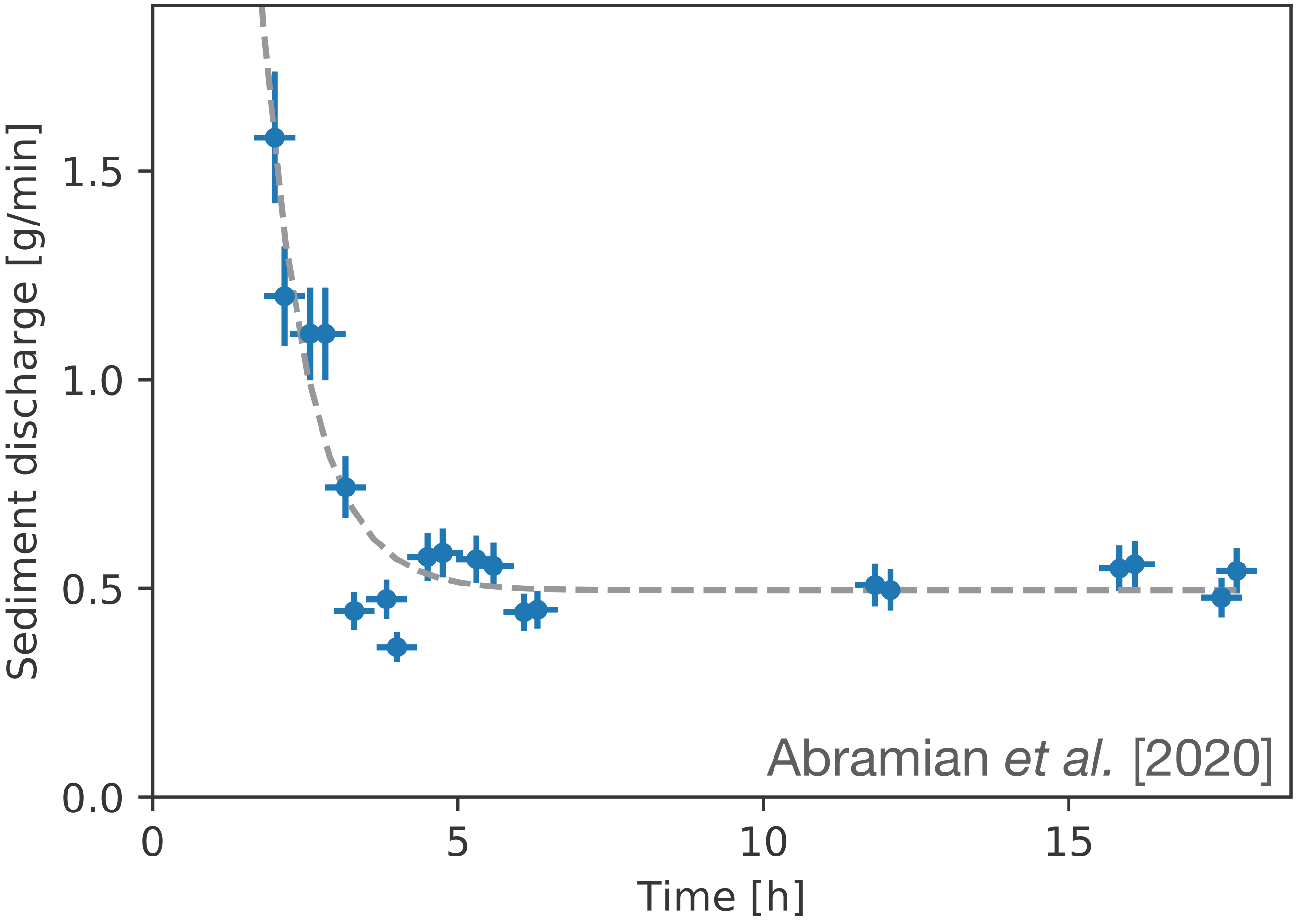


- constant flow discharge
- constant sediment discharge

Active laboratory river







Abramian *et al.* [2020]

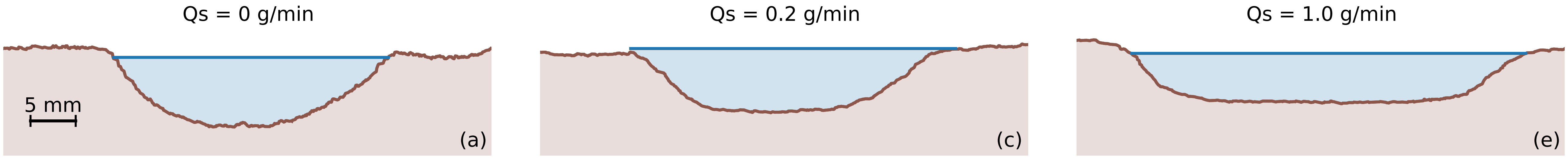
- constant flow discharge
- constant sediment discharge

Influence of sediment transport

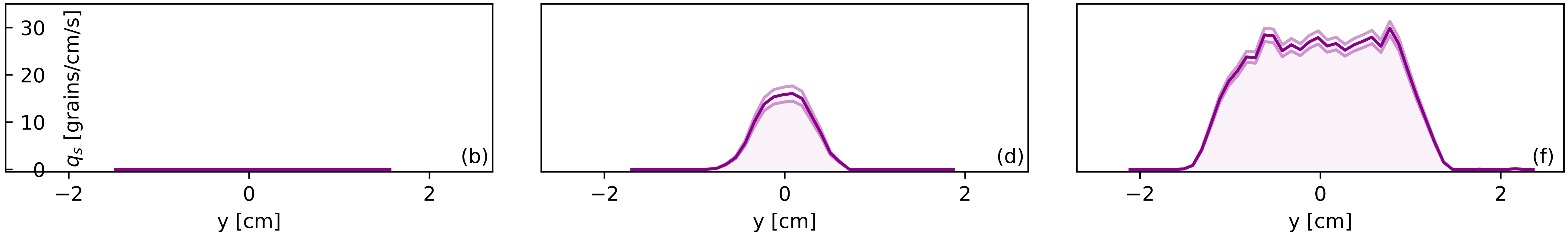
Abramian *et al.* [2020]

channel cross-section

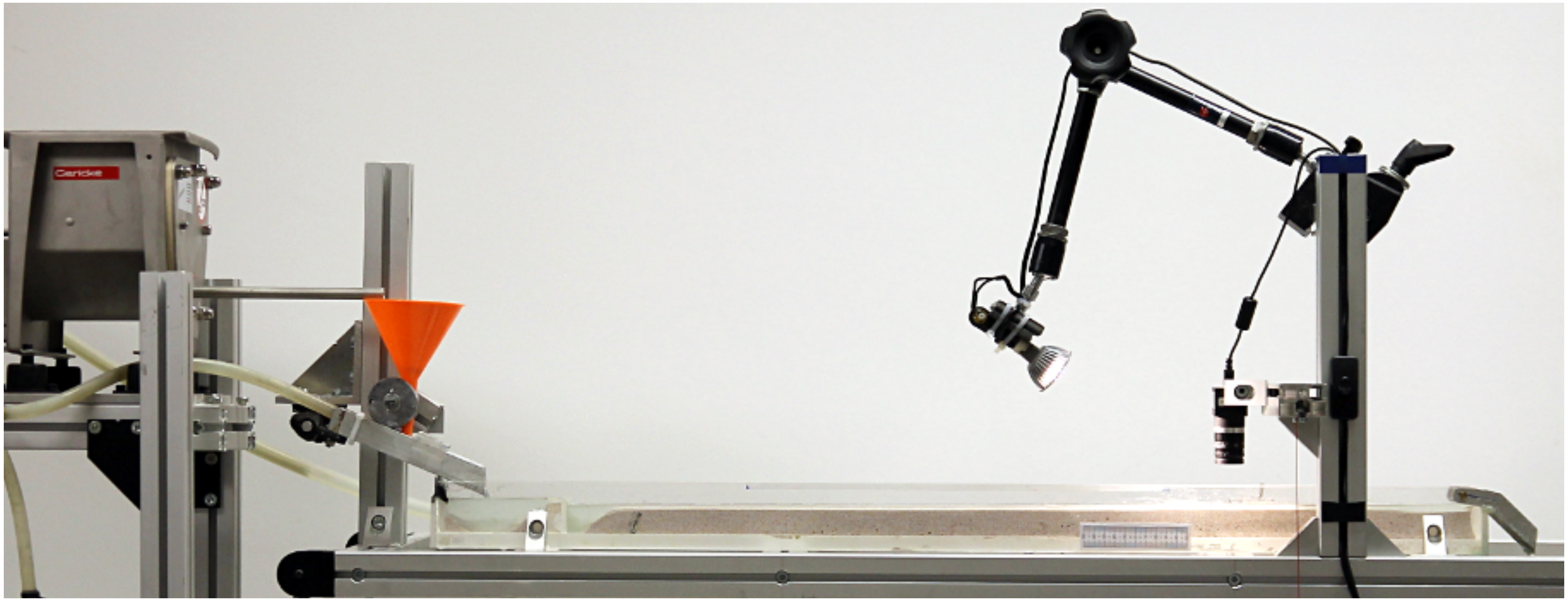
flow discharge = 0.97 L/min



profile of sediment flux



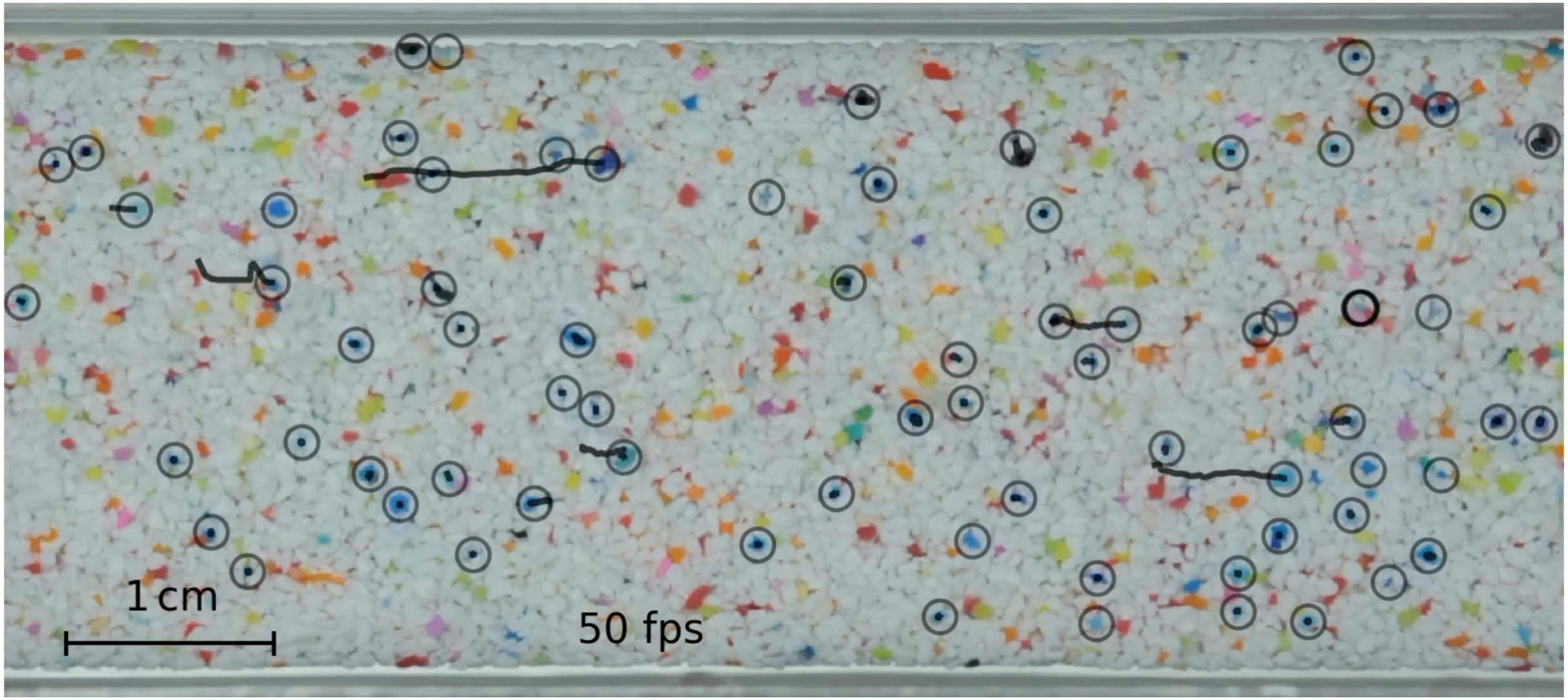
Bedload transport in a laboratory Flume



- plastic grains ($d_s \sim 0.830$mm)
- water-glycerol mixture ($Re \sim 10$)
- constant flow and sediment discharges

Seizilles *et al.* (2014), Abramian *et al.* (2019)

Bedload transport



Streamwise bedload flux

- **Near the threshold :**

tangential force

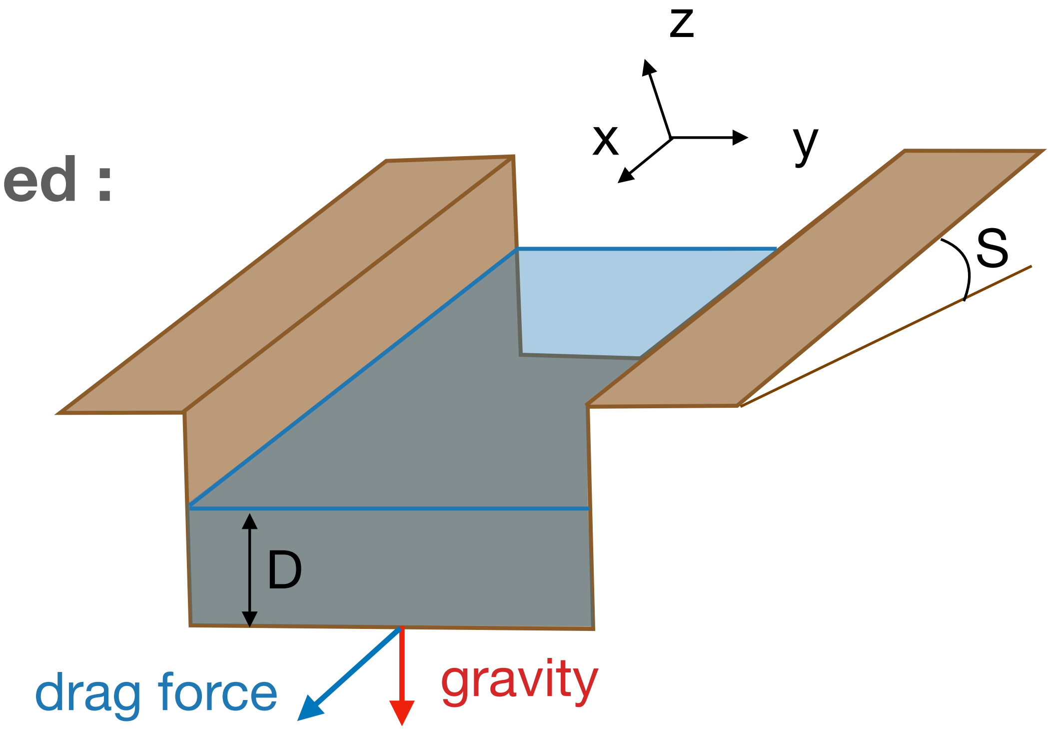
friction coefficient

$$q_s \propto \left(\frac{\mathrm{F_t}}{F_n} - \mu_t\right)$$

normal force

Charru *et al.* (2004), seizilles *et al.* (2014)

