Critical phenomena at the "permafrost-atmosphere" interface

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Permafrost

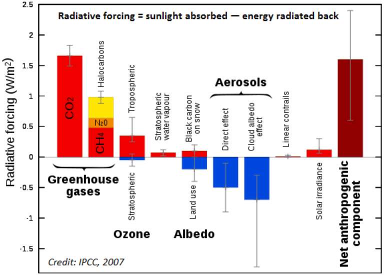
Permafrost is defined (by IPA) as ground (soil or rock and included ice or organic material) that remains at or below 0°C for at least two consecutive years.

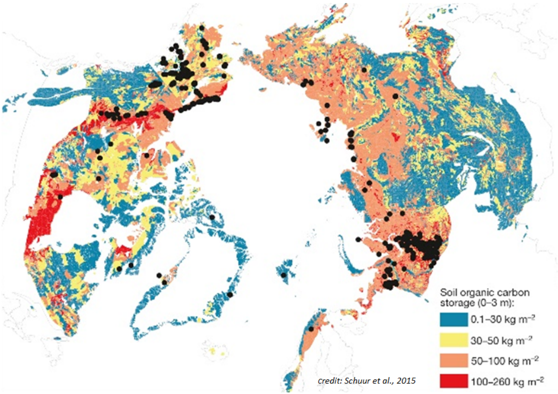
- In the Northern Hemisphere, regions in which permafrost occurs occupy approximately 25% (23 million km²) of the land area
- The thickness of permafrost varies from less than one meter to more than 1500 meters.
- Most of the permafrost existing today formed during cold glacial periods, and has persisted through warmer interglacial periods, including the Holocene (last 10,000 years).

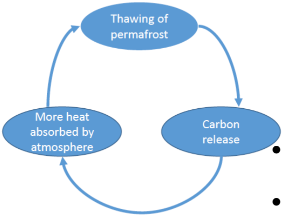
Arctic Armageddon Needs More Science Less Hype

Permafrost carbon - climate feedback

Radiative forcing components



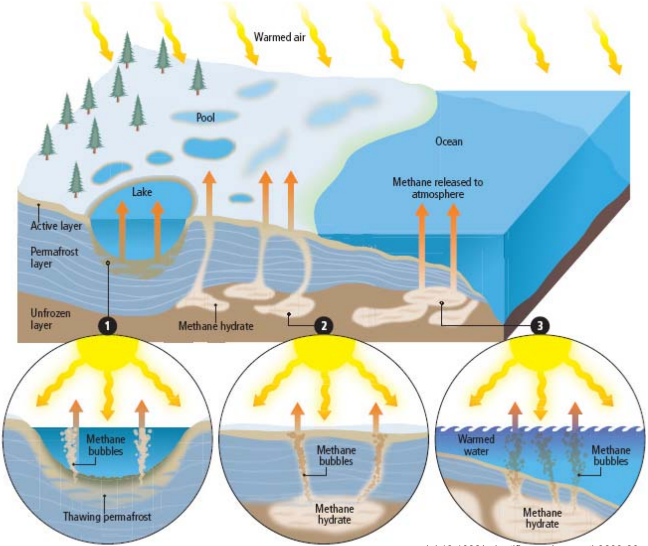




Schuur et al., Nature, 2017
Schaefer et al.,ERL, 2014

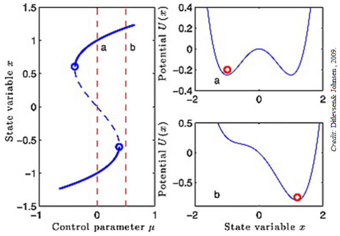
- The permafrost carbon feedback is the amplification of surface warming due to methane emissions from thawing permafrost.
- The surface permafrost carbon pool (0 – 3 m) is 1,035 ±150 Pg carbon (1 Pg = 1 billion tons).

