Logarithmic scaling and logarithmic correlations in critical phenomena

Victor Gurarie

KITP, December 2014

In this talk

- RG and logarithmic scaling
- Examples: 3D random bond Ising model and 2D percolation
- Symplectic fermions, the determinant of a Laplacian and conformal field theory with the central charge c=-2
- Quenched disorder, 2D random walks, percolation and conformal field theory with the central charge c=0

Critical phenomena

Critical phenomena

From Wikipedia, the free encyclopedia

In physics, **critical phenomena** is the collective name associated with the physics of critical points. Most of them stem from the divergence of the correlation length, but also the dynamics slows down. Critical phenomena include scaling relations among different quantities, power-law divergences of some quantities (such as the magnetic susceptibility in the ferromagnetic phase transition) described by critical exponents, universality, fractal behaviour, ergodicity breaking. Critical phenomena take place in second order phase transition, although not exclusively.

Critical phenomena

Critical phenomena

From Wikipedia, the free encyclopedia

In physics, **critical phenomena** is the collective name associated with the physics of critical points. Most of them stem from the divergence of the correlation length, but also the dynamics slows down. Critical phenomena include scaling relations among different quantities, power-law divergences of some quantities (such as the magnetic susceptibility in the ferromagnetic phase transition) described by critical exponents, universality, fractal behaviour, ergodicity breaking. Critical phenomena take place in second order phase transition, although not exclusively.

$$Z = \sum_{\sigma=\pm 1} \exp\left(K \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \sigma_{\mathbf{r}'}\sigma_{\mathbf{r}''}\right)$$ Ising model

Low temperature phase $K > K^*$

$$\langle \sigma_{\mathbf{r}} \rangle = M > 0$$

High temperature phase $K < K^*$

$$\langle \sigma_{\mathbf{r}} \rangle = 0 \qquad \langle \sigma_{\mathbf{r}_1}\sigma_{\mathbf{r}_2} \rangle \sim e^{-\frac{|\mathbf{r}_1 - \mathbf{r}_2|}{\xi}}$$

$$\xi \sim \frac{1}{(K^* - K)^{\nu}}$$

Critical point $K = K^*$

$$\langle \sigma_{\mathbf{r}_1}\sigma_{\mathbf{r}_2} \rangle \sim \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|^{d-2+\eta}}$$

critical exponents

RG group and logarithmic scaling

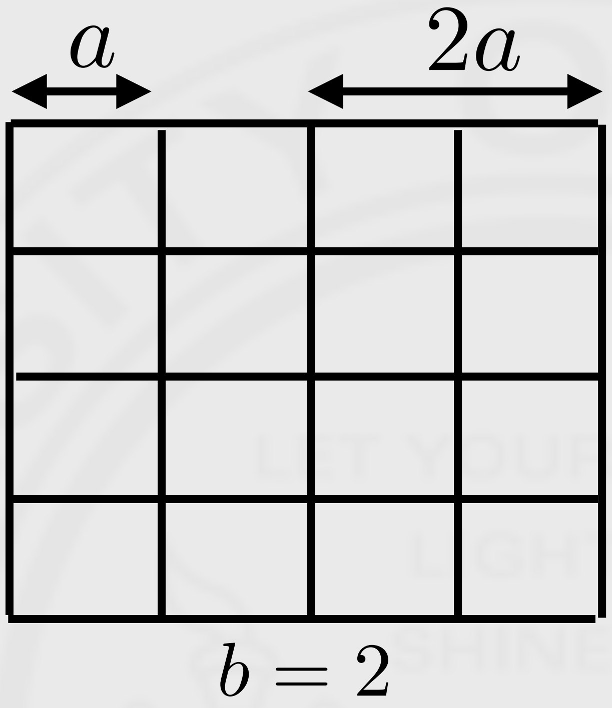
Renormalization group

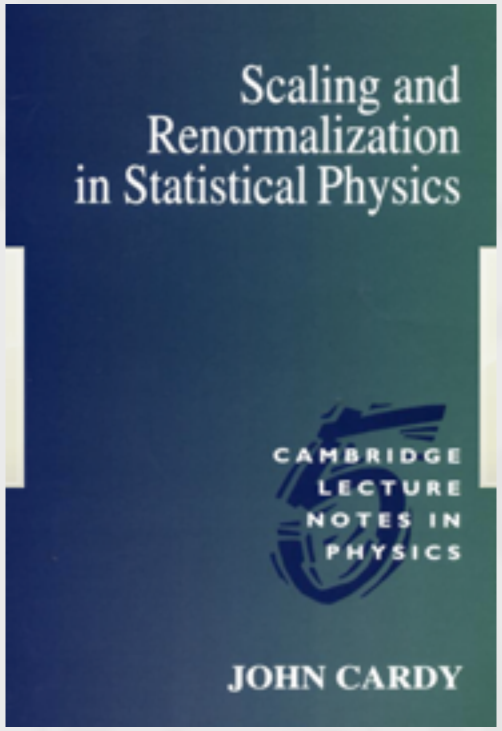
lattice spacing rescaling $a' = ba$

$$Z = \mathrm{Tr}_\sigma \, e^{-\mathcal{H}[K]} = \mathrm{Tr}_{\sigma'} \, e^{-\mathcal{H}[K']}$$

renormalization group transformation

$$K'_\alpha = R_\alpha[K]$$





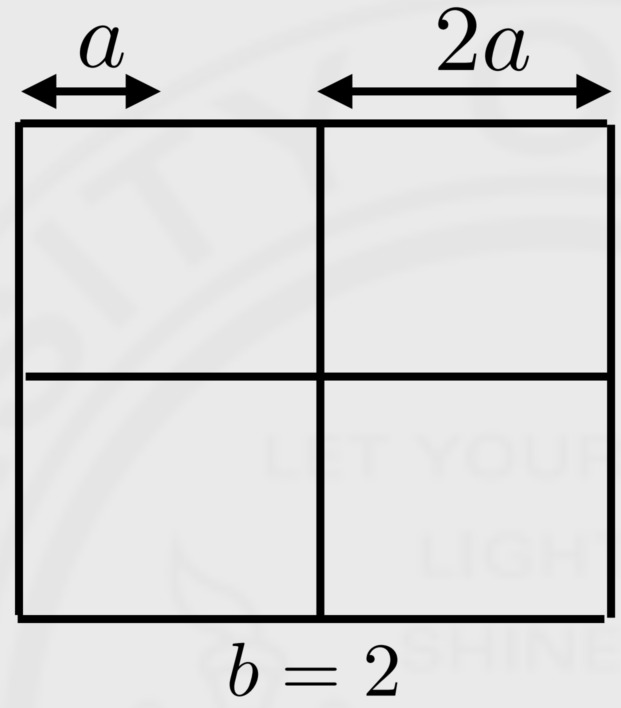
Renormalization group

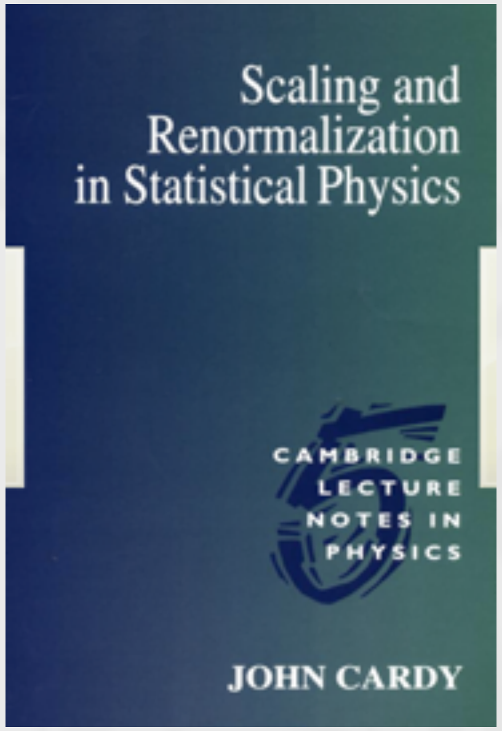
lattice spacing rescaling $a' = ba$

$$Z = \mathrm{Tr}_\sigma \, e^{-\mathcal{H}[K]} = \mathrm{Tr}_{\sigma'} \, e^{-\mathcal{H}[K']}$$

renormalization group transformation

$$K'_\alpha = R_\alpha[K]$$





Renormalization group

lattice spacing rescaling $a' = ba$

$$Z = \mathrm{Tr}_\sigma\, e^{-\mathcal{H}[K]} = \mathrm{Tr}_{\sigma'}\, e^{-\mathcal{H}[K']}$$