- **Flat bed :**



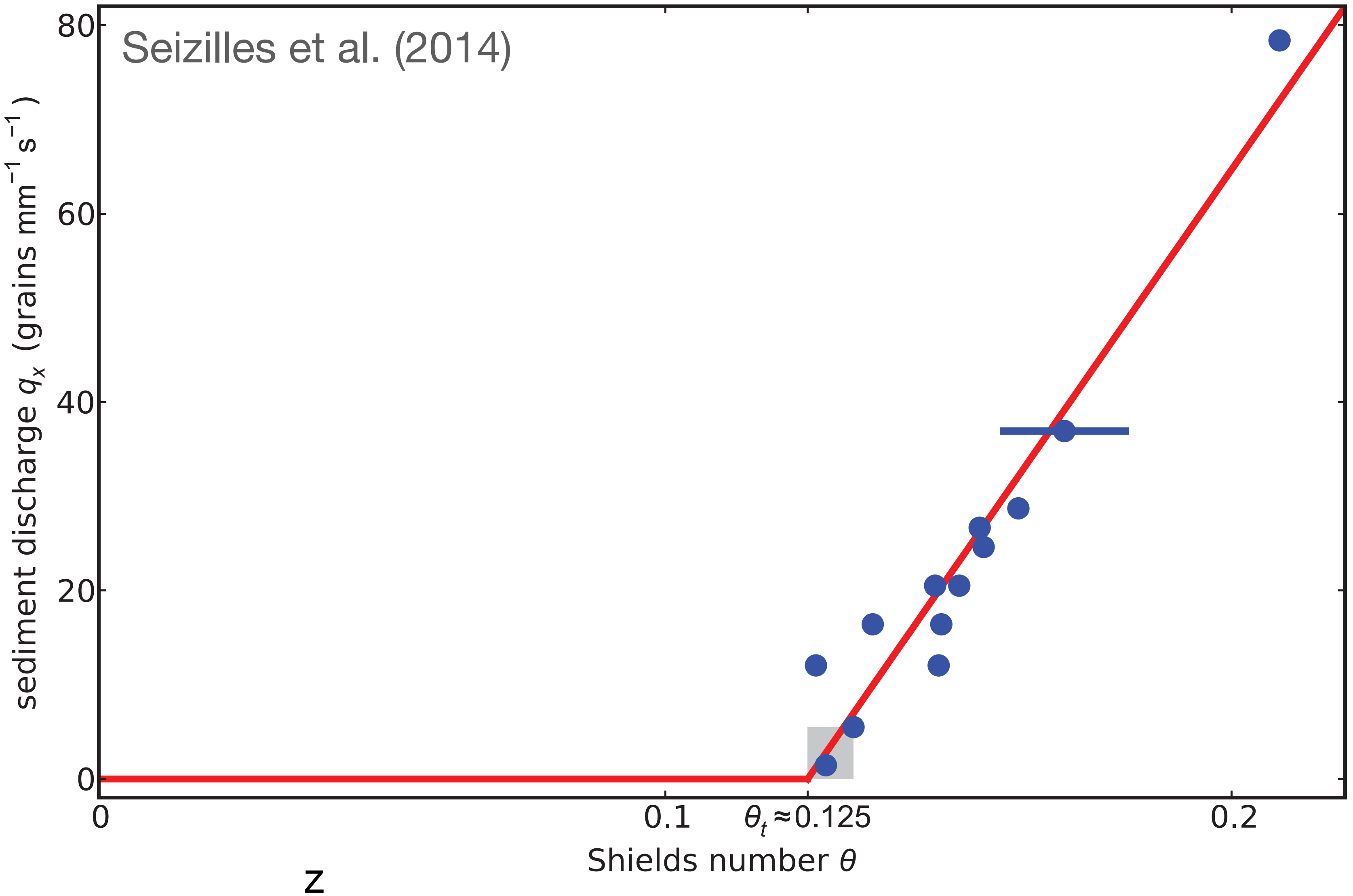
drag force

Shields number

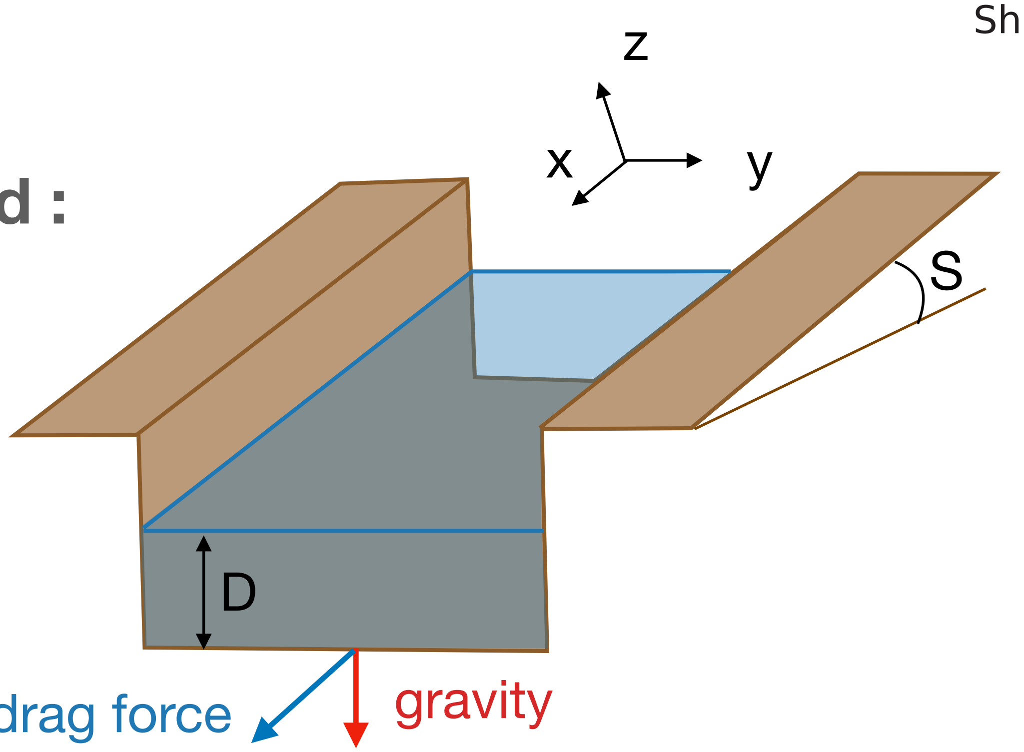
$$\frac{F_t}{F_n} \propto \frac{\boxed{\tau}}{\boxed{\Delta\rho g d_s}} = \theta$$

weight

Streamwise bedload flux



- **Flat bed :**



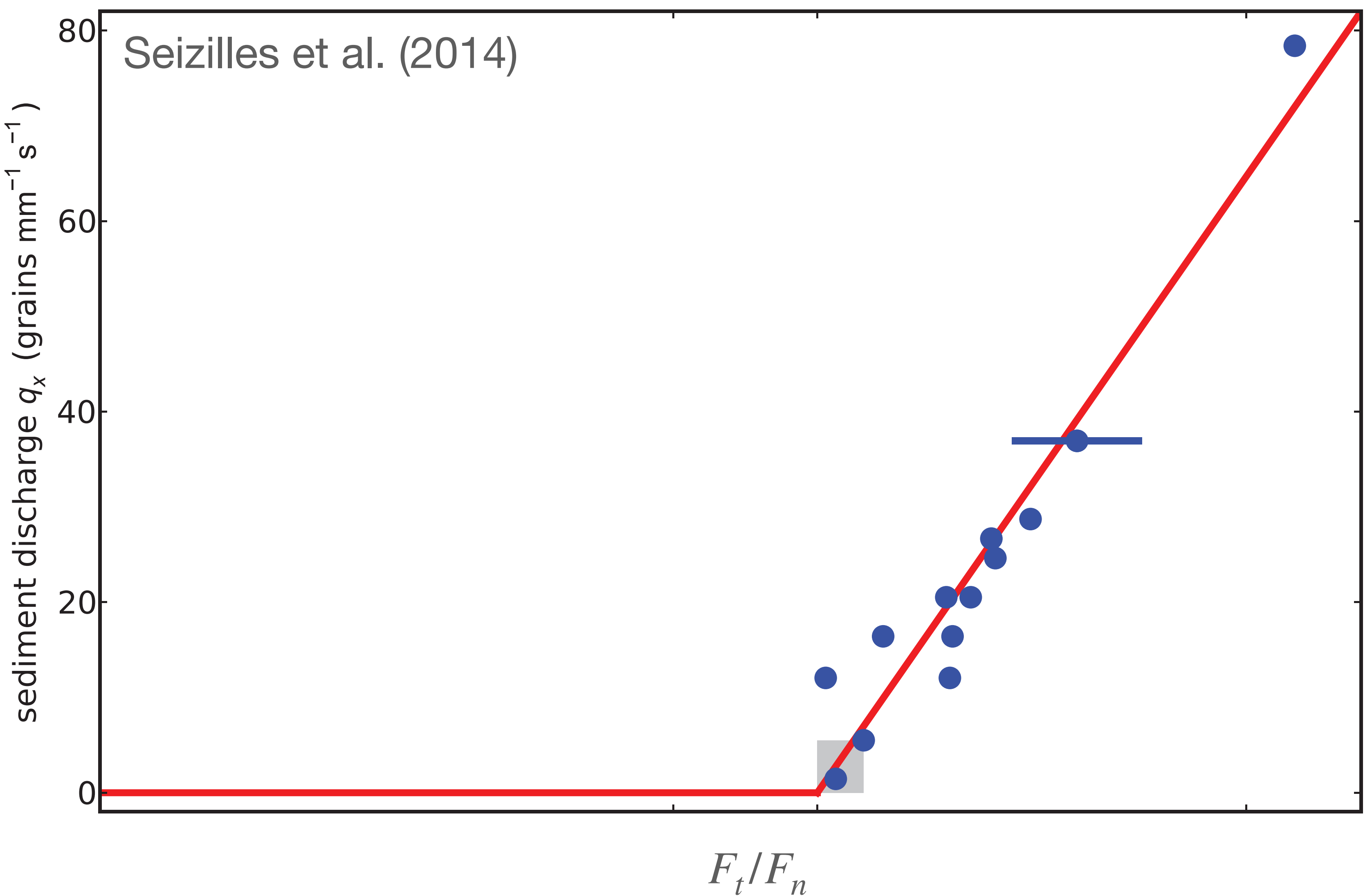
drag force

Shields number

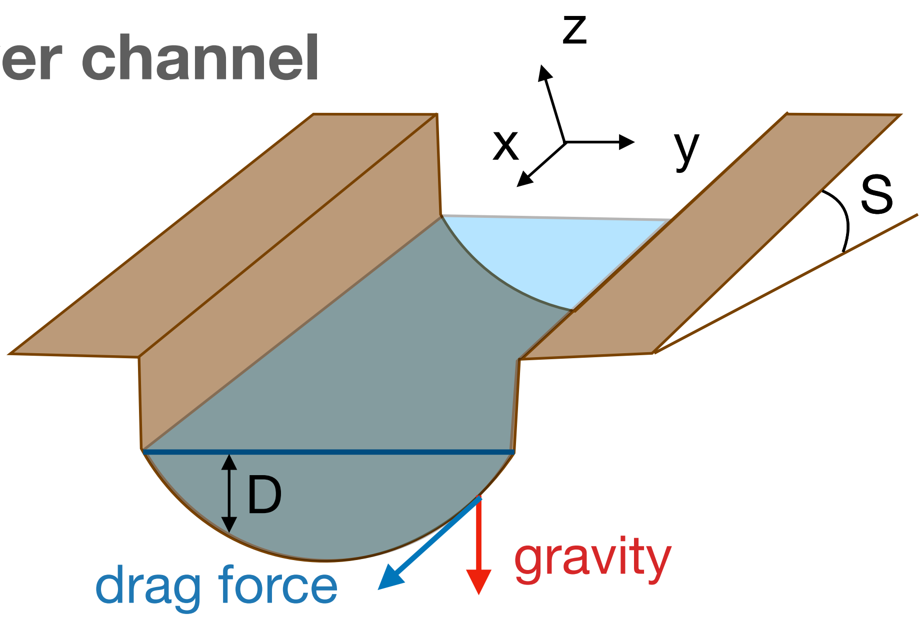
$$\frac{F_t}{F_n} \propto \frac{\boxed{\tau}}{\boxed{\Delta\rho g d_s}} = \theta$$

weight

Streamwise bedload flux

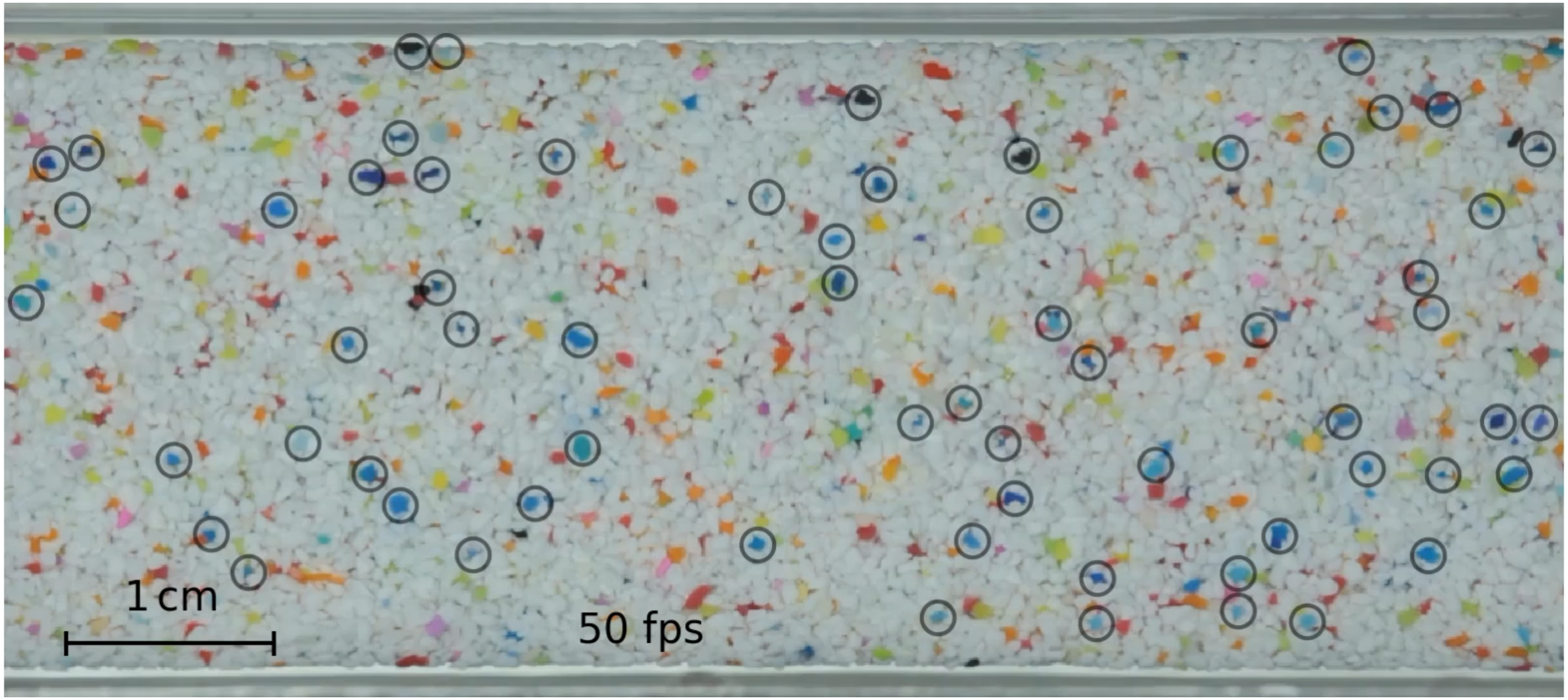


- **River channel**

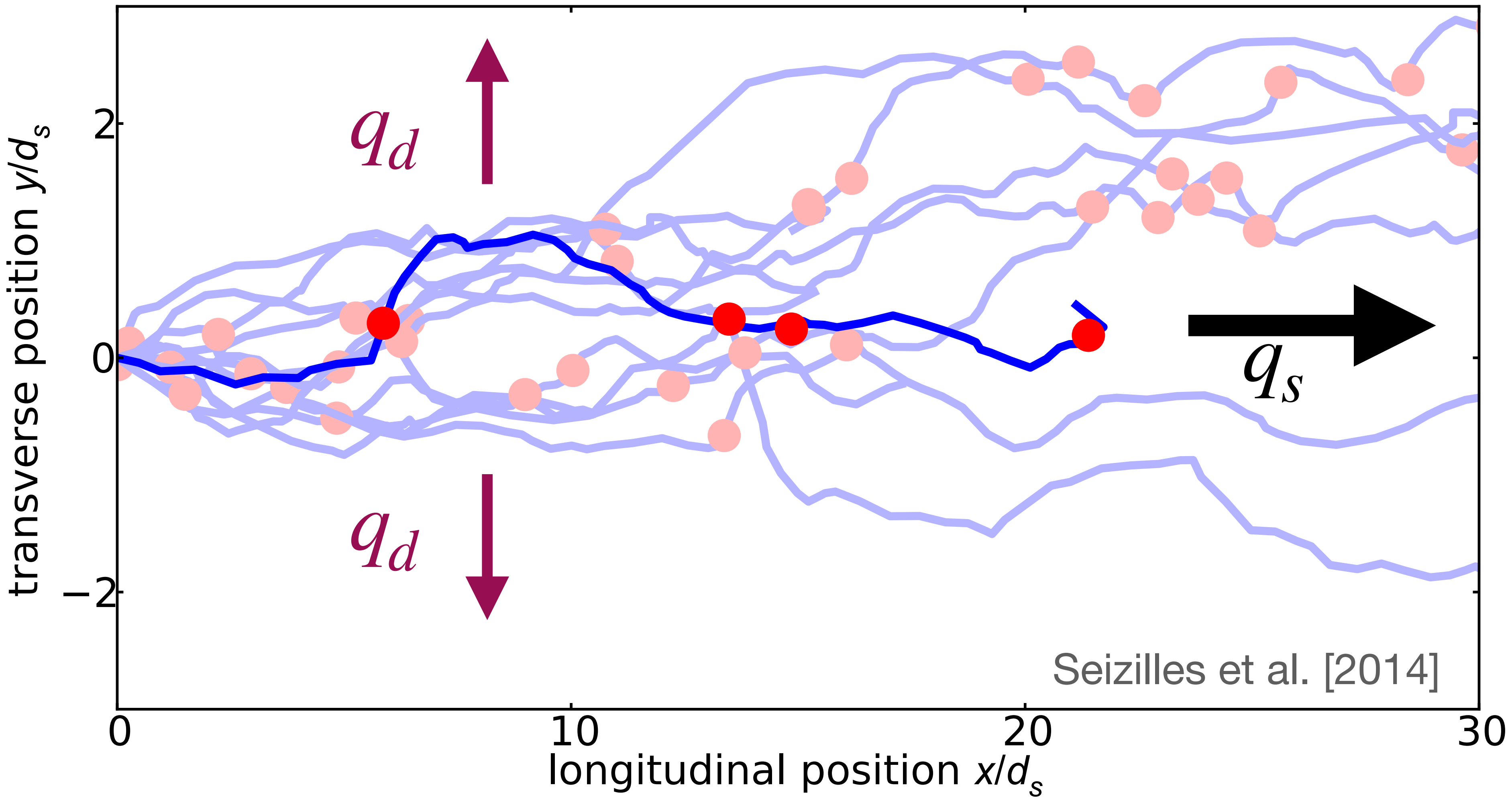


$$\frac{F_t}{F_n}=\sqrt{\underbrace{\boxed{\left(\frac{\mu_t}{\theta_t\cos\phi}\,\frac{\overset{\text{shear stress}}{\tau}}{\Delta\rho g d_s}\right)^2}}_{\text{drag force}}+\underbrace{\boxed{D'^2}}_{\text{gravity}}}$$