Carbon sources

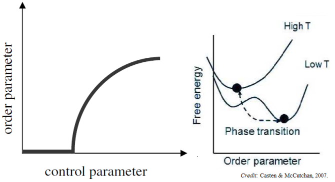


doi:10.1038/scientificamericanearth0609-30

Challenges in climate modeling



Bifurcations in nonlinear theory

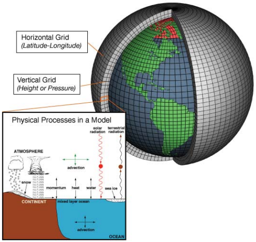


Phase transitions in statistical physics

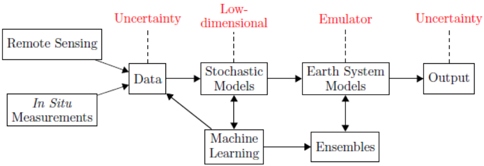
Key challenges

- *incorporating sub-grid scale processes*
- *linkage of scales*
- *critical behavior*

Global Climate Models

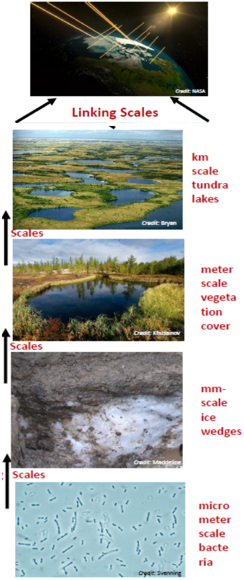


Data-Driven Climate Emulators

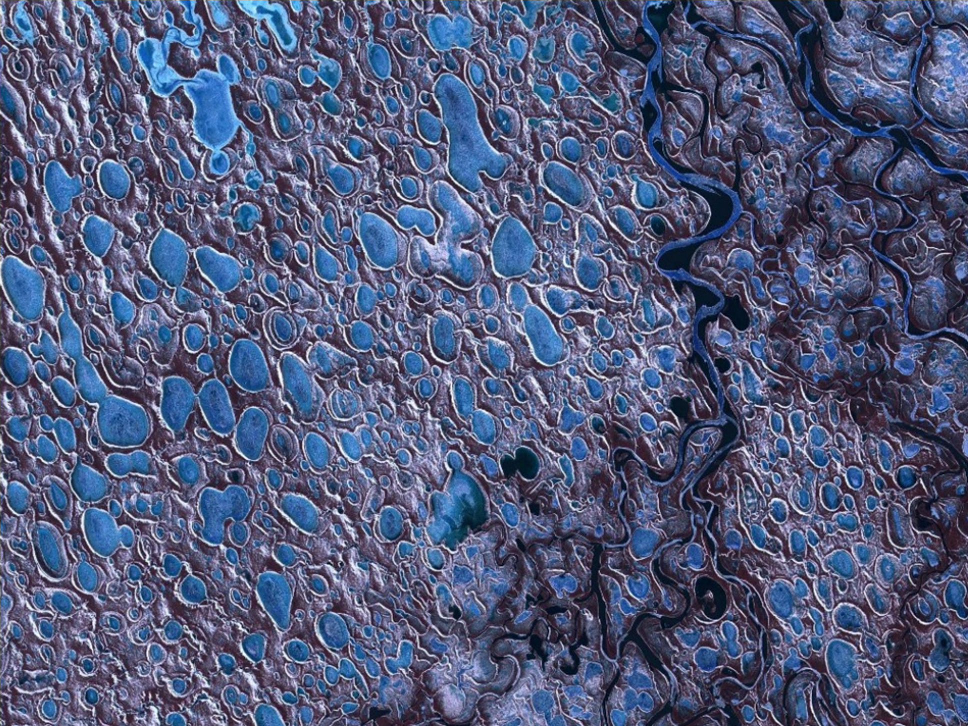


Lucarini, V. Intro to the Statistical Mechanics of Climate. J Stat Phys 179 (2020).
Ghil and Lucarini. The Physics of Climate Variability and Climate Change, Rev Mod Phys 92 (2020).
Sudakow et al. Statistical Mechanics in Climate Emulation: Challenges and Perspectives, Env. Data Sci., 10 (2022).