renormalization group transformation

$$K'_\alpha = R_\alpha[K]$$

fixed point of the renormalization group $K^*_\alpha = R_\alpha[K^*]$

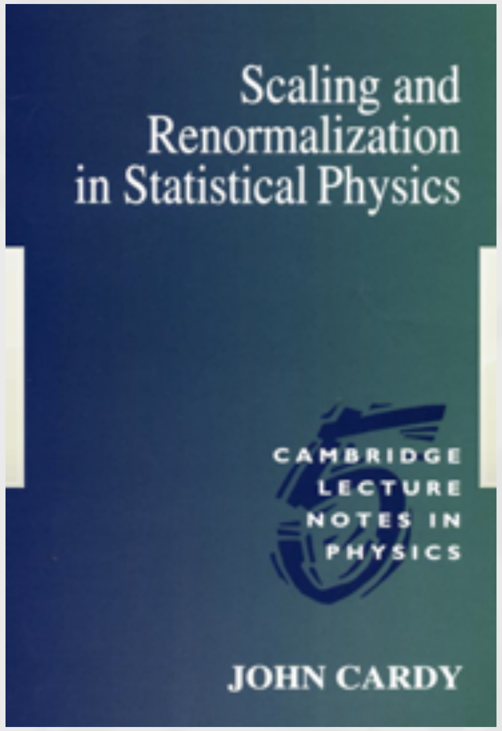
Taylor expansion about the fixed point

$$K'_\alpha - K^*_\alpha = R_\alpha[K] - K^*_\alpha \approx \sum_\beta \left.\frac{\partial R_\alpha[K]}{\partial K_\beta}\right|_{K=K^*} (K_\beta - K^*_\beta)$$

$$T_{\alpha\beta} = \left.\frac{\partial R_\alpha[K]}{\partial K_\beta}\right|_{K=K^*}$$

$$\sum_\alpha e^i_\alpha T_{\alpha\beta} = \lambda^i e^i_\beta$$

$$\lambda_i = b^{y_i}$$



Renormalization group

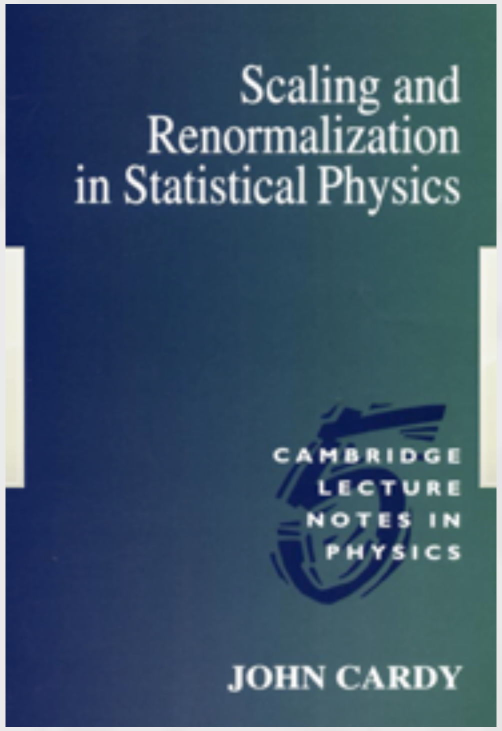
lattice spacing rescaling $a' = ba$

$$Z = \mathrm{Tr}_\sigma\, e^{-\mathcal{H}[K]} = \mathrm{Tr}_{\sigma'}\, e^{-\mathcal{H}[K']}$$

renormalization group transformation

$$K'_\alpha = R_\alpha[K]$$

fixed point of the renormalization group $K^*_\alpha = R_\alpha[K^*]$



Taylor expansion about the fixed point

$$K'_\alpha - K^*_\alpha = R_\alpha[K] - K^*_\alpha \approx \sum_\beta \left.\frac{\partial R_\alpha[K]}{\partial K_\beta}\right|_{K=K^*} (K_\beta - K^*_\beta)$$

$$T_{\alpha\beta} = \left.\frac{\partial R_\alpha[K]}{\partial K_\beta}\right|_{K=K^*} \qquad \sum_\alpha e^i_\alpha T_{\alpha\beta} = \lambda^i e^i_\beta$$

$$\lambda_i = b^{y_i}$$

scaling variables $u_i = \sum_\alpha e^i_\alpha (K_\alpha - K^*_\alpha)$

$$\mathcal{H}[K] \approx \mathcal{H}[K^*] + \sum_i u_i \sum_{\mathbf{r}} \phi_i(\mathbf{r})$$ scaling operators

scaling dimensions $x_i = d - y_i$

RG transformations

$$u'_i = b^{y_i} u_i$$

$$\sum_i u_i \int \frac{d^d r}{a^d} \phi_i(\mathbf{r}) = \sum_i u'_i \int \frac{d^d r}{a'^d} \phi'_i(\mathbf{r})$$

$$\phi'_i = b^{d-y_i} \phi_i$$

$$\langle \phi_i(\mathbf{r}_1)\phi_i(\mathbf{r}_2)\rangle_{K=K^*} \sim \frac{a^{2x_i}}{|\mathbf{r}_1 - \mathbf{r}_2|^{2x_i}}$$

The renormalization group matrix

$$K'_\alpha = R_\alpha[K]$$

$$T_{\alpha\beta} = \left.\frac{\partial R_\alpha[K]}{\partial K_\beta}\right|_{K=K^*} \qquad T_{\alpha\beta} \neq T_{\beta\alpha}$$

Are we sure that the eigenvalues of *T* actually exist?

$$\sum_\alpha e^i_\alpha T_{\alpha\beta} = \boxed{b^{y_i}} e^i_\beta$$

The renormalization group matrix

$$K'_\alpha = R_\alpha[K]$$

$$T_{\alpha\beta} = \left.\frac{\partial R_\alpha[K]}{\partial K_\beta}\right|_{K=K^*} \qquad T_{\alpha\beta} \neq T_{\beta\alpha}$$

Are we sure that the eigenvalues of *T* actually exist?

$$\sum_\alpha e^i_\alpha T_{\alpha\beta} = \boxed{b^{y_i}} e^i_\beta$$

What if they are complex? Possible, but far too exotic.

The renormalization group matrix

$$K'_\alpha = R_\alpha[K]$$

$$T_{\alpha\beta} = \left.\frac{\partial R_\alpha[K]}{\partial K_\beta}\right|_{K=K^*}$$

$$T_{\alpha\beta} \neq T_{\beta\alpha}$$

Are we sure that the eigenvalues of T actually exist?

$$\sum_\alpha e^i_\alpha T_{\alpha\beta} = \boxed{b^{y_i}} e^i_\beta$$

Wikipedia

$$\begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix}} & & & & \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ & \lambda_2 \end{matrix}} & & & \\ & & \boxed{\lambda_3} & & \\ & & & \ddots & \\ & & & & \boxed{\begin{matrix} \lambda_n & 1 \\ & \lambda_n \end{matrix}} \end{bmatrix}$$

An example of a matrix in a Jordan normal form. The grey blocks are called the Jordan blocks.

<u>What if the eigenvalues exist but coincide?</u>

Possible to bring T to the Jordan normal form

$$\boxed{\begin{aligned} u'_1 &= \lambda u_1 \\ u'_2 &= \lambda u_2 + \mu u_1 \end{aligned}}$$

$$T = \begin{pmatrix} \lambda(b) & 0 \\ \mu(b) & \lambda(b) \end{pmatrix}$$

This is what this talk is about.

