An elastic interface approach to failure in microcracked materials

Estelle Berthier, Laurent Ponson

Institut d'Alembert, Université Pierre et Marie Curie, Paris, France

Vincent Démery

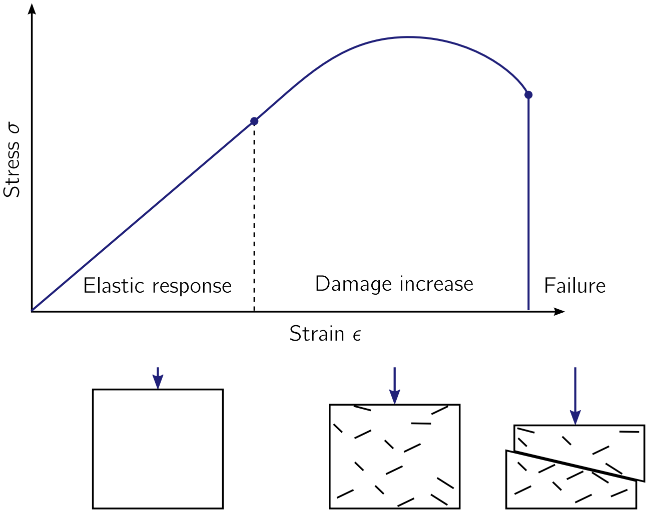
University of Massachusetts, Amherst, USA

$\rightarrow$ Gulliver, ESPCI, Paris, France

KITP 2014, October 20th

Thanks: ISCPIF, KECK Foundation

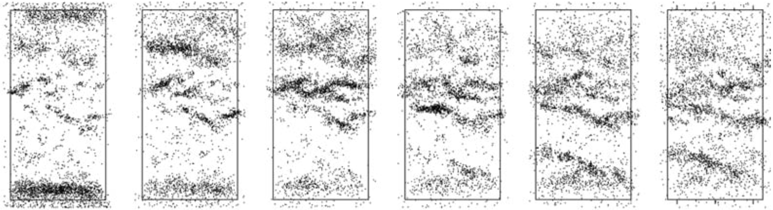
Phenomenology of damage failure



Characteristic features

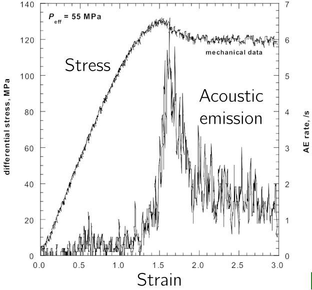
(acoustic emission in compressed sandstone)

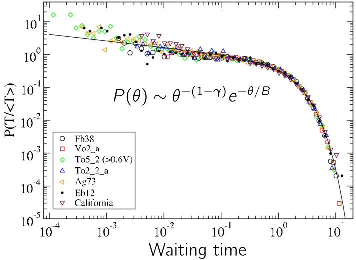
Localization



[Fortin *J. Geophys. Res.* 2006]

Avalanches





[Baud et al. *J. Struct. Geol* 2003, Davidsen *PRL* 2007]

Outline

Introduction to damage models

1d toy model: basic mechanisms

2d realistic model

Outline

Introduction to damage models

1d toy model: basic mechanisms

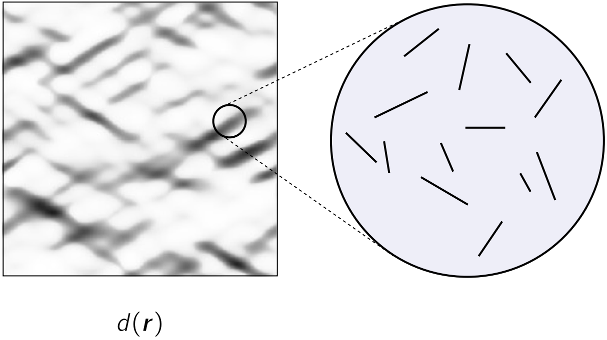
2d realistic model

Definition of damage

$d(\boldsymbol{r})$

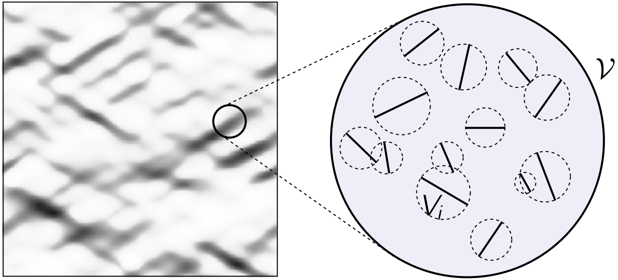
[Ponte-Castañeda, Willis *JMPS* 1995]