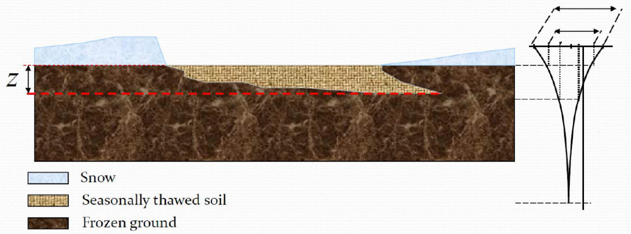
What is this talk about?



The thawing permafrost is a phase transition phenomenon ...on large regional scales ...that depends on environmental forcing and other factors. ... mathematical models.... to understand the processes on the interface "frozen ground-atmosphere" and ... their criticality.



The Stefan problem for permafrost



The classical Stefan Problem:

$$u_t = K_1 \Delta u, \quad 0 < z < l(t),$$

$$\tilde{u}_t = K_2 \Delta \tilde{u}, \quad l(t) < z < h$$

Boundary conditions on the depth: $\tilde{u}_z(x,y,h,t) + q(x,y)\tilde{u}(x,y,h,t) = V(x,y,t)$
on surface: $u(x,y,0,t) = U(x,y,t)$

at the thawing transition front: $\mathbf{n} \cdot (\nabla(u(x,y,z,t) - \tilde{u}(x,y,z,t))|_{(x,y,z)\in\Gamma}) = Qv$

n is the unit normal vector with respect to the thawing front,
v is the normal front velocity,
Q is a dimensionless latent heat.

Phase field model

The Stefan problem

$$u_t = K_1 \Delta u,\ 0 < z < l(t),$$

$$\tilde{u}_t = K_2 \Delta \tilde{u},\ l(t) < z < h,$$

the thawing transition front:

$$\mathbf{n} \cdot (\nabla(u(x,y,z,t) - \tilde{u}(x,y,z,t))) = QV$$

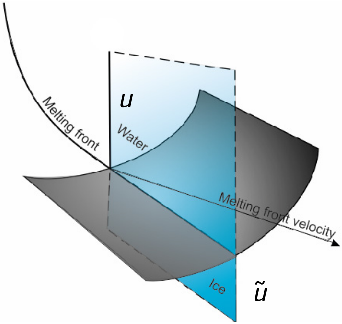
The phase field model

Landau,1937

Caginalp, Phys. Rev. A ,1989

$$K u_{ZZ} = \frac{bV}{2} \Phi_Z,$$

$$-V\xi^2\Phi_Z = \xi^2 \Phi_{ZZ} + a^{-1} g(\Phi)$$



"The order parameter": $\Phi = \begin{cases} +1, \text{thawing soil} \\ \Phi(z), \text{melting front} \\ -1, \text{frozen ground} \end{cases}$

$\epsilon = a^{-1/2}\xi^{-1}$ - a parameter that defines the front width.

Normal thawing front velocity may be found through the mean curvature flows.

Huisken, J. Differential Geom., 1984
Molotkov & Vakulenko, 1988

Ginzburg–Landau formalism - super-conductivity

The averaged lake growth

- We can formulate the modified free boundary problem [Caginalp, 1989] assuming that the parameters $a, \xi \to 0$, and $\xi a^{-1/2}, \alpha$ are fixed:

$$\Delta S u(x,y,z,t)\big|_{\Gamma} = -\sigma\kappa - \alpha\sigma v(x,y,z,t)$$

ΔS is the difference in entropy between liquid and solid;

σ is the surface tension;

κ – the sum of the principal curvatures at a point on the interface.

