MITMECHE

Nonlocal Continuum Modeling of Granular Flow

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Main Goal

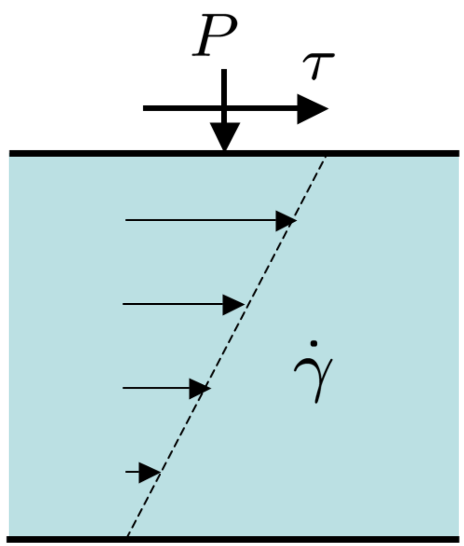
Q: What's the constitutive law for well-developed granular flow?

Inertial Flow Rheology: Simple Shear

- Inertial number: $I = \dot{\gamma} d \sqrt{\frac{\rho_s}{P}}$

  d = mean particle diameter

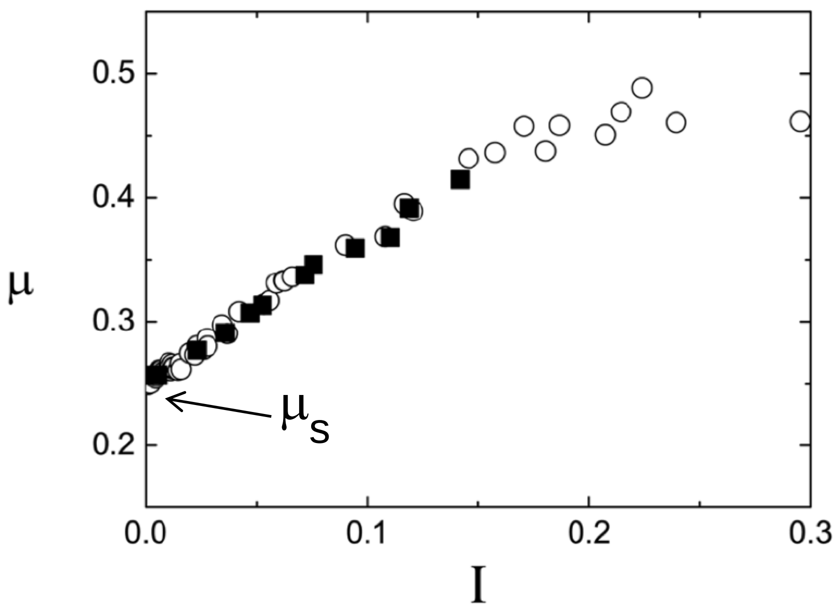
  ρ_s = the density of a grain



$$\mu = \tau / P$$

- Inertial rheology: $\mu = \mu(I)$

  [Da Cruz et al 2005, Jop et al 2006, related to “Bagnold Scaling” (1954)]



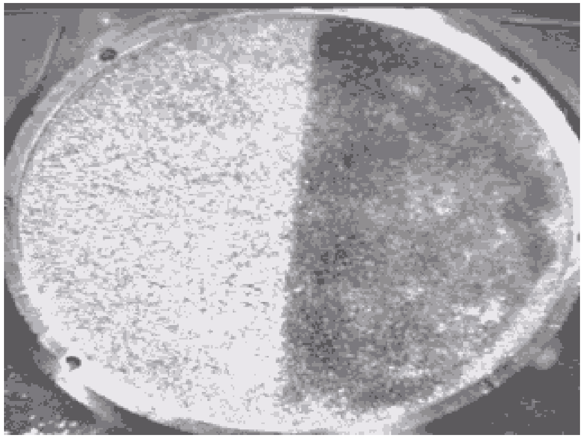
Fit to a straight line:

$$\mu(I) = \mu_s + bI$$

Dense granular flows are *cooperative.* Not captured by local rheology.

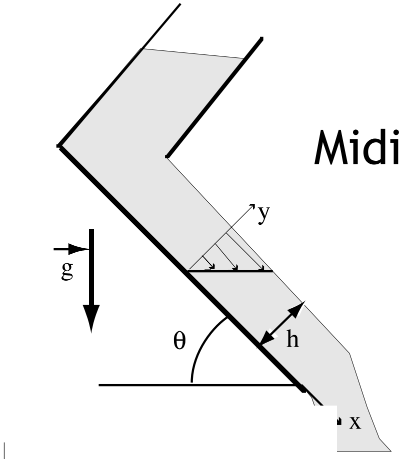
Grain-size-dependent flow fields

Split-bottom cell



van Hecke (2003, 2004, 2006)

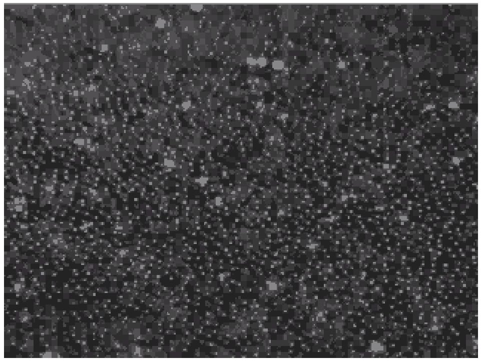
Thin layers - h_{stop} effect

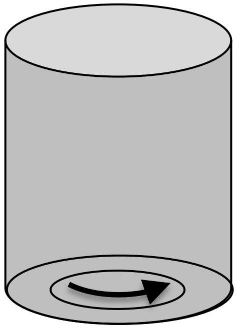


Midi (2004)

Drainage flows

Secondary Rheology



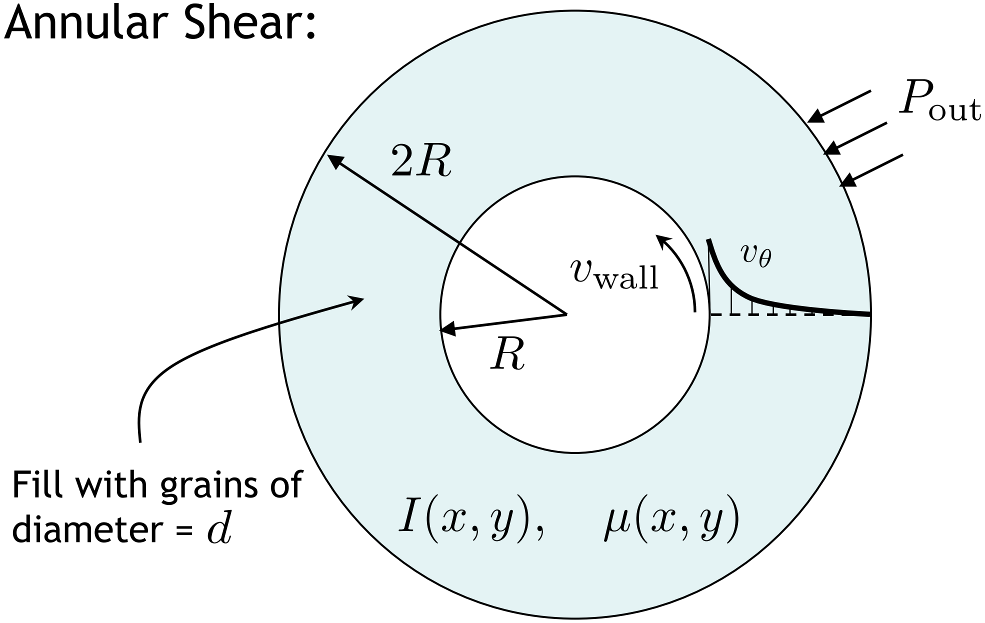


van Hecke (2010)

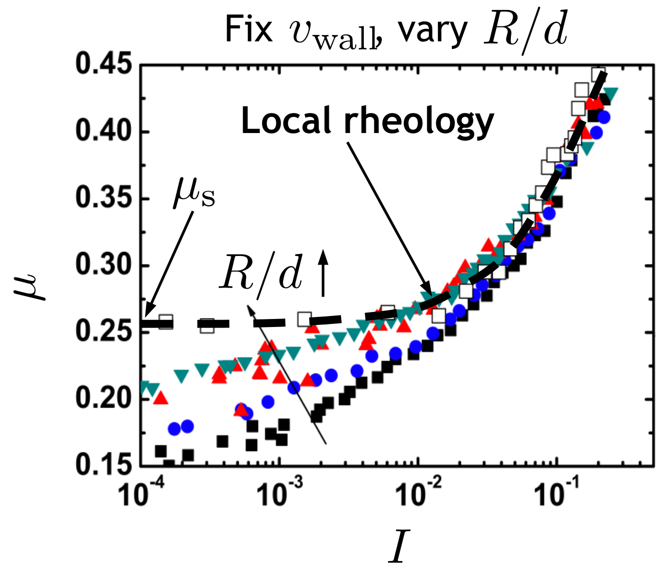
Grain size affects steady flow profiles

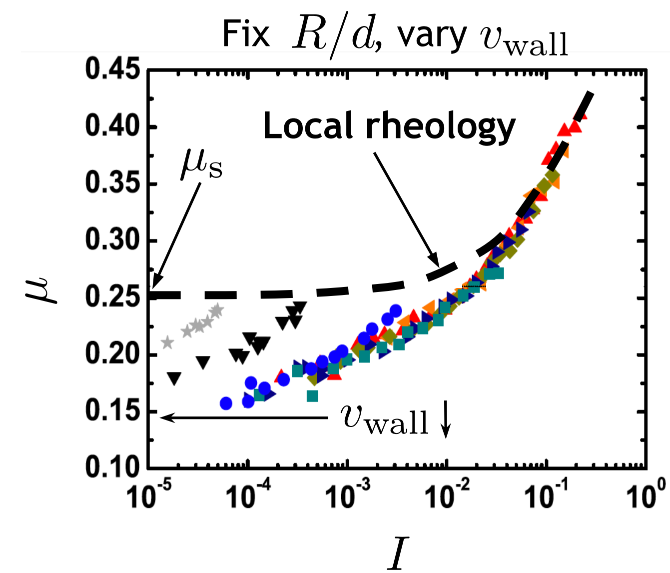
Koval et al., *PRE* (2009)

Annular Shear:



d and P_{out} held fixed



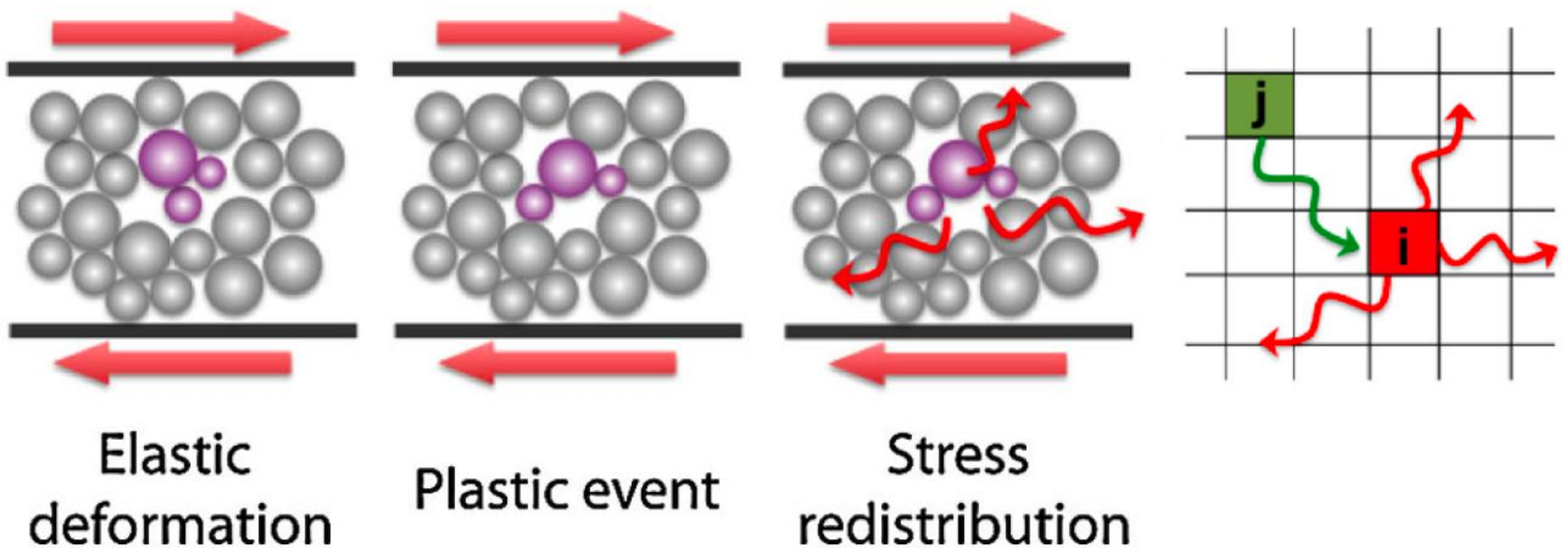


- μ vs. I is not one-to-one
- Flow occurs below μ_{s}

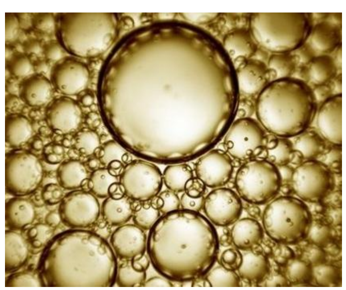
Physical picture of cooperativity

Basic idea: *Flow induces flow*. Plastic dynamics are spatially cooperative.

Kinetic Elasto-Plastic (KEP) model
Bocquet et al., *PRL* (2009)



Proposed for emulsions



Extend to granular materials

Nonlocal rheology background

The form of the nonlocal rheology is derived from a *Landau-like PDE* for the fluidity
(Related idea: "partial fluidization" Aranson and Tsimring, *PRE* (2001)):

Order parameter: Granular fluidity $= g = \dfrac{\dot{\gamma}}{\mu} = \dfrac{\text{Rearrangement rate}}{\text{Indicator of the bulk stress state}} =$ "Relative susceptibility to flow"

Dynamical PDE:

$$\frac{\partial g}{\partial \tilde{t}} = A^2 d^2 \nabla^2 g - \underbrace{\left[(\mu_s - \mu) g + b \sqrt{\frac{\rho_s d^2}{P}} \mu g^2 \right]}_{\text{Akin to a "coarse-grain energy derivative"}}$$

(Henann and Kamrin, *IJP* (2014))

Akin to a "coarse-grain energy derivative"

Reduce to ***steady-state-only*** model:

Expanding around steady, stable, homogeneous flow solutions

$$g = g_{\text{loc}}(\mu, P) + \xi(\mu)^2 \nabla^2 g$$

Local rheology (standard inertial flow relation):

$$g_{\text{loc}}(\mu, P) = \frac{\dot{\gamma}_{\text{loc}}}{\mu} = \begin{cases} \sqrt{P/\rho_s d^2}(\mu - \mu_s)/b\mu & \text{if } \mu > \mu_s \\ 0 & \text{if } \mu \le \mu_s \end{cases}$$

Cooperativity length:

$$\xi(\mu) = \frac{A}{\sqrt{|\mu - \mu_s|}} d$$

Cooperativity Length

Theoretical form for cooperativity length:

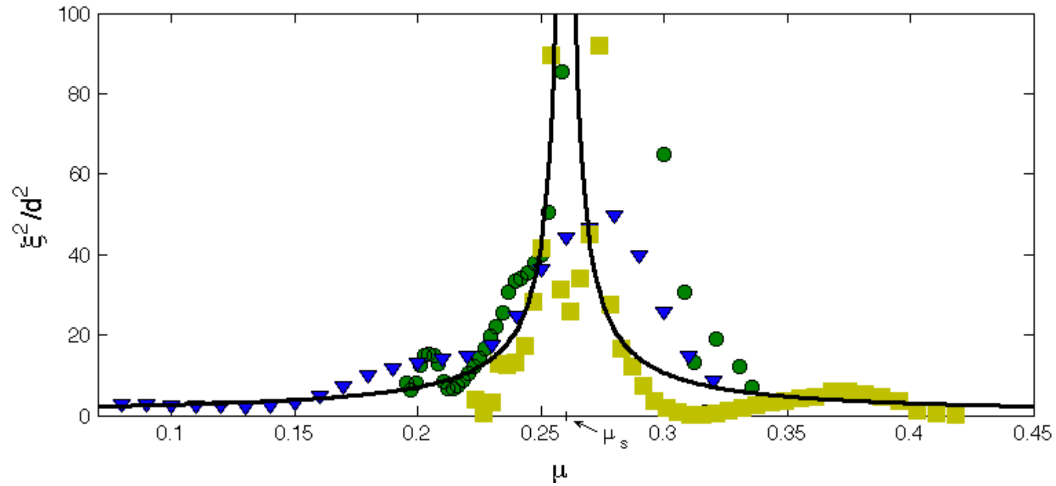
$$\xi(\mu) = \frac{A}{\sqrt{|\mu - \mu_s|}} d$$

[Similar to Bocquet PRL 2009]

Extract ξ directly from DEM tests using our proposed fluidity PDE:

$$g = g_{\mathrm{loc}}(\mu, P) + \xi(\mu)^2 \nabla^2 g$$

Direct tests: From steady-flow DEM data in the 3 geometries (annular shear, vertical chute, shear w/ gravity):

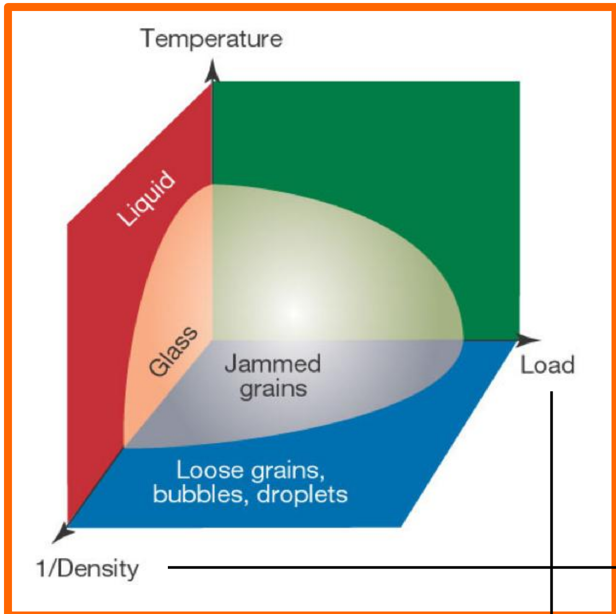


Kamrin & Koval PRL (2012)

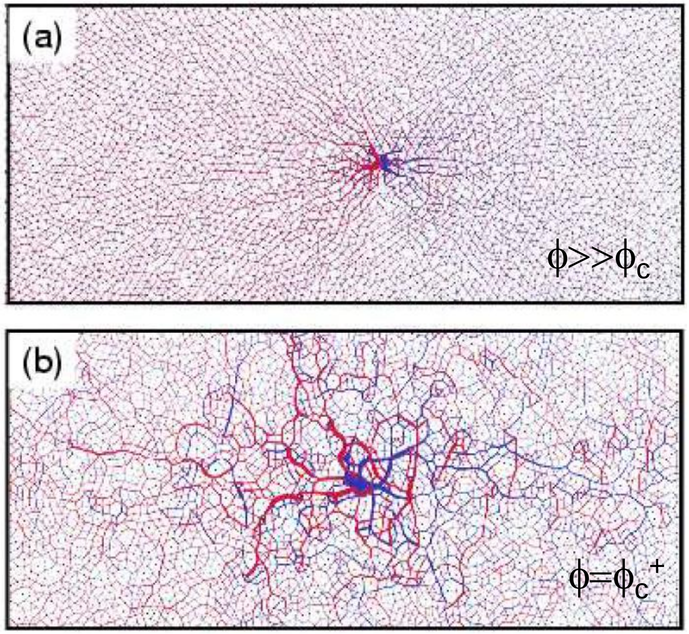
For our 2D DEM disks, we find: **A=0.70**

Only new material constant is A, the *nonlocal amplitude*. Local law constants all carry over.