Definition of damage



[Ponte-Castañeda, Willis *JMPS* 1995]

Definition of damage

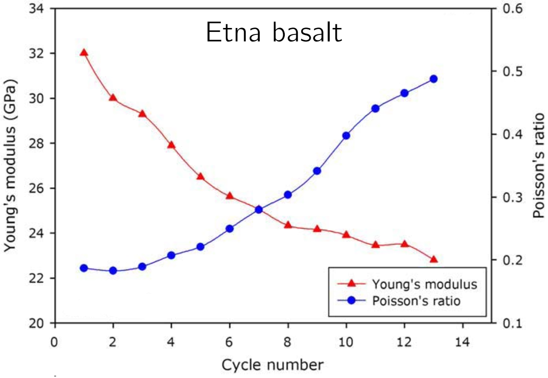


$d(\boldsymbol{r})$

$$d = \frac{1}{\mathcal{V}} \sum_i V_i$$

[Ponte-Castañeda, Willis *JMPS* 1995]

Damage affects the elastic properties



[Heap et al. *Tectonophysics* 2009]

- Simple model: $E(d) = E_0 \times (1 - d)$, $\nu = \nu_0$.
- More realistic expressions: Kachanov, Ponte-Castañeda and Willis.

Damage evolution

$$\dot{d} = \text{Max}\left(Y_{\text{dr}}(\boldsymbol{\sigma}, d) - Y_{\text{c}}(d), 0\right)$$

Driving force

Resistance

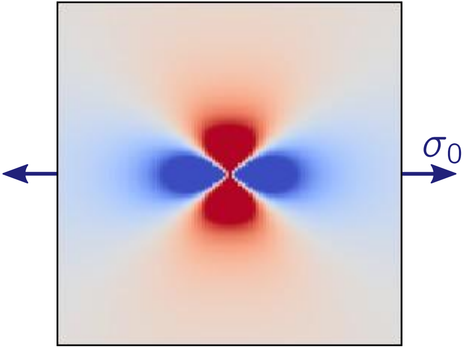
Damage evolution

$$\dot{d} = \mathrm{Max}\left(Y_{\mathrm{dr}}(\boldsymbol{\sigma}, d) - Y_{\mathrm{c}}(d), 0\right)$$

Damage criterion	Driving force	Resistance
Energetic	Elastic energy release $-\frac{\partial}{\partial d}\left(\frac{\boldsymbol{\sigma} : \boldsymbol{\epsilon}}{2}\right)\Big\vert_{\boldsymbol{\epsilon}}$	Surface energy
Mohr-Coulomb	Shear stress	Friction

Heterogeneous elastic properties redistribute the stress

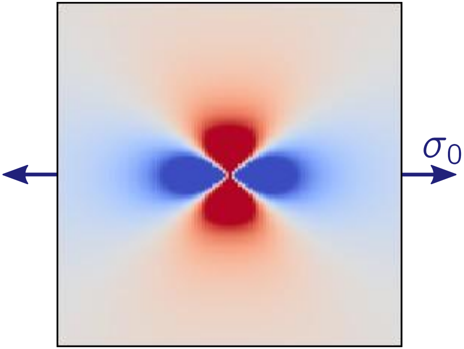
- The stress field around an ellipsoidal inclusion can be found. [Eshelby *PRSL A* 1957]
- General case $E(\boldsymbol{r})$: requires numerical resolution.