Cross-stream bedload diffusion



Cross-stream bedload diffusion



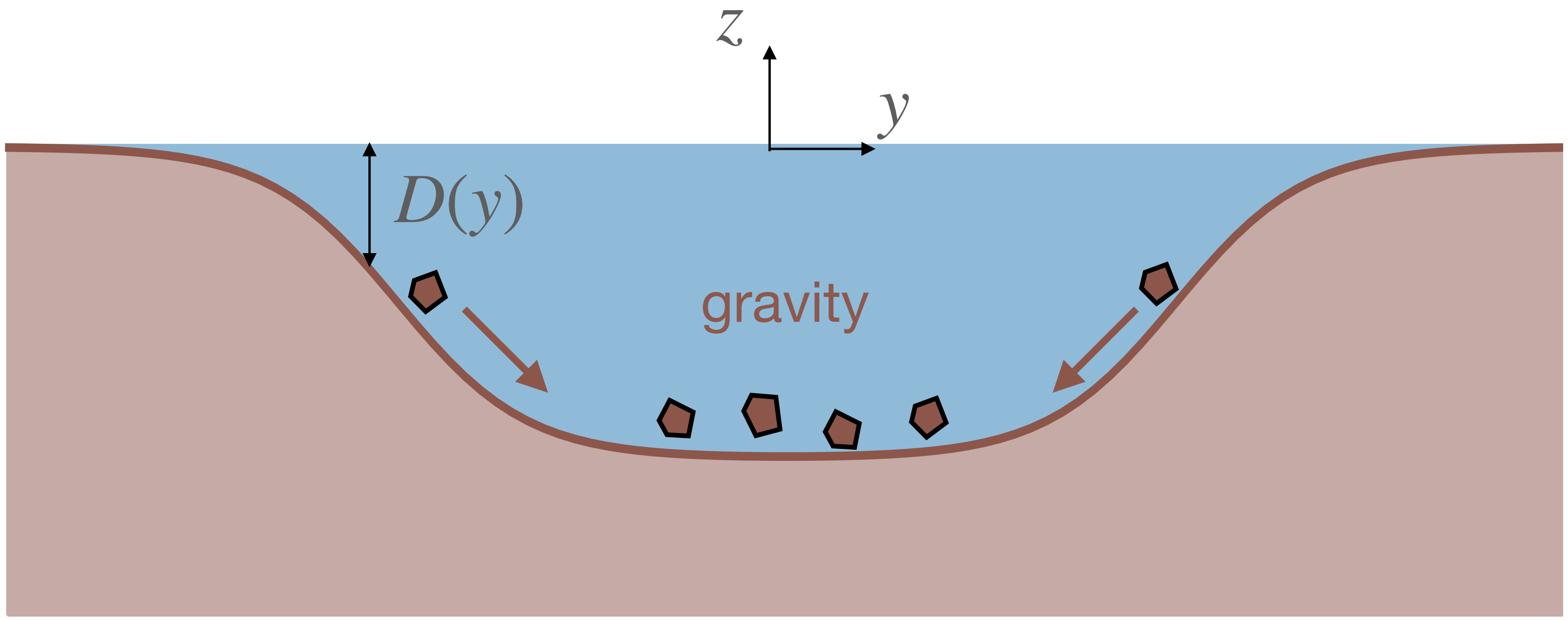
Seizilles et al. [2014]

gradient of streamwise flux

cross-stream diffusive flux : $q_d = -\,\ell_d \,\dfrac{\partial q_s}{\partial y}$

diffusion length $\approx 0.03\ d_s$

Cross-stream gravity flux



cross-stream gravity flux :

streamwise flux

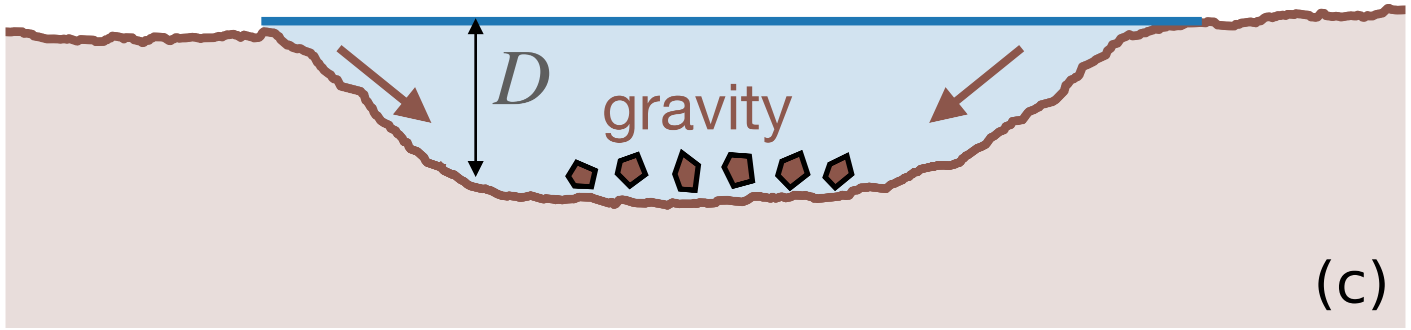
bed slope

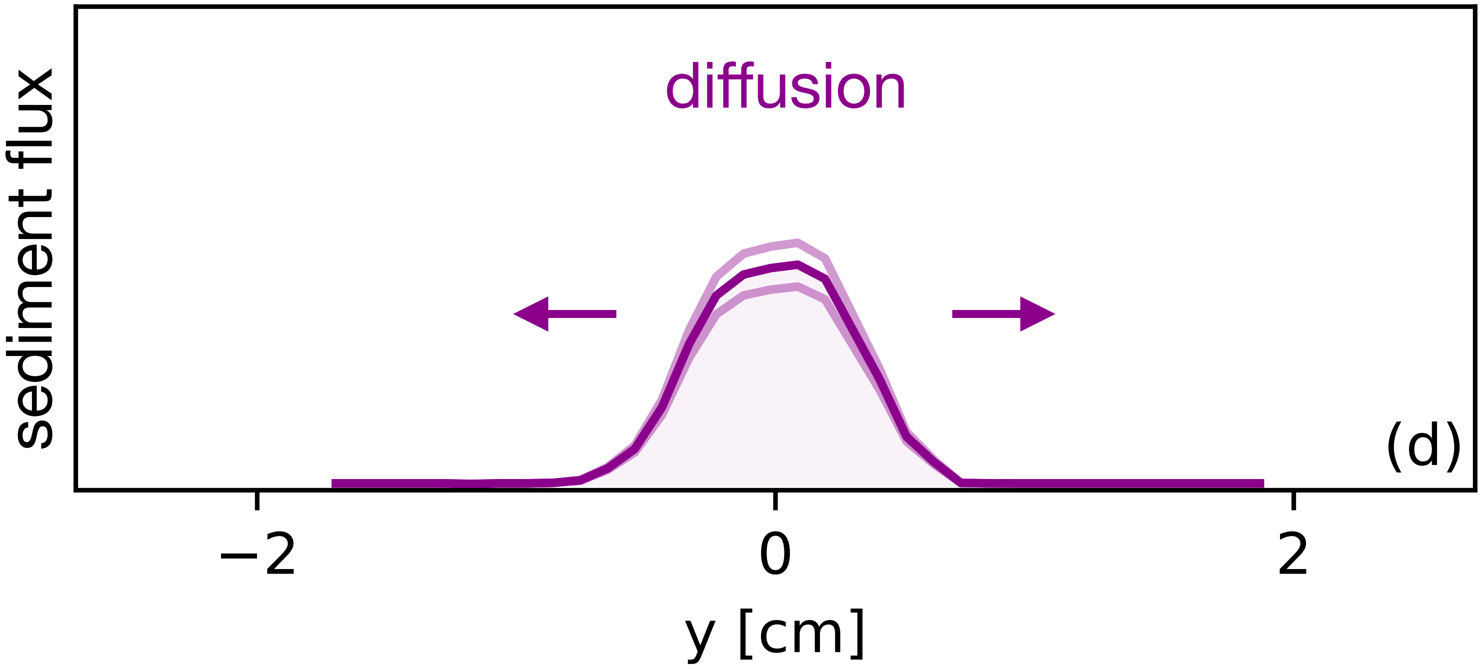
$$q_g = \alpha \; q_s \, \frac{\partial D}{\partial y}$$

constant

Ikeda et. al. [1988], Chen et al. [2010], Abramian et al. [2019, 2020]

Equilibrium condition in an active channel

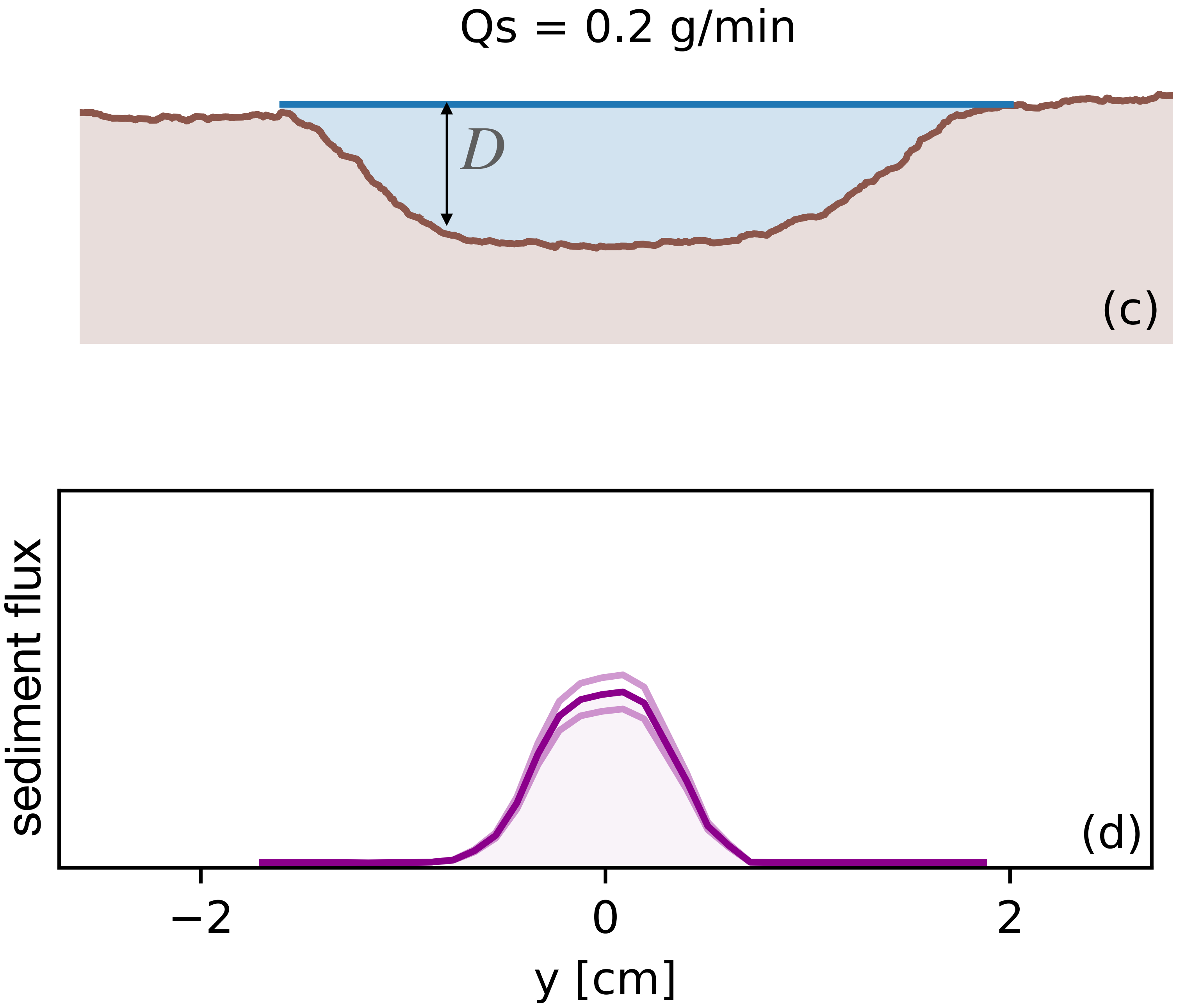




At equilibrium : $-\ell_d \frac{\partial q_s}{\partial y} + \alpha\, q_s \frac{\partial D}{\partial y} = 0$