The Diverging Length-Scale



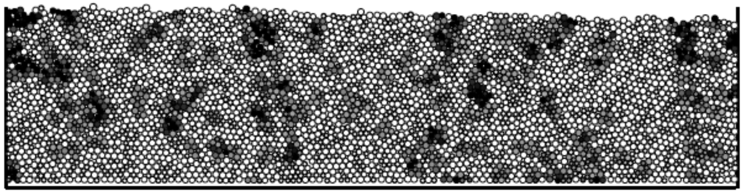
M. van Hecke,
Cond. Mat. (2009)



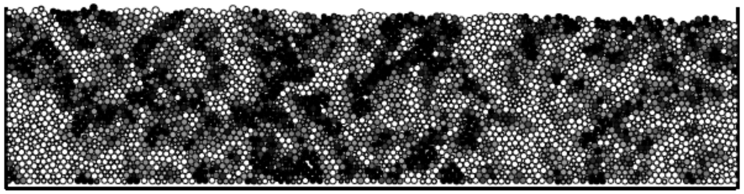
Staron et al,
PRL (2002)

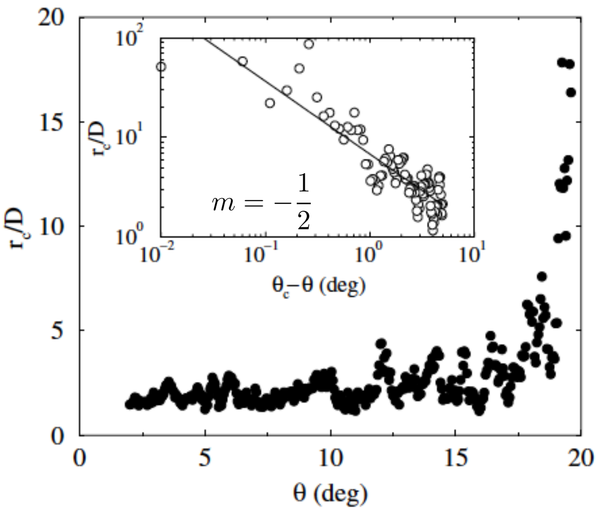
Pre-avalanche zone sizes for inclined plane flow:

$\theta = 5^o$



$\theta = 15^o$





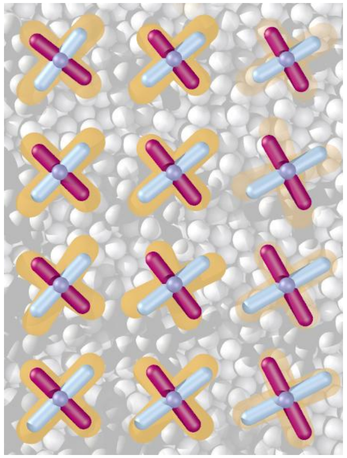
Model for steady dry flow

$$\boldsymbol{\sigma} = -P\mathbf{1} + 2\frac{P}{g}\mathbf{D} \quad \text{stress constitutive relation}$$

$$\mathrm{tr}(\mathbf{D}) = 0 \quad \text{constant-volume constraint}$$

$$\xi^2\nabla^2 g = g - g_{\mathrm{loc}} \quad \text{nonlocal rheology}$$

Coaxiality:



Rycroft, Kamrin, Bazant, *JMPS* (2009)

Local rheology (standard inertial flow relation):

$$g_{\mathrm{loc}} = \begin{cases} \sqrt{P/\rho_{\mathrm{s}} d^2}(\mu - \mu_{\mathrm{s}})/b\mu & \text{if } \mu > \mu_{\mathrm{s}} \\ 0 & \text{if } \mu \le \mu_{\mathrm{s}} \end{cases}$$

Cooperativity length:

$$\xi = \frac{A}{\sqrt{|\mu - \mu_{\mathrm{s}}|}} d$$

2 local material parameters: $\{\mu_{\mathrm{s}}, b\}$ + 2 grain parameters: $\{d, \rho_{\mathrm{s}}\}$ + 1 new *nonlocal amplitude*: $\{A\}$

Already known for many materials

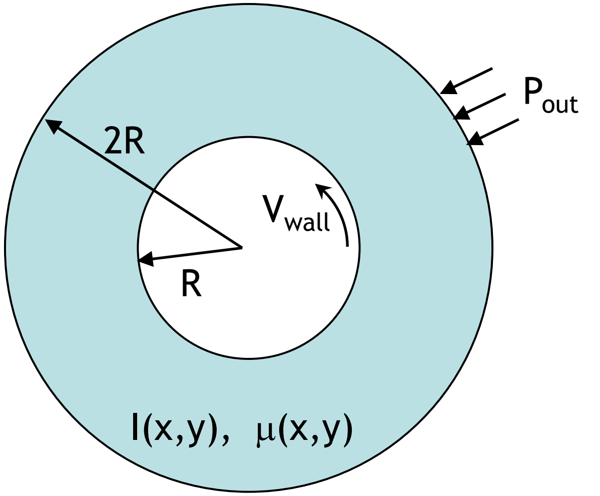
Our calibration: Glass beads A=0.48

DEM disks A=0.70

2D Results: Validating the Nonlocal Model

Check predictions against DEM data from Koval et al. (PRE 2009) in the annular couette cell.

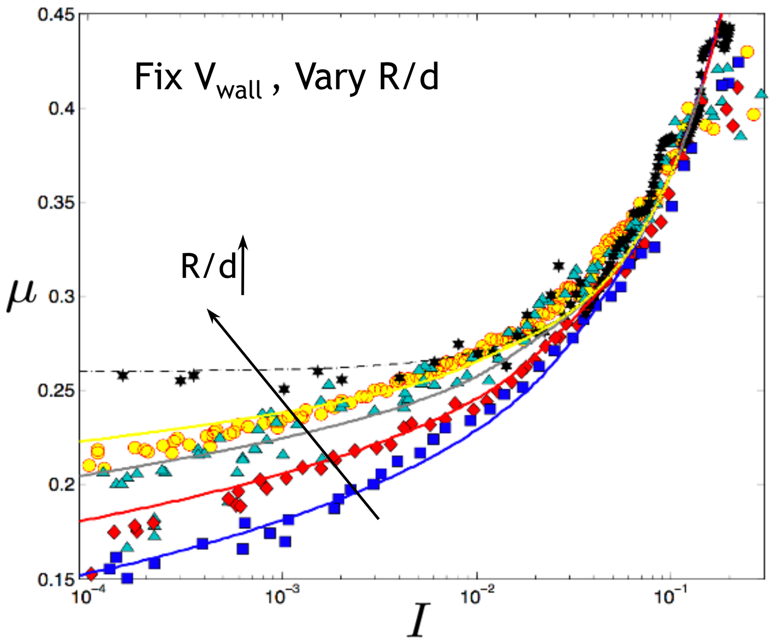
$$P(r) \cong P_{out}$$
$$\mu(r) = \mu_{wall}(R/r)^2$$

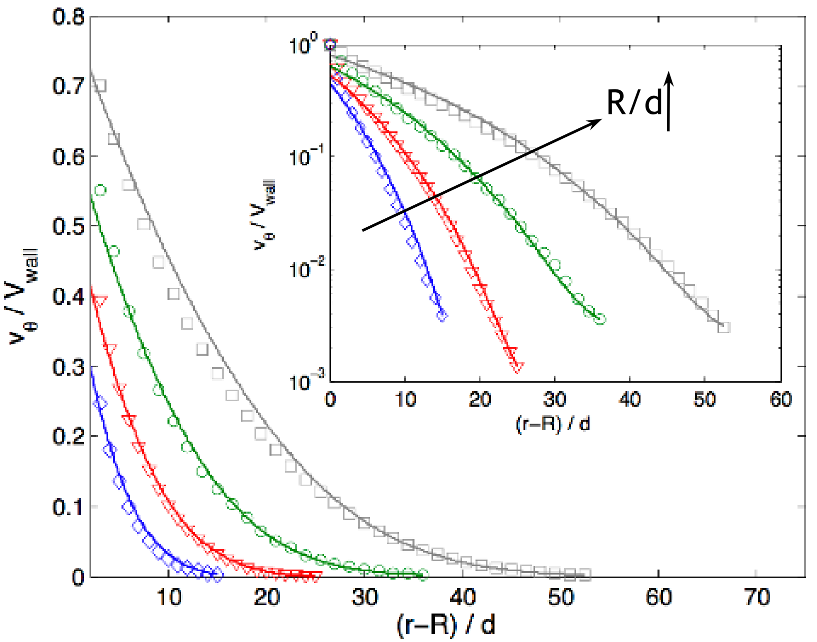


DEM (symbols)

Nonlocal Rheology (lines)

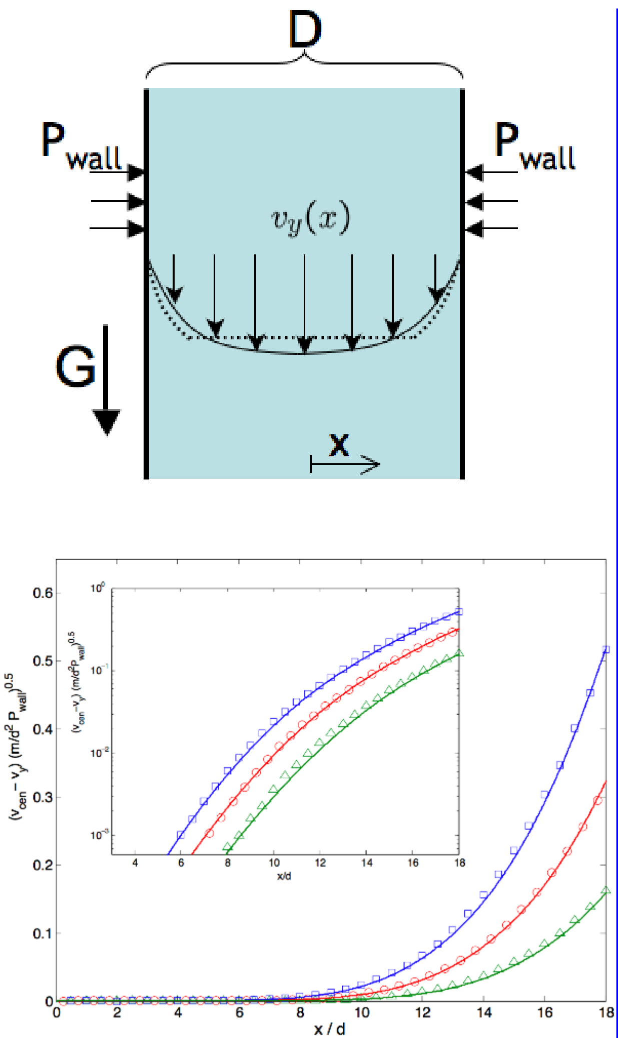
Local Rheology (- - -)

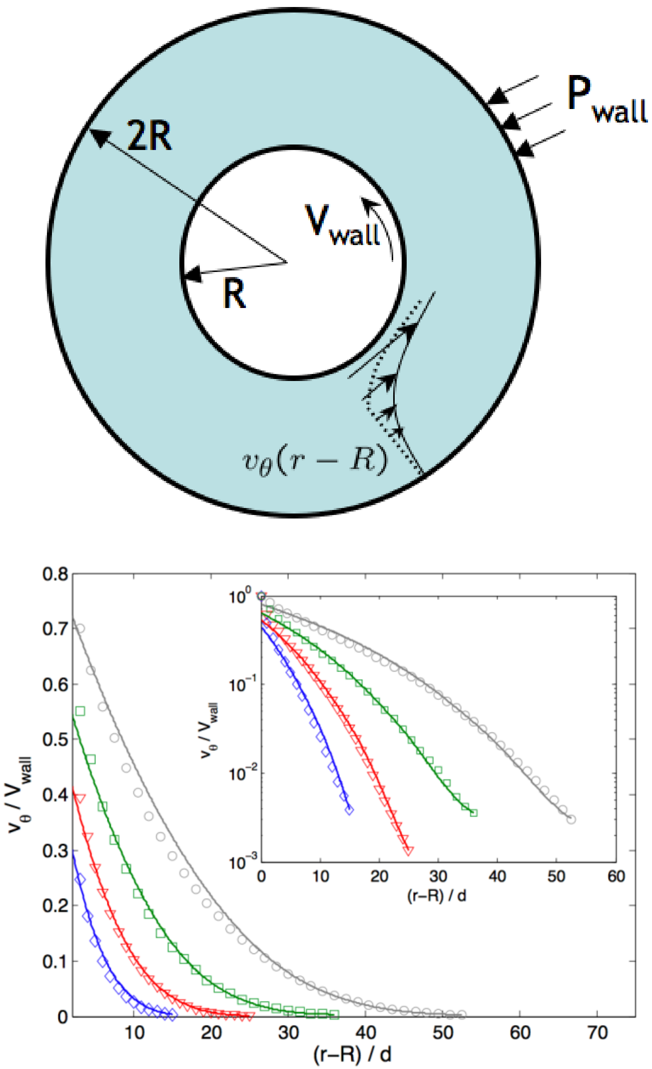


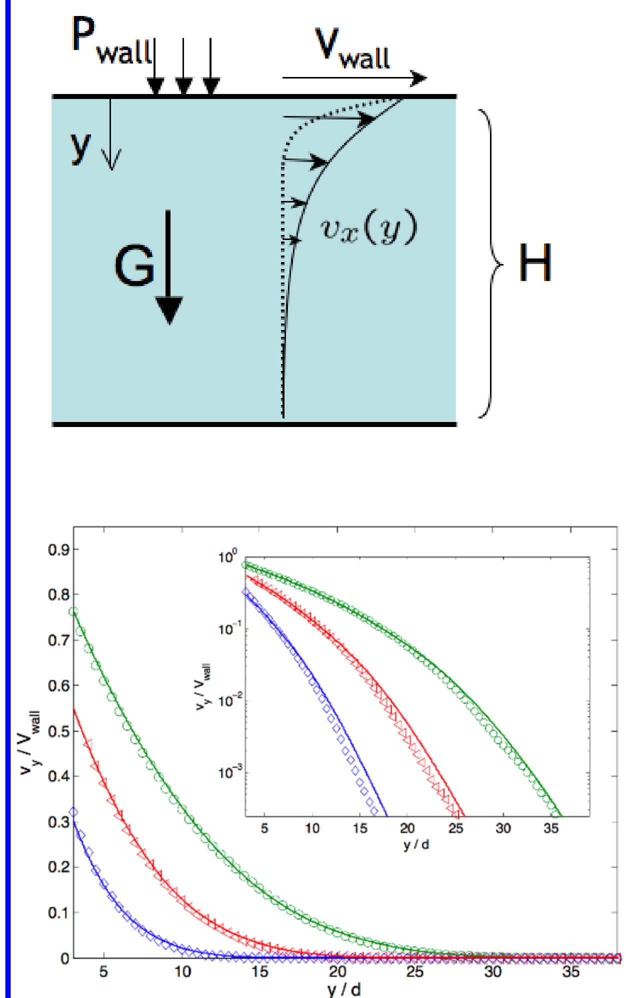


Kamrin & Koval PRL (2012)

2D Results: Same model in different geometries. Velocity profiles (theory compared to DEM disks)







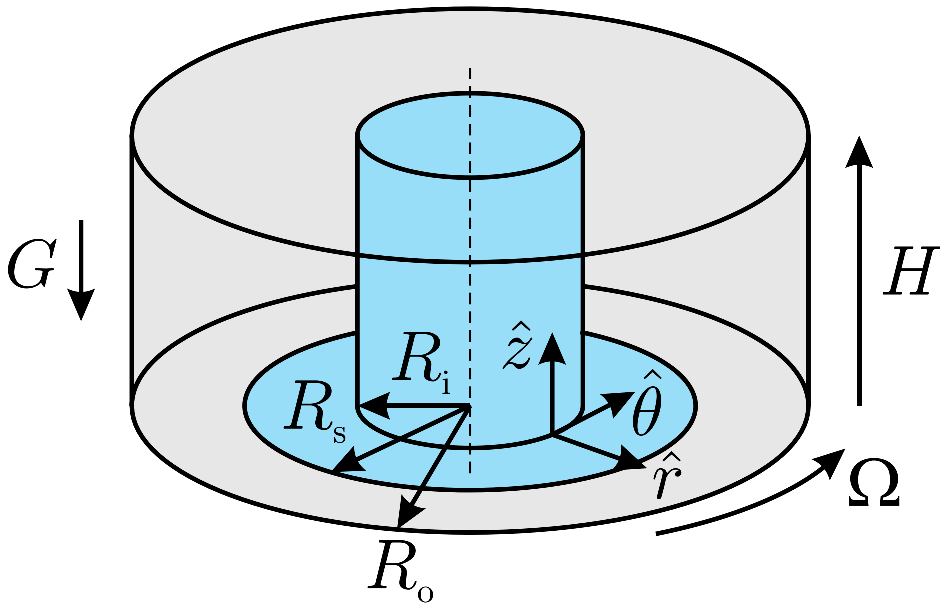
{Lines: Model predictions Symbols: DEM results}

Kamrin and Koval, *PRL* 2012

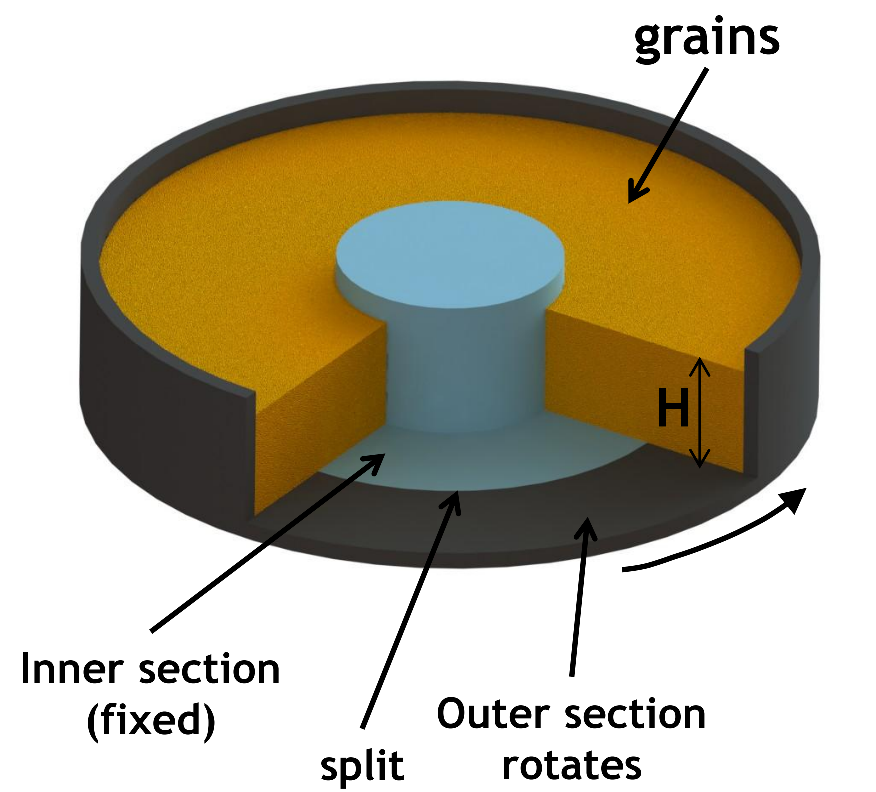
Comparison to 3D Experiments: Flow of beads in the split-bottom Couette cell