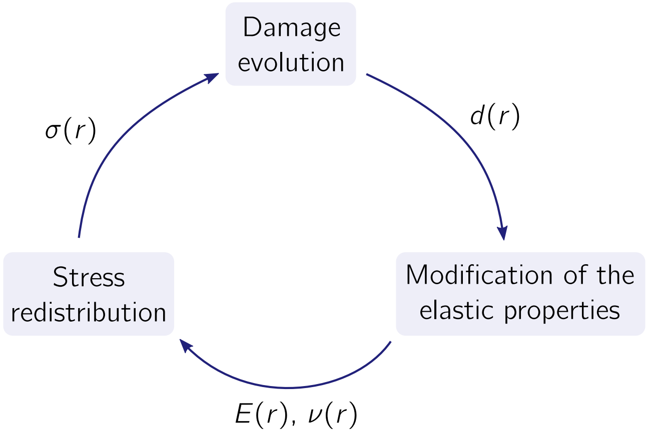
Heterogeneous elastic properties redistribute the stress

- The stress field around an ellipsoidal inclusion can be found. [Eshelby *PRSL A* 1957]
- General case $E(\boldsymbol{r})$: requires numerical resolution.
- If heterogeneities are weak, Eshelby result may be used as a Green function to compute the stress field to the first order in the heterogeneities (superposition principle)

$$\boldsymbol{\sigma}(\boldsymbol{r}) = \boldsymbol{\sigma}_0 + \boldsymbol{\Psi}_E * \delta E(\boldsymbol{r}).$$

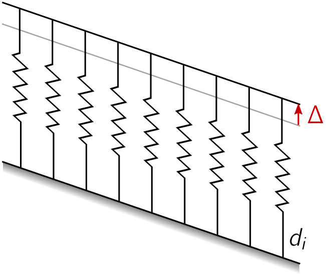


General structure of a damage model



Outline

1d model: fiber bundle with interactions

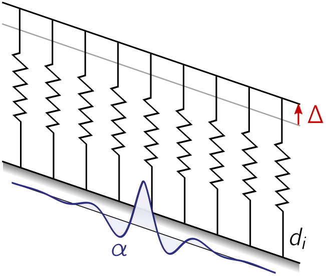


$$E = \frac{\Delta^2}{2}\int k(\underbrace{d(x)}_{\text{Elastic energy}})dx + \int \underbrace{\int_0^{d(x)} Y_c(d', x)dd'}_{\text{Fracture energy}}dx$$

Energetic criterion: damage at x if $Y(x) = -\frac{\delta E}{\delta d}(x) \geq 0.$

Stiffness:
$k(d) = ad^{\zeta} - (a+1)d + 1$

1d model: fiber bundle with interactions



$$E = \frac{\Delta^2}{2} \int k(\alpha * d(x)) dx + \int \int_0^{d(x)} Y_c(d', x) dd' dx$$

Elastic energy — Fracture energy

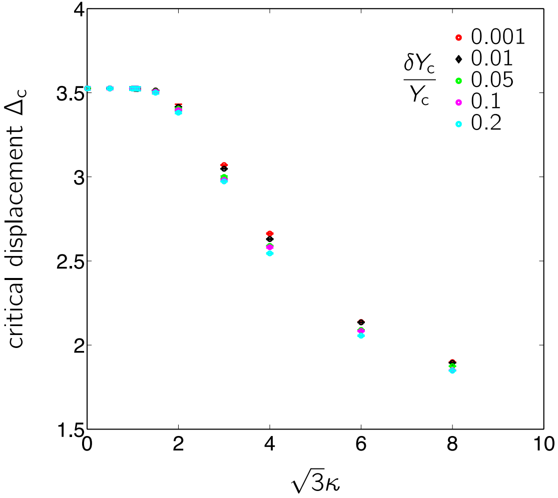
Energetic criterion: damage at x if $Y(x) = -\frac{\delta E}{\delta d}(x) \geq 0.$

Stiffness:
$k(d) = ad^\zeta - (a+1)d + 1$

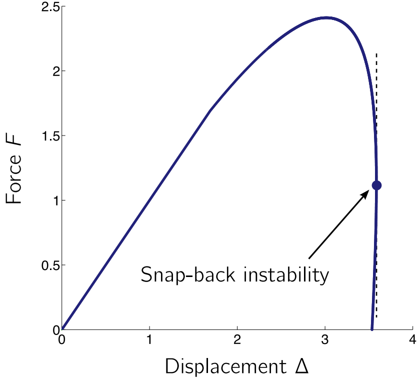
Load sharing:
$\alpha(x) = \exp\left(-\frac{|x|}{l}\right) \cos\left(\frac{\kappa x}{l}\right)$

1d model: numerical simulations

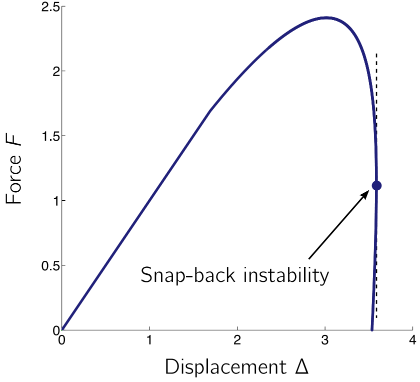
- Load sharing: $\alpha(x) = \exp\left(-\frac{|x|}{l}\right)\cos\left(\frac{\kappa x}{l}\right)$.
- Quasistatic evolution.