diffusion gravity

Equilibrium condition in an active channel

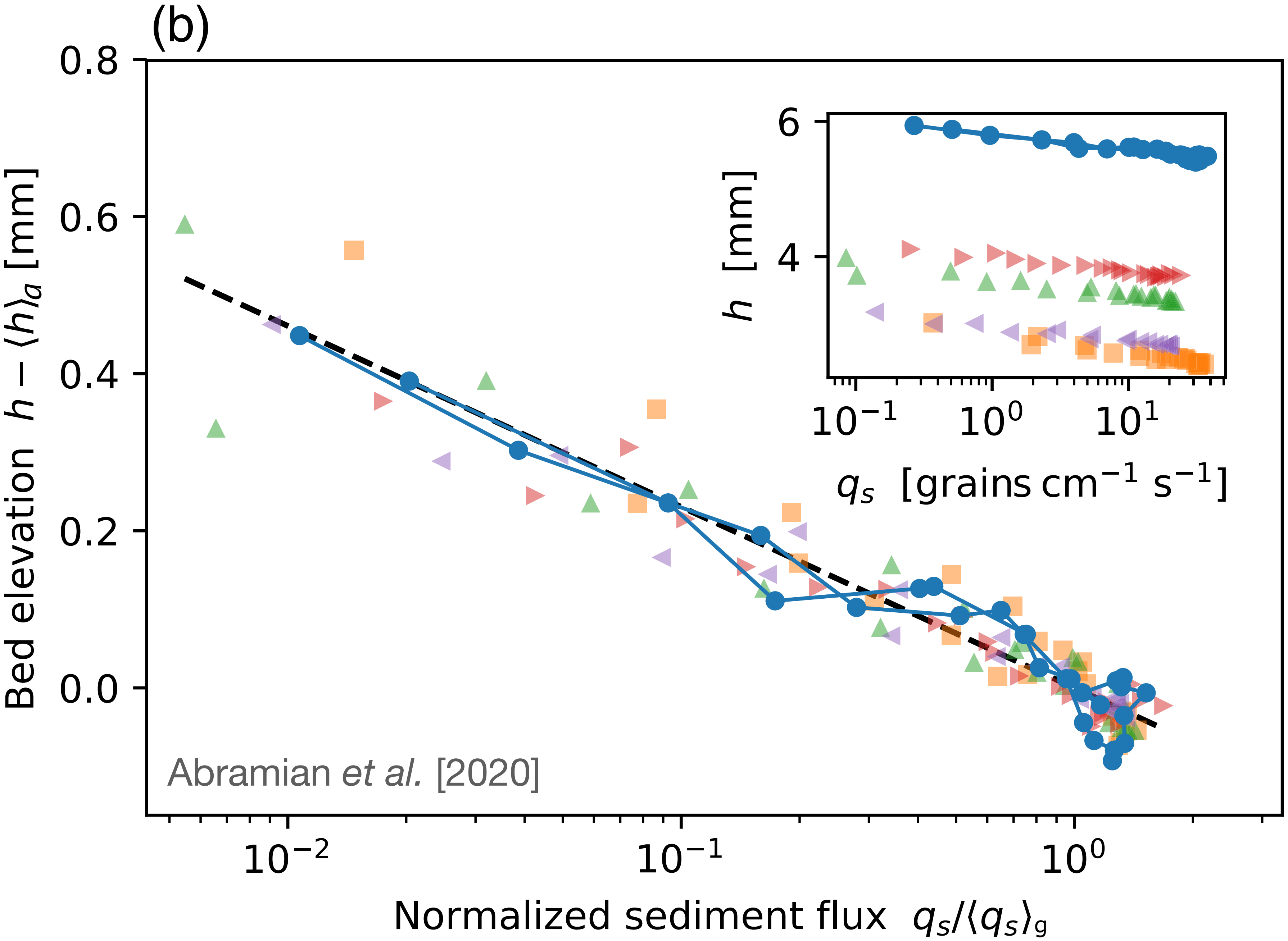


depth

$\lambda \sim$ diffusion length = damping height

At equilibrium : $q_s \propto e^{D/\lambda}$

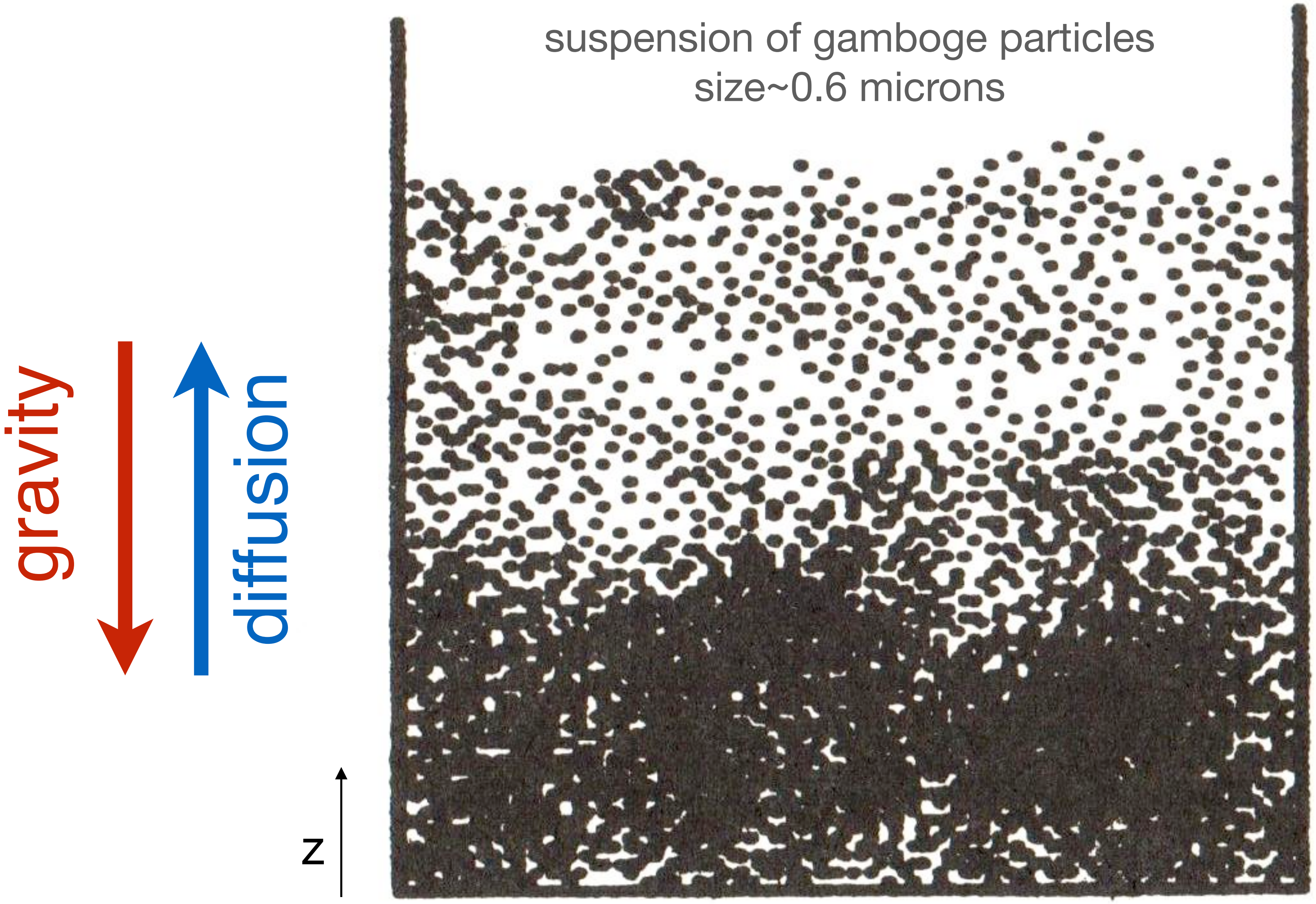
Boltzmann equilibrium

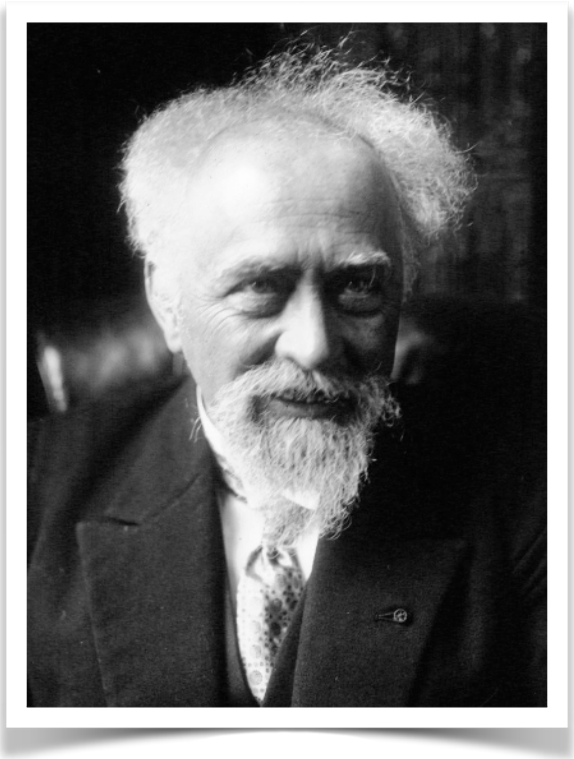


At equilibrium : $q_s \propto e^{D/\lambda}$

damping length $\lambda \sim 0.12\ d_S$

Gravity vs diffusion



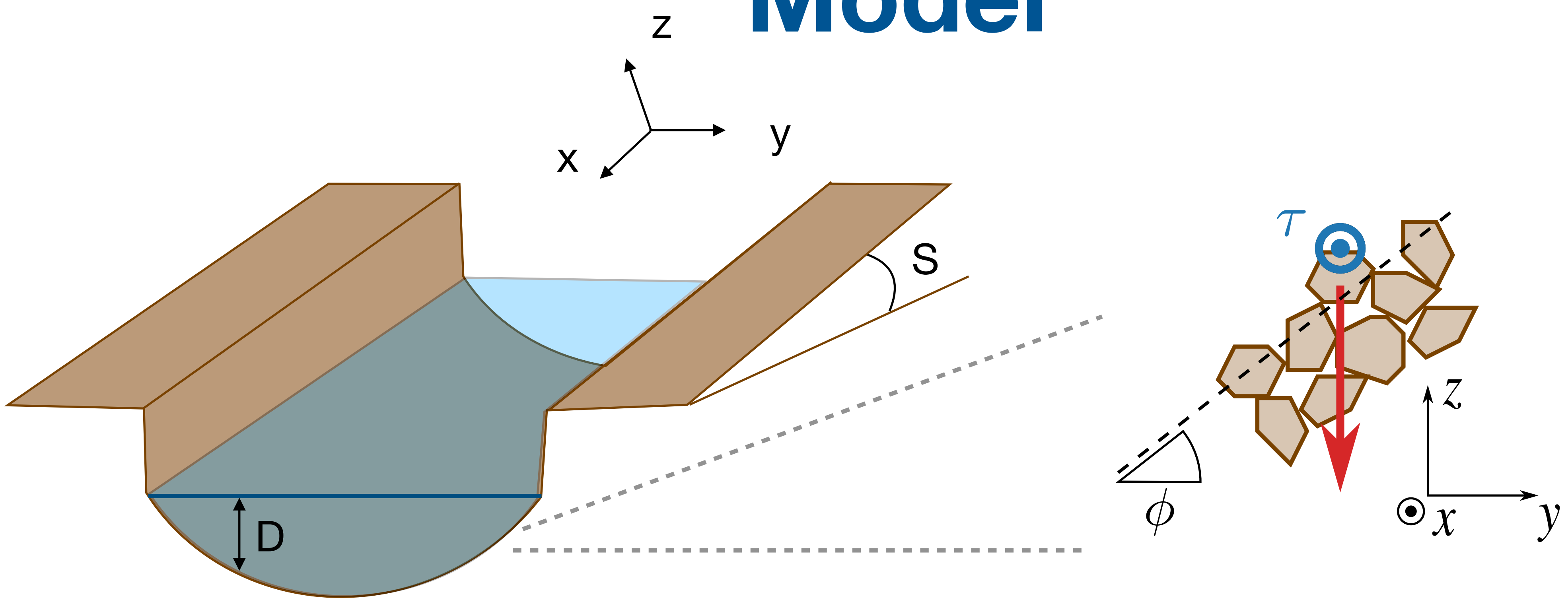


J. Perrin [1908]

At equilibrium, diffusion = gravity flux [Einstein, 1905]

$$c \sim e^{-z/\lambda}$$

Model



depth

- Diffusion vs gravity : $q_s \propto e^{D/\lambda}$

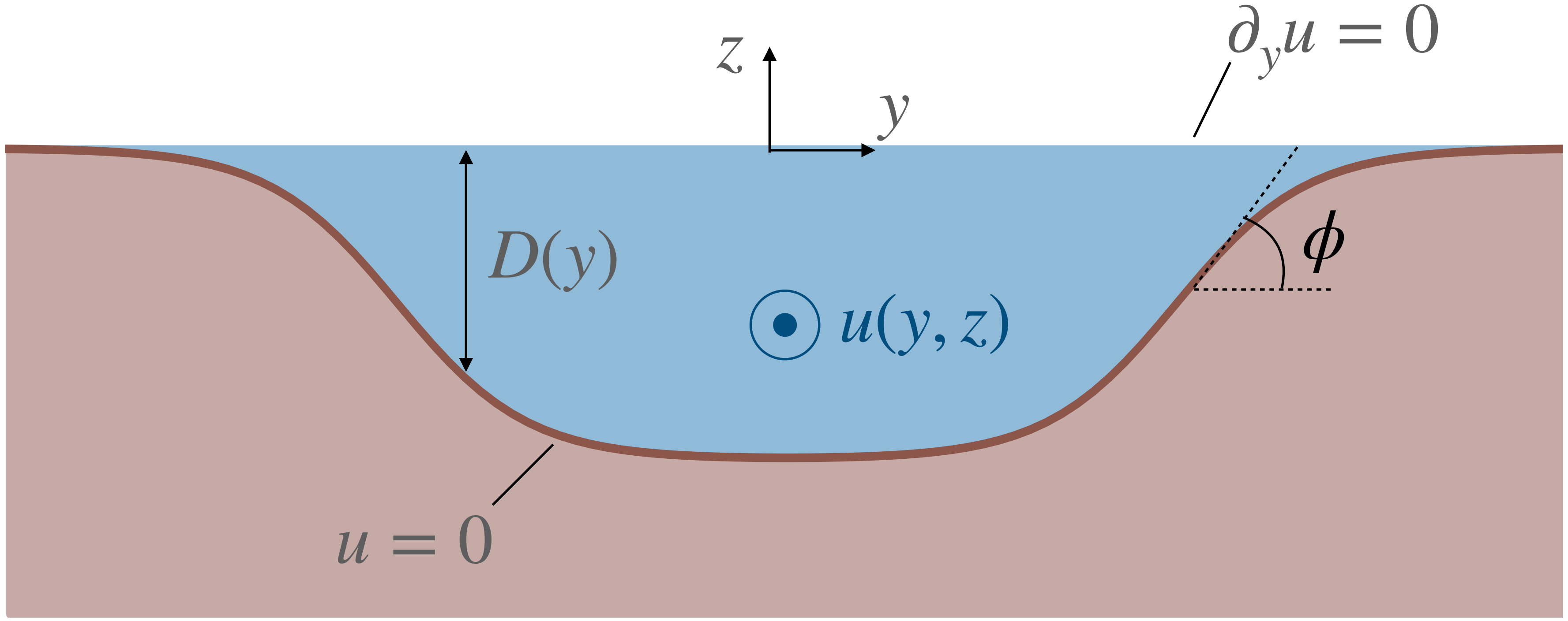
shear stress

- Transport law : $q_s \propto \left(\left(\frac{\mu_t}{\theta_t \cos\phi} \frac{\tau}{\Delta\rho g d_s} \right)^2 + D'^2 \right)^{1/2} - \mu_t$

- Shear stress : $\tau = f($channel shape$)$

→ Free boundary problem

Cross-stream diffusion of momentum



velocity

slope

Stokes equation : $\partial_{yy} u + \partial_{zz} u = -\dfrac{gS}{\nu}$

viscosity

for large aspect ratios $\rightarrow \tau = \rho g S \left(\boxed{D} + \boxed{\frac{1}{3}(D^3)''} \right) \cos\phi$

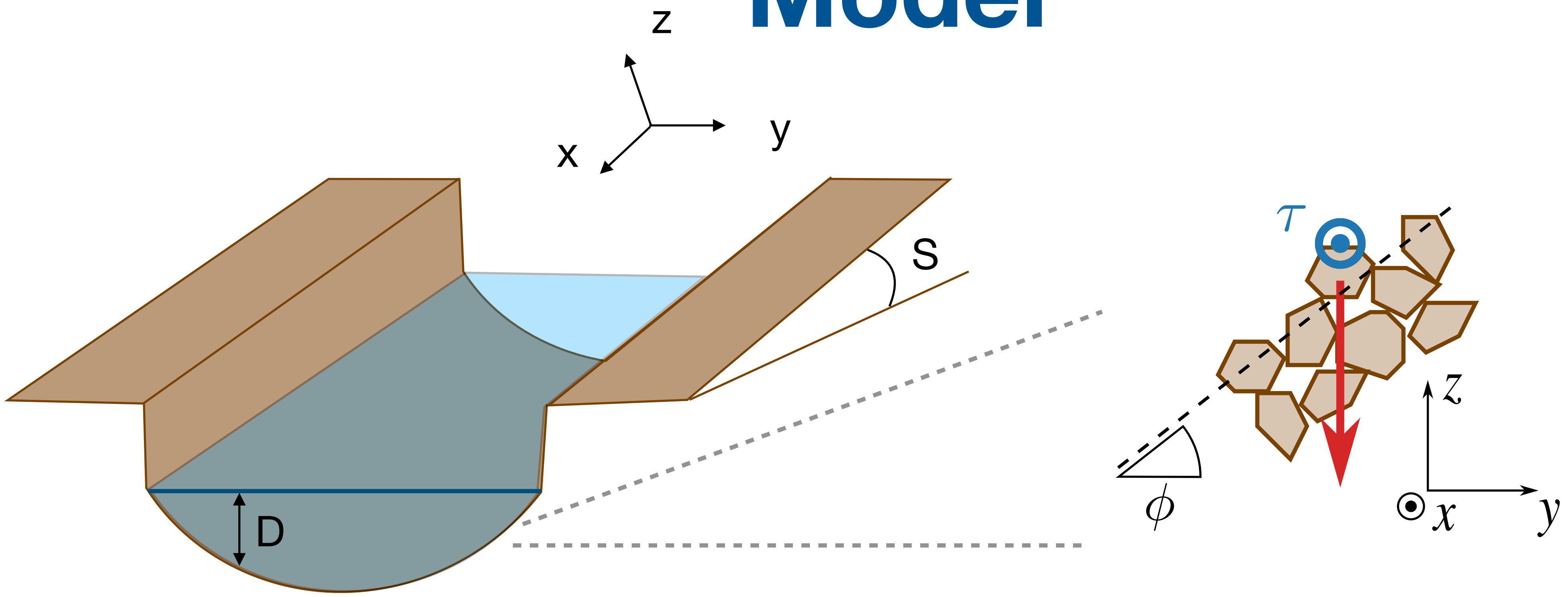
Devauchelle *et al.* [2021], Popovic *et al.* [2021]

weight of
water column
(shallow water)

cross-stream
diffusion of
momentum

orientation of
the bed surface

Model



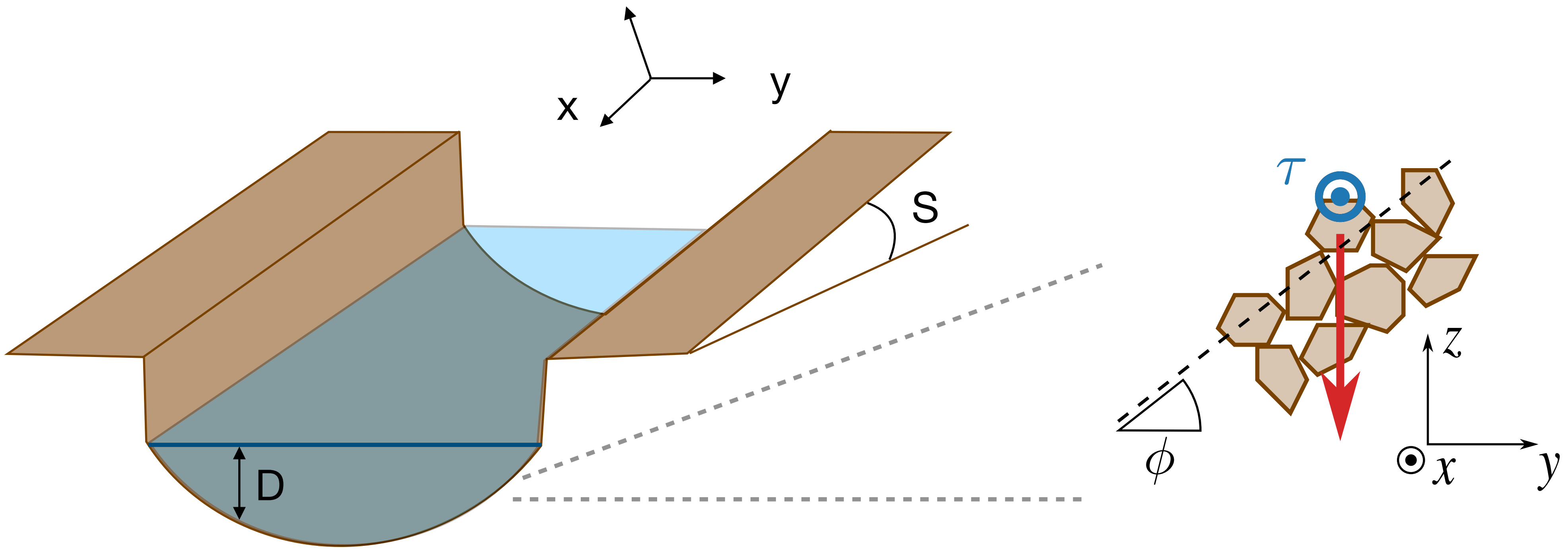
depth

- Diffusion vs gravity : $q_s \propto e^{D/\lambda}$

shear stress

- Transport law : $q_s \propto \left(\left(\frac{\mu_t}{\theta_t \cos\phi} \frac{\tau}{\Delta\rho g d_s} \right)^2 + D'^2 \right)^{1/2} - \mu_t$
- Shear stress : $\tau = \rho g S \left(D + \frac{1}{3}(D^3)'' \right) \cos\phi$

Model



slope

depth

friction coefficient

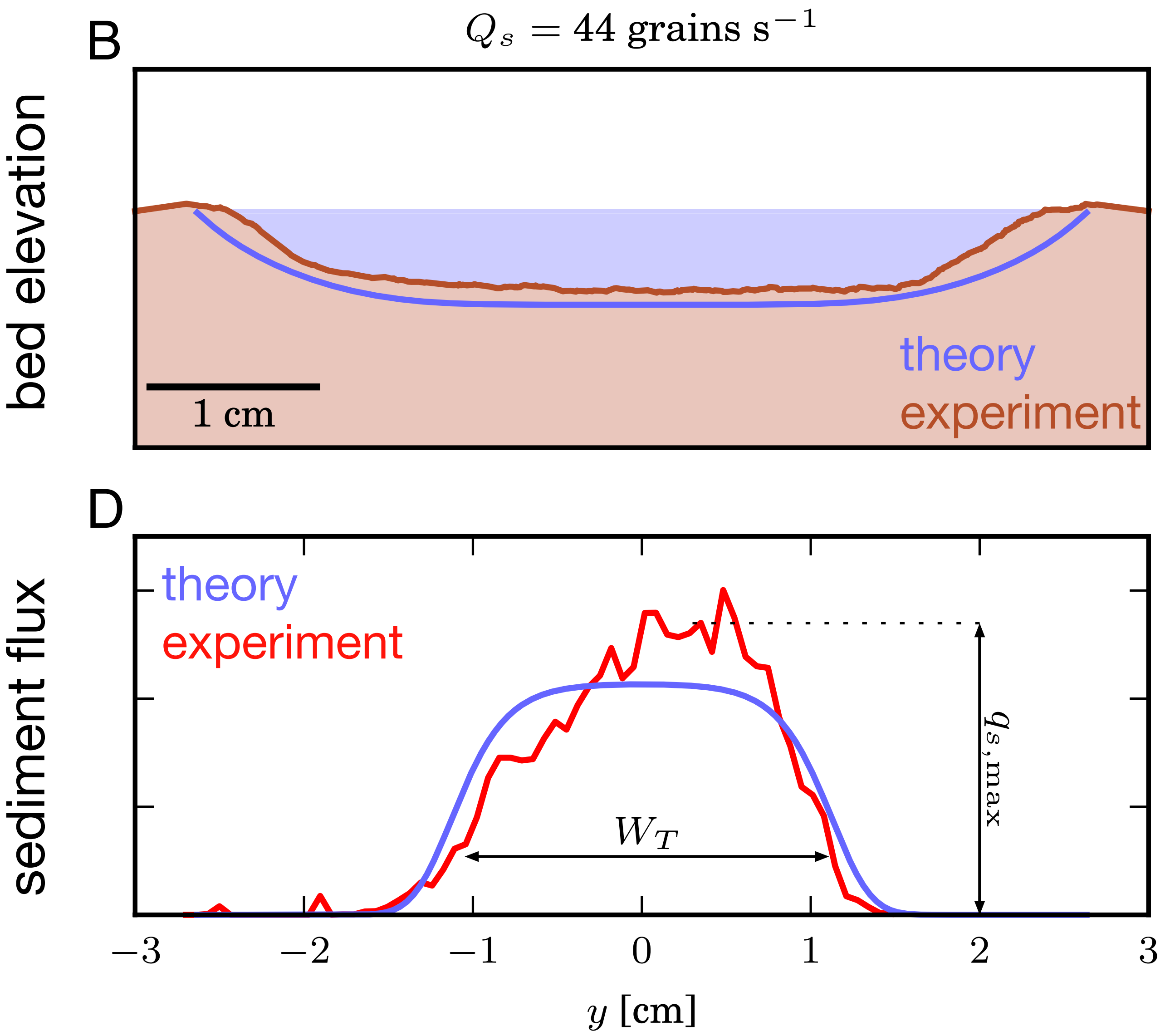
~ diffusion length

$$\sqrt{\frac{S^2}{L^2}\left(D+\frac{1}{3}(D^3)''\right)^2+D'^2}-\mu_t=e^{(D-\xi)/\lambda}$$