Correlation functions with Jordan forms

$$b = e^{\ell}$$
$$a' = e^{\ell} a$$

Infinitesimal RG equations $\longrightarrow$ finite RG transformations

$$\frac{\partial u_1}{\partial \ell} = y u_1 \qquad u_1' = b^y u_1$$

$$\frac{\partial u_2}{\partial \ell} = y u_2 - u_1 \qquad u_2' = b^y u_2 - \boxed{b^y \ln(b)}\, u_1$$

$$\frac{\partial a}{\partial \ell} = a$$

note the logarithm

invariant energy

$$\sum_i u_i \int \frac{d^d r}{a^d} \phi_i(\mathbf{r})$$

demands

$$\frac{\partial \phi_1}{\partial \ell} = x\phi_1 + \phi_2$$

$$\frac{\partial \phi_2}{\partial \ell} = x\phi_2$$

Correlation functions with Jordan forms

$$b = e^{\ell}$$
$$a' = e^{\ell} a$$

Infinitesimal RG equations $\longrightarrow$ finite RG transformations

$$\frac{\partial u_1}{\partial \ell} = y u_1 \qquad u_1' = b^y u_1$$

$$\frac{\partial u_2}{\partial \ell} = y u_2 - u_1 \qquad u_2' = b^y u_2 - \boxed{b^y \ln(b)}\, u_1$$

note the logarithm

$$\frac{\partial a}{\partial \ell} = a$$

invariant energy

$$\sum_i u_i \int \frac{d^d r}{a^d} \phi_i(\mathbf{r})$$

demands

$$\frac{\partial \phi_1}{\partial \ell} = x\phi_1 + \phi_2$$

$$\frac{\partial \phi_2}{\partial \ell} = x\phi_2$$

Invariance with respect to the above RG demands

$$\langle \phi_1(\mathbf{r}_1)\phi_1(\mathbf{r}_2)\rangle = \frac{2a^{2x}}{|\mathbf{r}_1 - \mathbf{r}_2|^{2x}} \ln \frac{a}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

$$\langle \phi_1(\mathbf{r}_1)\phi_2(\mathbf{r}_2)\rangle = \frac{a^{2x}}{|\mathbf{r}_1 - \mathbf{r}_2|^{2x}}$$

$$\langle \phi_2(\mathbf{r}_1)\phi_2(\mathbf{r}_1)\rangle = 0$$

$\phi_1(\mathbf{r})$ Logarithmic operator

$\phi_1(\mathbf{r}),\ \phi_2(\mathbf{r}_2)$ Logarithmic pair

How common are logarithmic operators?

- Require fine-tuning of the eigenvalues of the RG matrix.

- So perhaps do not appear except in some special fine-tuned models?

- In fact, that’s not true. Logarithmic operators are ubiquitous in certain models, especially in models with disorder.

Example: 3D random bond Ising model

3D random bond Ising model

$$Z = \sum_{\sigma=\pm 1} \exp\left(K \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \sigma_{\mathbf{r}'}\sigma_{\mathbf{r}''} + \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \delta K_{\mathbf{r}'\mathbf{r}''}\, \sigma_{\mathbf{r}'}\sigma_{\mathbf{r}''} \right)$$

random and Gaussian δK

$$P(\delta K) \sim e^{-\frac{\delta K^2}{4\gamma^2}}$$

3D random bond Ising model

$$Z = \sum_{\sigma=\pm 1} \exp\left(K \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \sigma_{\mathbf{r}'}\sigma_{\mathbf{r}''} + \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \delta K_{\mathbf{r}'\mathbf{r}''}\, \sigma_{\mathbf{r}'}\sigma_{\mathbf{r}''} \right)$$

random and Gaussian δK

$$P(\delta K) \sim e^{-\frac{\delta K^2}{4\gamma^2}}$$

Need to invoke the famous replica trick

$$Z^n = \sum_{\sigma=\pm 1} \exp\left(K \sum_{a=1}^{n} \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \sigma^a_{\mathbf{r}'}\sigma^a_{\mathbf{r}''} + \sum_{a=1}^{n} \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \delta K_{\mathbf{r}'\mathbf{r}''}\, \sigma^a_{\mathbf{r}'}\sigma^a_{\mathbf{r}''} \right)$$

and average over disorder

$$\langle Z^n \rangle = \sum_{\sigma=\pm 1} \exp\left(K \sum_{a=1}^{n} \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \sigma^a_{\mathbf{r}'}\sigma^a_{\mathbf{r}''} + \gamma \sum_{a,b=1}^{n} \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \sigma^a_{\mathbf{r}'}\sigma^a_{\mathbf{r}''}\sigma^b_{\mathbf{r}'}\sigma^b_{\mathbf{r}''} \right)$$

so that free energy can then be found if needed

$$F = -T \lim_{n\to 0} \frac{\langle Z^n \rangle - 1}{n}$$

3D random bond Ising model

$$Z = \sum_{\sigma=\pm 1} \exp\left(K \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \sigma_{\mathbf{r}'}\sigma_{\mathbf{r}''} + \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \delta K_{\mathbf{r}'\mathbf{r}''}\, \sigma_{\mathbf{r}'}\sigma_{\mathbf{r}''} \right)$$

random and Gaussian δK

$$P(\delta K) \sim e^{-\frac{\delta K^2}{4\gamma^2}}$$

Need to invoke the famous replica trick

$$Z^n = \sum_{\sigma=\pm 1} \exp\left(K \sum_{a=1}^{n} \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \sigma^a_{\mathbf{r}'}\sigma^a_{\mathbf{r}''} + \sum_{a=1}^{n} \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \delta K_{\mathbf{r}'\mathbf{r}''}\, \sigma^a_{\mathbf{r}'}\sigma^a_{\mathbf{r}''} \right)$$

and average over disorder

so that free energy can then be found if needed

$$\langle Z^n \rangle = \sum_{\sigma=\pm 1} \exp\left(K \sum_{a=1}^{n} \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \sigma^a_{\mathbf{r}'}\sigma^a_{\mathbf{r}''} + \gamma \sum_{a,b=1}^{n} \sum_{\langle \mathbf{r}'\mathbf{r}''\rangle} \sigma^a_{\mathbf{r}'}\sigma^a_{\mathbf{r}''}\sigma^b_{\mathbf{r}'}\sigma^b_{\mathbf{r}''} \right)$$

$$F = -T \lim_{n\to 0} \frac{\langle Z^n \rangle - 1}{n}$$

In the vicinity of the conventional Ising critical point $K = K^*,\ \gamma = 0$

$$\phi_t(\mathbf{r}') = \sum_{a=1}^{n} \sigma^a_{\mathbf{r}'}\sigma^a_{\mathbf{r}''}$$

this is the conventional thermal scaling operator, dimension

$$x_t = d - y_t = d - \frac{1}{\nu}$$

$$\sum_{a,b=1}^{n} \sigma^a_{\mathbf{r}'}\sigma^a_{\mathbf{r}''}\sigma^b_{\mathbf{r}'}\sigma^b_{\mathbf{r}''}$$

this is the scaling operator which is coupled to disorder strength

$$y_\gamma = d - 2x_t = d - 2\left(d - \frac{1}{\nu}\right) = \frac{2}{\nu} - d$$

Harris criterion

$$\langle Z^n \rangle = \sum_{\sigma=\pm 1} \exp\left(K \sum_{a=1}^{n} \sum_{\langle \mathbf{r}'\mathbf{r}'' \rangle} \sigma^a_{\mathbf{r}'} \sigma^a_{\mathbf{r}''} + \gamma \sum_{a,b=1}^{n} \sum_{\langle \mathbf{r}'\mathbf{r}'' \rangle} \sigma^a_{\mathbf{r}'} \sigma^a_{\mathbf{r}''} \sigma^b_{\mathbf{r}'} \sigma^b_{\mathbf{r}''} \right)$$

$$\sum_{a,b=1}^{n} \sigma^a_{\mathbf{r}'} \sigma^a_{\mathbf{r}''} \sigma^b_{\mathbf{r}'} \sigma^b_{\mathbf{r}''}$$

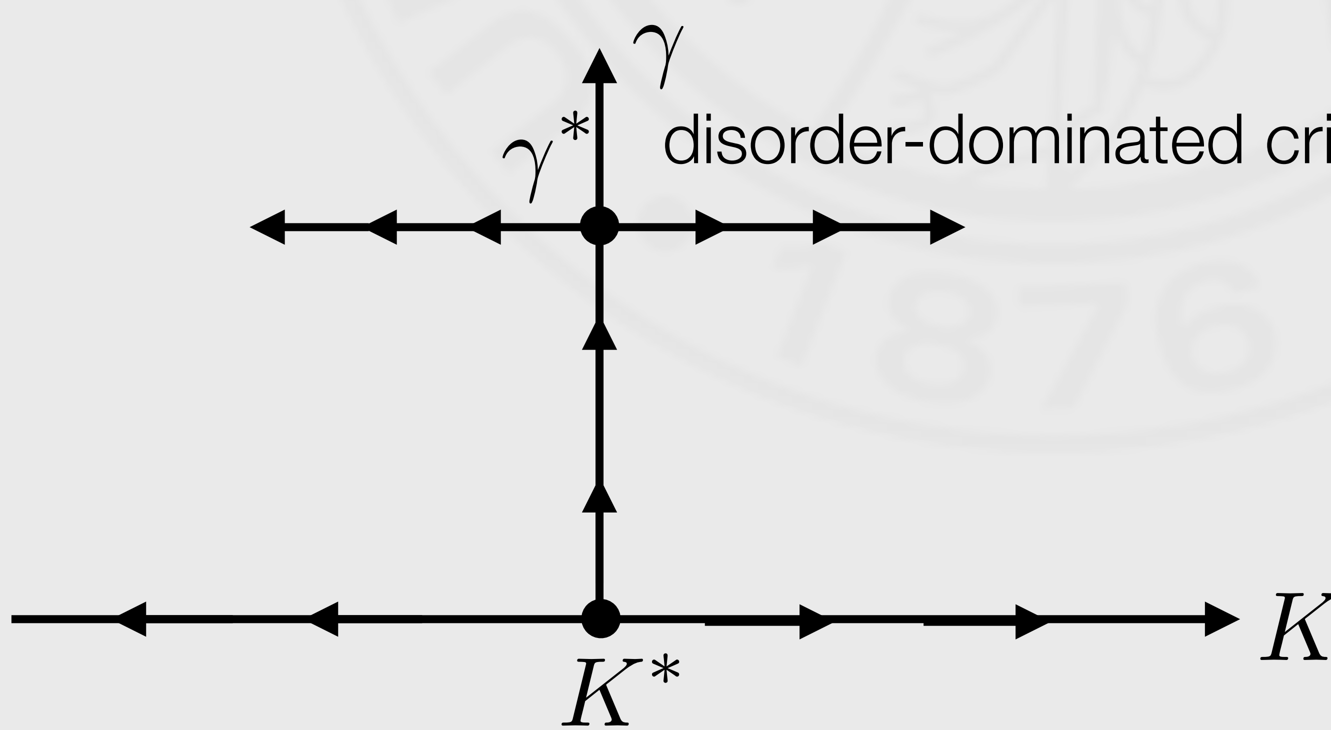
this is the scaling operator which is coupled to disorder strength