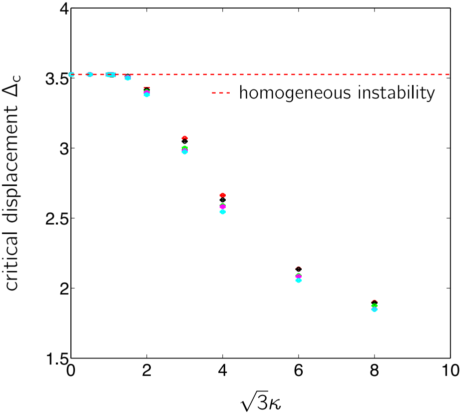


1d model: homogeneous damage evolution



1d model: homogeneous damage evolution





$$\alpha(x) = \exp\left(-\frac{|x|}{l}\right)\cos\left(\frac{\kappa x}{l}\right)$$

1d model: heterogeneous damage evolution

Expansion of the damage evolution around a homogeneous damage, $d(x) = \langle d \rangle + \delta d(x)$

$$Y[\Delta, d(x)] = Y(\Delta, \langle d \rangle) + \Psi_{\langle d \rangle} * \delta d(x)$$

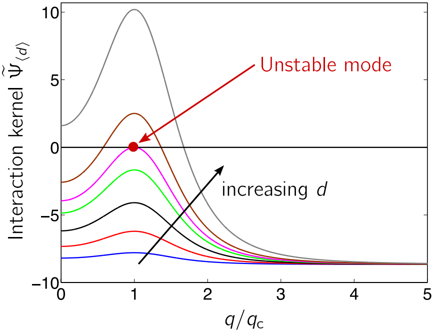
$\Psi_{\langle d \rangle}$ values in Fourier space are its eigenvalues:
$\tilde{\Psi}_{\langle d \rangle}(q) > 0 \Rightarrow$ the mode q is unstable.

1d model: heterogeneous damage evolution

Expansion of the damage evolution around a homogeneous damage, $d(x) = \langle d \rangle + \delta d(x)$

$$Y[\Delta, d(x)] = Y(\Delta, \langle d \rangle) + \Psi_{\langle d \rangle} * \delta d(x)$$

$\Psi_{\langle d \rangle}$ values in Fourier space are its eigenvalues: $\tilde{\Psi}_{\langle d \rangle}(q) > 0 \Rightarrow$ the mode q is unstable.

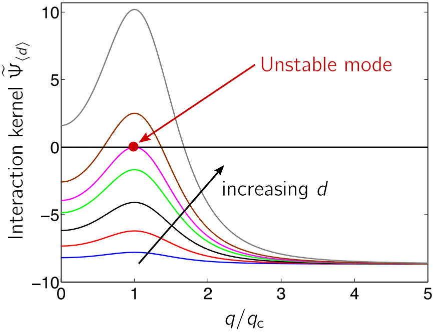


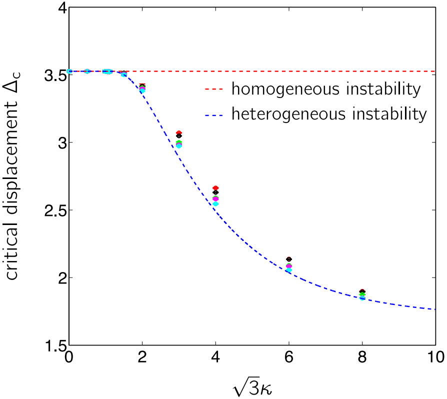
1d model: heterogeneous damage evolution

Expansion of the damage evolution around a homogeneous damage, $d(x) = \langle d \rangle + \delta d(x)$

$$Y[\Delta, d(x)] = Y(\Delta, \langle d \rangle) + \Psi_{\langle d \rangle} * \delta d(x)$$

$\Psi_{\langle d \rangle}$ values in Fourier space are its eigenvalues:
$\tilde{\Psi}_{\langle d \rangle}(q) > 0 \Rightarrow$ the mode q is unstable.