characteristic length
~ grain size

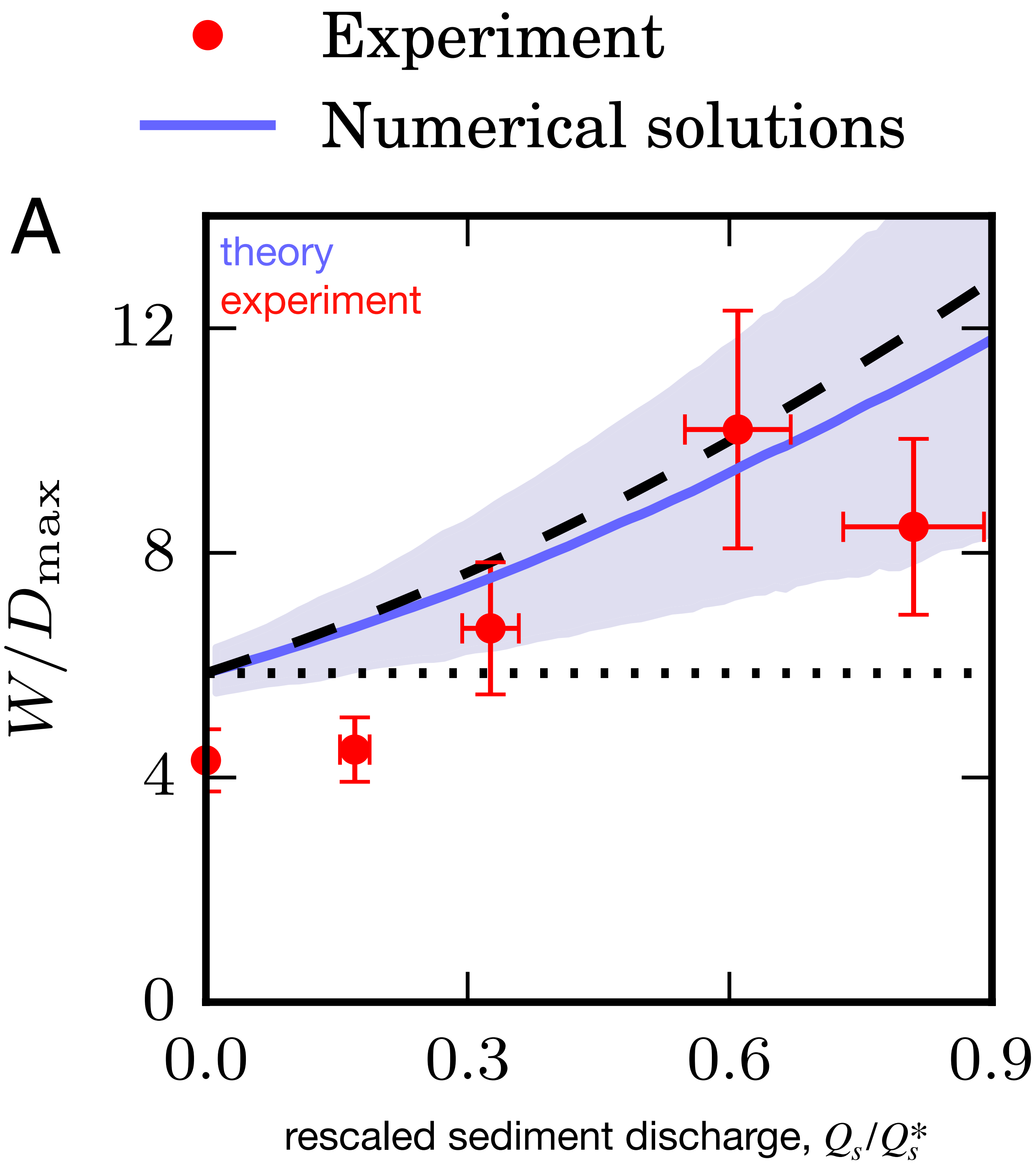
integration constant
related to the total
sediment discharge

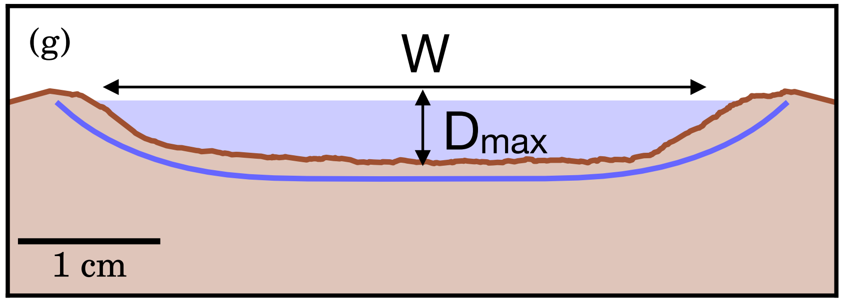
Comparison with experiments



Popovic *et al.* [2021]

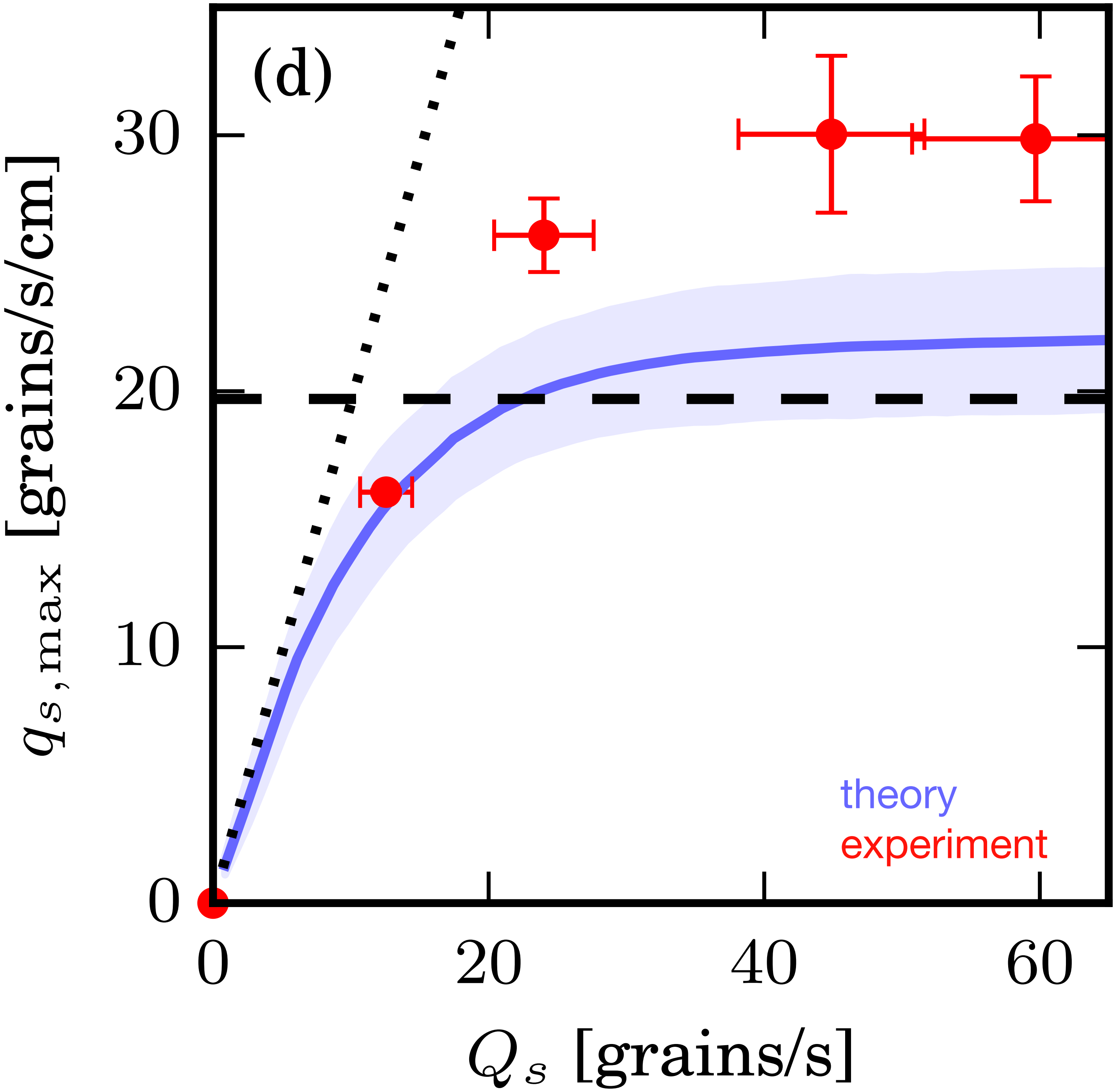
Aspect ratio

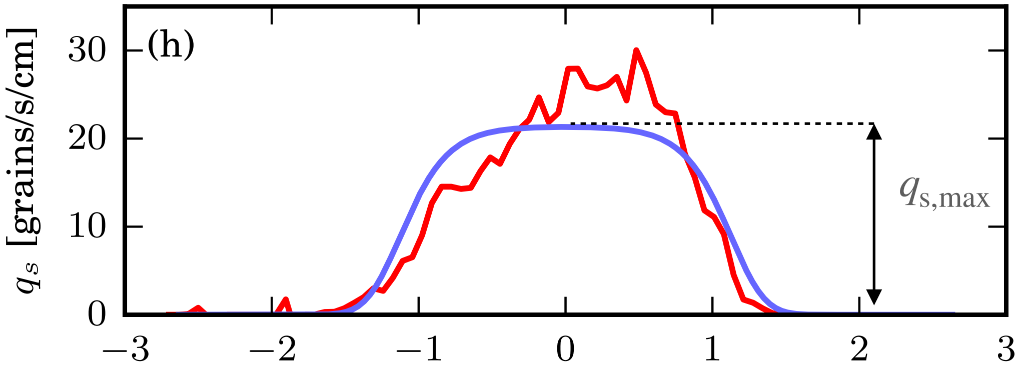




Popovic *et al.* [2021]

Sediment flux

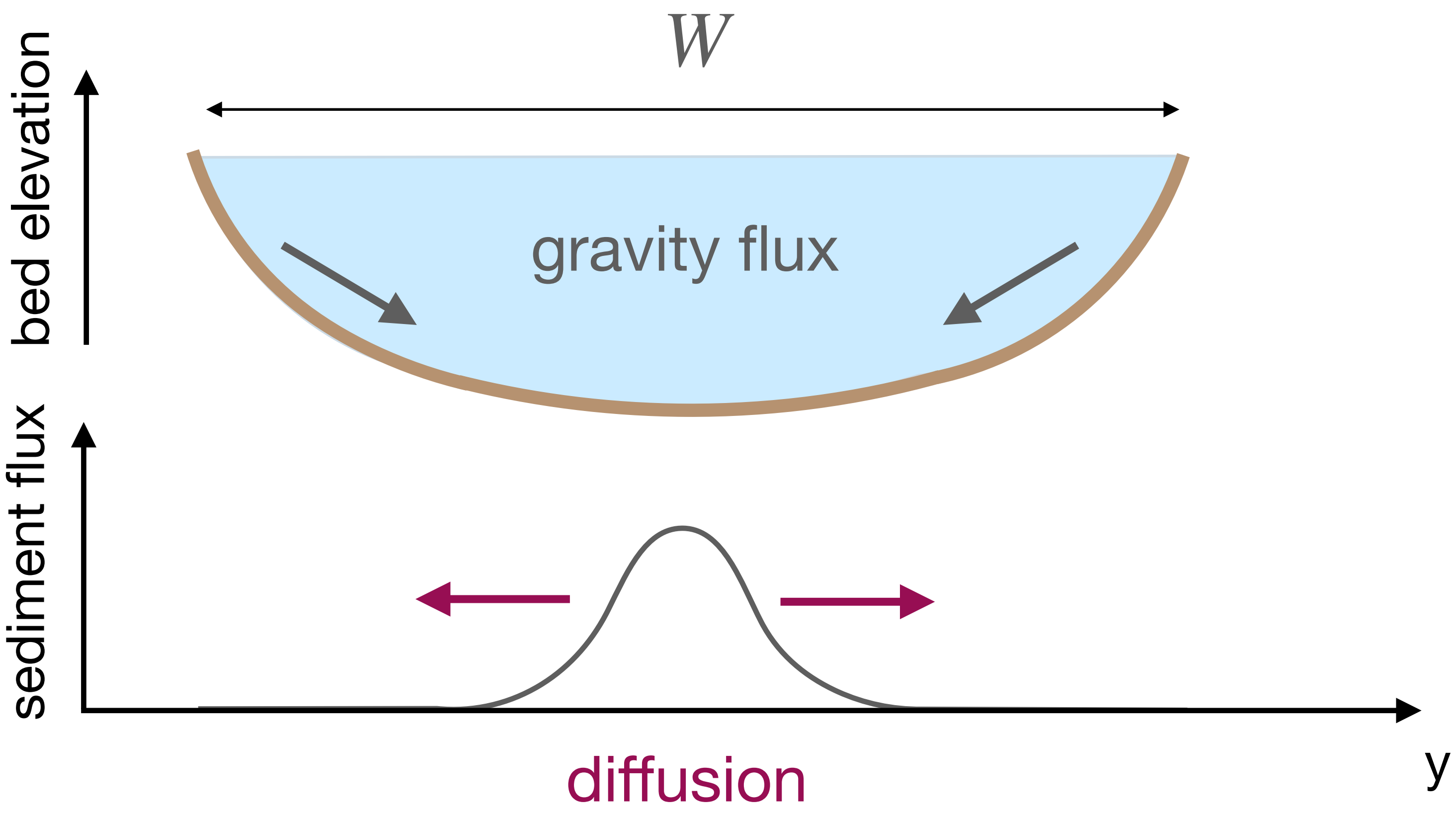




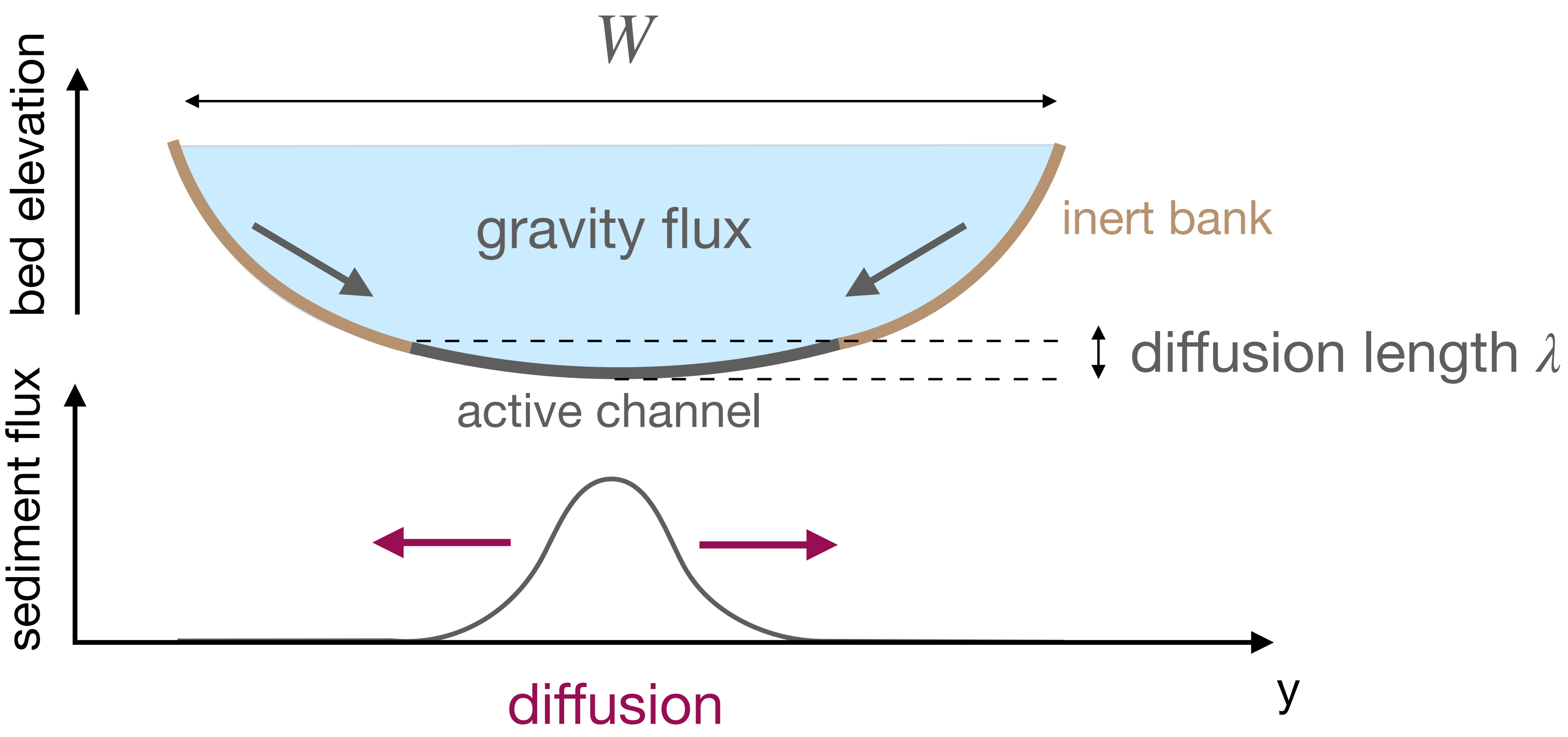
- The sediment flux saturates.