$$y_\gamma = d - 2x_t = d - 2\left(d - \frac{1}{\nu}\right) = \frac{2}{\nu} - d$$

$y_t > 0 \rightarrow \nu < \dfrac{2}{d}$ disorder is relevant

equivalently $\alpha > 0$

RG flow for 3D Ising, with α>o



disorder-dominated critical point

Ising critical point

Disorder-dominated point is logarithmic

J. Cardy, 1999

Let's look at the energy operator $E^a(\mathbf{r}) = \sigma^a_{\mathbf{r}}\sigma^a_{\mathbf{r}'}$

Two physical correlators:

$\lim_{n\to 0}\langle E^1(0)E^1(\mathbf{r})\rangle$ thermal correlator of two energies averaged over disorder

$\lim_{n\to 0}\langle E^1(0)E^2(\mathbf{r})\rangle$ product of two thermal averaged energies further averaged over disorder

However these are complicated operators from the RG point of view. Simple correlators are irreducible representations of the replica permutation group.

$$E_{\text{sym}} = \sum_{a=1}^{n} E^a$$

$$E^a_{\text{irr}} = E^a - \frac{E_{\text{sym}}}{n}$$

with dimensions $x_{\text{sym}}(n)$, $x_{\text{irr}}(n)$

$$\frac{1}{n}\langle E_{\text{sym}}(0)E_{\text{sym}}(\mathbf{r})\rangle = \langle E^1(0)E^1(\mathbf{r})\rangle + (n-1)\langle E^1(0)E^2(\mathbf{r})\rangle = \frac{S(n)}{r^{2x_{\text{sym}}(n)}}$$

$$\frac{n}{n-1}\langle E^1_{\text{irr}}(0)E^1_{\text{irr}}(\mathbf{r})\rangle = \langle E^1(0)E^1(\mathbf{r})\rangle - \langle E^1(0)E^2(\mathbf{r})\rangle = \frac{I(n)}{r^{2x_{\text{irr}}(n)}}$$

$$x_{\text{sym}}(0) = x_{\text{irr}}(0) = x$$
$$S(0) = I(0)$$

$$\lim_{n\to 0}\langle E^1(0)E^1(\mathbf{r})\rangle = \lim_{n\to 0}\frac{1}{n}\left(\frac{S(n)}{r^{2x_{\text{sym}}(n)}} + (n-1)\frac{I(n)}{r^{2x_{\text{irr}}(n)}}\right) \sim \frac{\ln(r)}{r^{2x}}$$

Distinct features of the logarithms in this context

- Logarithms appear directly at the critical point

- No fine tuning is needed to get logarithms, and no fine tuning can eliminate them

- This is a generic feature of problems with quenched disorder (to be discussed later)

Example: 2D percolation

Percolation as Q-state Potts model

Q-state Potts model

$$Z = \sum_{\sigma=1,2,\ldots,Q} \exp\left(K \sum_{\langle \mathbf{r}'\mathbf{r}'' \rangle} \delta_{\sigma_{\mathbf{r}'}\sigma_{\mathbf{r}''}} \right)$$

$Q = 2$ corresponds to the Ising model

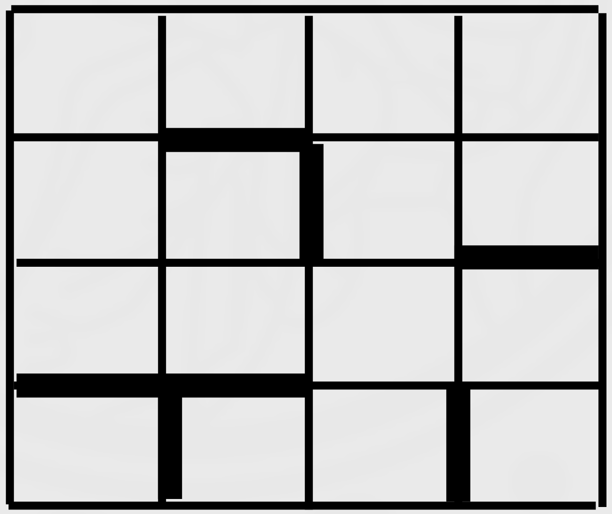
More generally, known to have a second order phase transition in 2D for $1 \leq Q \leq 4$

Standard map from Q=1 Potts model to percolation

$$Z = \frac{1}{(1-p)^{\#\ \text{bonds}}} \sum_{\sigma=1,2,\ldots,Q} \prod_{\langle \mathbf{r}'\mathbf{r}'' \rangle} (1 - p + p\,\delta_{\sigma_{\mathbf{r}'}\sigma_{\mathbf{r}''}})$$

$$p = 1 - e^{-K}$$

percolation (Fortuin-Kasteleyn) clusters



$$Z = \frac{1}{(1-p)^{\#\ \text{bonds}}} \sum_{\text{clusters}} Q^{\#\ \text{clusters}} p^{\#\ \text{cluster bonds}} (1-p)^{\#\text{remaining bonds}}$$

limit $Q \to 1$ counts all clusters equally and is equivalent to studying percolation

$Q \to 0$ counts spanning trees and is equivalent to computing the determinant of the lattice laplacian