1d model: conclusion

- Initial model: no linear elastic kernel.
- Linear elastic kernel:
  - Obtained by expanding the initial model for a weakly heterogeneous damage $d(x) = \langle d \rangle + \delta d(x)$:

$$Y[\Delta, d(x)] = Y(\Delta, \langle d \rangle) + \Psi_{\langle d \rangle} * \delta d(x)$$

  - Stability analysis in Fourier space: sign of $\tilde{\Psi}_{\langle d \rangle}(q)$.

1d model: conclusion

- Initial model: no linear elastic kernel.
- Linear elastic kernel:
  - Obtained by expanding the initial model for a weakly heterogeneous damage $d(x) = \langle d \rangle + \delta d(x)$:
  $$Y[\Delta, d(x)] = Y(\Delta, \langle d \rangle) + \Psi_{\langle d \rangle} * \delta d(x)$$
  - Stability analysis in Fourier space: sign of $\tilde{\Psi}_{\langle d \rangle}(q)$.
- 2 possible cases:
  - $\frac{\partial Y(\Delta, d)}{\partial d} \geq 0$ first: homogeneous instability.
  - $\tilde{\Psi}_{\langle d \rangle}(q_c) \geq 0$ first: localization with wavelength $\lambda_c = 2\pi / q_c$.
- Different from usual depinning transitions (brittle failure. . .):
  - Kernel evolves with damage: $\Psi_{\langle d \rangle}$.
  - Kernel instability, weak effect of the disorder.

Outline

Introduction to damage models

1d toy model: basic mechanisms

2d realistic model

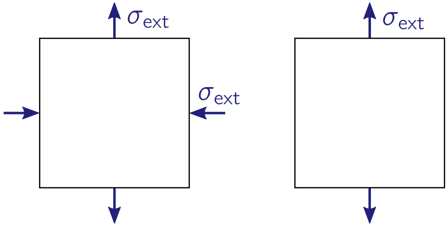
2d model

- Energetic criterion for damage evolution (no friction, applies to traction, shear),

$$Y(\boldsymbol{\epsilon}, d) = -\frac{\partial}{\partial d}\left(\frac{\mathrm{Tr}(\boldsymbol{\sigma\epsilon})}{2}\right)\Big|_{\boldsymbol{\epsilon}} - Y_{c0}.$$

- Stress redistribution from continuum mechanics (2d - plane strain),

$$\mathrm{div}(\sigma) = 0,$$
$$\epsilon_{ij,kl} - \epsilon_{jk,li} + \epsilon_{kl,ij} - \epsilon_{li,jk} = 0 \quad \forall i, j, k, l.$$



shearing | uniaxial traction

2d model: stress redistribution

Expansion of the driving force for a weakly heterogeneous damage $d(\boldsymbol{r}) = \langle d \rangle + \delta d(\boldsymbol{r})$

$$Y(\boldsymbol{r}) = Y_0 + \Psi_{\langle d \rangle} * \delta d(\boldsymbol{r})$$

2d model: stress redistribution

Expansion of the driving force for a weakly heterogeneous damage $d(\boldsymbol{r}) = \langle d \rangle + \delta d(\boldsymbol{r})$

$$Y(\boldsymbol{r}) = Y_0 + \Psi_{\langle d \rangle} * \delta d(\boldsymbol{r})$$

$$\tilde{\Psi}_{\langle d \rangle}(\boldsymbol{q}) = \frac{a''}{2}\mathrm{Tr}(\boldsymbol{\sigma}^2) + \frac{b''}{2}\mathrm{Tr}(\boldsymbol{\sigma})^2 - \frac{1}{a+b}\left[a'\mathrm{Tr}(\boldsymbol{\sigma}\mathcal{O}(\boldsymbol{q})) + b'\mathrm{Tr}(\boldsymbol{\sigma})\right]^2$$

$$\boldsymbol{\varepsilon} = a\boldsymbol{\sigma} + b\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

2d model: stress redistribution

Expansion of the driving force for a weakly heterogeneous damage $d(\boldsymbol{r}) = \langle d \rangle + \delta d(\boldsymbol{r})$