- The river must widen to accommodate an increase of sediment discharge

Waving hand explanation

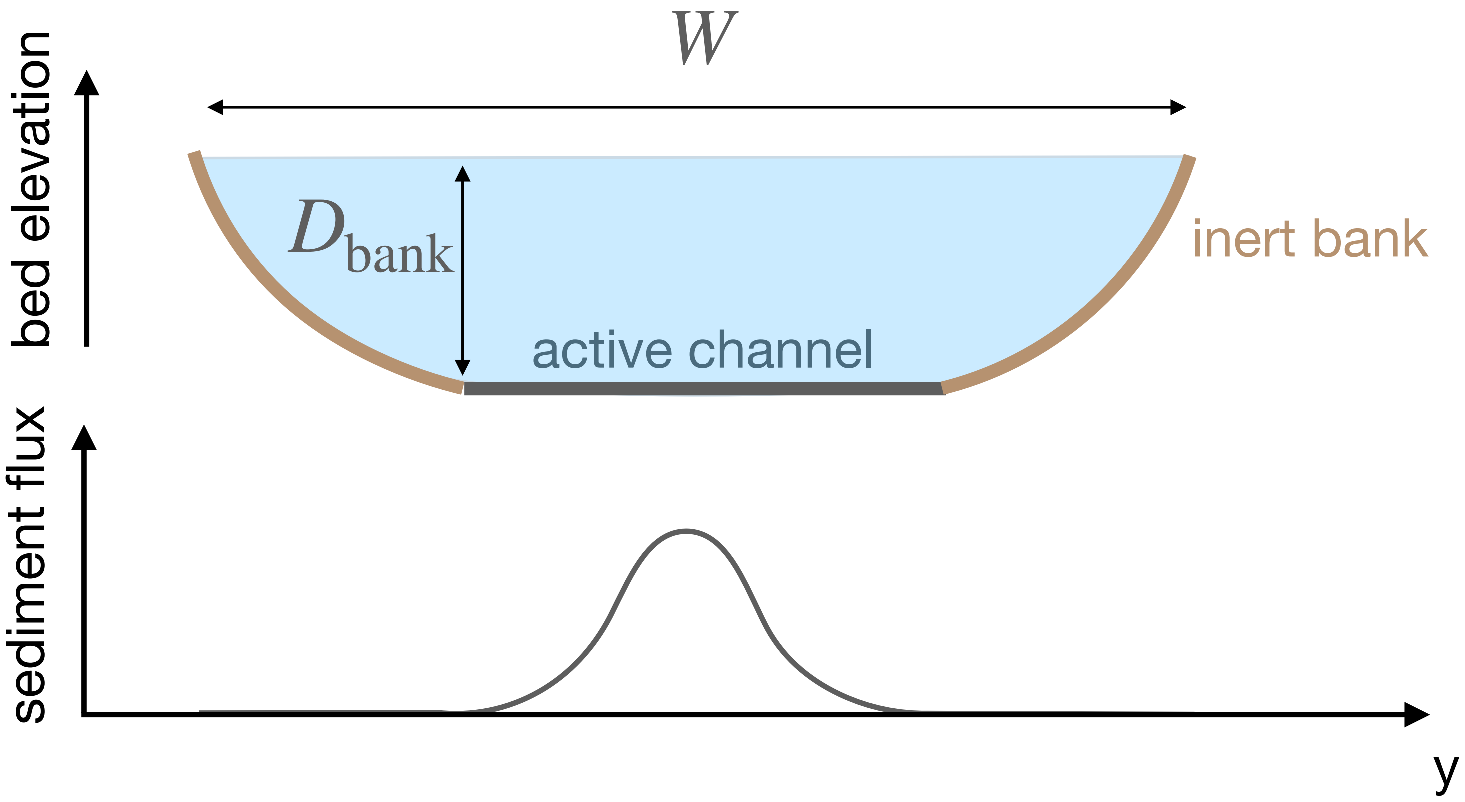


Waving hand explanation



- Equilibrium → flux $\sim e^{D/\lambda} \rightarrow$ inert bank
- Active channel → lateral slope $D' \sim \dfrac{\lambda}{\mathrm{W}}$

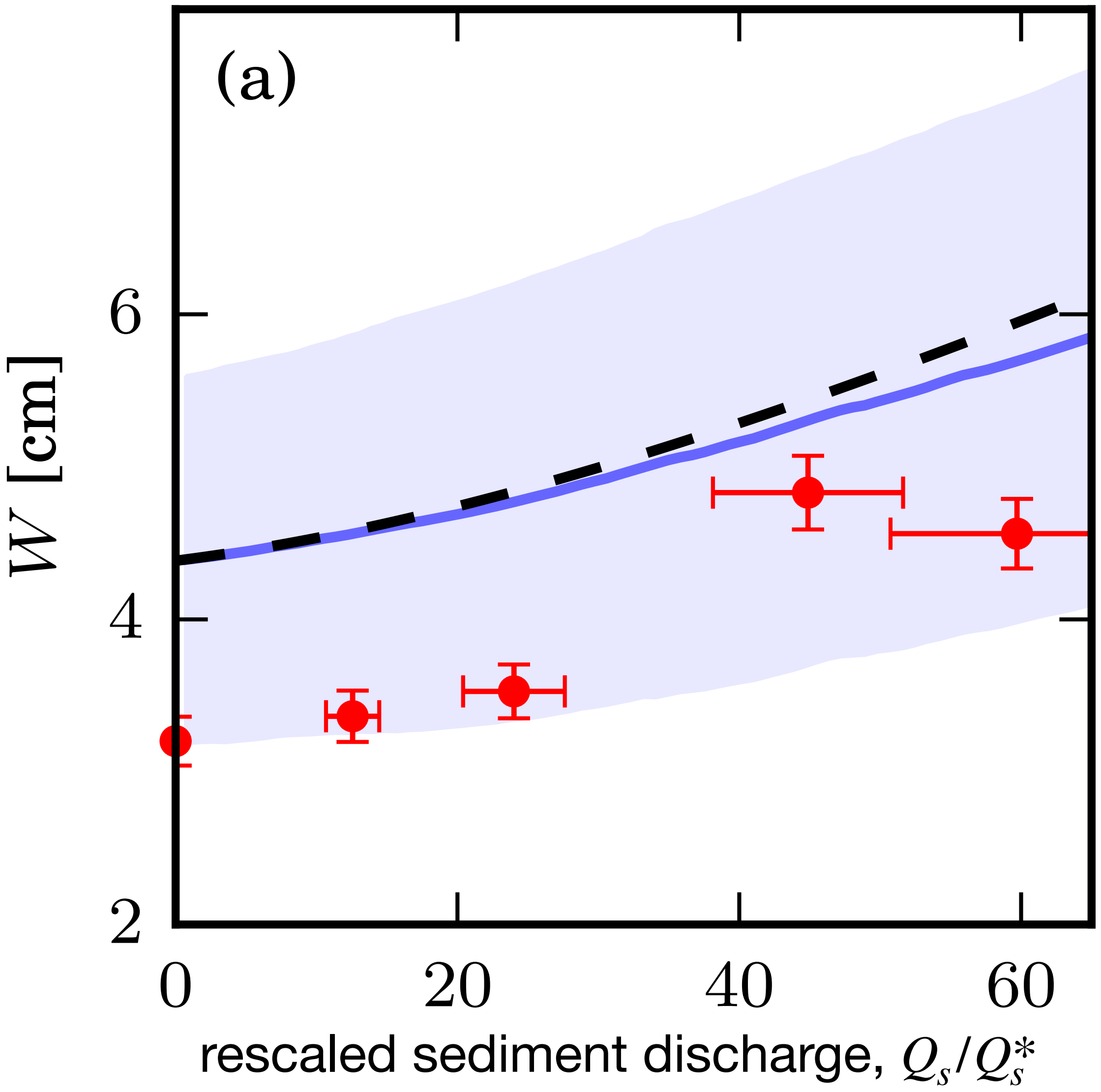
Waving hand explanation



- Equilibrium → flux $\sim e^{D/\lambda}$ → inert bank
- Active channel → lateral slope $D' \sim \dfrac{\lambda}{\mathrm{W}}$
- $\lambda \sim d_s$ → flat active channel
- Force → $\dfrac{F_t}{F_n} \sim D_{\text{bank}}\ S\ /\ L$
- inert river → $D_{\text{bank}} \sim \dfrac{d_s}{S}$

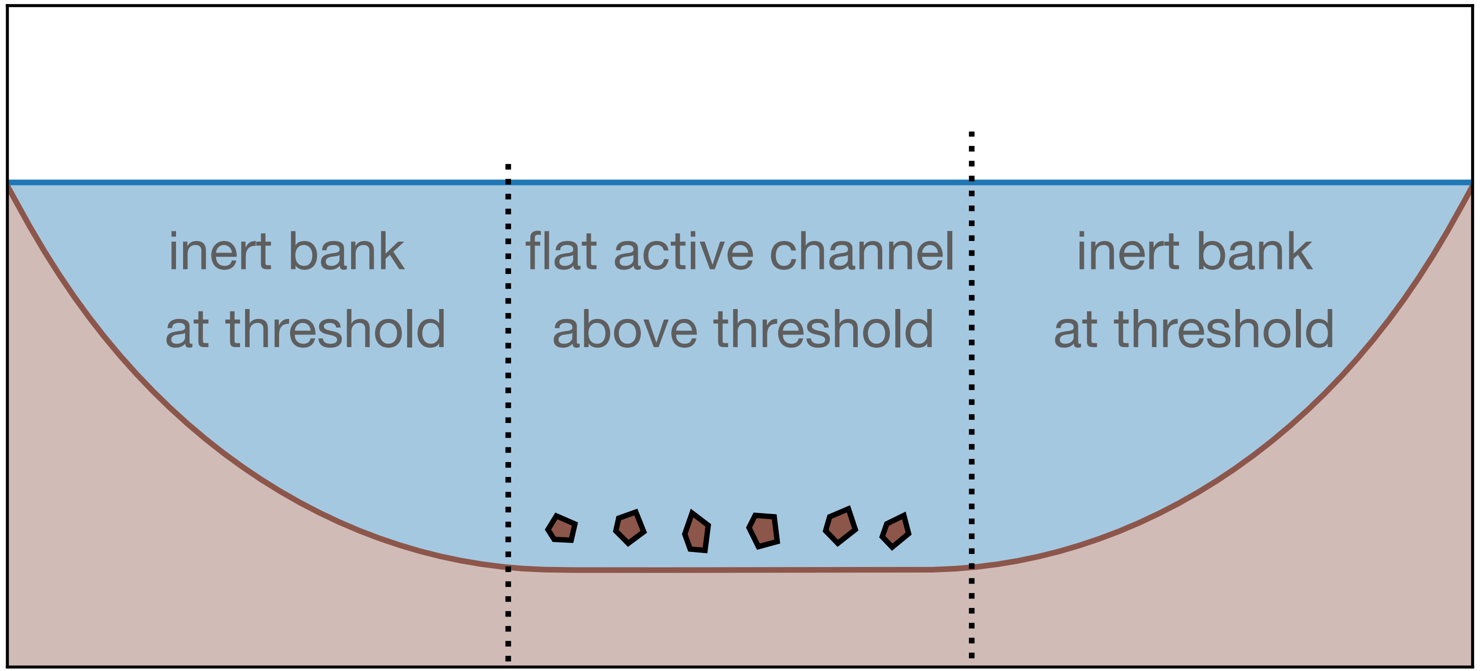
→ $\dfrac{F_t}{F_n} \sim 1.2\ \mu_t$

Waving hand explanation



- As the river widens, the force exerted on the grains saturates to a value about 20% higher than the threshold of entrainment.
- Rivers self organize near the threshold of sediment transport.
- Saturation of force → saturation of maximum sediment flux
- The river widens to transport more sediment.

Why is momentum diffusion so important ?



inert bank
at threshold

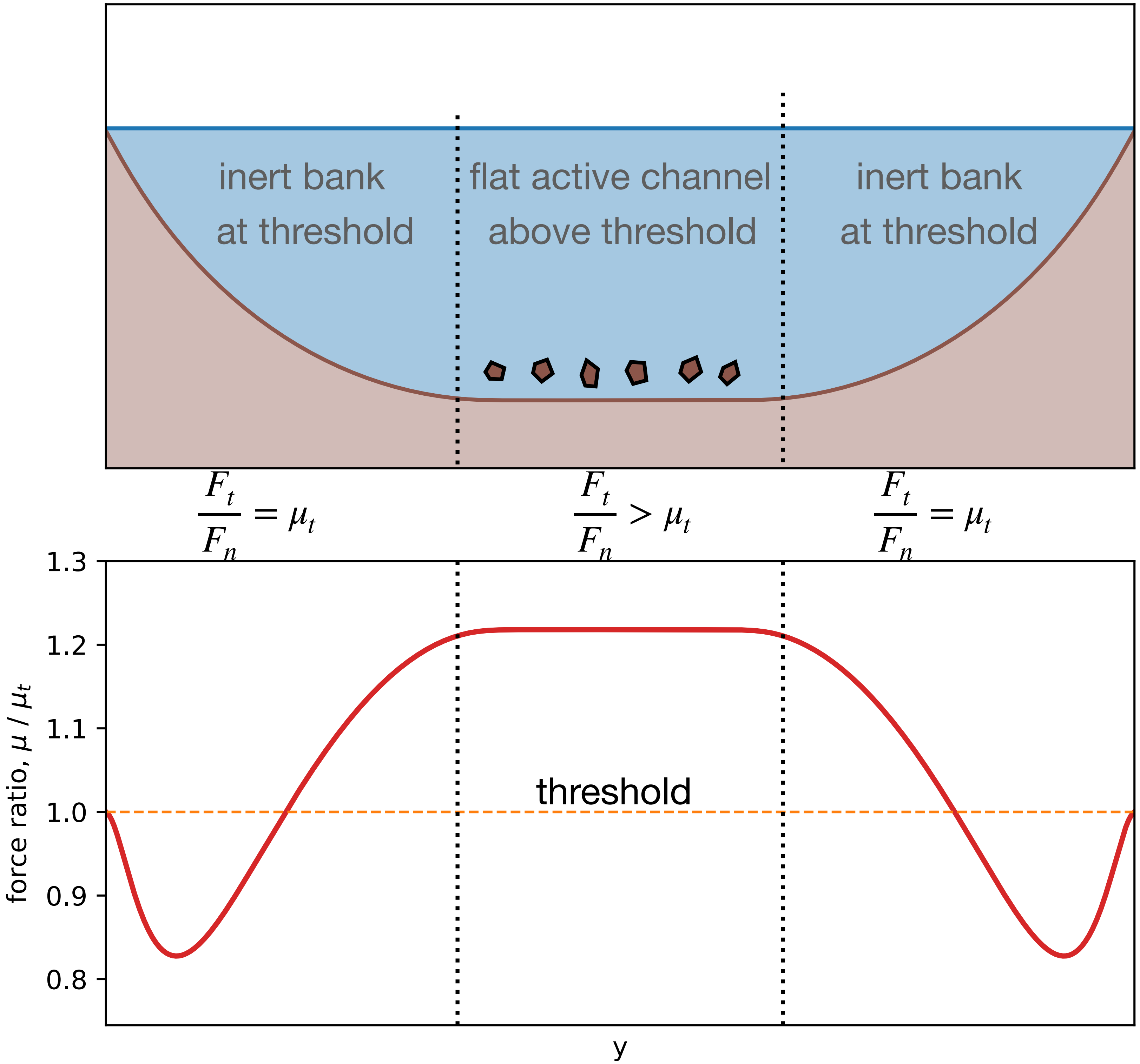
flat active channel
above threshold

inert bank
at threshold

$\frac{F_t}{F_n} = \mu_t$ $\frac{F_t}{F_n} > \mu_t$ $\frac{F_t}{F_n} = \mu_t$

$$\frac{F_t}{F_n} = \sqrt{\frac{S^2}{L^2}\left(\underbrace{\boxed{D}}_{\text{shallow water}} + \underbrace{\boxed{\frac{1}{3}(D^3)''}}_{\text{momentum diffusion}}\right)^2 + \underbrace{\boxed{D'^2}}_{\text{gravity}}}$$

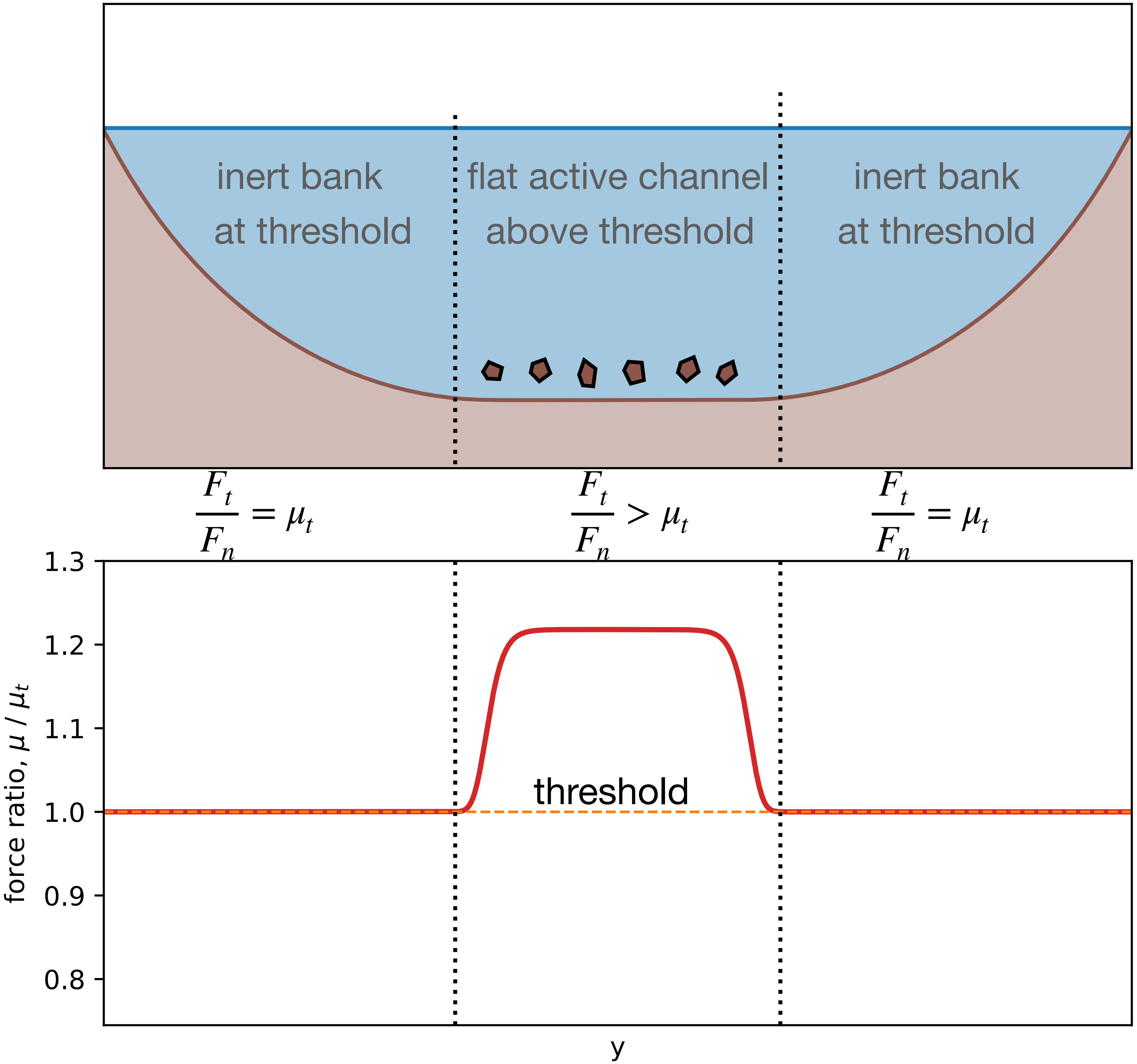
Why is momentum diffusion so important ?