No previous continuum model has correctly predicted the flows in this geometry

Schematic:

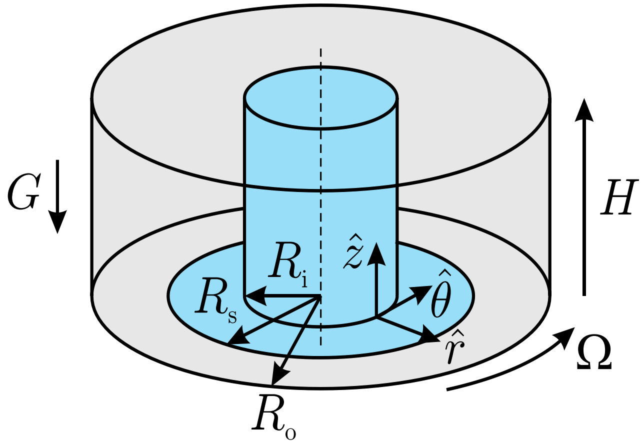


Filled with grains:



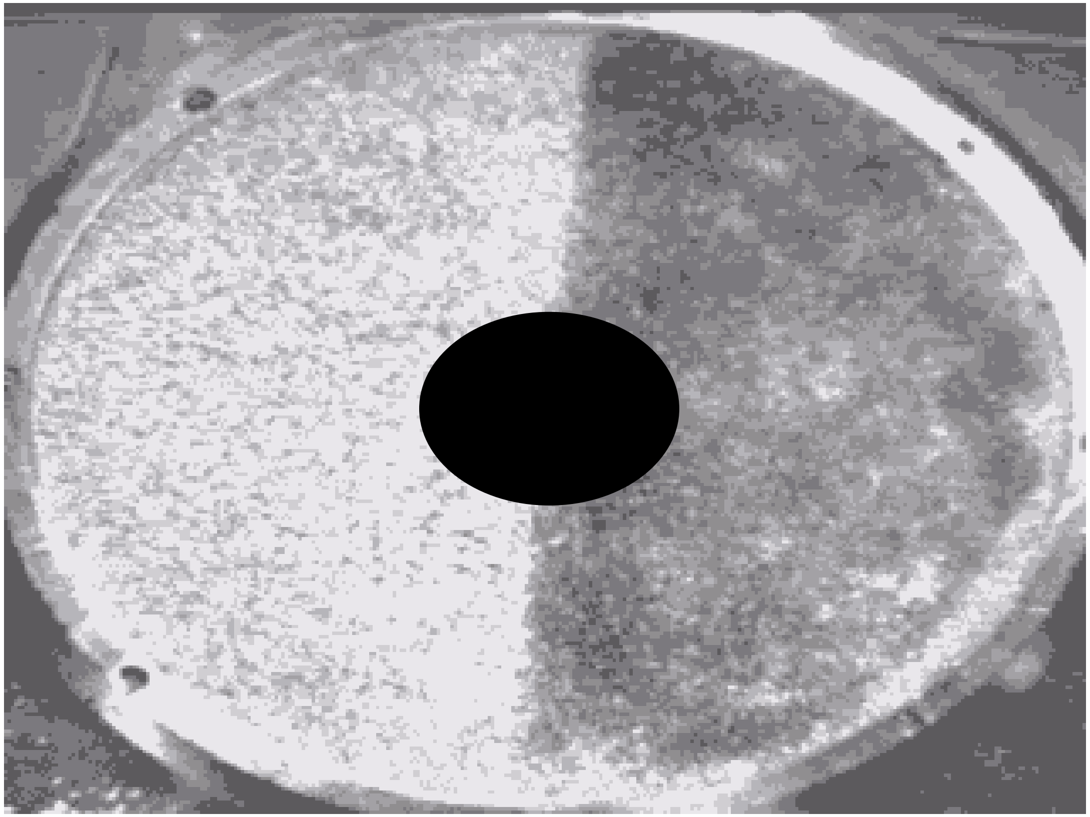
van Hecke (2003, 2004, 2006)

Flow in the split-bottom Couette cell – surface flow



The normalized revolution-rate: $\omega = (v_\theta / r)/\Omega$

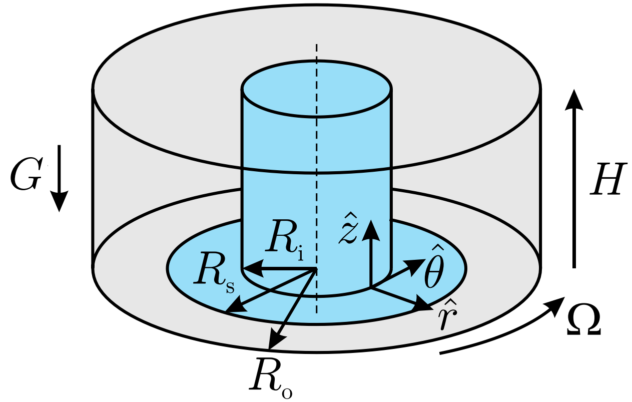
Viewed from the top down:

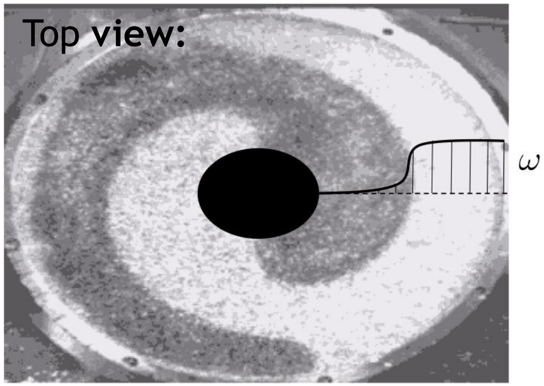


ω

van Hecke (2003, 2004, 2006)

Flow in the split-bottom Couette cell – surface flow

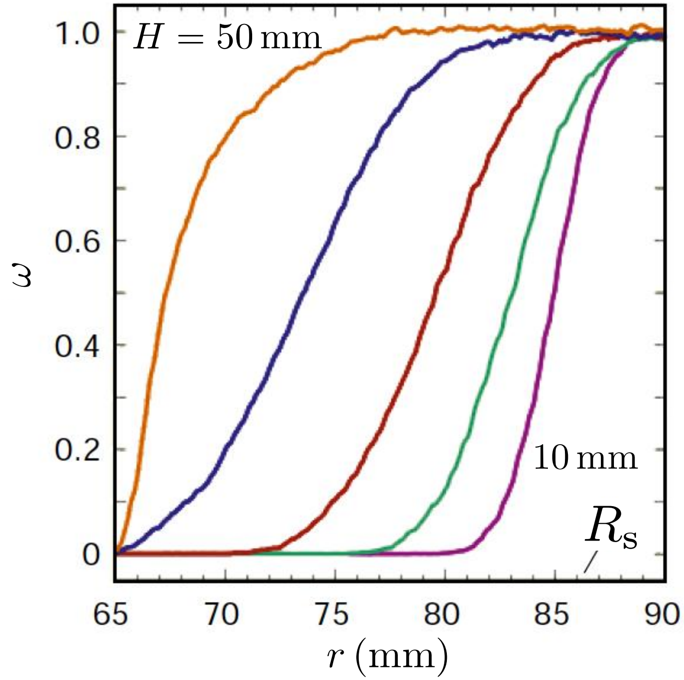




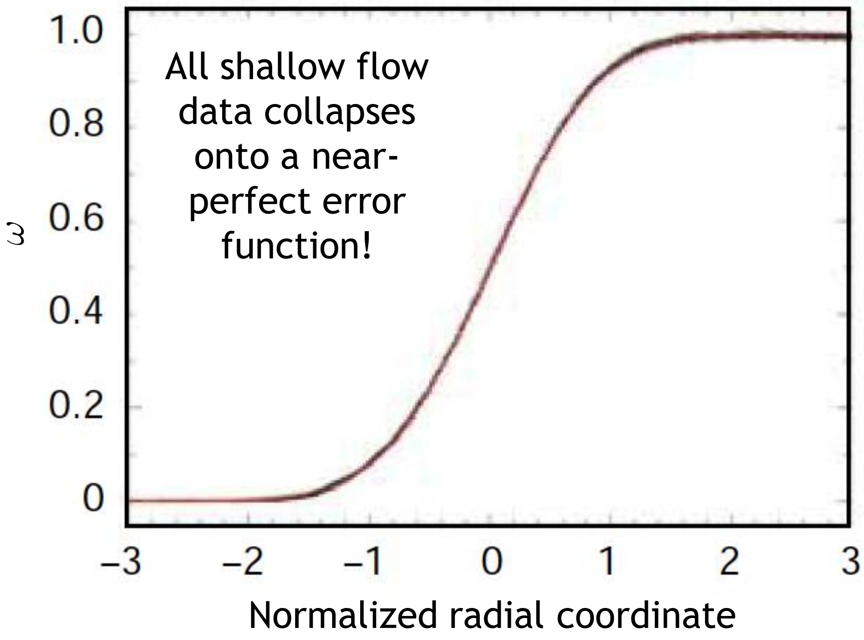
$$(d = 0.35\,\mathrm{mm},\ R_s = 85\,\mathrm{mm})$$

Exp data from: Fenistein et al., *Nature* (2003)

Flow fields on the top surface for five values of H:

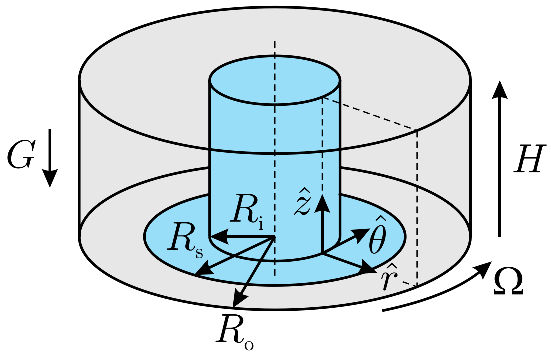


Universal flow profile:

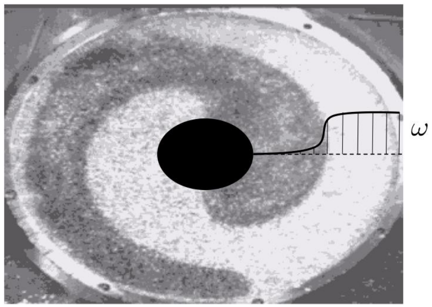


Nonlocal model in the split-bottom Couette cell

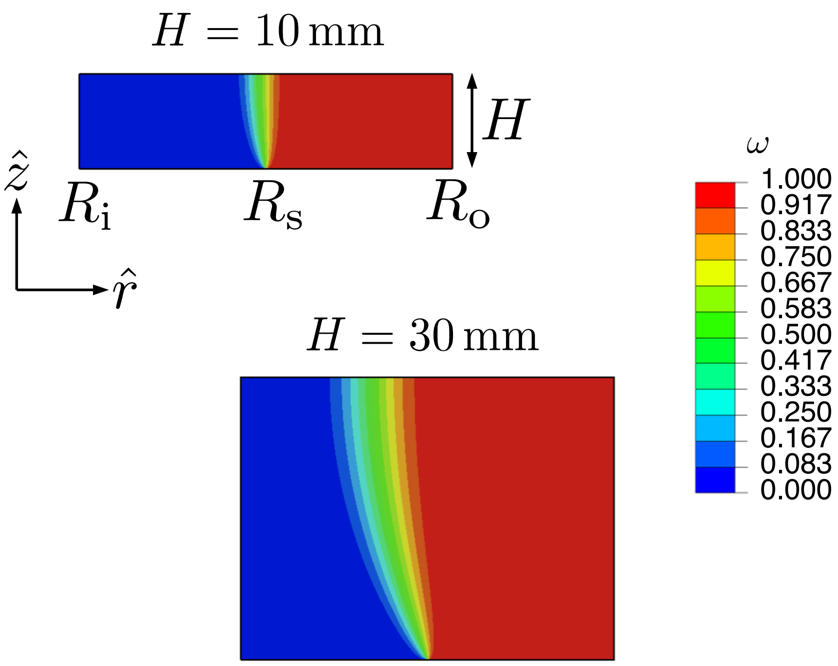
- Custom-wrote an FEM subroutine to simulate the nonlocal model via Abaqus User-Element (UEL).
- Performed many sims varying H and d.



Top view:

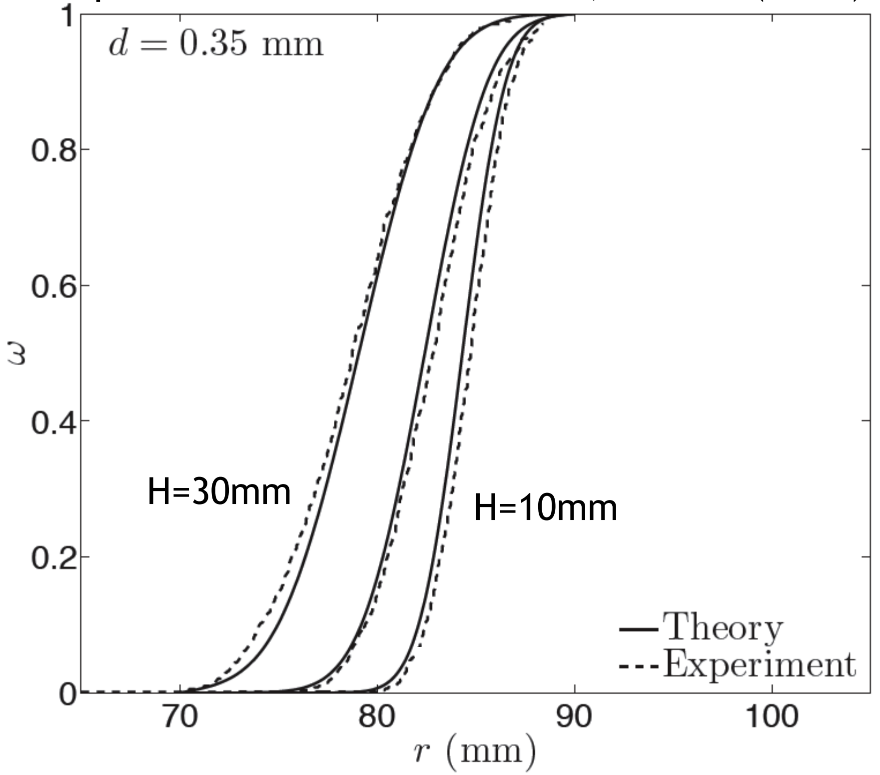


Simulated flow field in the r-z plane:



Flow field on top surface

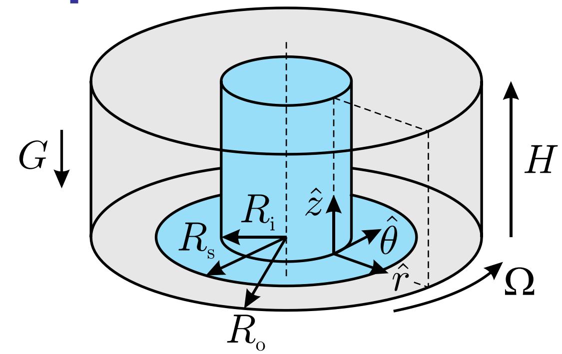
Exp data from: Fenistein et al, *Nature* (2003)



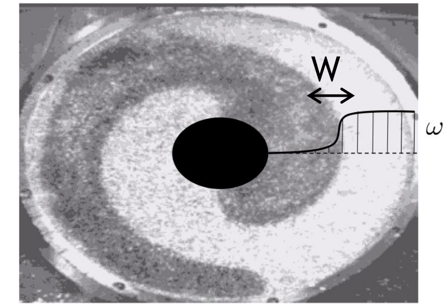
Henann and Kamrin, *PNAS* 2013

Nonlocal model in the split-bottom Couette cell

Define:
R_c = Location of shear-band center
W = Width of shear-band
$\lambda = (r-R_c)/W$ = Normalized position

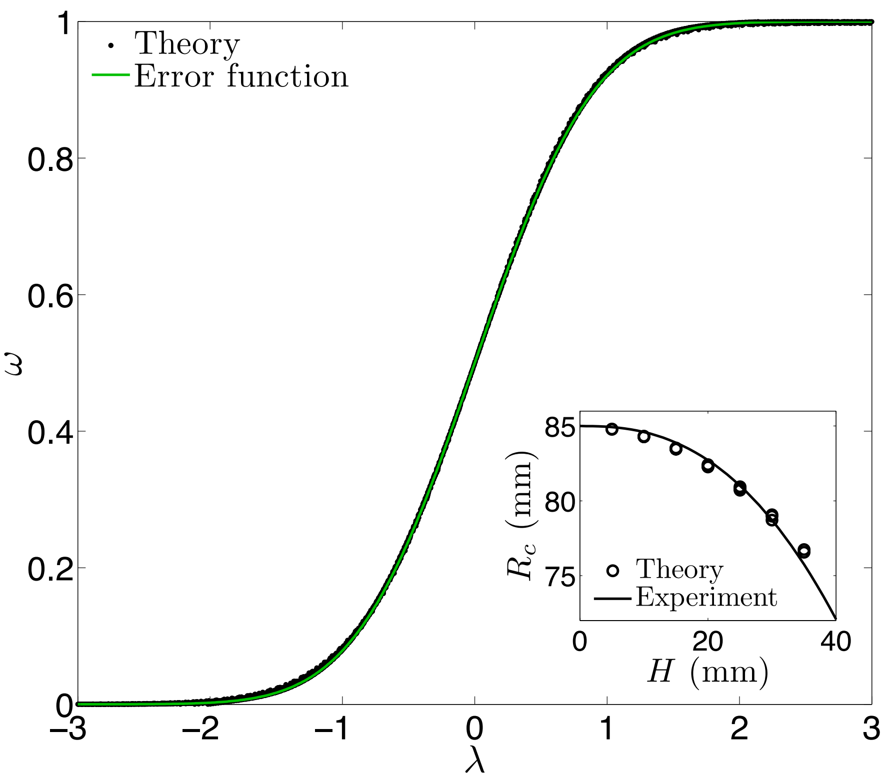


Top view:

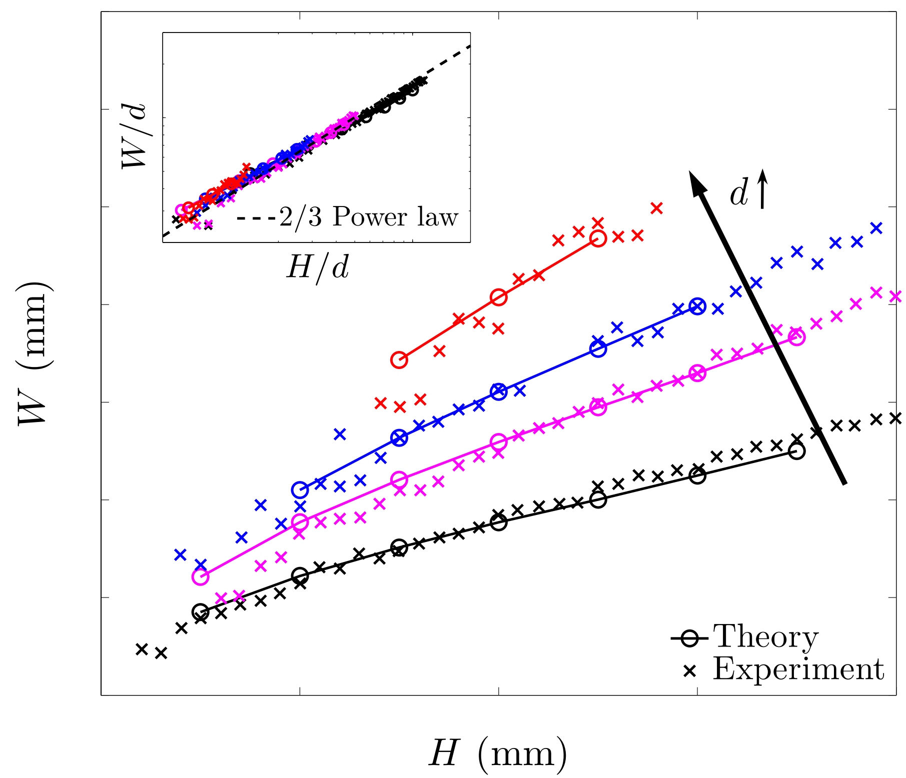


Surface flow results for all 22 combinations of H and d tried:

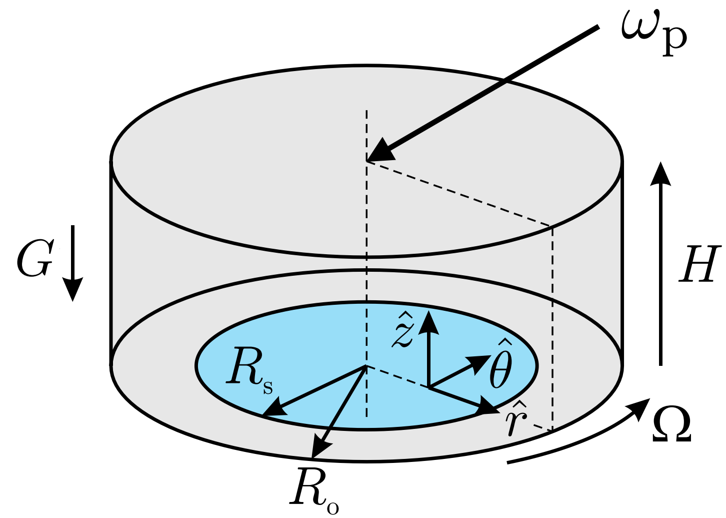
Collapse to error-function flow field



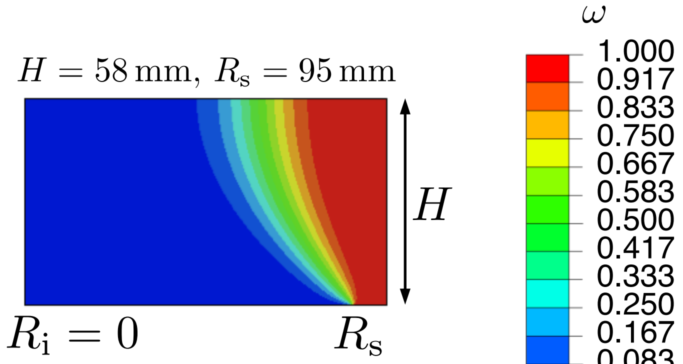
Non-diffusive scaling of W with H

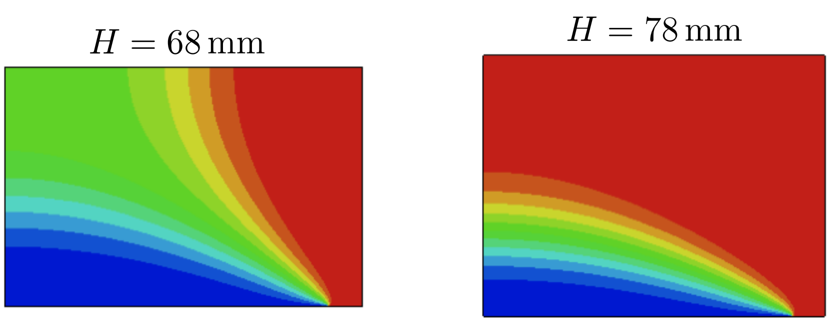


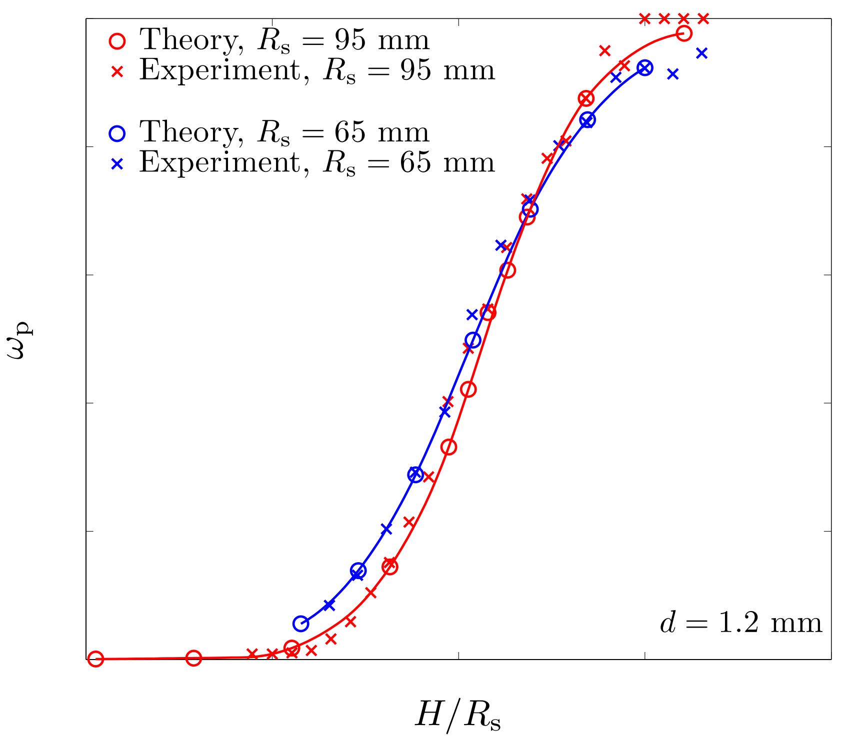
Remove the inner wall and let H increase



Rotation of the top-center point
Exp data from: Fenistein et al., *PRL* (2006)





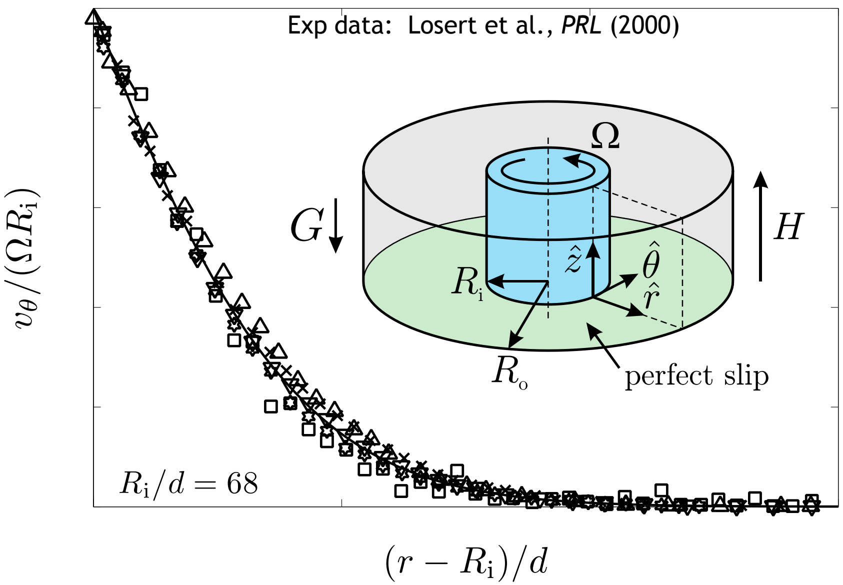


The nonlocal model correctly captures the transition from shallow to deep behavior.

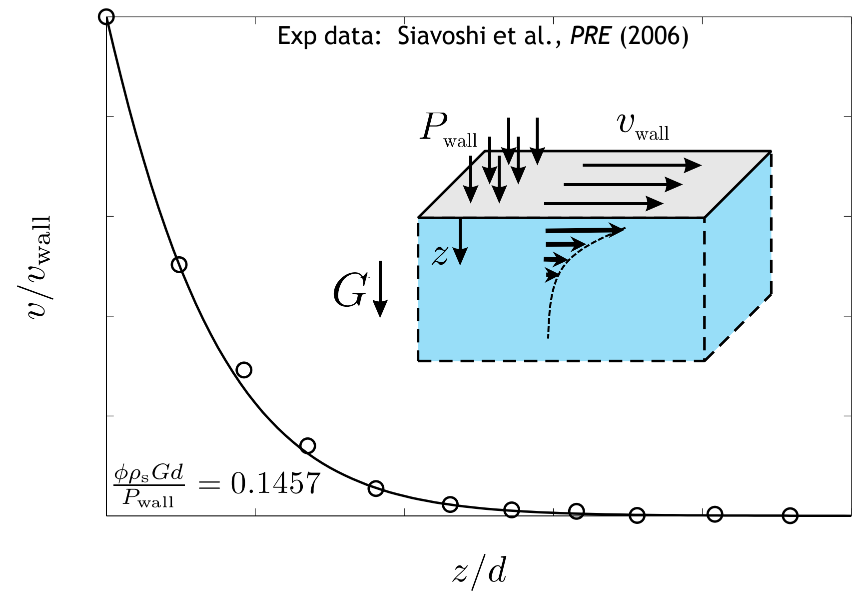
Henann and Kamrin, *PNAS* 2013

Existing 3D wall-shear data using glass beads

Annular shear flow:



Linear shear flow with gravity:



Same exact model and parameters used in split-bottom cases also captures these flows.

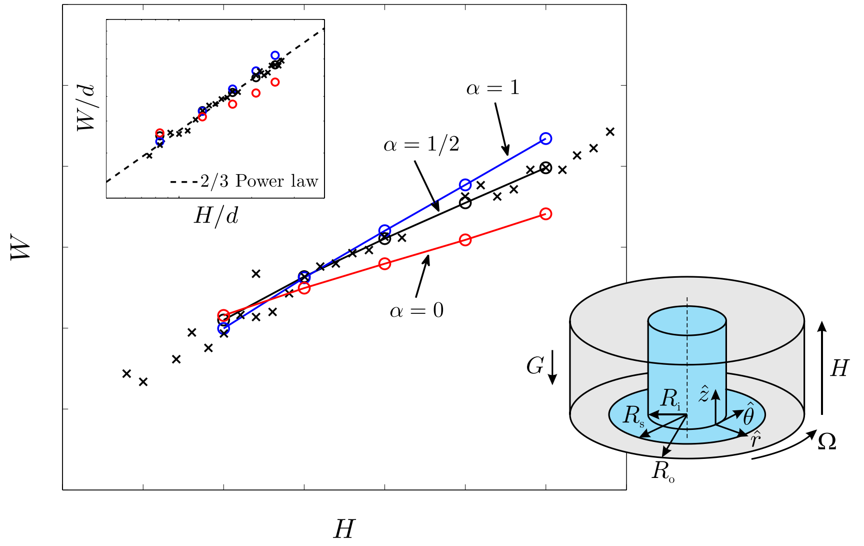
Henann and Kamrin, *PNAS* 2013

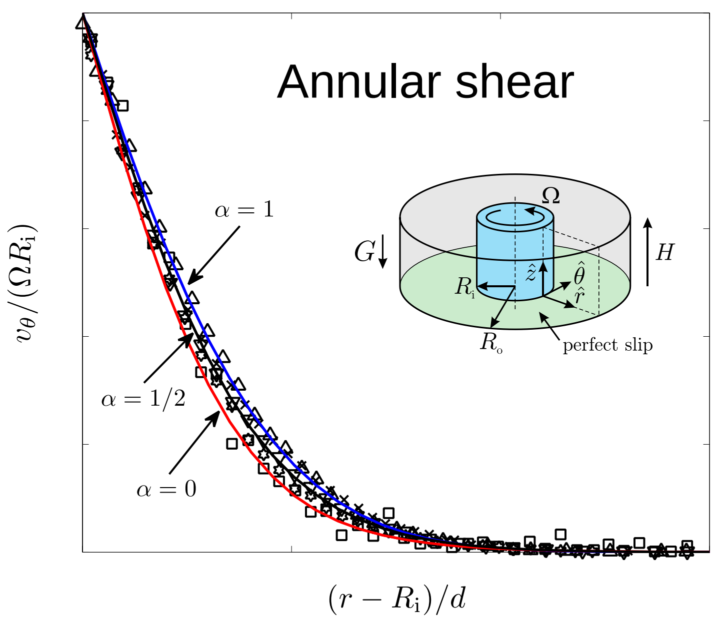
More on the cooperativity length

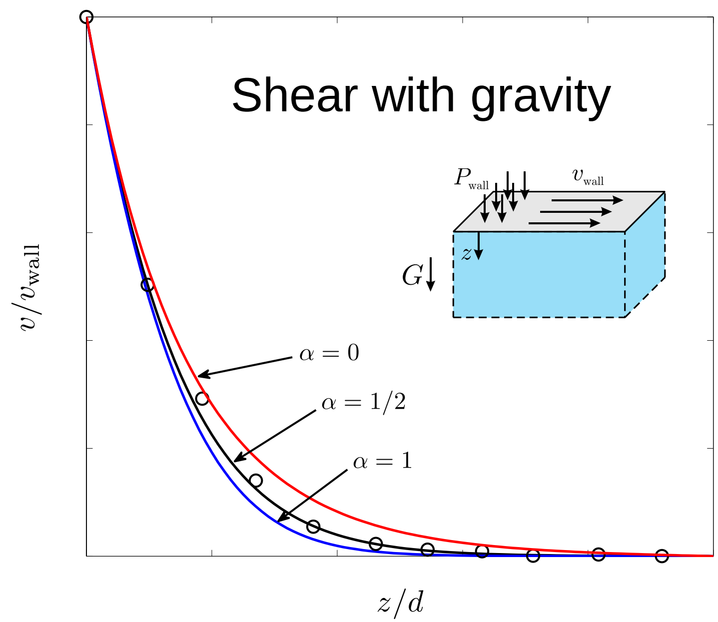
Suppose we try different power laws in the cooperativity length formula:

$$\xi(\mu) = \frac{A}{(|\mu - \mu_s|)^\alpha} d$$

Split-bottom cell





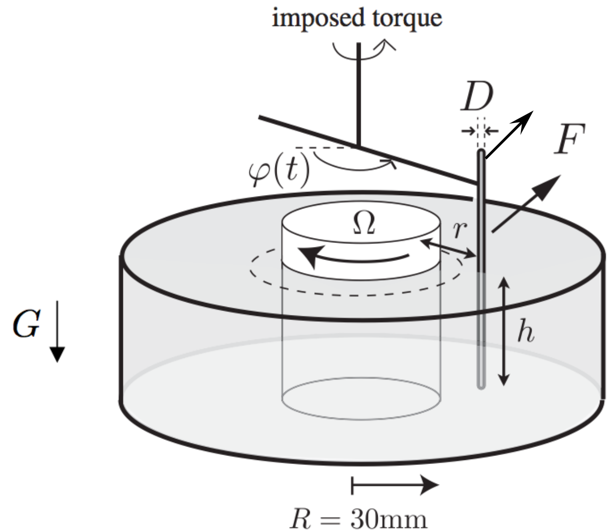


Secondary Rheology

Henann and Kamrin *PRL*, *(In Press)*

Experiment:

Reddy et. al PRL, 2009



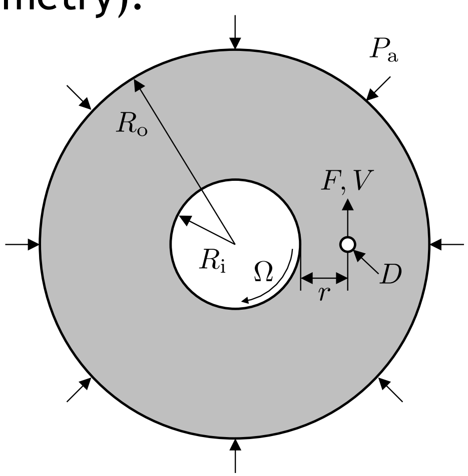
- When inner wall is stationary, rod has a yield force:

$$v_{creep} = 0 \quad \text{unless} \quad F > F_c$$

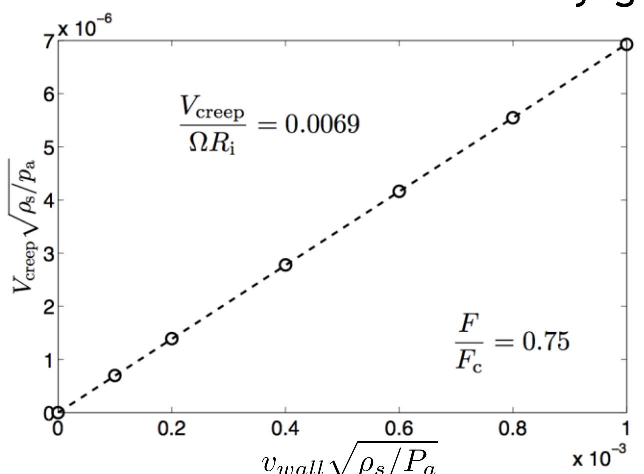
- When wall is moving, rod creeps for any F. Given r/d,

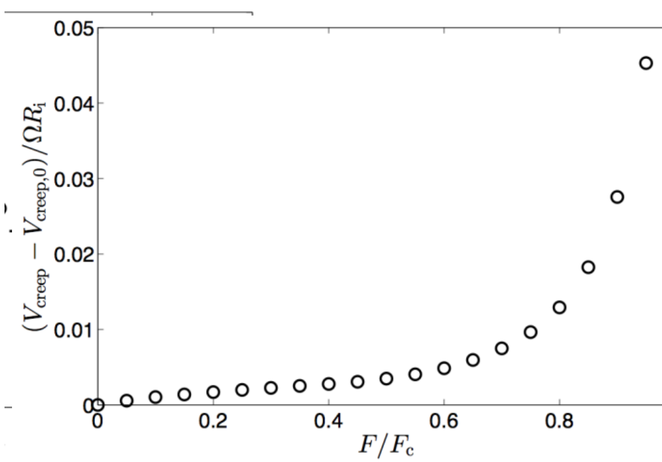
$$v_{creep} \approx C_1 v_{wall} \sinh(C_2 F)$$

Nonlocal model (Simplified 2D geometry):



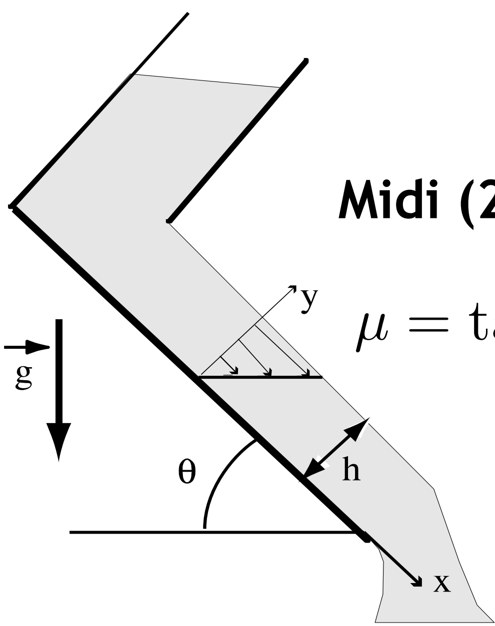
Stationary inner wall / Moving inner wall





H_{stop} Phenomenon: A different kind of particle-size dependence

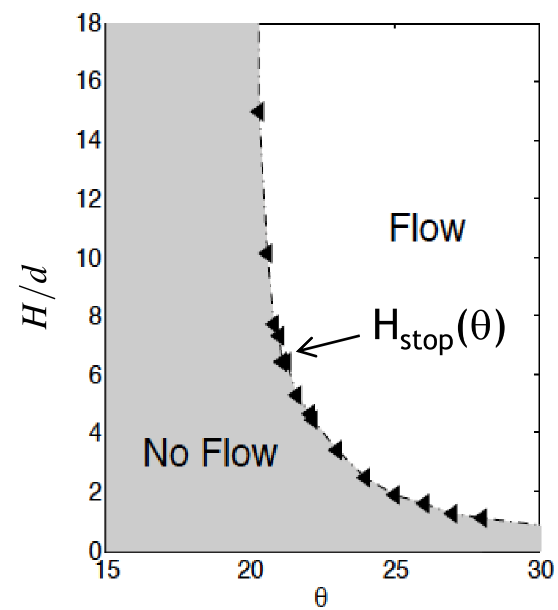
- Another famous granular size-effect is the "extra strengthening" granular layers gain when they are thin. Effect is easy to see in the inclined chute geometry.
- Produces so-called "H_{stop} curve".