Logarithmic observables in percolation

R. Vasseur, J. L. Jacobsen, H. Saleur (2012)

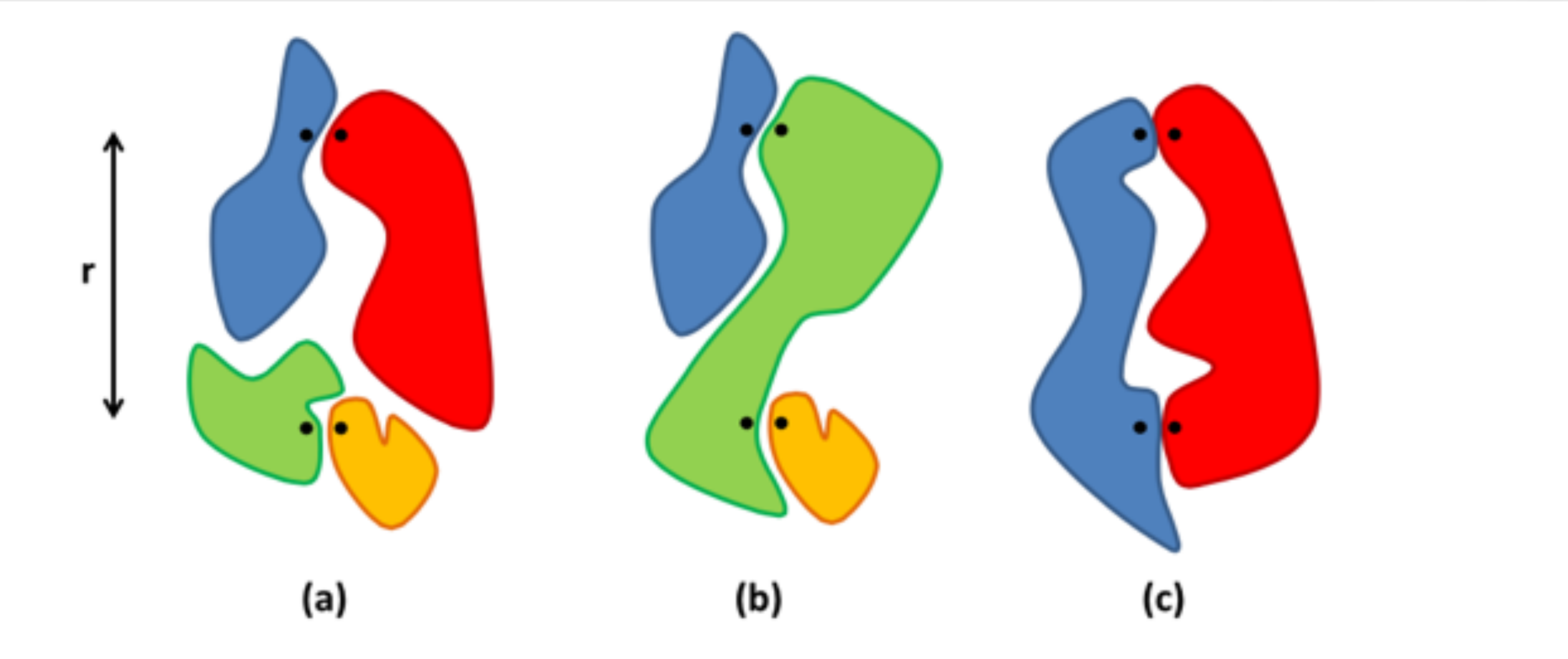
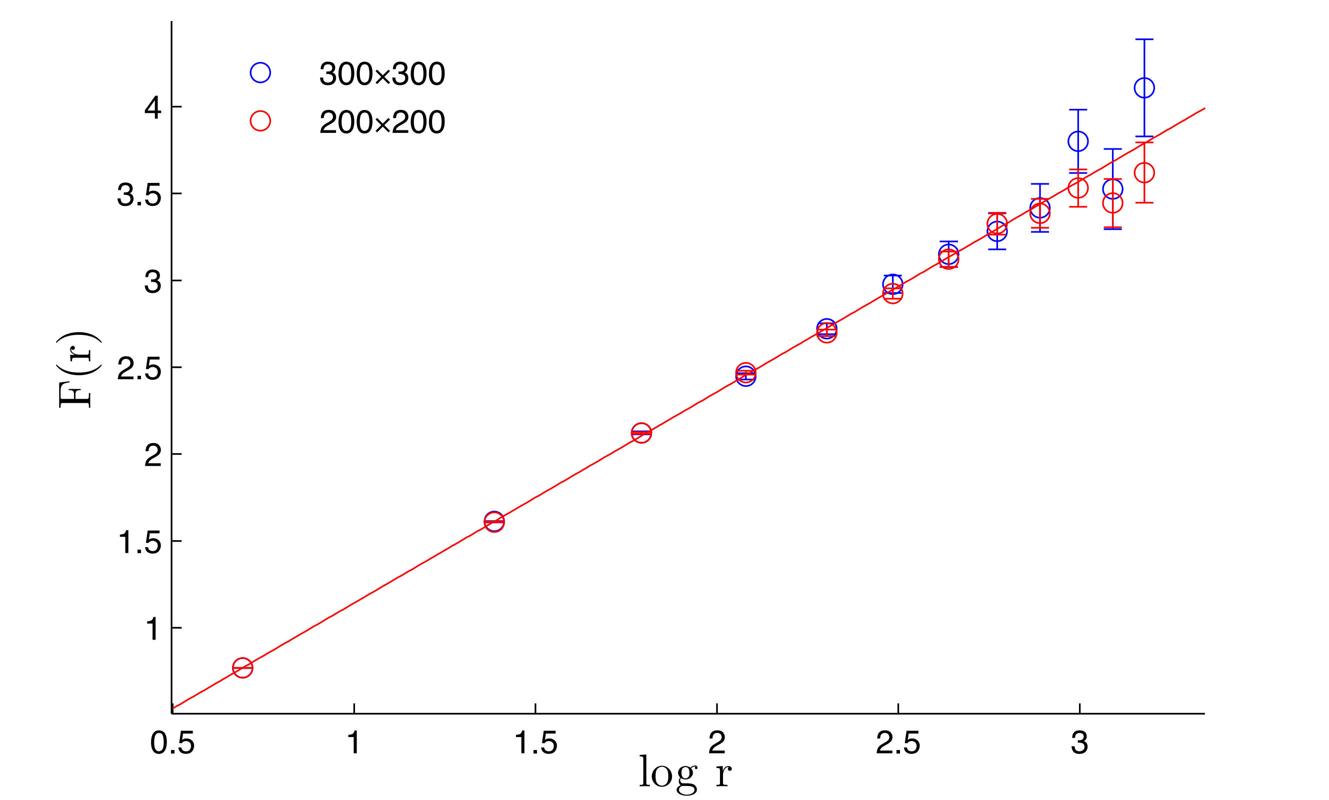


Figure 1. Percolation configurations contributing to (a) $\mathbb{P}_0(r)$, (b) $\mathbb{P}_1(r)$ (one cluster propagating a distance $r = |r_1 - r_2|$), (c) $\mathbb{P}_2(r)$ (two propagating clusters).

$\mathbb{P}_{\neq}$ probability that two nearby points belong to different clusters

$$F(r) \equiv \frac{\mathbb{P}_0(r) + \mathbb{P}_1(r) - \mathbb{P}_{\neq}^2}{\mathbb{P}_2(r)} \sim \theta + \frac{2\sqrt{3}}{\pi} \log r$$



Symplectic fermions

Determinant of a Laplacian

VG, M. Flohr, C. Nayak, 1996

defined via a Grassmann functional integral

$$\det\left(\Delta\right) \sim \int \mathcal{D}\theta \mathcal{D}\bar{\theta}\, \theta(\mathbf{r})\bar{\theta}(\mathbf{r})\, e^{-\frac{1}{4\pi}\int d^2r\, \partial_\mu \theta \partial_\mu \bar{\theta}} = \prod_{n,\lambda_n \neq 0} \lambda_n$$

necessary to get rid of the "zero mode"

with proper normalization

$$\left\langle \theta(\mathbf{r})\bar{\theta}(\mathbf{r}) \right\rangle = 1$$

$$\langle I \rangle = 0$$

identity

$$\tilde{I}(\mathbf{r}) = \theta(\mathbf{r})\bar{\theta}(\mathbf{r})$$

This is a logarithmic pair:

$$\langle I(\mathbf{r}_1) I(\mathbf{r}_2) \rangle = 0$$

$$\left\langle I(\mathbf{r}_1) \tilde{I}(\mathbf{r}_2) \right\rangle = 1$$

$$\left\langle \tilde{I}(\mathbf{r}_1) \tilde{I}(\mathbf{r}_2) \right\rangle = -4 \ln\left(|\mathbf{r}_1 - \mathbf{r}_2|\right)$$

Models related to the determinant of Δ

- Any model involving counting of spanning trees, or $Q \to 0$ Potts model

- In particular, dense polymers (self-avoiding random walks passing through every point of a lattice)

- Abelian sandpile model

CFT approach to logarithms

VG, 1993

$$T(z) = \sum_n \frac{L_n}{z^{n+2}}$$

L_0 generates scale transformations

Definition of the primary operators in CFT

$$L_0 A = hA$$

$$L_n A = 0 \qquad n > 0$$

$$\langle A(0)A(z)\rangle = \frac{1}{z^{2h}}$$

CFT approach to logarithms

VG, 1993

$$T(z) = \sum_n \frac{L_n}{z^{n+2}}$$

L_0 generates scale transformations

Definition of the primary operators in CFT

$$L_0 A = hA$$

$$L_n A = 0 \qquad n > 0$$

$$\langle A(0)A(z)\rangle = \frac{1}{z^{2h}}$$

Definition of a logarithmic primary in CFT

$$L_0 C = hC$$

$$L_0 D = hD + C$$

$$L_n C = 0,\ L_n D = 0 \qquad n > 0$$

$$\langle C(0)C(z)\rangle = 0$$

$$\langle C(0)D(z)\rangle = \frac{1}{z^{2h}}$$

$$\langle D(0)D(z)\rangle = \frac{-2\ln z}{z^{2h}}$$

CFT approach to logarithms

VG, 1993

$$T(z) = \sum_n \frac{L_n}{z^{n+2}}$$

L_0 generates scale transformations

Definition of the primary operators in CFT

$$L_0 A = hA$$
$$L_n A = 0 \qquad n > 0$$

$$\langle A(0)A(z)\rangle = \frac{1}{z^{2h}}$$

Definition of a logarithmic primary in CFT

$$L_0 C = hC$$
$$L_0 D = hD + C$$
$$L_n C = 0, \; L_n D = 0 \qquad n > 0$$

$$\langle C(0)C(z)\rangle = 0$$
$$\langle C(0)D(z)\rangle = \frac{1}{z^{2h}}$$
$$\langle D(0)D(z)\rangle = \frac{-2\ln z}{z^{2h}}$$