without diffusion of momentum

$$\frac{F_t}{F_n} = \sqrt{\frac{S^2}{L^2}\left(\underbrace{\boxed{D}}_{\text{shallow water}} + \underbrace{\boxed{\frac{1}{3}(D^3)''}}_{\text{momentum diffusion}}\right)^2 + \underbrace{\boxed{D'^2}}_{\text{gravity}}}$$

Why is momentum diffusion so important ?



with diffusion of momentum

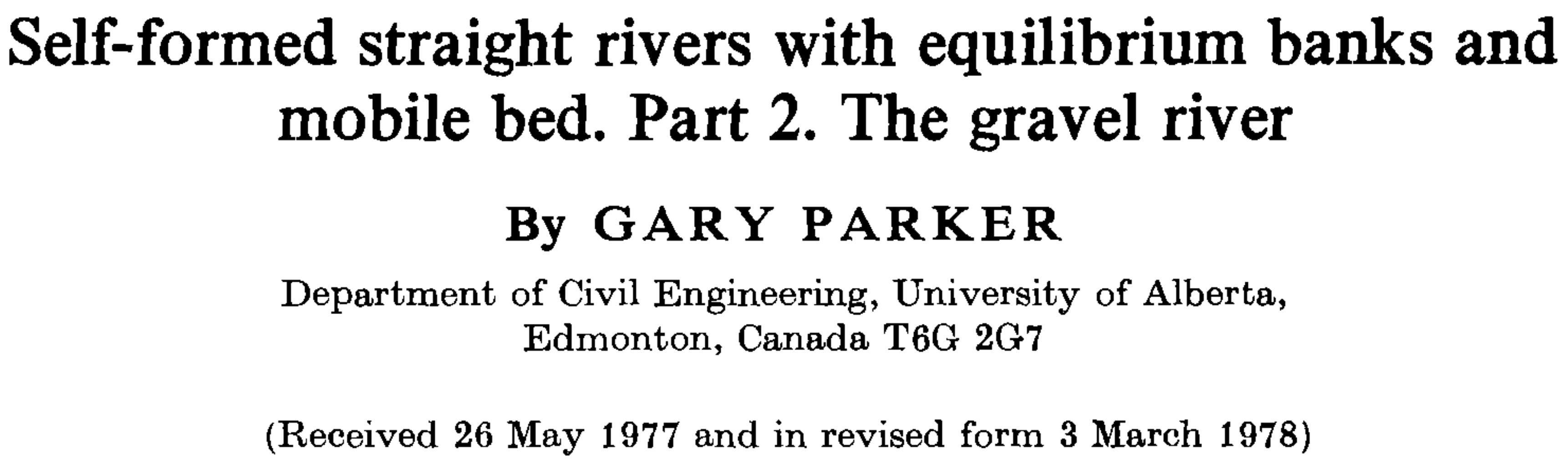
$$\frac{F_t}{F_n} = \sqrt{\frac{S^2}{L^2}\left(\boxed{D} + \boxed{\frac{1}{3}(D^3)''}\right)^2 + \boxed{D'^2}}$$

shallow water

momentum diffusion

gravity

Why is momentum diffusion so important ?

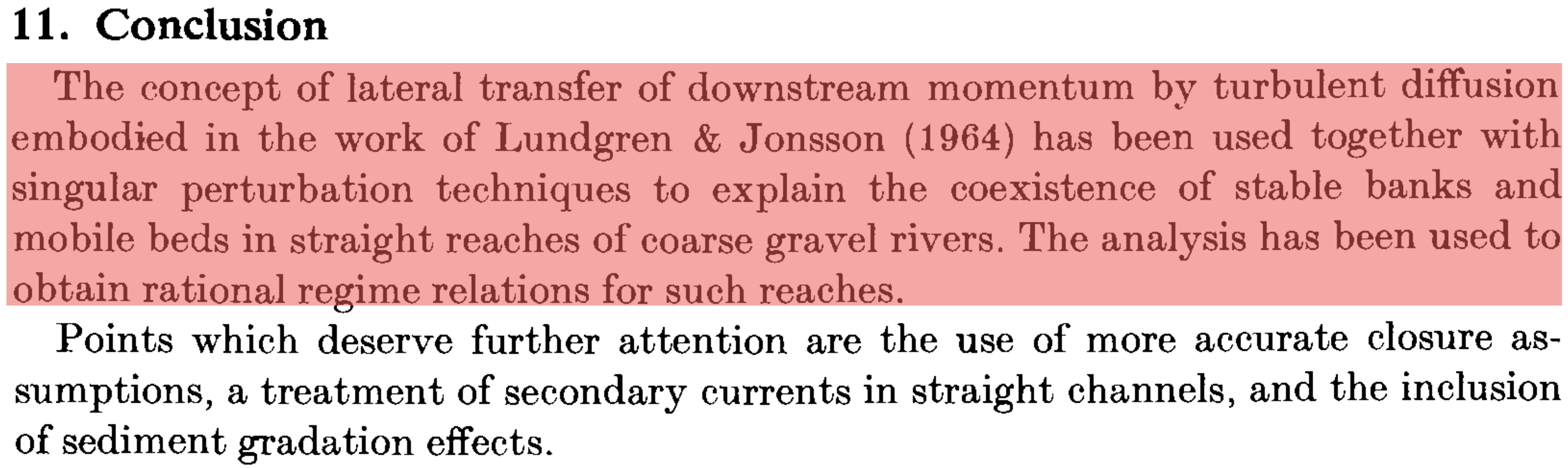


Self-formed straight rivers with equilibrium banks and mobile bed. Part 2. The gravel river

By GARY PARKER
Department of Civil Engineering, University of Alberta,
Edmonton, Canada T6G 2G7

(Received 26 May 1977 and in revised form 3 March 1978)

G. Parker [1978]

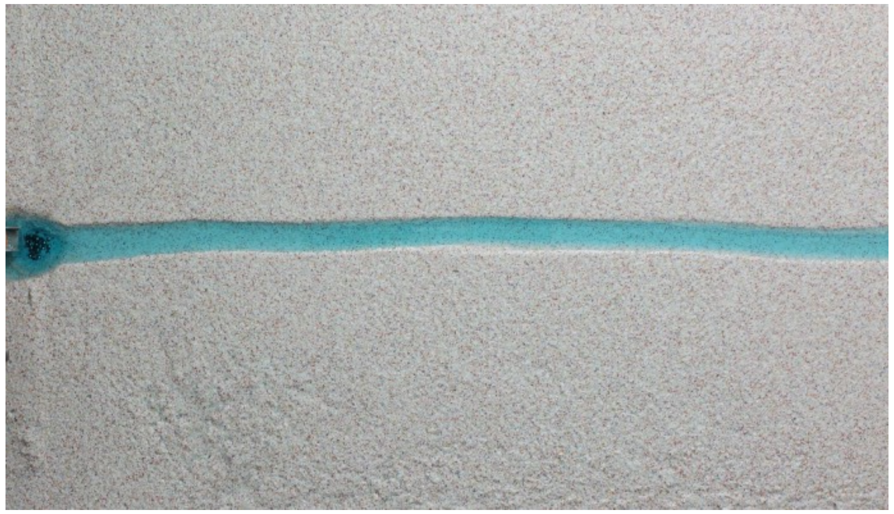


11. Conclusion

The concept of lateral transfer of downstream momentum by turbulent diffusion embodied in the work of Lundgren & Jonsson (1964) has been used together with singular perturbation techniques to explain the coexistence of stable banks and mobile beds in straight reaches of coarse gravel rivers. The analysis has been used to obtain rational regime relations for such reaches.

Points which deserve further attention are the use of more accurate closure assumptions, a treatment of secondary currents in straight channels, and the inclusion of sediment gradation effects.

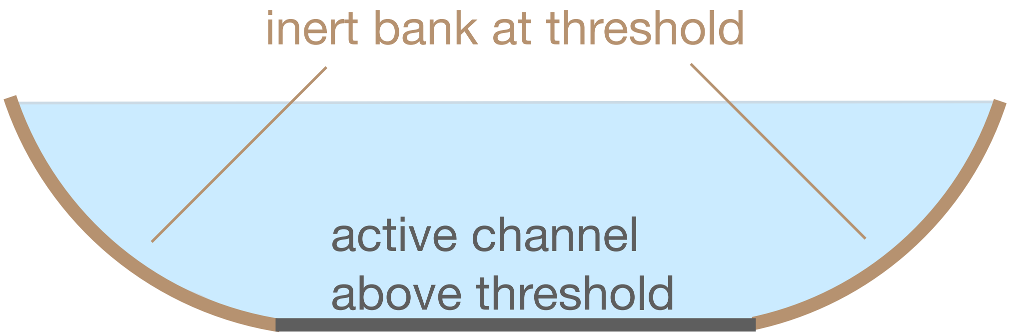
Take home messages



- Laboratory rivers construct their bed near the threshold of entrainment

- Result of the combination of 2 diffusion processes.

- Diffusion of bedload particles



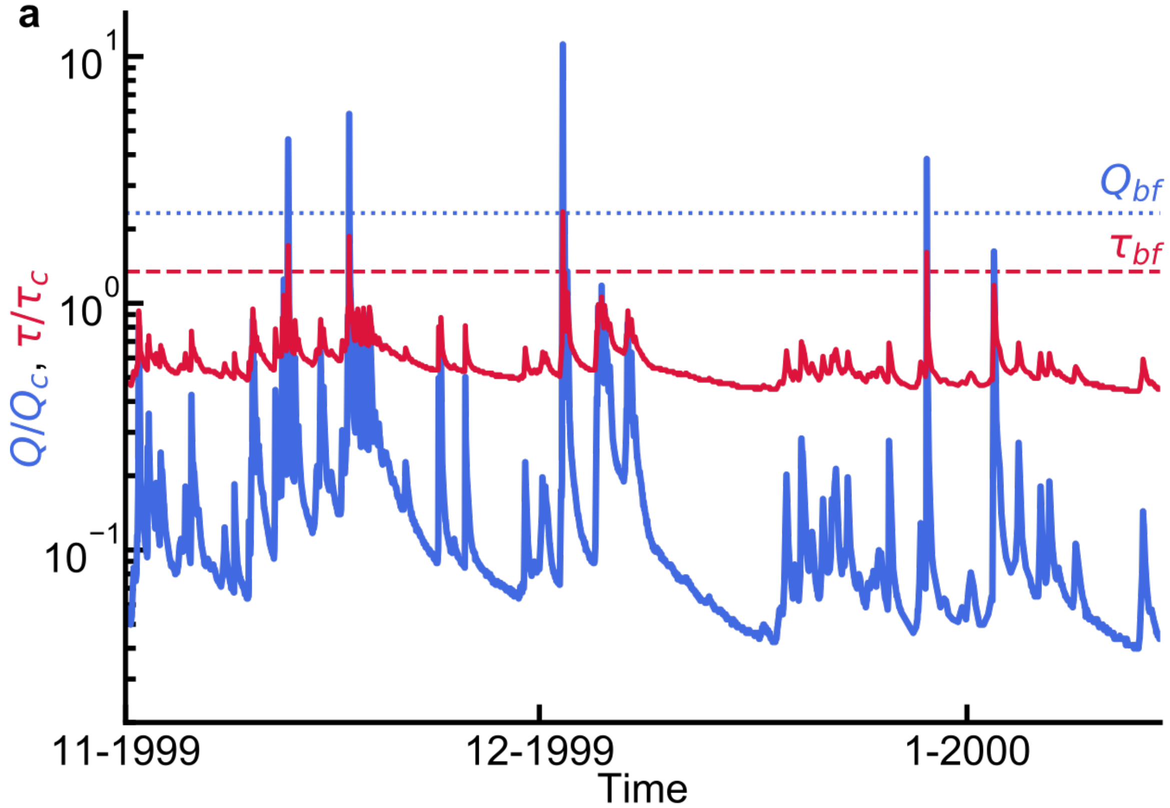
- Diffusion of momentum

Open questions

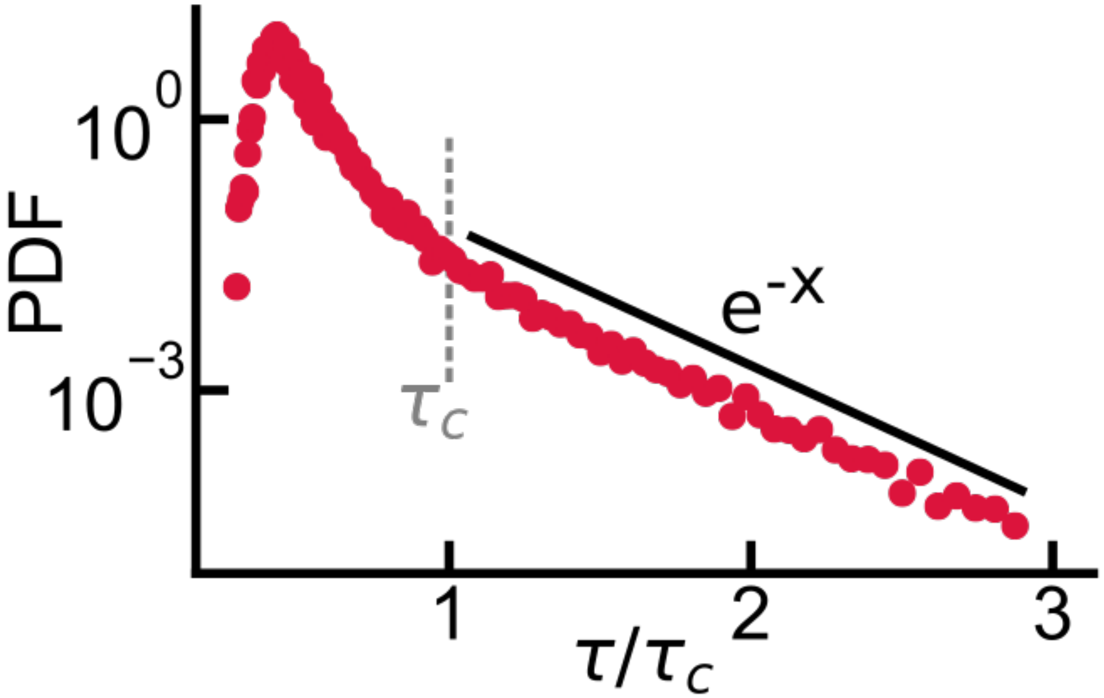
Tian-Shan, China

Do alluvial rivers self-organize their bed near the threshold of motion ?

Mameyes river, Puerto-Rico

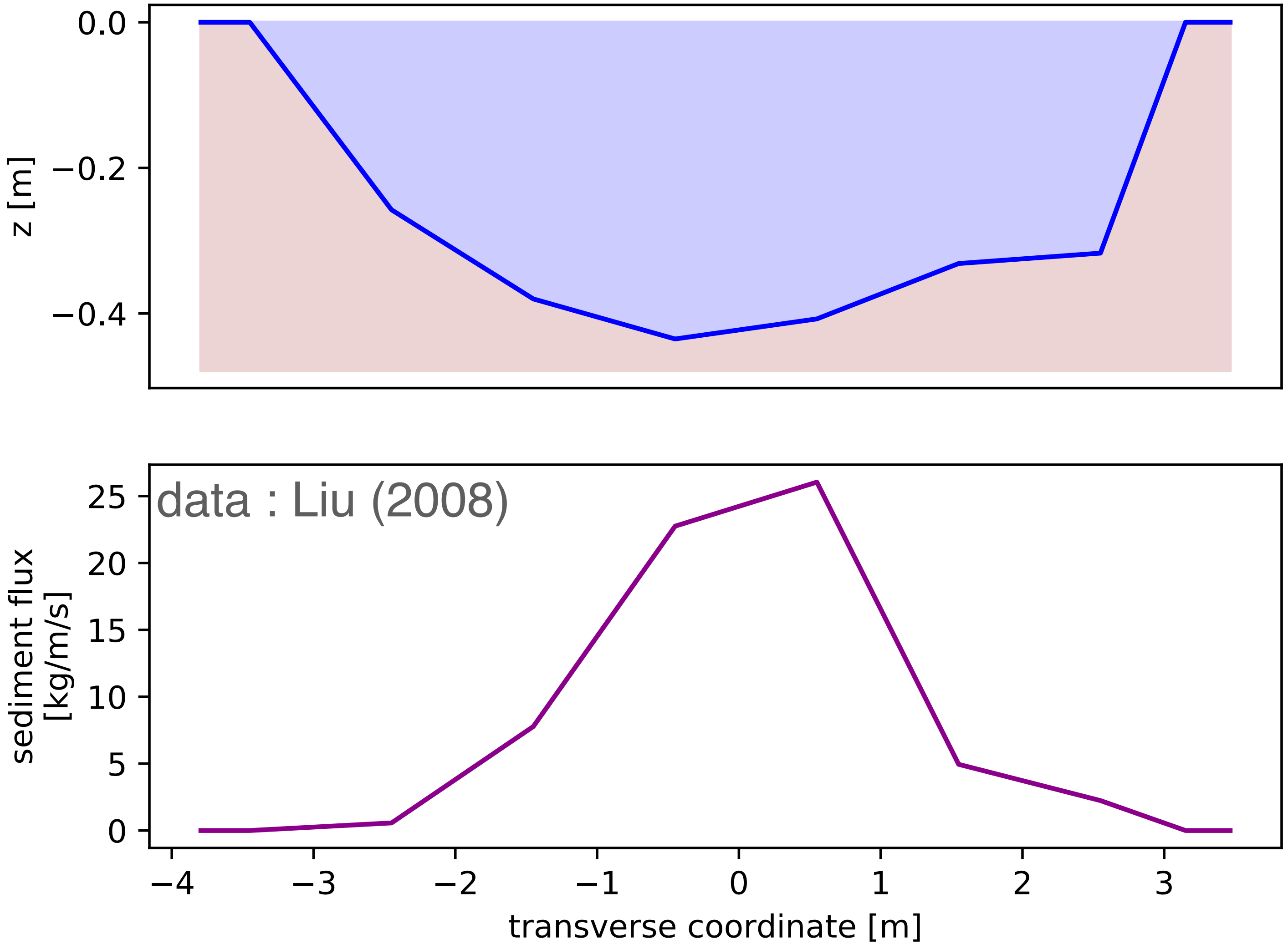


Phillips *et al.* [2016, 2022]