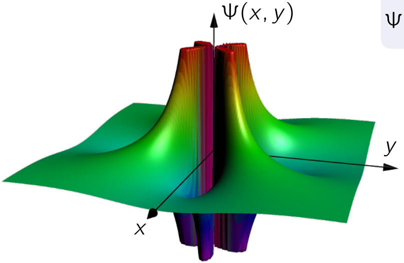
$$Y(\boldsymbol{r}) = Y_0 + \Psi_{\langle d \rangle} * \delta d(\boldsymbol{r})$$

$$\tilde{\Psi}_{\langle d \rangle}(\boldsymbol{q}) = \frac{a''}{2}\mathrm{Tr}(\boldsymbol{\sigma}^2) + \frac{b''}{2}\mathrm{Tr}(\boldsymbol{\sigma})^2 - \frac{1}{a+b}\left[a'\mathrm{Tr}(\boldsymbol{\sigma}\mathcal{O}(\boldsymbol{q})) + b'\mathrm{Tr}(\boldsymbol{\sigma})\right]^2$$

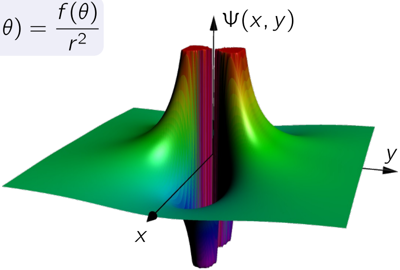
$$\boldsymbol{\varepsilon} = a\boldsymbol{\sigma} + b\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

$$\mathcal{O}(\boldsymbol{q}) = \frac{1}{\boldsymbol{q}^2}\begin{pmatrix} q_y^2 & -q_x q_y \\ -q_x q_y & q_x^2 \end{pmatrix} = \begin{pmatrix} \sin(\omega)^2 & -\sin(\omega)\cos(\omega) \\ -\sin(\omega)\cos(\omega) & \cos(\omega)^2 \end{pmatrix}$$

$$\Psi(r,\theta) = \frac{f(\theta)}{r^2}$$



shearing



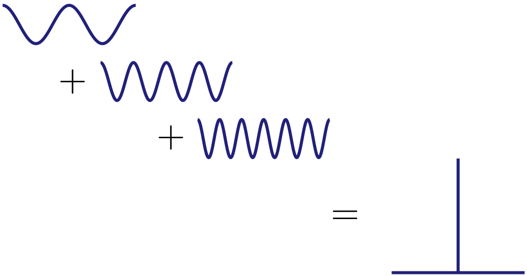
uniaxial traction

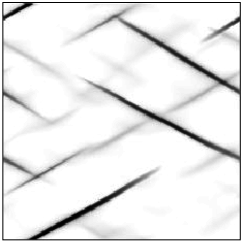
2d model: kernel behavior

- The kernel depends only on the angle ω of the wavevector $\boldsymbol{q}$:

$$\tilde{\Psi}_d(q, \omega) = \tilde{\psi}_d(\omega).$$

- At instability, all the wavelength with the same angle diverge: localization along bands.



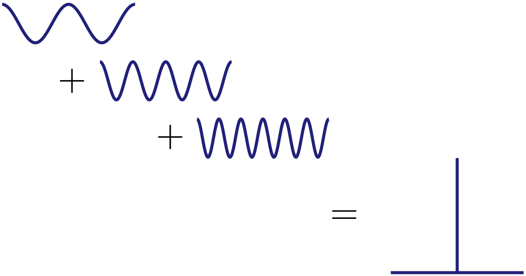


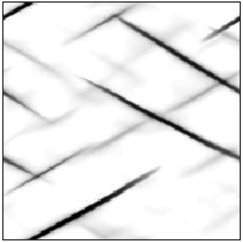
2d model: kernel behavior

- The kernel depends only on the angle ω of the wavevector $\boldsymbol{q}$:

$$\tilde{\Psi}_d(q,\omega) = \tilde{\psi}_d(\omega).$$

- At instability, all the wavelength with the same angle diverge: localization along bands.





- The critical damage for localization and the orientation of the localization band can be predicted.
  - 45° for shearing, 57° for uniaxial tension.

Conclusion

For any damage model, the elastic kernel can be computed

$\Rightarrow$ Elastic interface model for damage evolution.

Localization comes from an elastic kernel instability.

Future work

How is this new kind of transition related to the observed avalanches statistics?

Avalanches in the 1d model