- The normal melting front velocity at the point (*mean curvature flows*):

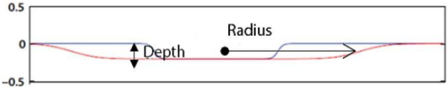
$$v(x,y,z,t) = \delta - \mu\kappa(x,y,z,t)$$

δ is the nonperturbed kink velocity,

μ is a positive coefficient.

- For circular fronts of the radius $R(t)$ the thawing front velocity becomes

$$\frac{dR}{dt} = \delta - \mu R^{-1}$$



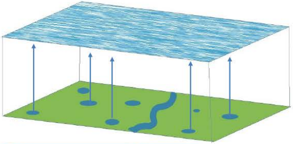
Stochastic effects

- The lake growth is more complicated process (e.g. the water drainage leads to decreasing of lake size). To take into account possible stochastic effects we can use the Fokker-Planck equation:

$$\frac{\partial (f(R)\rho)}{\partial R} = d\frac{\partial^2 \rho}{\partial R^2}$$

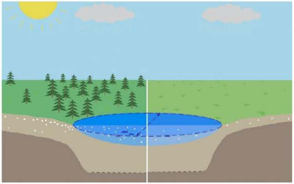
- This result is consistent with some experimental observations [Downing et al., 2006]. In fact, the lake areas in different regions are distributed according to the Pareto law:

$$\rho(R) = constR^{-m/d}$$



- The total rate of the methane emission per unit time:

$$V = \rho \int_D \exp\left(-\frac{U_0}{k_b u}\right) dxdydz$$



- The permafrost lake methane emission rate:

$$V = \beta(BR_{av} - 1)\exp(-b_0 / u_{av}(t))$$

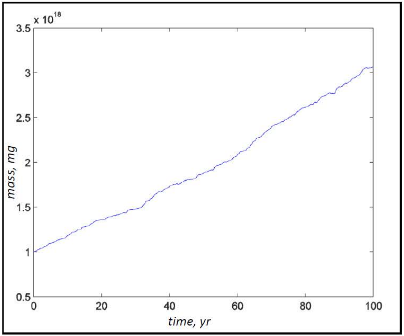
Methane Gun

- The positive feedback in the climate system:

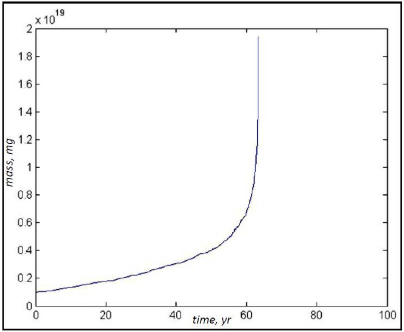
$$u(t) = \bar{u} + \gamma X$$

- The total methane mass in atmosphere:

$$\frac{dX}{dt} = H(t, X) + \beta X$$



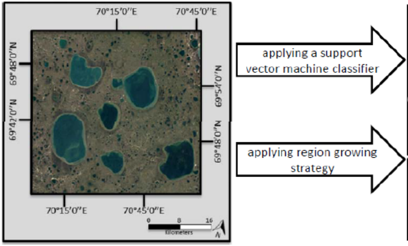
A linear growth of methane in the atmosphere

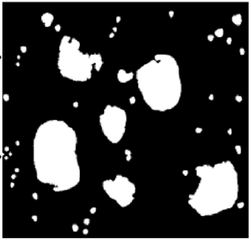


A sharp increasing of methane mass in the atmosphere

I. Sudakov, S. Vakulenko. A mathematical model for a positive permafrost carbon-climate feedback // *IMA Journal of Applied Mathematics*, 2014. doi: 10.1093/imamat/hxu010

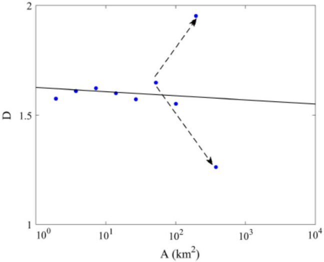
Shapes

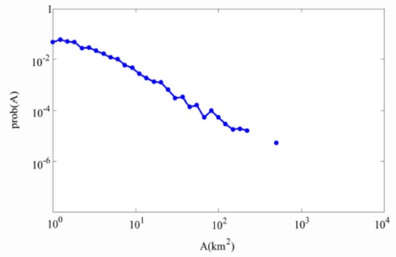




The fractal dimension of lakes is between 1.6 and 1.7.