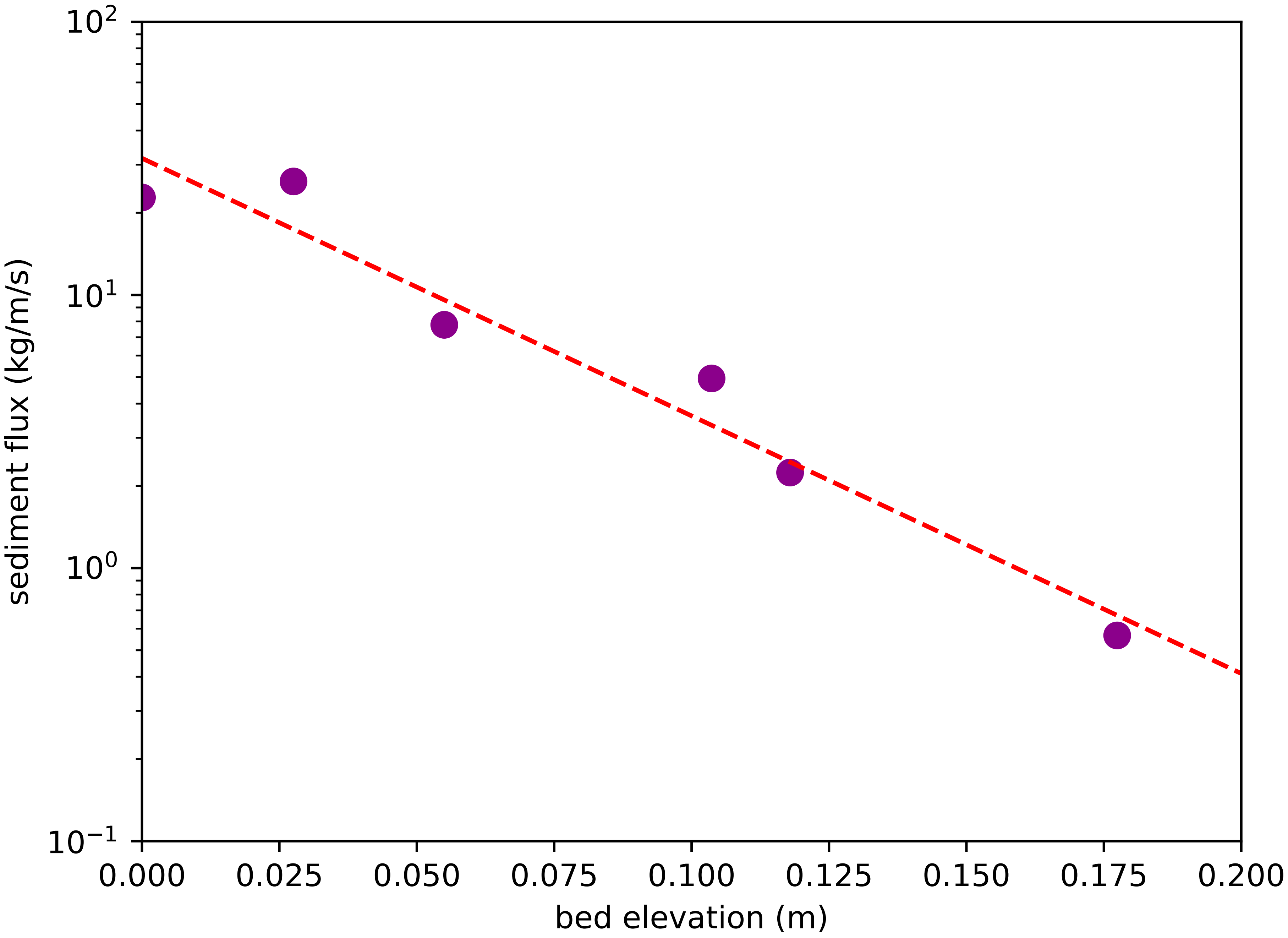
Do alluvial rivers obey the Boltzmann-like equilibrium condition ?

Urumqi He,
Tian-Shan, China

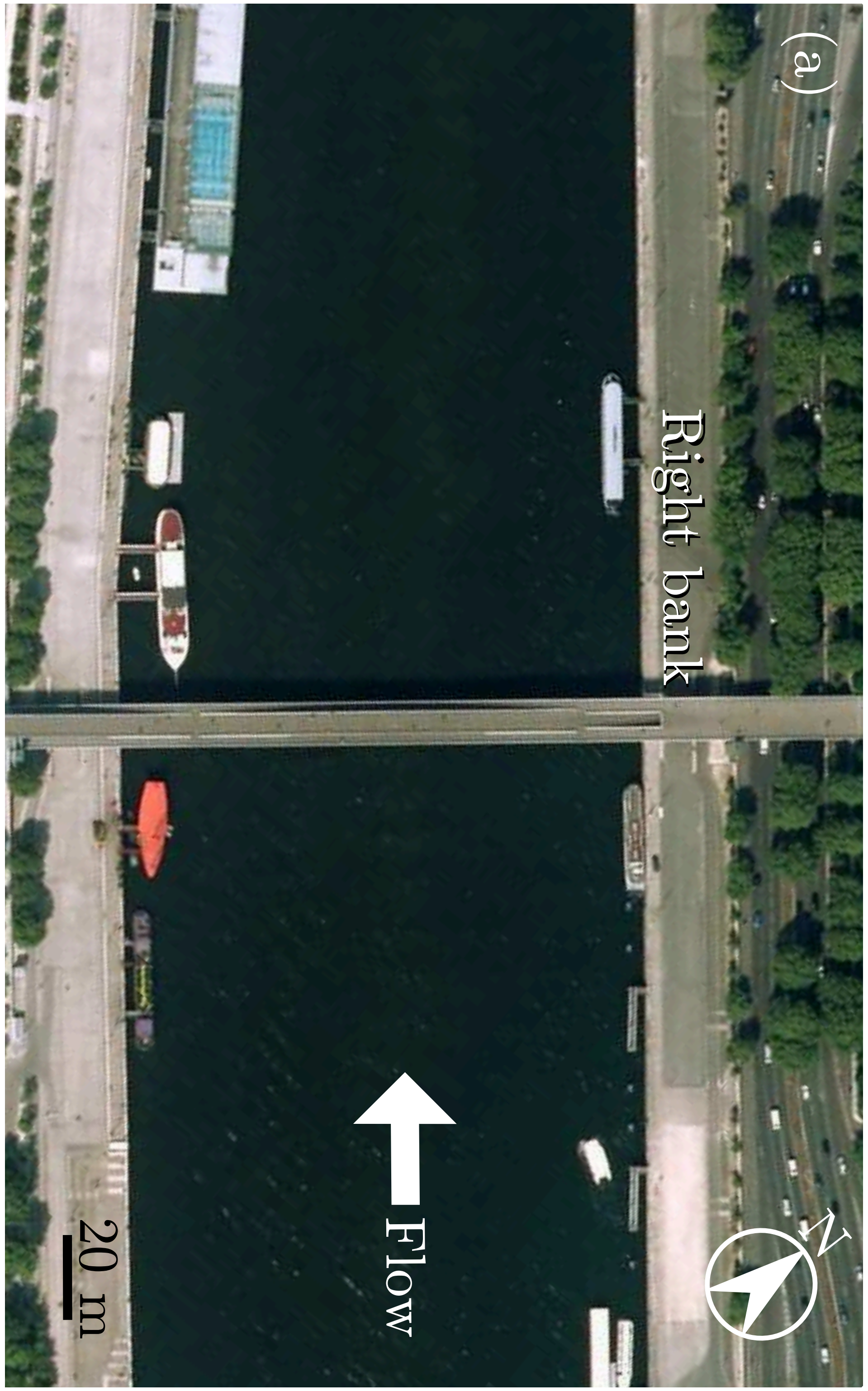


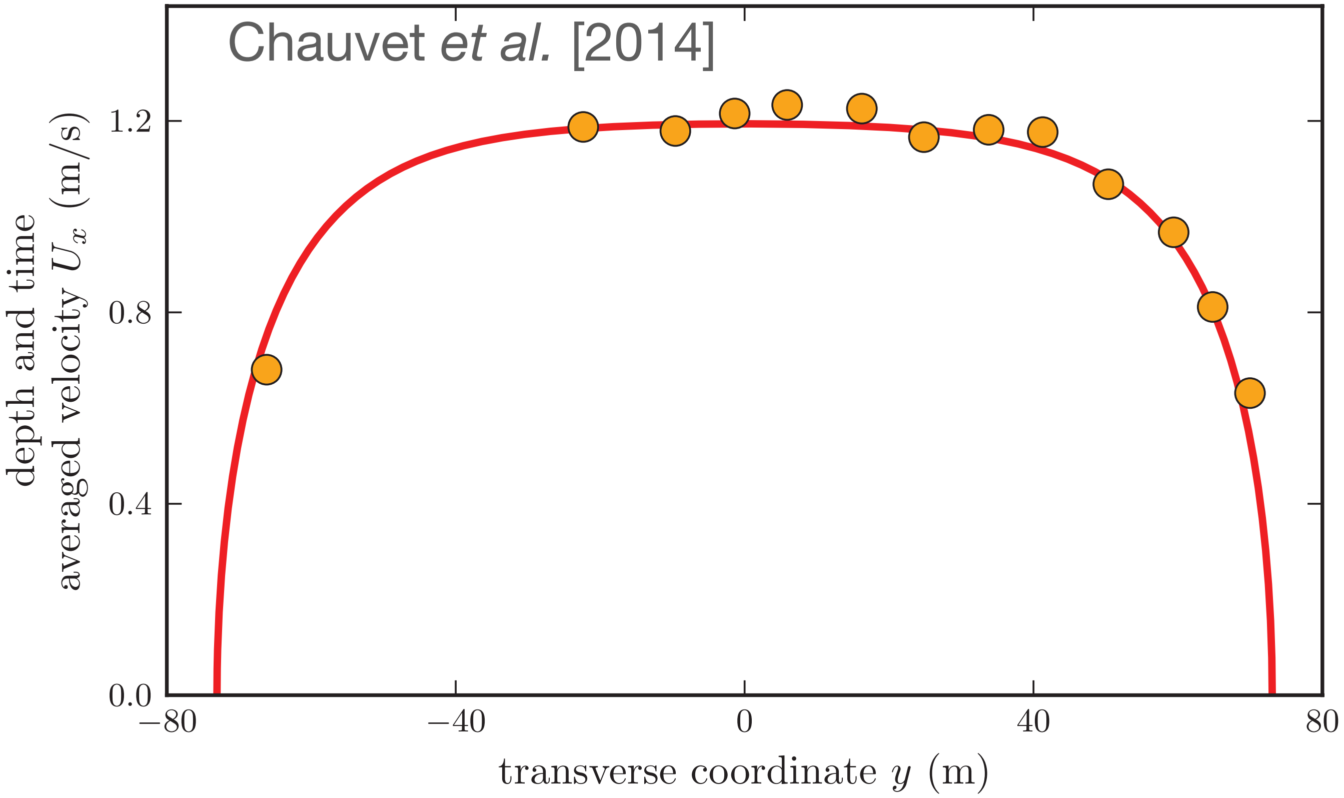
data : Liu (2008)

Do alluvial rivers obey the Boltzmann-like equilibrium condition ?



Do alluvial rivers diffuse momentum in the cross-stream direction ?



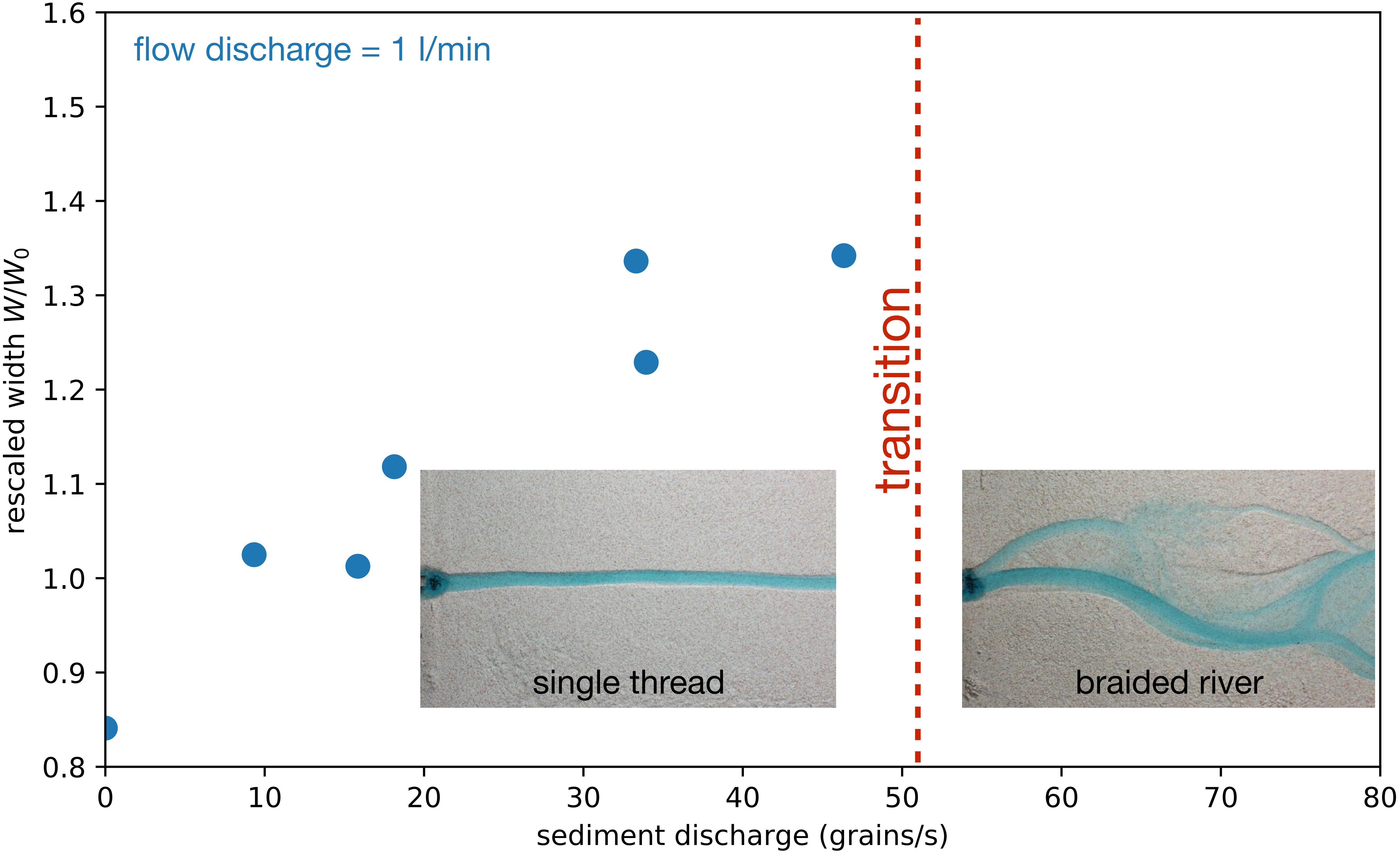


- flat river bed
- curved depth-averaged velocity profile

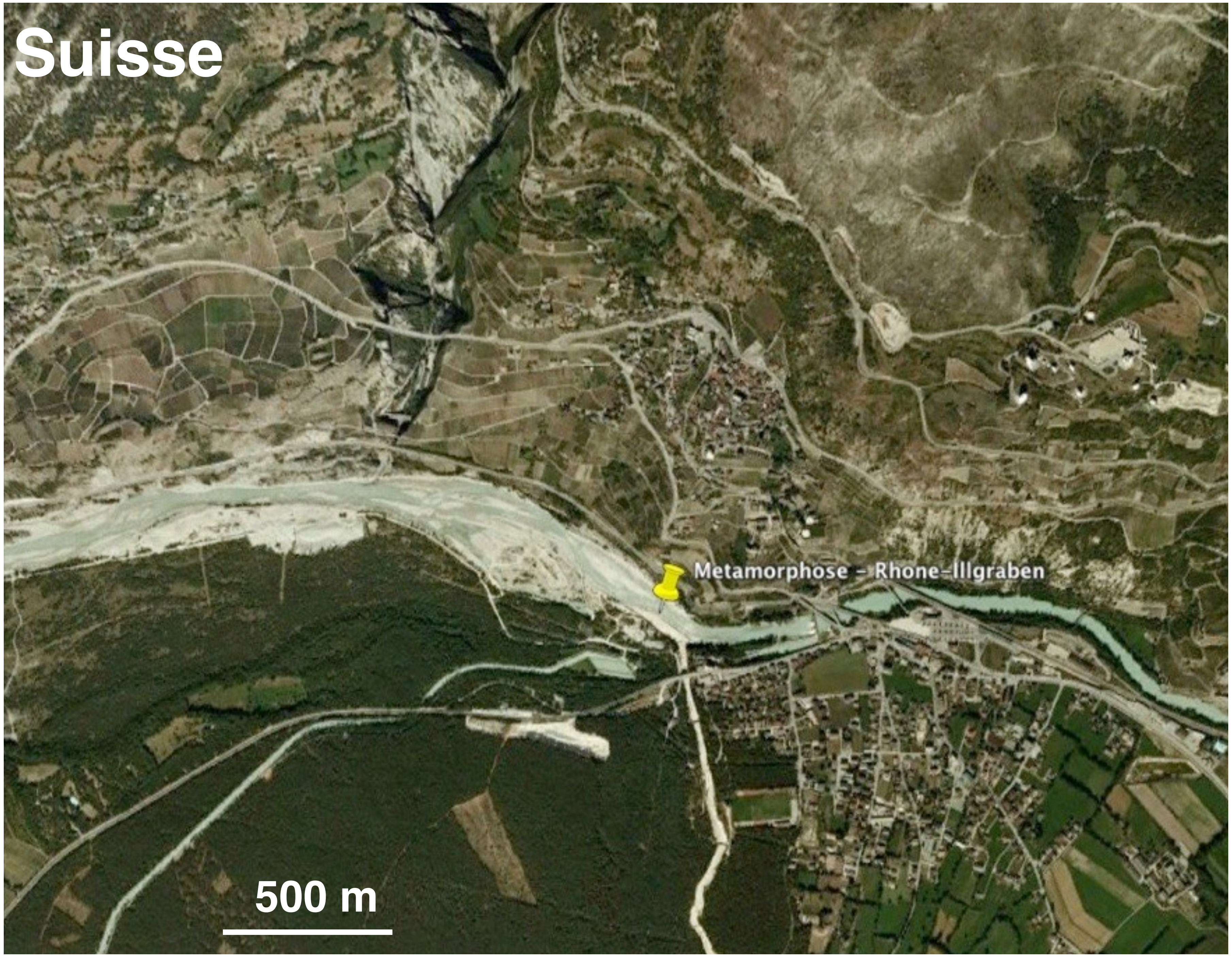
→ cross-stream diffusion of momentum

(Popović et al., sub)

Stability of alluvial rivers ?



Stability of alluvial rivers ?



Stability of alluvial rivers ?