c=-2 theory has a zero dimension primary

$$L_0 \tilde{I} = I$$
$$L_0 I = 0$$

$$\left\langle \tilde{I}(0)\tilde{I}(z)\right\rangle = -2\ln z$$

Conformal field theory for symplectic fermions

$$S = \frac{1}{4\pi} \int d^2 z \, \partial\theta \bar{\partial}\bar{\theta}$$

$$T \sim \partial\theta\partial\bar{\theta} \qquad c = -2$$

Kac table of degenerate operators

$n \setminus m$	1	2	3	4	$\ldots$
1	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	$\ldots$
2	1	$\frac{3}{8}$	0	$-\frac{1}{8}$	$\ldots$
3	3	$\frac{15}{8}$	1	$\frac{3}{8}$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

$$\Delta_{n,m} = \frac{(2n-m)^2 - 1}{8}$$

Conformal field theory for symplectic fermions

$$S = \frac{1}{4\pi} \int d^2 z \, \partial\theta \bar{\partial}\bar{\theta}$$

$$T \sim \partial\theta\partial\bar{\theta} \qquad c = -2$$

Kac table of degenerate operators

$n \setminus m$	1	2	3	4	$\dots$
1	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	$\dots$
2	$\boxed{1}$	$\frac{3}{8}$	0	$-\frac{1}{8}$	$\dots$
3	3	$\frac{15}{8}$	1	$\frac{3}{8}$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

$\partial\theta$

$$\Delta_{n,m} = \frac{(2n-m)^2 - 1}{8}$$

Conformal field theory for symplectic fermions

$$S = \frac{1}{4\pi}\int d^2z\,\partial\theta\bar{\partial}\bar{\theta}$$

$$T \sim \partial\theta\partial\bar{\theta} \qquad c = -2$$

Kac table of degenerate operators

$\sigma(z)$

$n \setminus m$	1	2	3	4	...
1	0	$\boxed{-\frac{1}{8}}$	0	$\frac{3}{8}$	...
2	$\boxed{1}$	$\frac{3}{8}$	0	$-\frac{1}{8}$	...
3	3	$\frac{15}{8}$	1	$\frac{3}{8}$	...
...	...	...	...	...	...

$\partial\theta$

$$\Delta_{n,m} = \frac{(2n-m)^2 - 1}{8}$$

$$\sigma(z)\sigma(0) = z^{\frac{1}{8}}\left(\tilde{I} + \ln z\, I + \dots\right)$$

Conformal field theory for symplectic fermions

$$S = \frac{1}{4\pi}\int d^2 z\, \partial\theta\bar{\partial}\bar{\theta}$$

$$T \sim \partial\theta\partial\bar{\theta} \qquad c = -2$$

Kac table of degenerate operators

$\sigma(z)$

$n \setminus m$	1	2	3	4	...
1	0	$\boxed{-\frac{1}{8}}$	0	$\frac{3}{8}$	...
2	$\boxed{1}$	$\frac{3}{8}$	0	$-\frac{1}{8}$	...
3	3	$\frac{15}{8}$	1	$\frac{3}{8}$	...
...	...	...	...	...	...

$\partial\theta$

$$\Delta_{n,m} = \frac{(2n-m)^2 - 1}{8}$$

$$\sigma(z)\sigma(0) = z^{\frac{1}{8}}\left(\tilde{I} + \ln z\, I + \ldots\right)$$

$$\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = [(z_1 - z_3)(z_2 - z_4)x(1-x)]^{\frac{1}{4}}\, F_i(x)$$

$$F_1(x) = F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right)$$

$$F_2(x) = F\left(\frac{1}{2}, \frac{1}{2}, 1; 1-x\right)$$

$$F_1(1-x) \approx \ln x$$
$$x \to 0$$

Conformal field theory for symplectic fermions

$$S = \frac{1}{4\pi}\int d^2z\,\partial\theta\bar{\partial}\bar{\theta}$$

$$\langle\sigma(\infty)\partial\theta(z)\partial\bar{\theta}(w)\sigma(0)\rangle = \frac{\frac{1}{2}\left(\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}\right)}{(z-w)^2}$$

$$T \sim \partial\theta\partial\bar{\theta} \qquad c=-2$$

Kac table of degenerate operators

$\sigma(z)$

$n \setminus m$	1	2	3	4	…
1	0	$\boxed{-\frac{1}{8}}$	0	$\frac{3}{8}$	…
2	$\boxed{1}$	$\frac{3}{8}$	0	$-\frac{1}{8}$	…
3	3	$\frac{15}{8}$	1	$\frac{3}{8}$	…
…	…	…	…	…	…

$\partial\theta$

$$\Delta_{n,m} = \frac{(2n-m)^2-1}{8}$$

$$\sigma(z)\sigma(0) = z^{\frac{1}{8}}\left(\tilde{I} + \ln z\, I + \dots\right)$$

$$\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = [(z_1-z_3)(z_2-z_4)x(1-x)]^{\frac{1}{4}}\,F_i(x)$$

$$F_1(x) = F\left(\frac{1}{2},\frac{1}{2},1;x\right)$$

$$F_2(x) = F\left(\frac{1}{2},\frac{1}{2},1;1-x\right)$$

$$F_1(1-x) \approx \ln x$$
$$x \to 0$$

Relevance to the stat mech models?

- Identification of observables related to the operator $\tilde{I}$

- Meaning of this observable in the Abelian sandpile model?

V.S. Poghosyan, S. Y. Grigorev,
V. B. Priezzhev, P. Ruelle, 2010

P. Ruelle, 2013

CFT at c=0 and problems involving averaging over disorder

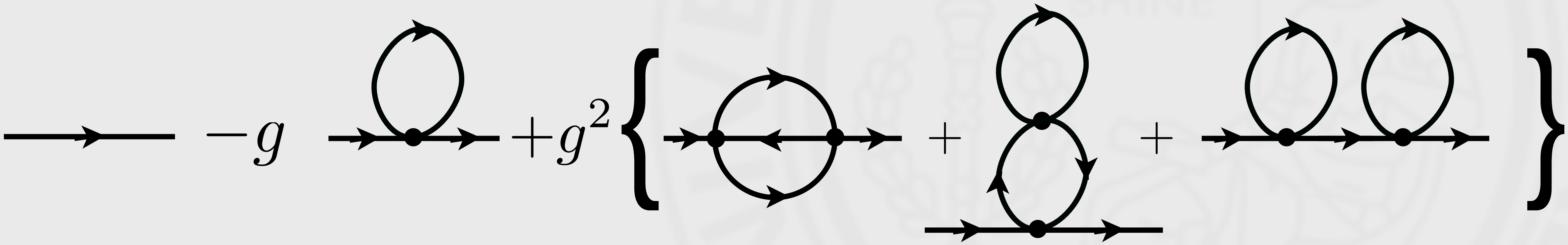
SARW: Effective field theory

$$P(t,x) = \int_{x(0)=0}^{x(t)=x} \mathcal{D}x(t)\, e^{-\frac{1}{D}\int_0^t dt\, \dot{x}_\mu^2 - \frac{g}{2}\int dt dt'\, \delta\left(\vec{x}(t) - \vec{x}'(t)\right)}$$

SARW: Effective field theory

$$P(t,x) = \int_{x(0)=0}^{x(t)=x} \mathcal{D}x(t)\, e^{-\frac{1}{D}\int_0^t dt\, \dot{x}_\mu^2 - \frac{g}{2}\int dt dt'\, \delta\left(\vec{x}(t)-\vec{x}'(t)\right)}$$