The lakes are ***power-law distributed.***

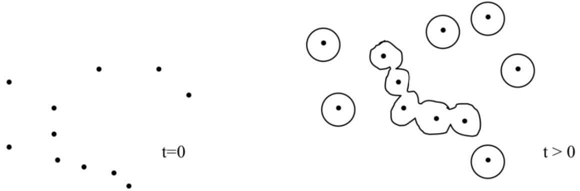


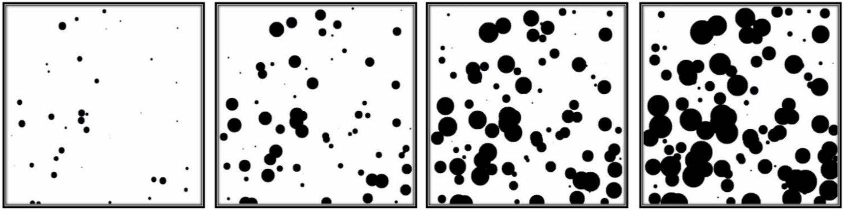


Nucleation-and-growth of tundra lakes

The area fraction ϕ of the unstable phase is defined by Johnson–Mehl–Avrami–Kolmogorov (JMAK) equation

$$\phi(t) = 1 - \rho(t) = exp[-\frac{1}{3}\gamma\pi v^4 t^5]$$





The atmosperic dynamics model

Fluid Velocity:

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = \sigma\Delta\mathbf{v} - \nabla P + \sigma_1(\theta - \Theta_0)\mathbf{z}$$

Temperature:

$$\theta_t + (\mathbf{v}\cdot\nabla)\theta + w\Gamma = \Delta\theta - 3\alpha\theta$$

Incompressible fluid:

$$\nabla\cdot\mathbf{v} = 0$$

Gas concentration:

$$C_t + (\mathbf{v}\cdot\nabla)C = d\Delta C - b_0^2 C$$

$\boldsymbol{v}$ – *the fluid velocity with the vertical component* w;

$\theta(x, y, z, t)$ – *the temperature field;*

Θ_0 – *the reference temperature;*

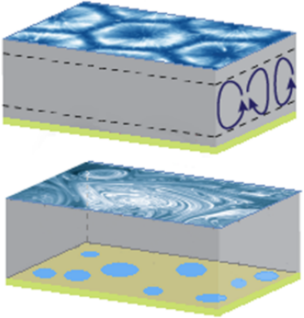
P – *the pressure;*

σ – *the Prandtl number,*

σ_1 – *the buoyancy parameter;*

Γ – *a dimensionless adiabatic lapse rate;*

α – *absorption of radiation per unit volume.*



Goody's model is similar to Rayleigh–Benard convection except that thermal radiative transfer is included, thereby altering the basic state temperature profile and introducing radiative damping.

We consider the motion in the flat layer Ω defined by $0 < z < h$, $(x, y) \in \Pi$, where Π is a rectangle.

Goody model: gas concentration

$$C_t + (\mathbf{v} \cdot \nabla) C = d \Delta C - b_0^2 C$$

- $C(x, y, z, t)$ – gas concentration;
- d – the diffusion coefficient;
- $-b_0^2$ describes a natural degradation of gas molecules in the system due to chemical reactions.

To describe gas influence on fluid circulation, we assume that the coefficient α depends on C: $\alpha = \alpha(C)$.