Midi (2004)

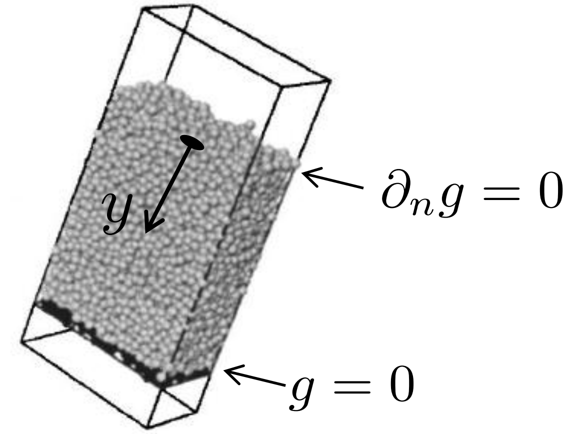
$$\mu = \tan\theta$$

Experiment
(glass beads, rough floor):

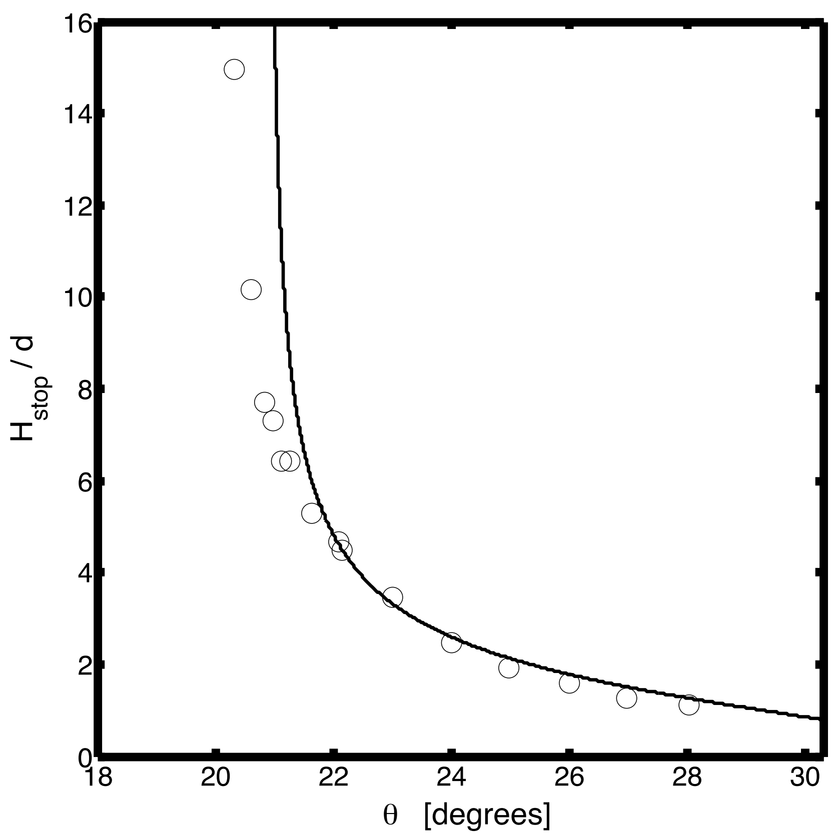


Pouliquen, Phys Fluids, 1999

H_{stop} phenomenon.
Stability of the no-flow solution



Let the fluidity be a small perturbation from the global g=0 solution and check stability of global no-flow solution.



— Theory

○ Experiment, glass beads (Pouliquen 1999)

Note: Same continuum parameters are used here that were used for split-bottom flow field predictions.

Kamrin and Henann *Soft Matter, (In Press)*

Concluding remarks

- The nonlocal fluidity model is capable of describing multiple incarnations of nonlocal phenomenology in granular media.
- Model has captured complex phenomena untenable to local rheological models, such as:
  1. Grain-size dependent shear features in steady flows
  2. Secondary rheology
  3. “Smaller-is-stronger” H_{stop} size-effect
- Work in progress: Predicting silo flow rates? More conclusive theoretical understanding?

Relevant papers:

K. Kamrin and G. Koval, PRL (2012), D. Henann and K. Kamrin, PNAS (2013),
K. Kamrin and G. Koval, JCPM (2014), D. Henann and K. Kamrin, IJP (2014),
K. Kamrin and D. Henann, Soft Matter (In press),
D. Henann and K. Kamrin, PRL (In press)

Massachusetts Institute of Technology
NSF
Schlumberger

Flow and flow-stoppage in drainage geometries

Big open questio
Can you predict l
hourglass will flo

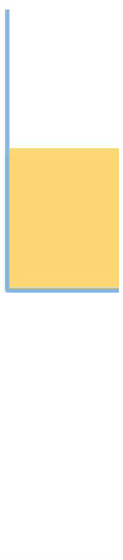
constants

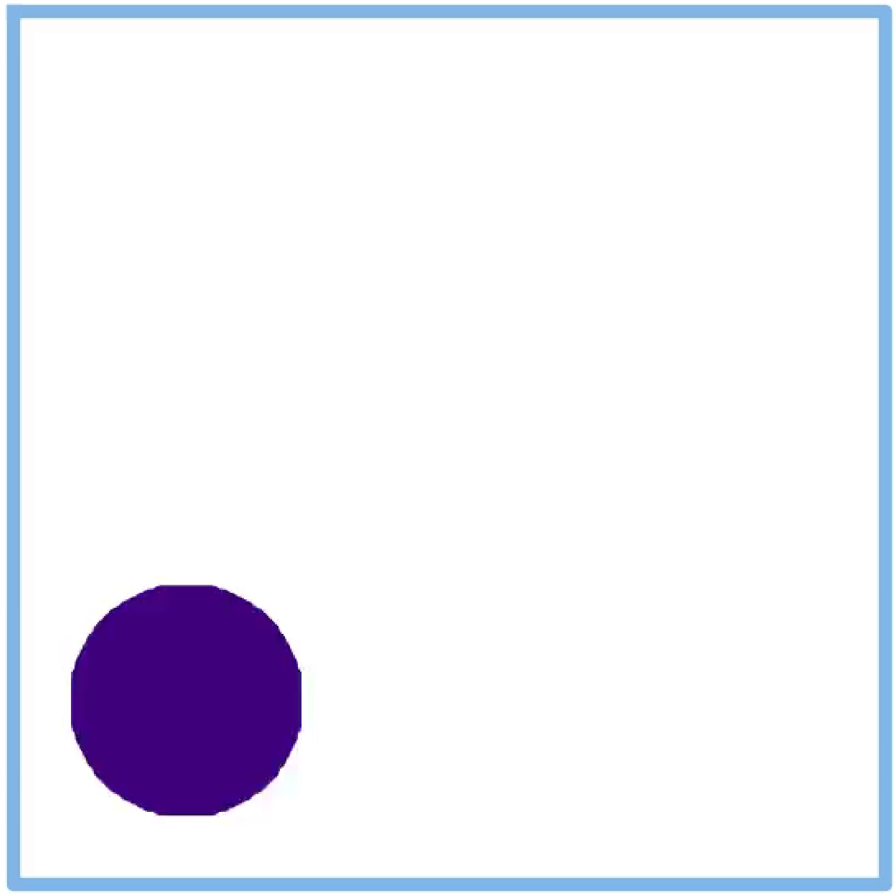
"Beverloo Correlati
(in quasi-2D):

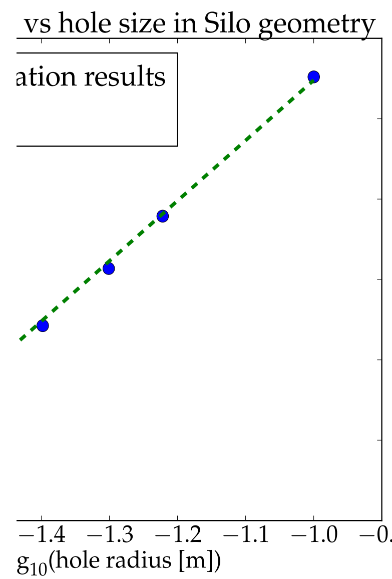
$$- kd)^{3/2}$$

Continuum simulation (Material Point Method) of *local* law:

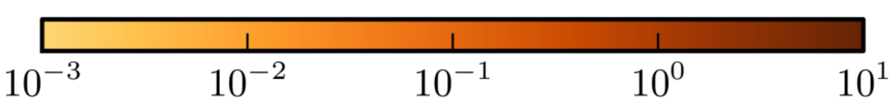
[c/o Sachith Dunatunga]







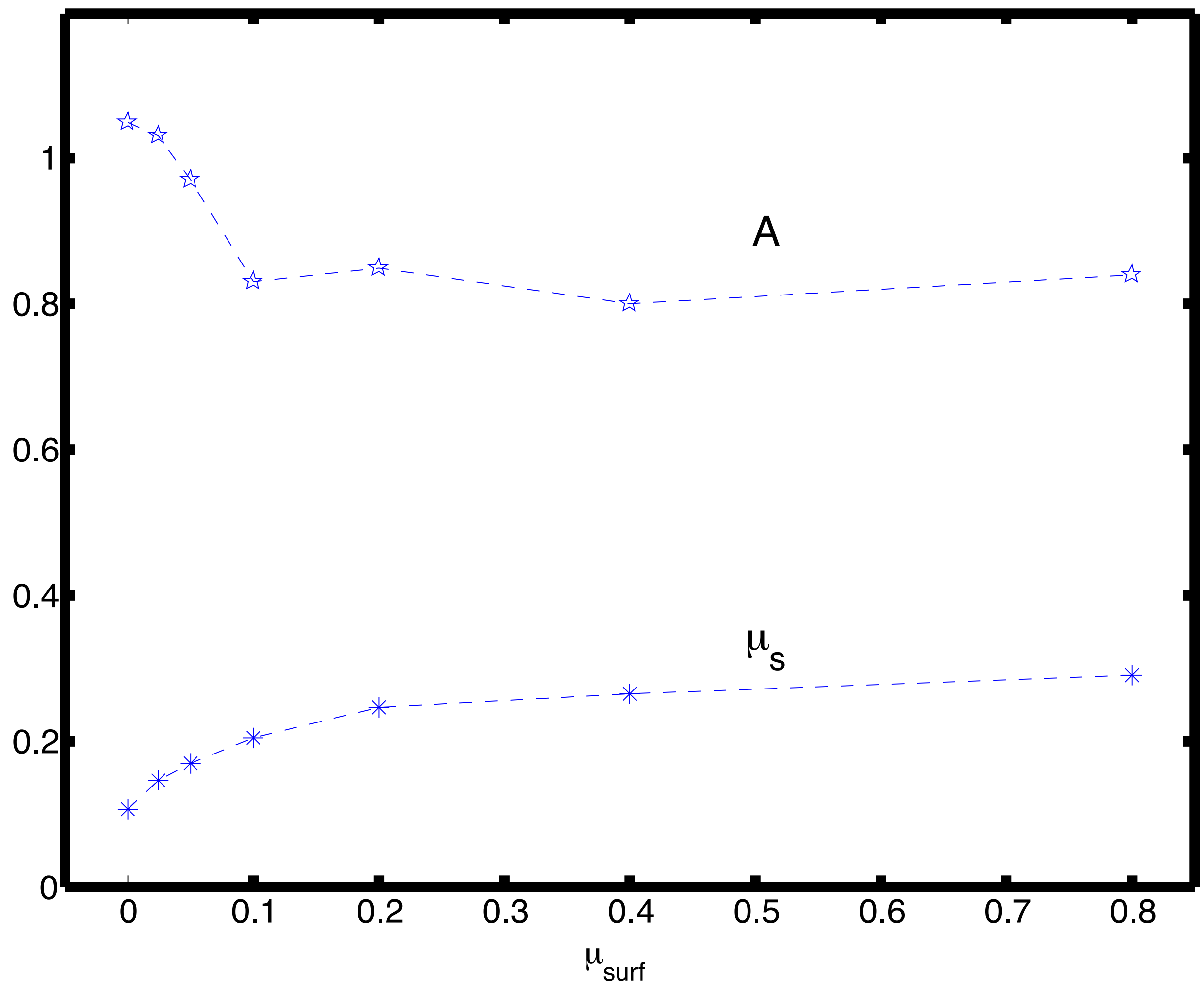
Plastic strain-rate:



Results in progress: Toward a formula for A

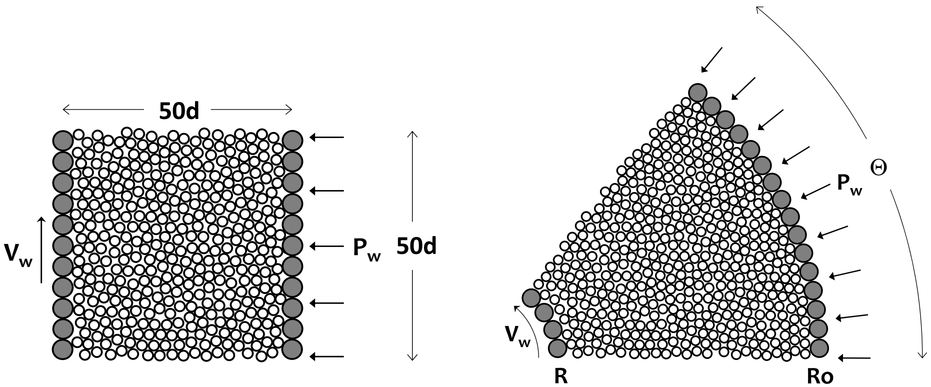
New tests with DEM disks:

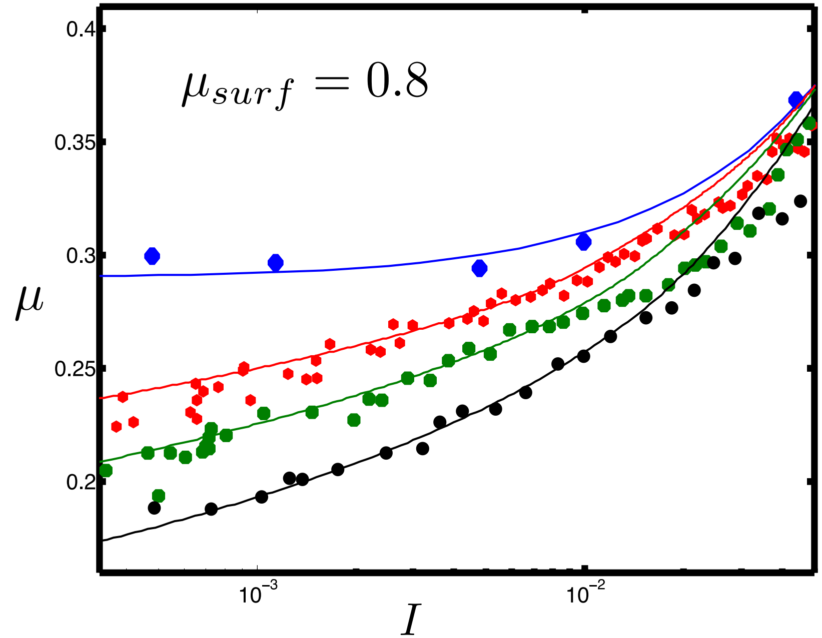
Dependence of continuum model variables on the surface friction of the grains, μ_{surf}

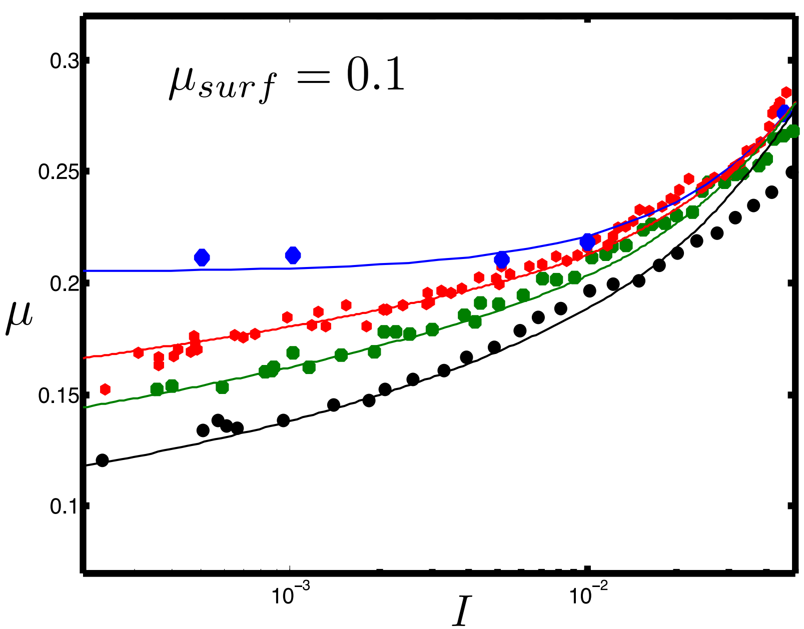


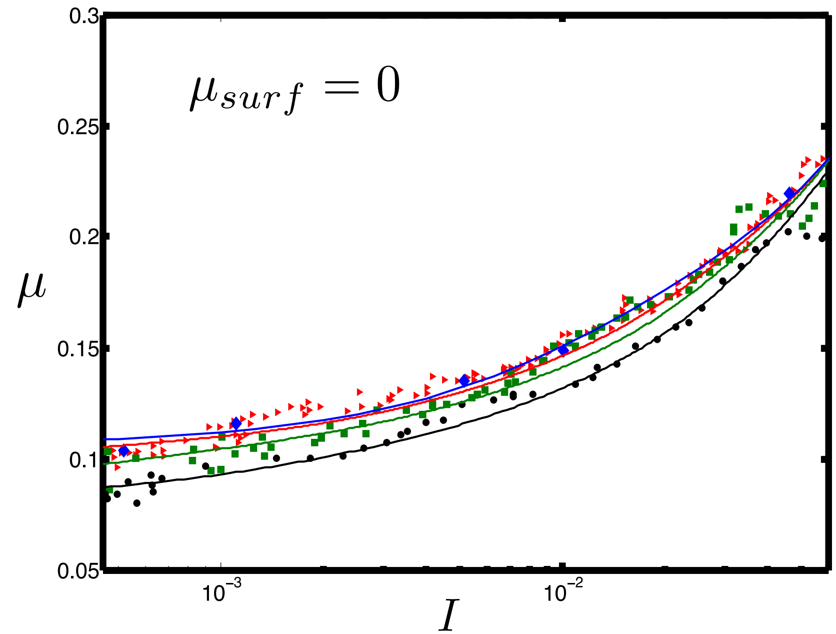
Kamrin and Koval, Comp. Part. Mech. (2014)

Local law dominates when particle surface friction vanishes?