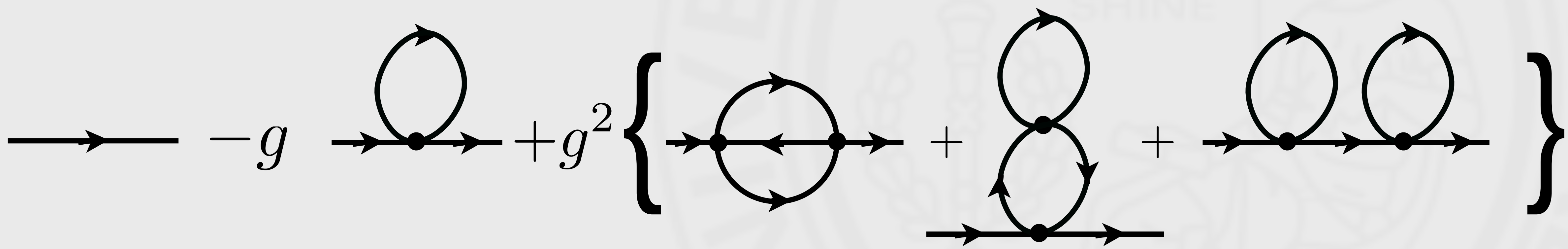
Perturbative expansion



SARW: Effective field theory

$$P(t,x) = \int_{x(0)=0}^{x(t)=x} \mathcal{D}x(t)\, e^{-\frac{1}{D}\int_0^t dt\, \dot{x}_\mu^2 - \frac{g}{2}\int dt dt'\, \delta\left(\vec{x}(t) - \vec{x}'(t)\right)}$$

Perturbative expansion



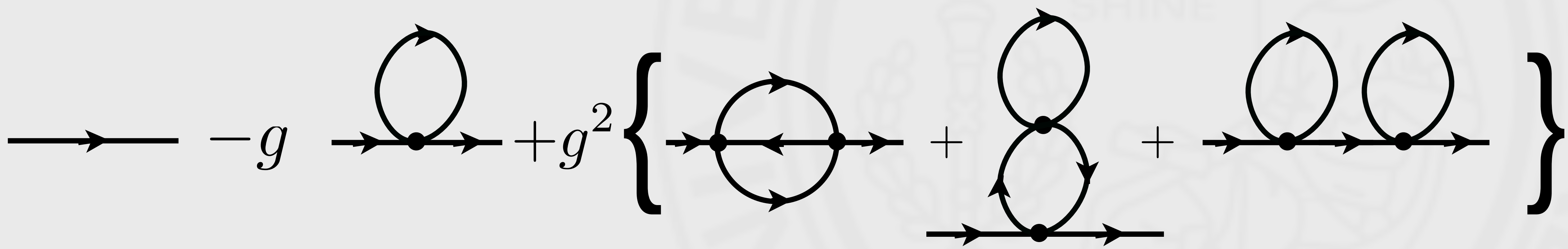
is reproduced by the expansion of this Green's function with a random imaginary potential *i V(x)* in powers of *V(x)*

$$\frac{1}{i\omega + D\frac{\partial^2}{\partial x^2} - iV(x)} \qquad \langle V(x)V(y)\rangle = g\,\delta(x-y)$$

SARW: Effective field theory

$$P(t,x) = \int_{x(0)=0}^{x(t)=x} \mathcal{D}x(t)\, e^{-\frac{1}{D}\int_0^t dt\, \dot{x}_\mu^2 - \frac{g}{2}\int dt dt'\, \delta\left(\vec{x}(t) - \vec{x}'(t)\right)}$$

Perturbative expansion



is reproduced by the expansion of this Green's function with a random imaginary potential *i V(x)* in powers of *V(x)*

$$\frac{1}{i\omega + D\frac{\partial^2}{\partial x^2} - iV(x)} \qquad \langle V(x)V(y)\rangle = g\,\delta(x-y)$$

$$P(\omega,x) = \frac{\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\,\phi(x)\,\bar{\phi}(0)\,e^{\int d^2x\,\bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}{\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\,e^{\int d^2x\,\bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}$$

Random potentials: replica approach

$$P(\omega, x) = \frac{\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, \phi(x)\, \bar{\phi}(0)\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}{\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}$$

Random potential

$$\langle V(x) V(y) \rangle = g\, \delta(x - y)$$

De Gennes, 1972

Random potentials: replica approach

$$P(\omega, x) = \frac{\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, \phi(x)\, \bar{\phi}(0)\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}{\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}$$

Random potential

$$\langle V(x) V(y)\rangle = g\,\delta(x-y)$$

Introduce *n* replicas

$$P(\omega, x) = \frac{\int \prod_{i=1}^n \mathcal{D}\phi_i \mathcal{D}\bar{\phi}_i\, \phi_1(x)\, \bar{\phi}_1(0)\, e^{\sum_{i=1}^n \int d^2x\, \bar{\phi}_i\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi_i}}{\left[\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}\right]^n}$$

De Gennes, 1972

Random potentials: replica approach

$$P(\omega, x) = \frac{\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\,\phi(x)\,\bar{\phi}(0)\,e^{\int d^2x\,\bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}{\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\,e^{\int d^2x\,\bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}$$

Random potential

$$\langle V(x)V(y)\rangle = g\,\delta(x-y)$$

Introduce *n* replicas

$$P(\omega, x) = \frac{\int \prod_{i=1}^{n} \mathcal{D}\phi_i\mathcal{D}\bar{\phi}_i\,\phi_1(x)\,\bar{\phi}_1(0)\,e^{\sum_{i=1}^{n}\int d^2x\,\bar{\phi}_i\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi_i}}{\left[\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\,e^{\int d^2x\,\bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}\right]^n}$$

take *n* to zero

$$P(\omega, x) = \lim_{n\to 0}\int \prod_{i=1}^{n} \mathcal{D}\phi_i\mathcal{D}\bar{\phi}_i\,\phi_1(x)\,\bar{\phi}_1(0)\,e^{\sum_{i=1}^{n}\int d^2x\,\bar{\phi}_i\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi_i}$$

De Gennes, 1972

Random potentials: replica approach

$$P(\omega,x)=\frac{\int\mathcal{D}\phi\mathcal{D}\bar\phi\,\phi(x)\,\bar\phi(0)\,e^{\int d^2x\,\bar\phi\left(D\frac{\partial^2}{\partial x^2}-iV+i\omega\right)\phi}}{\int\mathcal{D}\phi\mathcal{D}\bar\phi\,e^{\int d^2x\,\bar\phi\left(D\frac{\partial^2}{\partial x^2}-iV+i\omega\right)\phi}}$$

Random potential

$$\langle V(x)V(y)\rangle = g\,\delta(x-y)$$

Introduce *n* replicas

$$P(\omega,x)=\frac{\int\prod_{i=1}^n\mathcal{D}\phi_i\mathcal{D}\bar\phi_i\,\phi_1(x)\,\bar\phi_1(0)\,e^{\sum_{i=1}^n\int d^2x\,\bar\phi_i\left(D\frac{\partial^2}{\partial x^2}-iV+i\omega\right)\phi_i}}{\left[\int\mathcal{D}\phi\mathcal{D}\bar\phi\,e^{\int d^2x\,\bar\phi\left(D\frac{\partial^2}{\partial x^2}-iV+i\omega\right)\phi}\right]^n}$$

take *n* to zero

$$P(\omega,x)=\lim_{n\to0}\int\prod_{i=1}^n\mathcal{D}\phi_i\mathcal{D}\bar\phi_i\,\phi_1(x)\,\bar\phi_1(0)\,e^{\sum_{i=1}^n\int d^2x\,\bar\phi_i\left(D\frac{\partial^2}{\partial x^2}-iV+i\omega\right)\phi_i}$$

and finally average over random potential

$$P(\omega,x)=\lim_{n\to0}\int\prod_{i=1}^n\mathcal{D}\phi_i\mathcal{D}\bar\phi_i\,\phi_1(x)\,\bar\phi_1(0)\,e^{-\int d^2x\left[\sum_{i=1}^n D\partial_\mu\bar\phi_i\partial_\mu\phi_i-i\omega\bar\phi_i\phi_i+\frac{g}{2}\left(\sum_{i=1}^n\bar\phi_i\phi_i\right)^2\right]}$$

De Gennes, 1972

Random potentials: replica approach

$$P(\omega,x) = \frac{\int \mathcal{D}\phi\mathcal{D}\bar\phi\,\phi(x)\,\bar\phi(0)\,e^{\int d^2x\,\bar\phi\left(D\frac{\partial^2}{\partial x^2}-iV+i\omega\right)\phi}}{\int \mathcal{D}\phi\mathcal{D}\bar\phi\,e^{\int d^2x\,\bar\phi\left(D\frac{\partial^2}{\partial x^2}-iV+i\omega\right)\phi}}$$