If gas concentration C is small

$$\alpha \approx \alpha_0 + \alpha_1 C$$

- α_0, α_1 – constants;
- α_1 – a phenomenological coefficient that can be defined by an analysis of experimental data on gas mass in the system.

Boundary conditions: gas sources

- The boundary condition for C describes absence of gas flux at $z = h$ and gas production at the bottom

$$C_z(x, y, z, t)\big|_{z=h} = 0$$

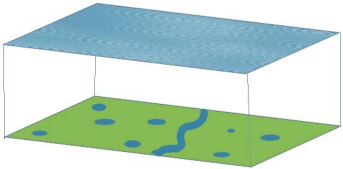
$$C_z(x, y, z, t)\big|_{z=0} = \mu(x, y, \theta(x, y, 0, t))$$

- We assume that this flux is inhomogeneous in space and depends on the temperature.
- The function μ describes gas flux intensity at the bottom:

$$\mu(x, y, t) = g(x, y) \exp(-\frac{V_0}{k_B \theta})$$

- V_0 is a potential barrier, k_B is the Boltzmann constant.

Boundary conditions: gas sources



- We consider the circles as the main gas source, we can assume that

$$g(x, y, t) = -c_g \sum_{i=1}^{N} \chi_{\Omega_i(t)}(x, y)$$

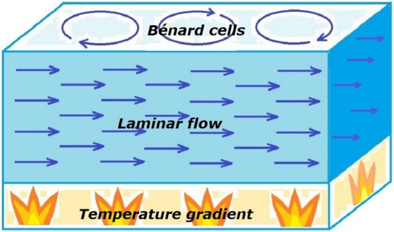
- $c_g > 0$ is a constant, Ω_i is a two dimensional domain occupied by the i-th circle and χ_V denotes the characteristic function of the set V: $\chi_V = 1$ if $(x, y) \in V$ and $\chi_V = 0$ otherwise.
- We can assume that Ω_i are fixed domains.

Control of bifurcations by space inhomogeneities

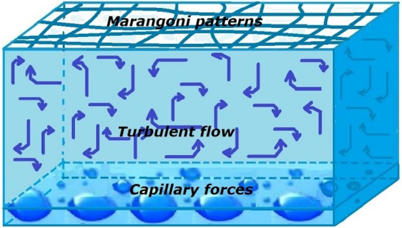
- For simple linear temperature profiles $U(y)$ the bifurcation is a result of a single mode instability.This mode is periodic in x with period $T = 2\pi/k$.
- The instability arises if, for a given k, the real part $r_k(b) = Re\lambda_k(b)$ of the eigenvalue λ_k corresponding to this mode goes through 0 as a bifurcation parameter b passes through a critical point $b = b_c$.
- For $b < b_c$ we determine that the trivial zero solution of fluid equations is stable, and for b close to b_c we find stable solutions describing periodical patterns.
- For the Marangoni case, an analysis of this system shows that the system bifurcates into two steady state solutions, which are local attractors.

S. Vakulenko and I. Sudakov. Complex bifurcations in Bénard-Marangoni convection, J. Phys. A: Math. Theor., 49 424001, 2016.

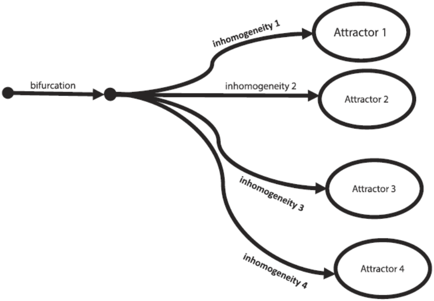
Control of bifurcations by space inhomogeneities



convection+conduction+temperature



convection+conduction+capillarity



- The number of slow modes is controllable by the space inhomogeneity.
- These slow modes are associated with eigenfunctions of a linear operator that describe linearization of system.