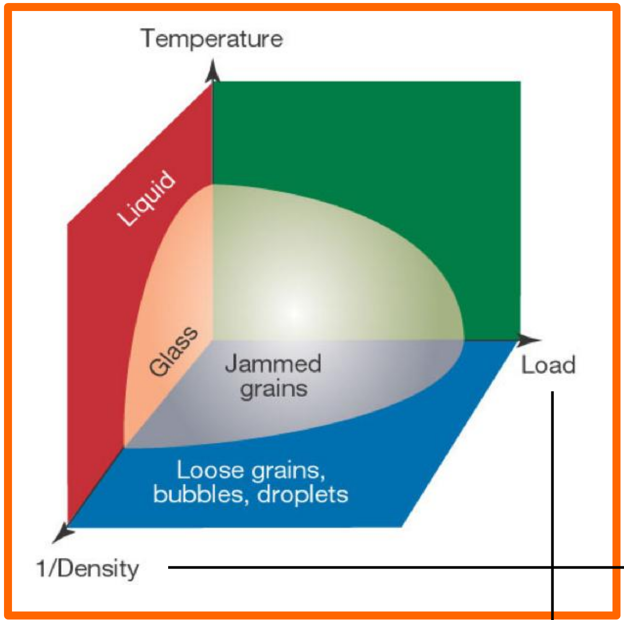




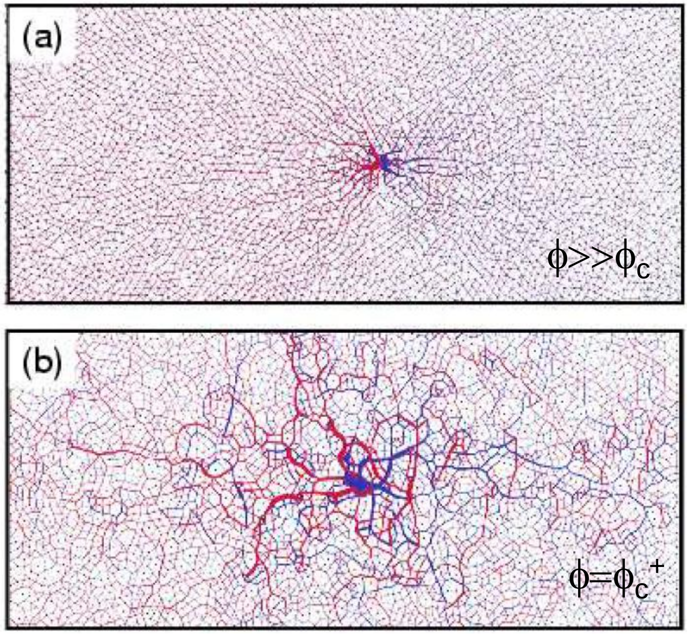




The Diverging Length-Scale

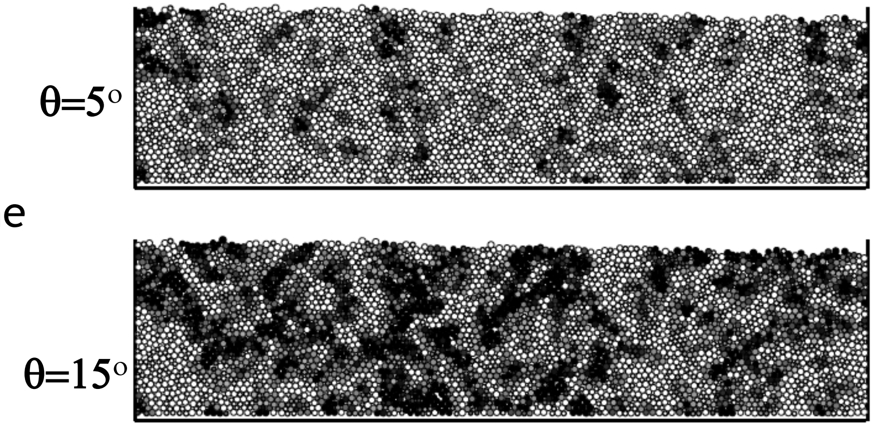


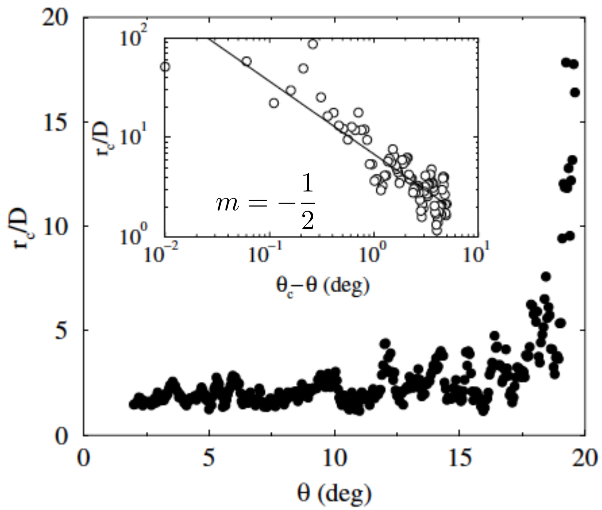
M. van Hecke,
Cond. Mat. (2009)



Staron et al,
PRL (2002)

Pre-avalanche zone sizes for inclined plane flow:



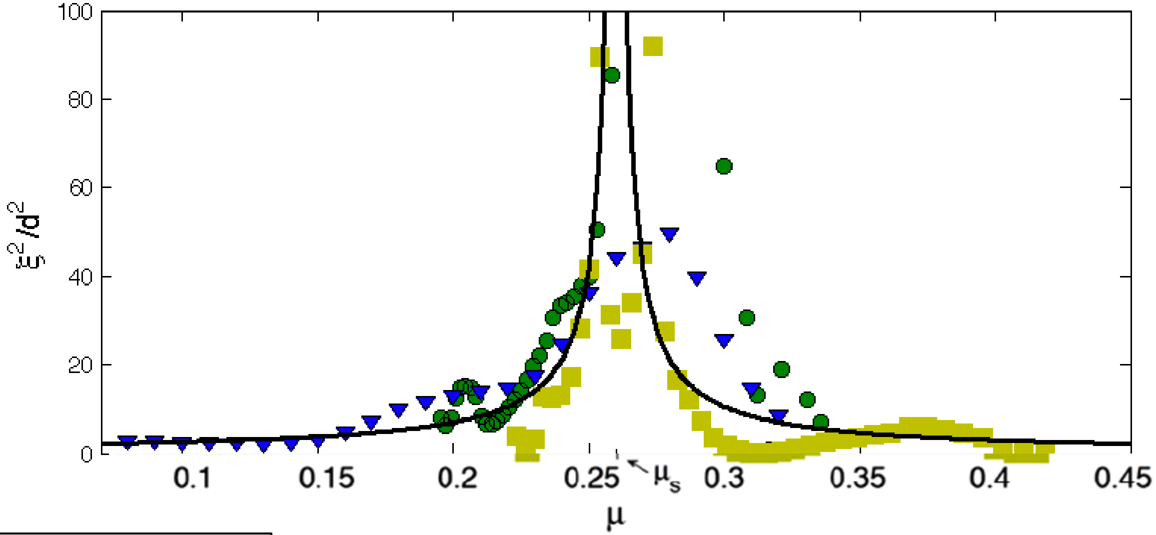


Cooperativity Length

Theoretical form:

$$\xi(\mu) = \frac{A}{\sqrt{|\mu - \mu_s|}} d$$

Direct tests: Steady-flow DEM data in 3 geometries
(annular shear, vertical chute, shear w/ gravity):

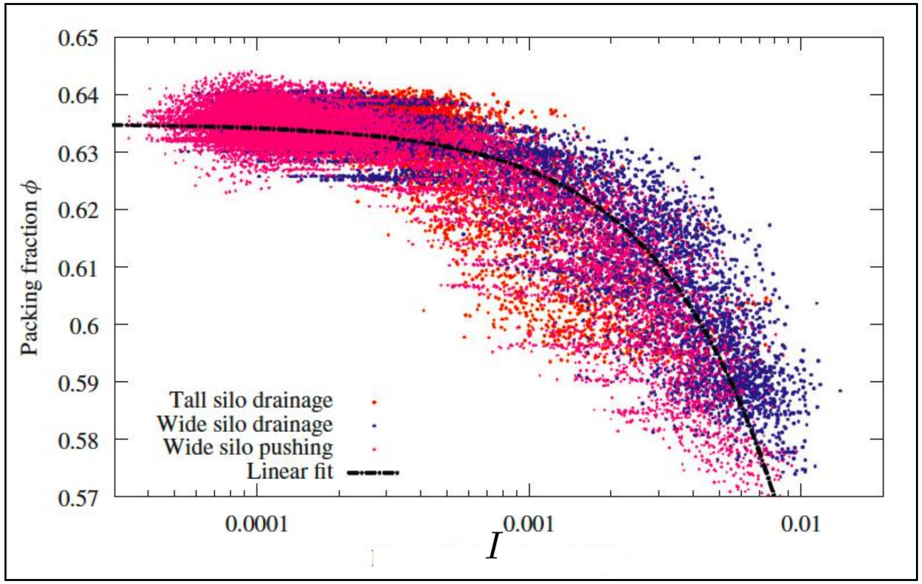


Extract » from DEM using: $g = g_{\mathrm{loc}}(\mu, P) + \xi^2 \nabla^2 g$

For our 2D DEM disks, we find: **A=0.70**

Only new material constant is A, the *nonlocal amplitude*. Local law constants all carry over.

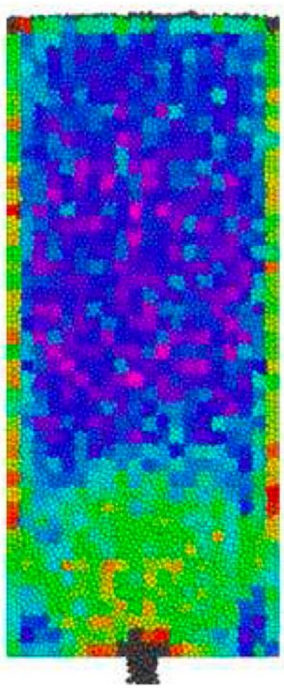
Developed packing fraction on 5d scale



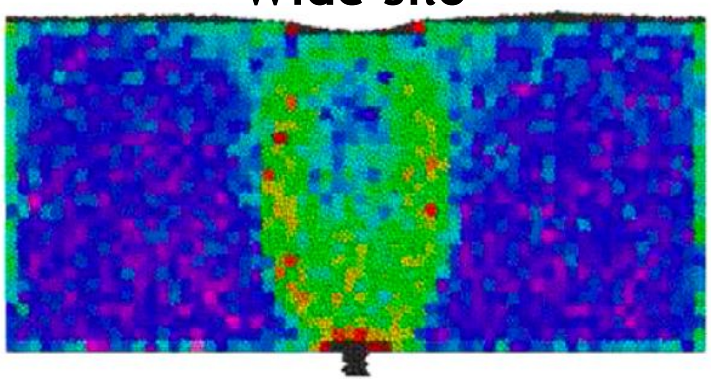
$$\phi \cong \phi(I)$$

Rycroft, Kamrin, and Bazant (JMPS 2009)

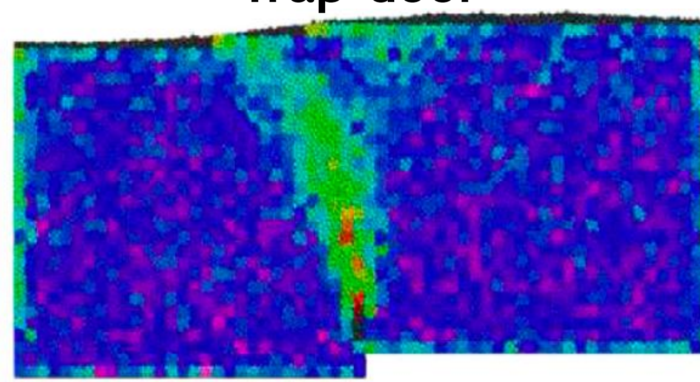
Tall silo

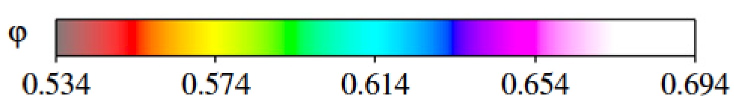


Wide silo



Trap-door



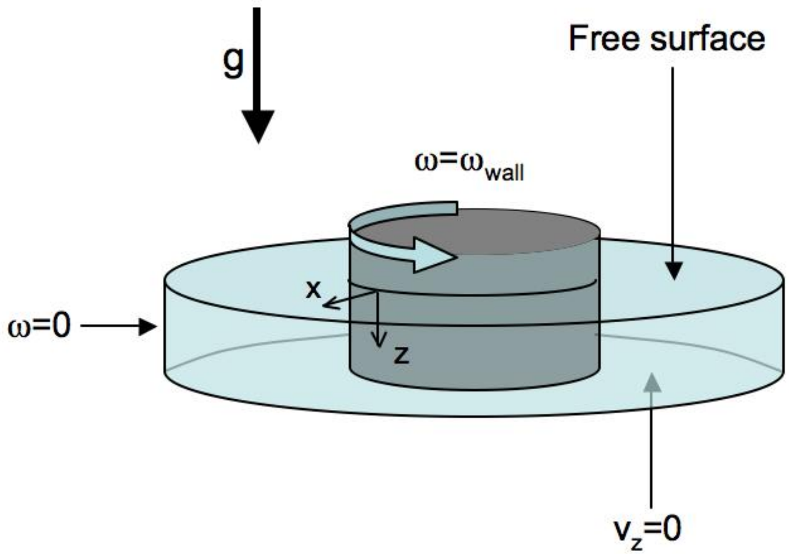


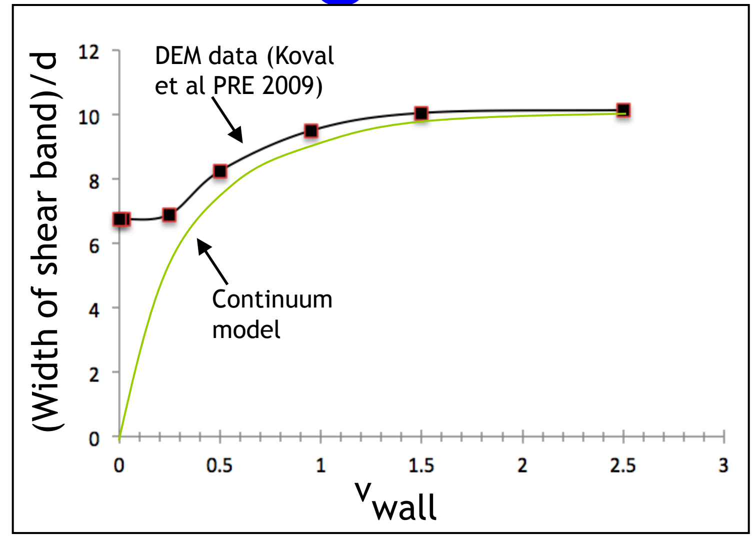
Stage 2: Fixing the problem. Accounting for size-effects

Past nonlocal granular flow approaches:

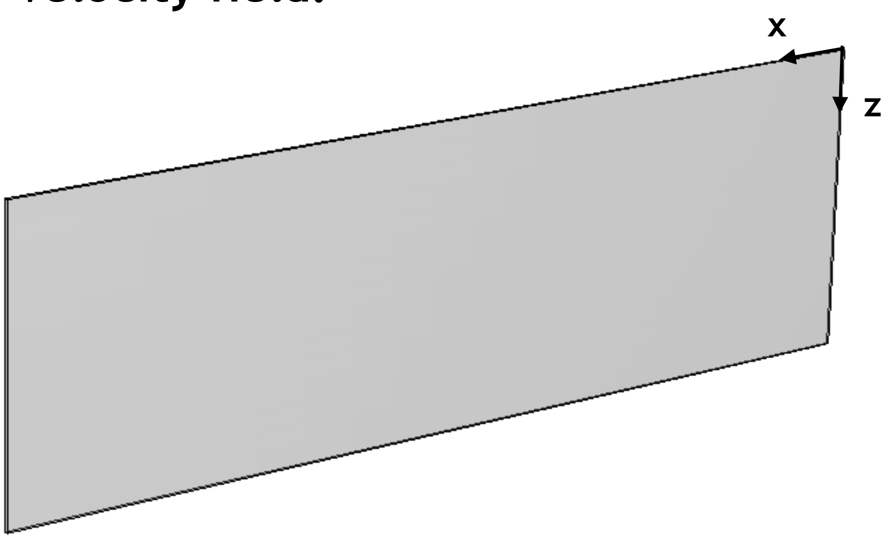
- Self-activated process (Pouliquen & Forterre 2009)
- Cosserat continuum (Mohan et.al 2002)
- Partial fluidization (Aronson & Tsimring 2001)

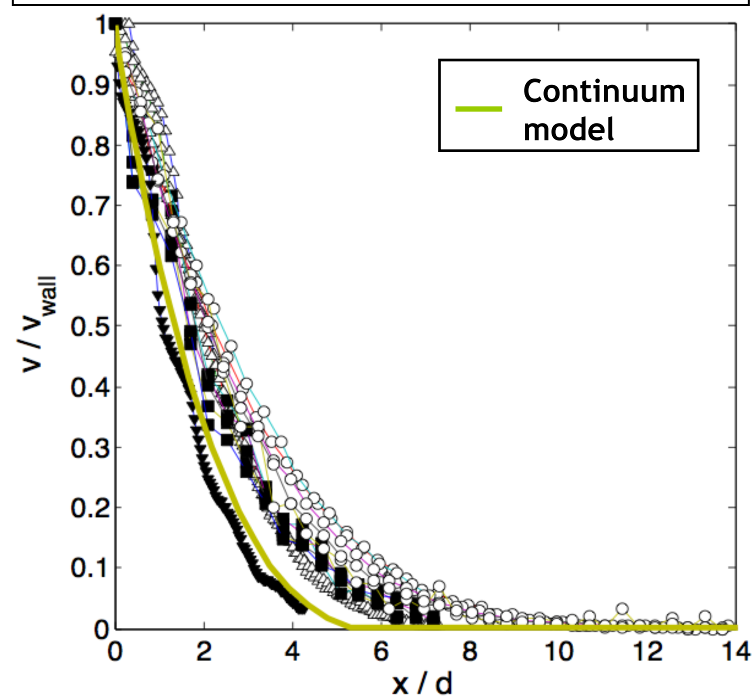
Flow not spreading enough





Velocity field:

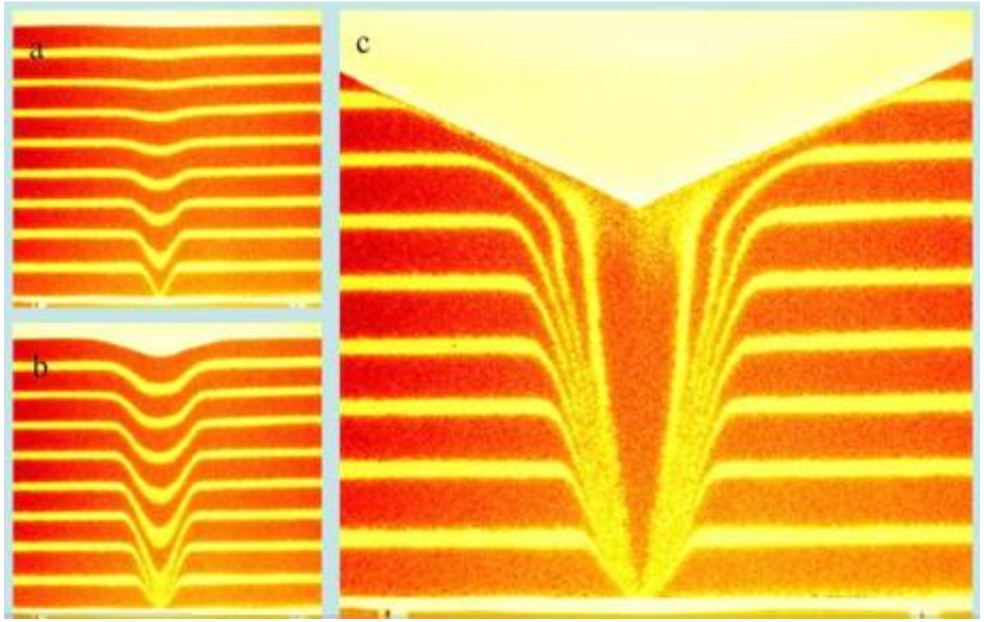




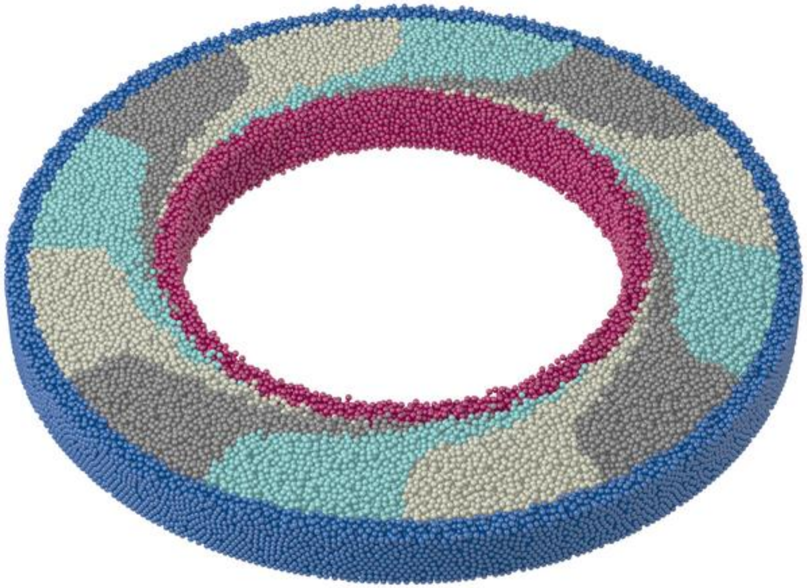
G.D.R. Midi, *Euro. Phys. Journ. E*, (2004)

Stage 1: Local modeling.

Static zones and flowing zones.



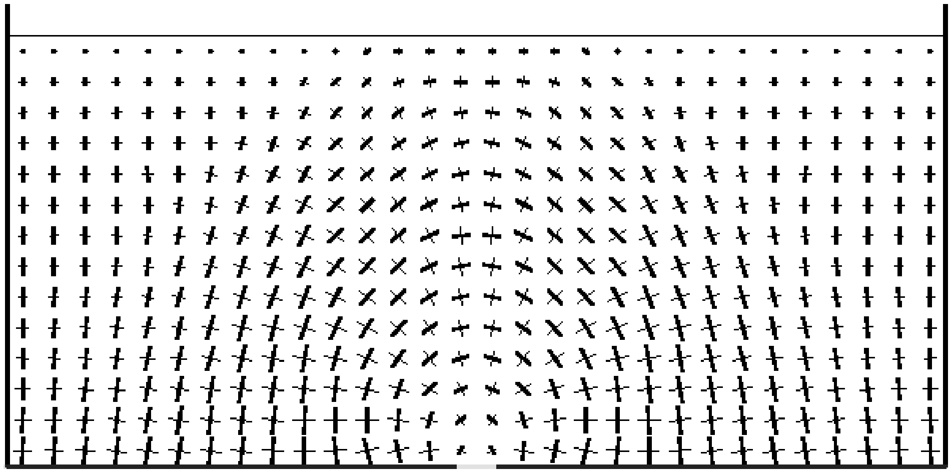
Samadani & Kudrolli (2002)
(Experiment)



Kamrin, Rycroft, & Bazant (2007)
(Discrete Element Method [DEM])

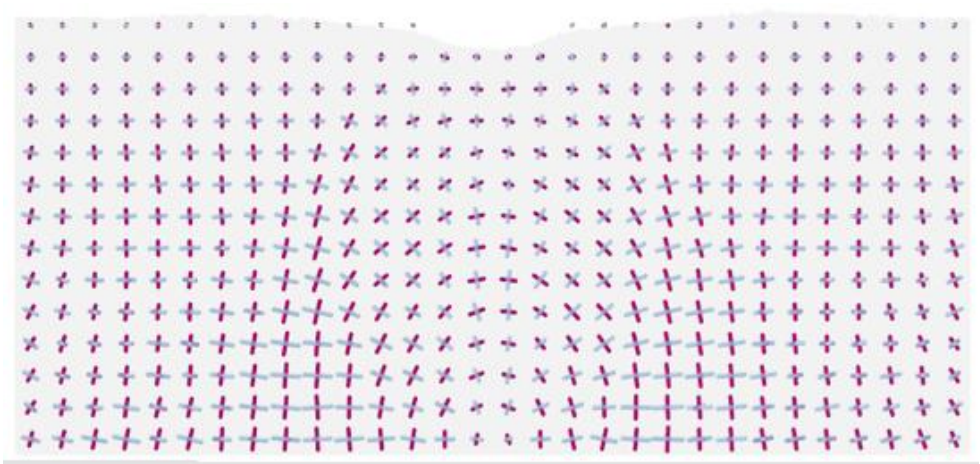
Cauchy stress field

Eigendirections of stress:



Theory

Kamrin, *Int. J Plasticity*, (2010)



DEM simulation

Rycroft, Kamrin, and Bazant (JMPS 2009)

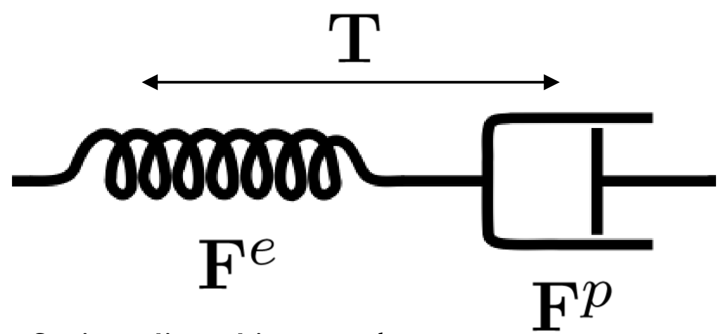
Constitutive Process

Conservation of linear and angular momentum require

$$\nabla \cdot \mathbf{T} + \rho \mathbf{g} = \rho \frac{D\mathbf{v}}{Dt} \qquad \mathbf{T} = \mathbf{T}^T$$

To close the system:

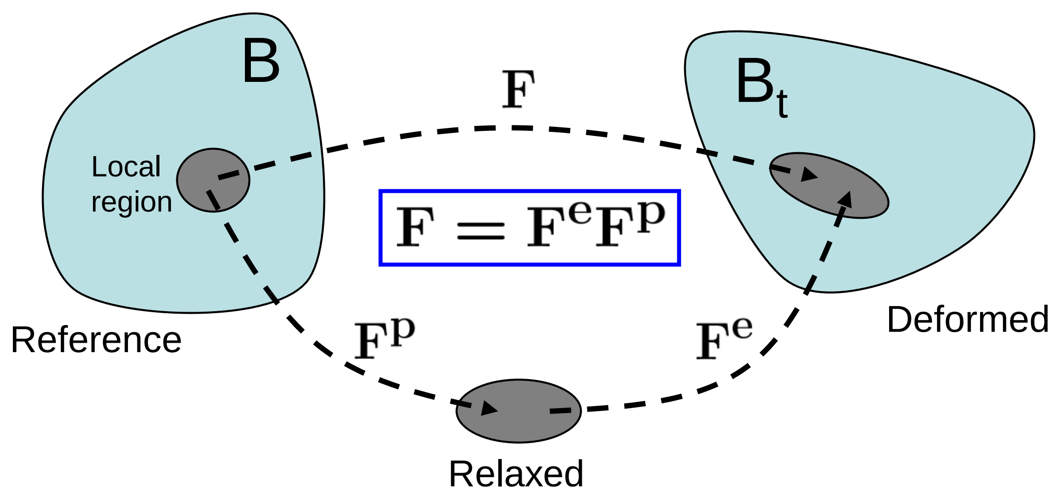
(1D picture)



Spring: Jiang-Liu granular elasticity law [Jiang and Liu, *PRL*, 2003]

Dashpot: Inertial rheology [Jop et. al, *Nature*, 2006]

(3D version, Kroner-Lee decomposition)



Results
FEM Simulations

- Simulations performed using ABAQUS/Explicit finite element package.
- Constitutive model applied as a user material subroutine (VUMAT).
- All flow parameters in the model are from the papers of Jop et. al. (2006) and Jiang and Liu (2003).

Continuing on

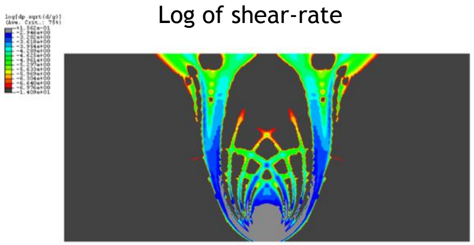
• *More 3D tests:* Need to check the relation in more geometries with less symmetry; e.g. silo/hopper flow. This will help determine if there are any non-negligible anisotropic size-effects.

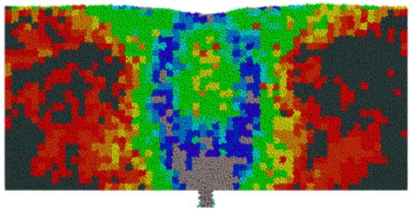
• *Transient strengthening/dilation:* With well-developed flow response quantified, should include a model for the transient behavior. Has a long history in the soils literature (i.e. Critical State Theory).

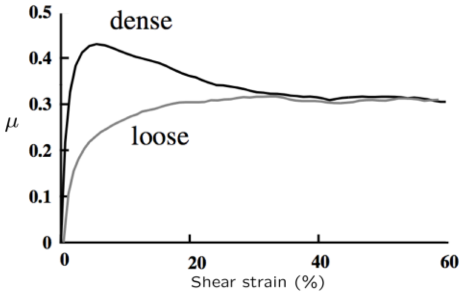
• *Improved understanding of fluidity BC's:* Sims have used transparent fluidity BC's, but better theory is needed to inform BC assignment.

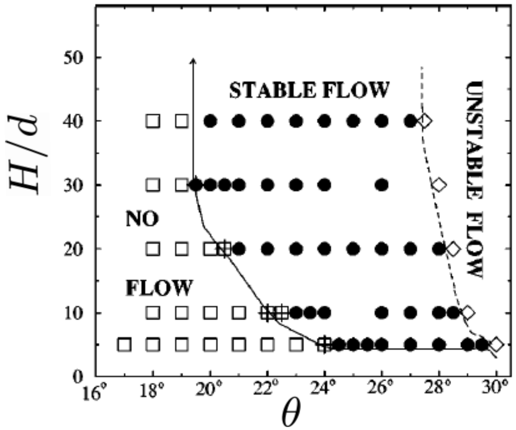
• *Dynamic nonlocality:* So far have studied well-developed flows. There are known nonlocal dynamic effects in thin layers, such as thickness-dependent startup angle for inclined chutes.

• *More concrete theoretical understanding:* Statistical arguments have been proposed, but a rigorous continuum mechanical derivation is lacking. Possibly an order-parameter in the free-energy function?

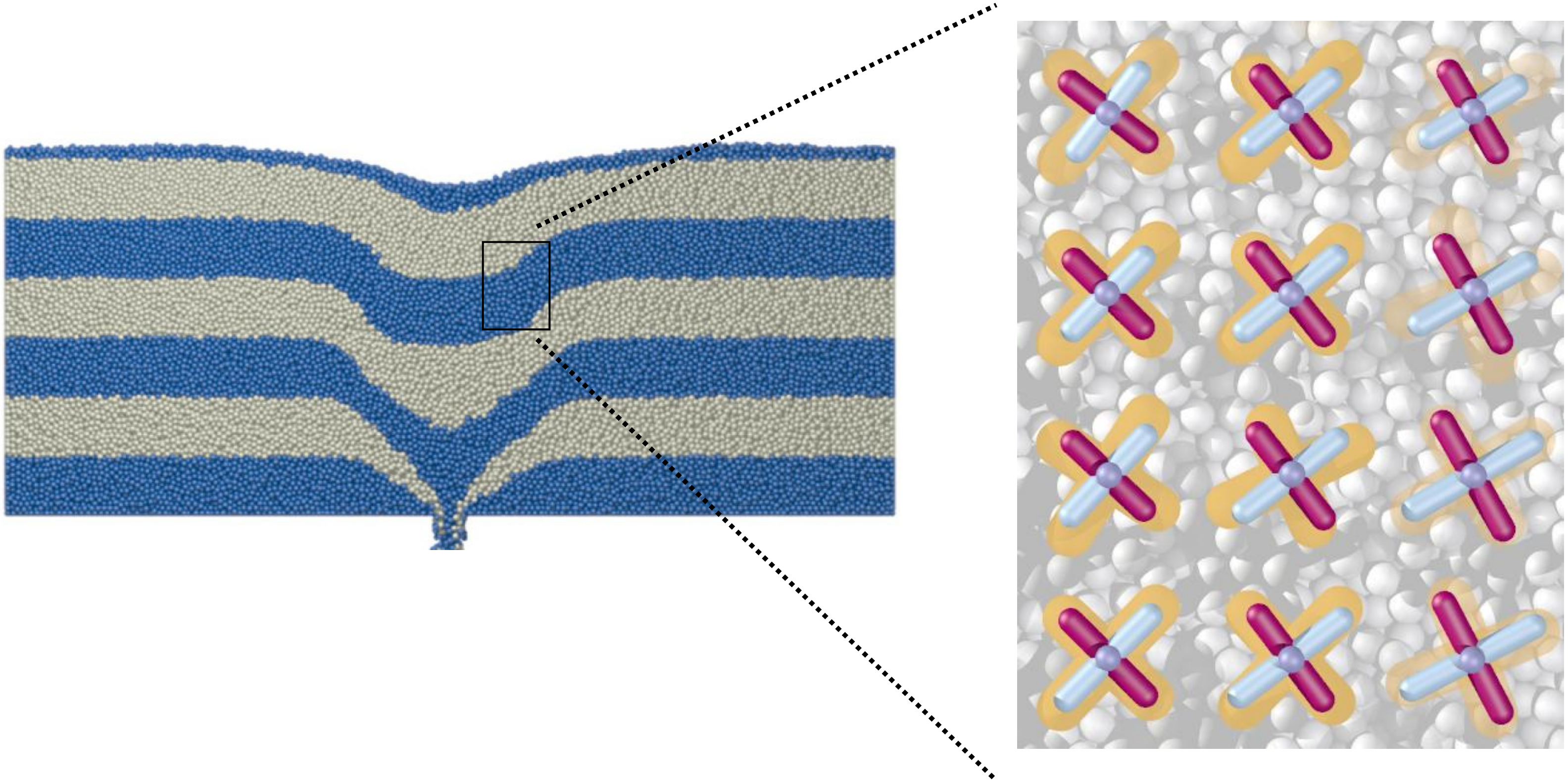








Granular Continuum?



(Blue crosshairs = Cauchy stress evecs)

(Orange crosshairs = Strain-rate evecs)

Rycroft, Kamrin, and Bazant (JMPS 2009)

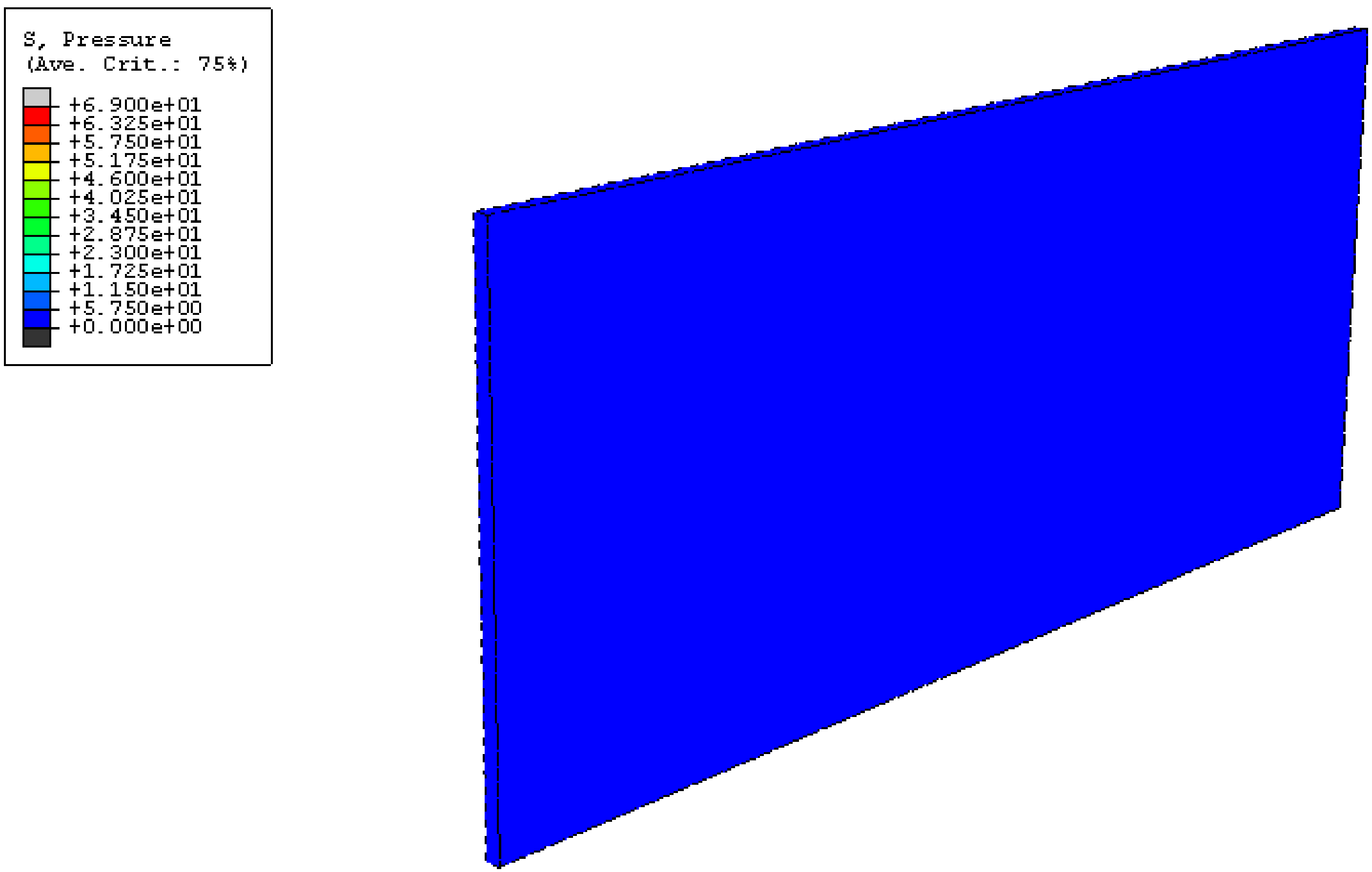
Some evidence for continuum treatment:

- Space-averaged fields vary smoothly when coarse-grained at width 5d
- Certain deterministic relationships between fields appear in elements 5d wide

Rough Inclined Chute

Since this model does have elasticity, we always know all stresses, even in regions of 0 plastic flow.

Pressure field:



Mathematical Form

"Mandel stress"
↓

$$\mathbf{T} = \mathbf{R^e M R^{e}}^T/(\det\mathbf{F})$$

$$\mathbf{M} = \frac{\partial\psi(\mathbf{E^e})}{\partial\mathbf{E^e}}$$

$$\mathbf{D^p} = d^p(\mathbf{M})\frac{\mathbf{M}_0}{\tau}$$

Enforce:
- **Frame indifference**
- **2nd law of thermodynamics**
- **Coaxiality of flow and stress***

Definitions:

$$\mathbf{R^e} \equiv \mathbf{F^e}\left(\mathbf{F^{e}}^T\mathbf{F^e}\right)^{-1/2}$$

$$\mathbf{M}_0 \equiv \mathbf{M} - (1/3)(\mathrm{tr}\mathbf{M})\mathbf{1}$$

$$\tau \equiv \sqrt{(\mathbf{M}_0 : \mathbf{M}_0)/2}$$

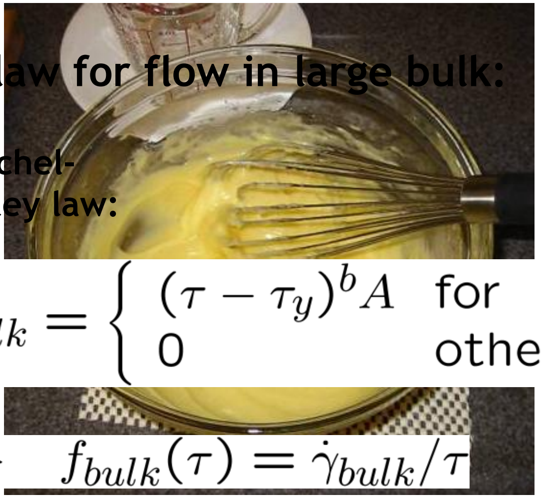
To close, must choose isotropic scalar functions ψ and d^p with d^p ¸ 0.

Nonlocal Fluidity Model

Existing theory for emulsions

[Goyon et al, Nature (2007), L. Bocquet et al, PRL (2009)]

Define: $\text{Fluidity} \equiv f = \dot{\gamma}/\tau$ (Inverse viscosity)



Local law for flow in large bulk:

Herschel-Bulkley law:

$$\dot{\gamma}_{bulk} = \begin{cases} (\tau - \tau_y)^b A & \text{for} \quad \tau > \tau_y \\ 0 & \text{otherwise} \end{cases}$$

$$\rightarrow \quad f_{bulk}(\tau) = \dot{\gamma}_{bulk}/\tau$$

Nonlocal add-on:

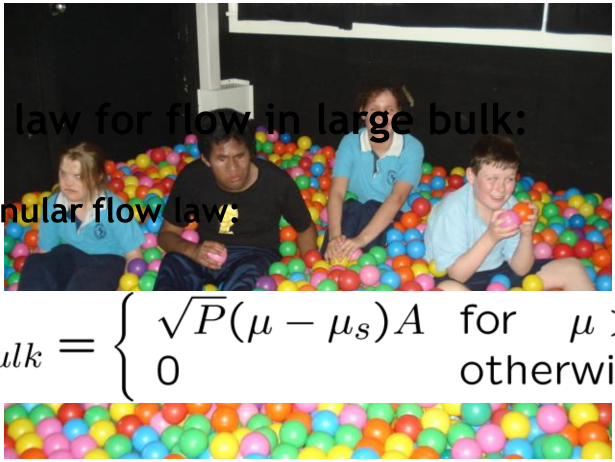
$$f = f_{bulk}(\tau)$$

0 when $\tau < \tau_y$

Micro-level length-scale proportional to d

Extend to granular media

Define: $\text{Fluidity} \equiv f = \dot{\gamma}/\mu$



Local law for flow in large bulk:

Granular flow law:

$$\dot{\gamma}_{bulk} = \begin{cases} \sqrt{P}(\mu - \mu_s)A & \text{for} \quad \mu > \mu_s \\ 0 & \text{otherwise} \end{cases}$$

$$\rightarrow \quad f_{bulk}(\mu, P) = \dot{\gamma}_{bulk}/\mu$$

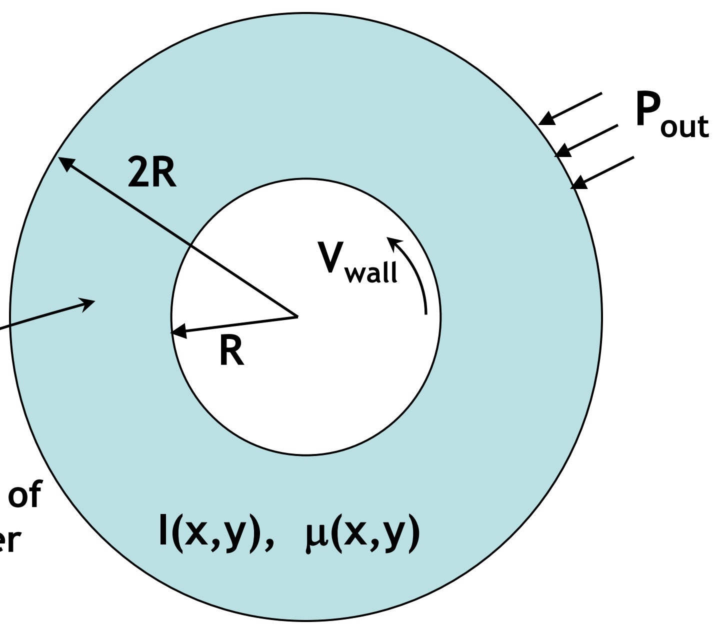
Nonlocal law:

$$f = f_{bulk}(\mu, P) + \xi(\mu)^2 \nabla^2 f$$

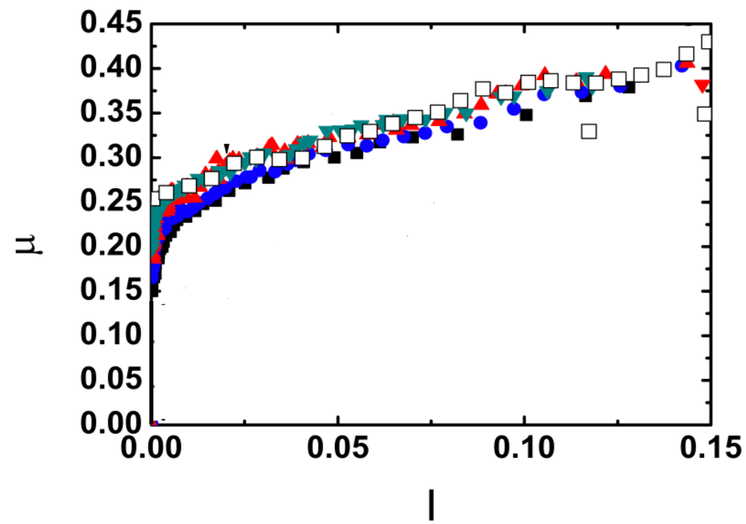
Quantifying Steady Slow Flow (2D)

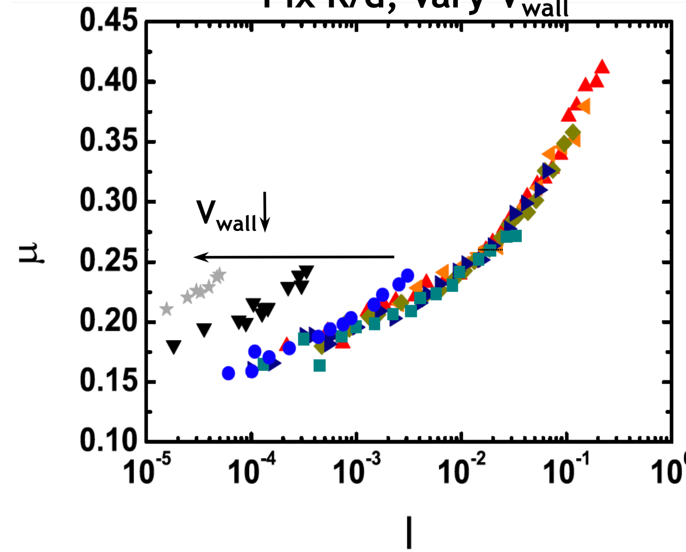
- Inertial number: $I = \dot{\gamma} d \sqrt{\frac{\rho_s}{P}}$
- Inertial flow rheology says: $\mu = f(I)$

Koval et al. (PRE 2009)



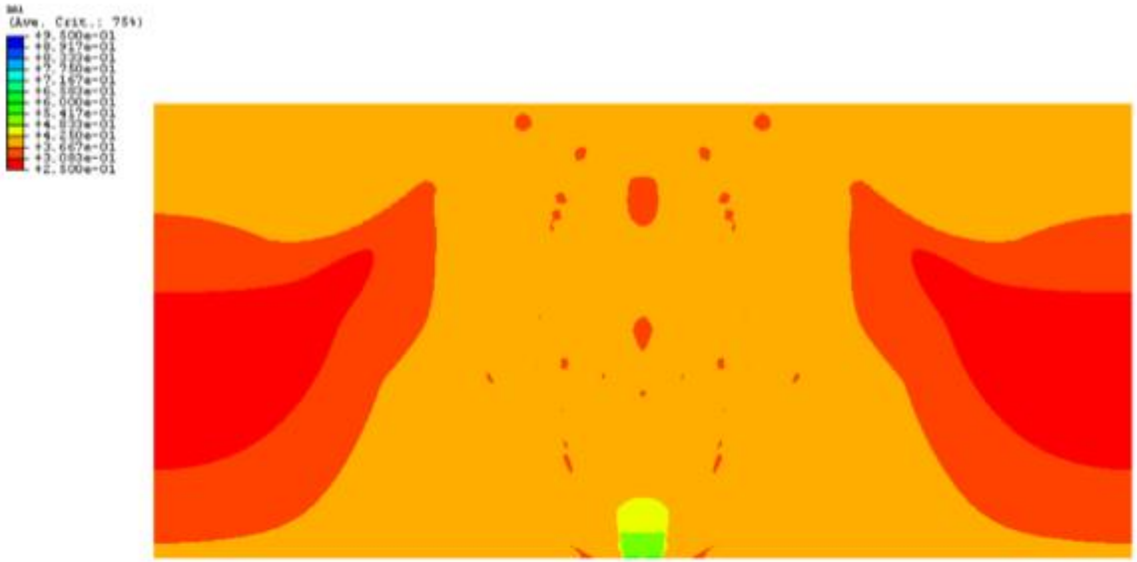
d and P_{out} held fixed





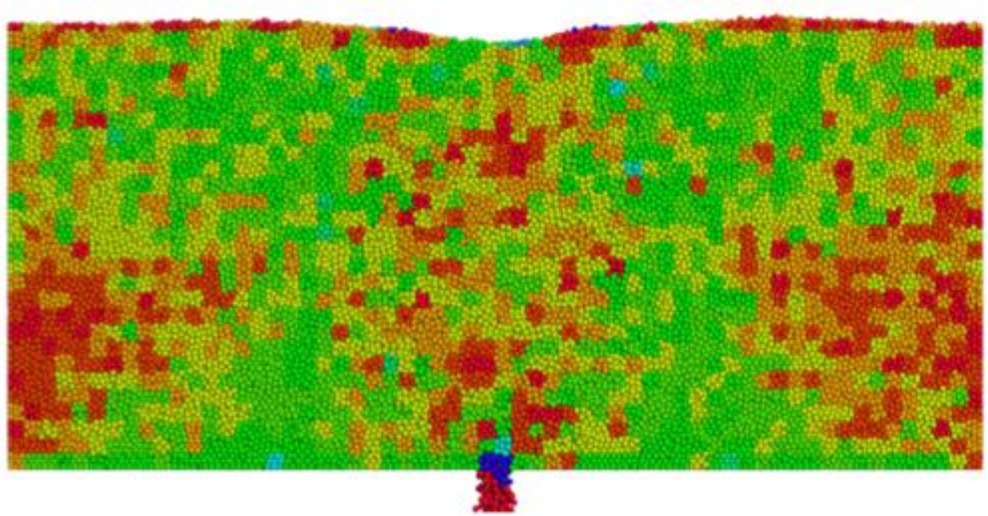
Flow *and* Stresses

Stress ratio (μ):



Theory

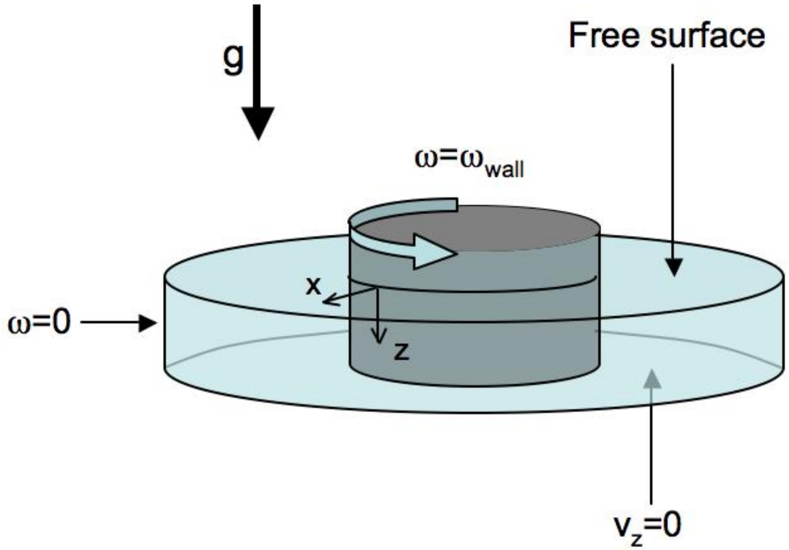
Kamrin, *Int. J Plasticity*, (2010)



DEM simulation

Rycroft, Kamrin, and Bazant (JMPS 2009)

Flow not spreading enough

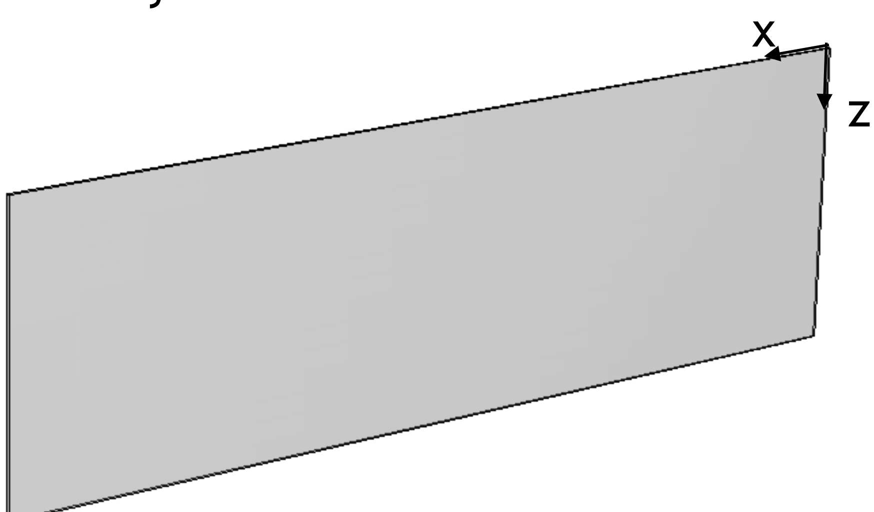


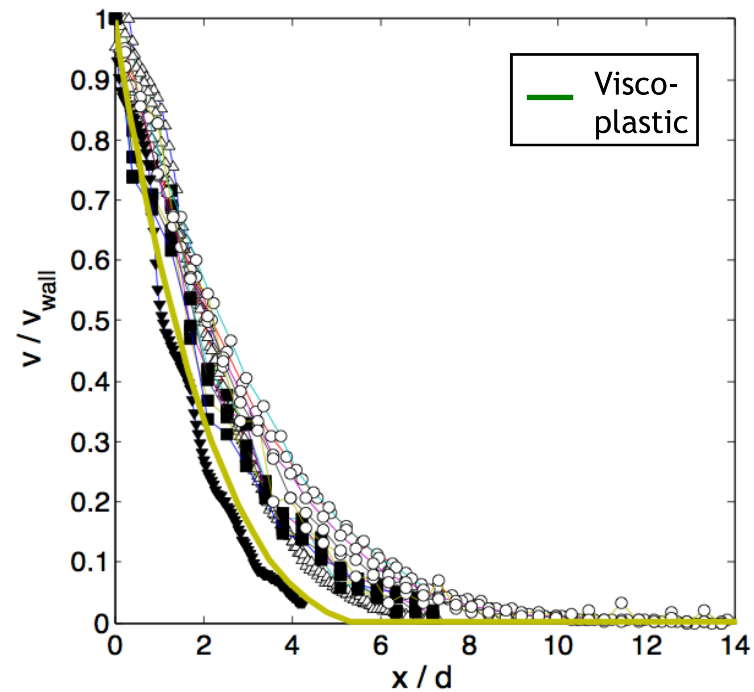
Elasto-plastic model predicts:

1. Shear band at inner wall
2. Flow roughly invariant in the vertical direction
3. Sharp flow/no-flow interface.

Known experimental data verify 1) and 2) but contradict 3).

Velocity field:





G.D.R. Midi, *Euro. Phys. Journ. E*, (2004)