Extended Goody model

- We linearize main equations at zero state $\mathbf{v} = \mathbf{0}, \theta = T_0(z), C = C_0$ for small perturbations θ, C assuming, for simplicity, that μ is a function of θ and independent of x, y:

$$\theta_t = K\Delta\theta - 3\alpha_0\theta - 3\alpha_1 C T_0$$

$$C_t = d\Delta C - b_0^2 C$$

- $\alpha_0, b_0 > 0$;
- We denote by θ the temperature deviation with respect to the base state T_0;
- We consider these equations in the layer $\Omega = \{0 < z < h,\ x \in (-L_1, L_1), y \in (-L_2, L_2)\}$ and assume that $h << L_1, L_2$.

The simplest boundary conditions

$$T_z(x,y,z)|_{z=0} = r_0 T(x,y,0)$$
$$T_z(x,y,z)|_{z=h} = 0$$
$$C_z(x,y,z)|_{z=h} = 0$$
$$C_z(x,y,z)|_{z=0} = \beta T(x,y,0)$$

- $\beta = \frac{d\mu(\theta)}{d\theta}|_{\theta=T_0(0)}$;
- The coefficient r_0 is a result of linearization: $r_0 = \frac{dq}{d\theta}(T_0(0))$.
- The simplest boundary conditions describing a uniform gas emission.
- The last condition means that we consider a simple linear approximation: emission is proportional to the temperature deviation.
- We have the gas flux inside the system if $\beta < 0$.

Critical level of methane emissions

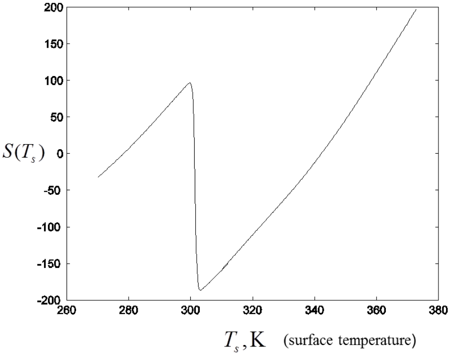
- We obtain an explicit relation for the emission critical value β_c

$$|\beta_c| \approx \frac{\alpha_0}{3\alpha_1 K T_0}\sqrt{b_0/d}$$

- The parameter K is thermal diffusivity, $K \approx 0.15 \cdot 10^{-8} m \cdot sec^{-2}$
- b_0^2 defines a rate of the natural greenhouse gas concentration C decay. $b_0^2 \approx 0.3 - 0.5 * 10^{-9} sec^{-1}$;
- d is the greenhouse gas diffusion coefficient; $d \approx 10^{-6} - 10^{-7} m^2 \cdot sec^{-1}$;

- **Estimates show that methane emission level, that observed now, should be increased minimum in $10^3 - 10^4$ (maybe, more) times to attain the critical level**

Bistable regime



We derive a nonlinear equation for the surface temperature and show that, for sufficiently large methane emissions from the soil, the climate system becomes bistable.

The corresponding pattern is always the same, with our "toy planet" sinking slowly into a homogeneous greenhouse gas fog.

Sudakov I, Vakulenko S. Bifurcations of the climate system and greenhouse gas emissions // Philos Trans A Math Phys Eng Sci. 2013: 371(1991): 20110473.

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∉ The processes on the "permafrost - atmosphere" interface are critical at all scales.

∉ The complex shape of tundra lakes and phase transitions in there are important factors of greenhouse gas emission intensity from them.

∉ There is a possible tipping point in atmospheric dynamics resulting from greenhouse gas emission from tundra lakes, where the climate system becomes bistable under sufficiently intensive greenhouse gas emissions.

THANK YOU!

RSF Russian Science Foundation
The Research Council of Norway
Federal Ministry of Education and Research
Microsoft
RFBR RUSSIAN FOUNDATION FOR BASIC RESEARCH
DYNASTY
NSF
MCRN Mathematics and Climate Research Network
STINT The Swedish Foundation for International Cooperation in Research and Higher Education
THE ROYAL SOCIETY
Kavli Institute for Theoretical Physics