Random potential

$$\langle V(x)V(y)\rangle = g\,\delta(x-y)$$

Introduce *n* replicas

$$P(\omega,x) = \frac{\int \prod_{i=1}^n \mathcal{D}\phi_i\mathcal{D}\bar\phi_i\,\phi_1(x)\,\bar\phi_1(0)\,e^{\sum_{i=1}^n \int d^2x\,\bar\phi_i\left(D\frac{\partial^2}{\partial x^2}-iV+i\omega\right)\phi_i}}{\left[\int \mathcal{D}\phi\mathcal{D}\bar\phi\,e^{\int d^2x\,\bar\phi\left(D\frac{\partial^2}{\partial x^2}-iV+i\omega\right)\phi}\right]^n}$$

take *n* to zero

$$P(\omega,x) = \lim_{n\to 0}\int \prod_{i=1}^n \mathcal{D}\phi_i\mathcal{D}\bar\phi_i\,\phi_1(x)\,\bar\phi_1(0)\,e^{\sum_{i=1}^n \int d^2x\,\bar\phi_i\left(D\frac{\partial^2}{\partial x^2}-iV+i\omega\right)\phi_i}$$

and finally average over random potential

$$P(\omega,x) = \lim_{n\to 0}\int \prod_{i=1}^n \mathcal{D}\phi_i\mathcal{D}\bar\phi_i\,\phi_1(x)\,\bar\phi_1(0)\,e^{-\boxed{\int d^2x\left[\sum_{i=1}^n D\partial_\mu\bar\phi_i\partial_\mu\phi_i - i\omega\bar\phi_i\phi_i + \frac{g}{2}\left(\sum_{i=1}^n\bar\phi_i\phi_i\right)^2\right]}}$$

This is the famous *O(n)* model in the *n*→*0* limit

De Gennes, 1972

Random potentials: "supersymmetry approach"

$$P(\omega, x) = \frac{\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, \phi(x)\, \bar{\phi}(0)\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}{\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}$$

$$\langle V(x) V(y) \rangle = g\, \delta(x-y)$$

Random potentials: "supersymmetry approach"

$$P(\omega, x) = \frac{\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, \phi(x)\, \bar{\phi}(0)\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}{\int \mathcal{D}\phi \mathcal{D}\bar{\phi}\, e^{\int d^2x\, \bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi}}$$

$$\langle V(x) V(y) \rangle = g\, \delta(x - y)$$

Introduce fermionic fields ψ

$$P(\omega, x) = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \mathcal{D}\bar{\psi} \mathcal{D}\psi\, \phi(x)\, \bar{\phi}(0)\, e^{\int d^2x \left[\bar{\phi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\phi + \bar{\psi}\left(D\frac{\partial^2}{\partial x^2} - iV + i\omega\right)\psi\right]}$$

Random potentials: "supersymmetry approach"

$$P(\omega,x) = \frac{\int \mathcal{D}\phi \mathcal{D}\bar\phi \, \phi(x)\, \bar\phi(0)\, e^{\int d^2x \, \bar\phi \left(D \frac{\partial^2}{\partial x^2} - iV + i\omega \right)\phi}}{\int \mathcal{D}\phi \mathcal{D}\bar\phi \, e^{\int d^2x \, \bar\phi \left(D \frac{\partial^2}{\partial x^2} - iV + i\omega \right)\phi}}$$

$$\langle V(x) V(y) \rangle = g\, \delta(x-y)$$

Introduce fermionic fields ψ

$$P(\omega,x) = \int \mathcal{D}\phi \mathcal{D}\bar\phi \mathcal{D}\bar\psi \mathcal{D}\psi \, \phi(x)\, \bar\phi(0)\, e^{\int d^2x \left[\bar\phi \left(D \frac{\partial^2}{\partial x^2} - iV + i\omega \right)\phi + \bar\psi \left(D \frac{\partial^2}{\partial x^2} - iV + i\omega \right)\psi \right]}$$

Average over random potential, to find effective field theory with the action

$$S = \int d^2x \left[D \left(\partial_\mu \bar\phi \, \partial_\mu \phi + \partial_\mu \bar\psi \, \partial_\mu \psi \right) + \frac{g}{2} \left(\bar\phi \phi + \bar\psi \psi \right)^2 \right]$$

<u>We would like to study CFTs corresponding to the field theories of this type. All have c=0.</u>

“Supersymmetric” critical theories

- Supersymmetric effective field theories describe a variety of interesting critical behavior in 2 dimensions. Most have not been understood.

- Examples include self-avoiding random walks and percolation (mostly understood, although not completely) and quantum motion in random potentials under various conditions (mostly not understood).

- Most famous example, the quantum Hall transition, has been extensively studied, and yet is not understood.

Supersymmetry

A typical action

$$S = \int d^2x \left[D \left(\partial_\mu \bar{\phi}\, \partial_\mu \phi + \partial_\mu \bar{\psi}\, \partial_\mu \psi \right) + \frac{g}{2} \left(\bar{\phi}\phi + \bar{\psi}\psi \right)^2 \right]$$

$$\begin{pmatrix} \phi' \\ \psi' \end{pmatrix} = \begin{pmatrix} \alpha_1 & \epsilon \\ \bar{\epsilon} & \alpha_2 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Superunitary (more precisely, in this example, orthosymplectic) group is the symmetry group of this action

Supersymmetry

A typical action

$$S = \int d^2x \left[D \left(\partial_\mu \bar{\phi}\, \partial_\mu \phi + \partial_\mu \bar{\psi}\, \partial_\mu \psi \right) + \frac{g}{2} \left(\bar{\phi}\phi + \bar{\psi}\psi \right)^2 \right]$$

$$\begin{pmatrix} \phi' \\ \psi' \end{pmatrix} = \begin{pmatrix} \alpha_1 & \epsilon \\ \bar{\epsilon} & \alpha_2 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Superunitary (more precisely, in this example, orthosymplectic) group is the symmetry group of this action

<u>Strange reducible but indecomposable representations of the superunitary group</u>

$$\begin{tikzcd}
& \bar{\phi}\phi - \bar{\psi}\psi \arrow[dl, "\epsilon"'] \arrow[dr, "\bar{\epsilon}"] & \\
\bar{\phi}\psi \arrow[dr, "\bar{\epsilon}"'] & & \phi\bar{\psi} \arrow[dl, "\epsilon"] \\
& \bar{\phi}\phi + \bar{\psi}\psi &
\end{tikzcd}$$

scalar at the bottom

Logarithms and the indecomposable reps

$$
\begin{tikzcd}
 & D \arrow[dl] \arrow[dr] & \\
\zeta \arrow[dr] & & \bar{\zeta} \arrow[dl] \\
 & C &
\end{tikzcd}
$$

Logarithmic operators love indecomposable multiplets

Z. Masarani, D. Serban, 1996

Logarithms and the indecomposable reps

$$\begin{tikzcd} & D \arrow[dl] \arrow[dr] & \\ \zeta \arrow[dr] & & \bar{\zeta} \arrow[dl] \\ & C & \end{tikzcd}$$

Logarithmic operators love indecomposable multiplets

Z. Masarani, D. Serban, 1996

$\langle C(z)\, C(w) \rangle = 0$ Used to be mysterious, now natural $\delta \langle \zeta(z) C(w) \rangle = 0$

Logarithms and the indecomposable reps

$$\begin{tikzcd} & D \arrow[dl] \arrow[dr] & \\ \zeta \arrow[dr] & & \bar{\zeta} \arrow[dl] \\ & C & \end{tikzcd}$$

Logarithmic operators love indecomposable multiplets

Z. Masarani, D. Serban, 1996

$$\langle C(z)\, C(w)\rangle = 0$$ Used to be mysterious, now natural $$\delta \left\langle \zeta(z) C(w) \right\rangle = 0$$

$$\langle C(z)\, D(w)\rangle = \frac{1}{(z-w)^{2\lambda}}$$

$$\delta \left\langle D(z)\bar{\zeta}(w)\right\rangle = \left\langle \zeta(z)\bar{\zeta}(w)\right\rangle - \langle D(z)C(w)\rangle = 0$$

So ζ are just the usual primary fields

$$\left\langle \zeta(z)\bar{\zeta}(w)\right\rangle = \frac{1}{(z-w)^{2\lambda}}$$

Logarithms and the indecomposable reps

$$\begin{tikzcd} & D \arrow[dl] \arrow[dr] & \\ \zeta \arrow[dr] & & \bar{\zeta} \arrow[dl] \\ & C & \end{tikzcd}$$

Logarithmic operators love indecomposable multiplets

Z. Masarani, D. Serban, 1996

$$\langle C(z)\, C(w)\rangle = 0$$

Used to be mysterious, now natural $\delta \left\langle \zeta(z) C(w) \right\rangle = 0$

$$\langle C(z)\, D(w)\rangle = \frac{1}{(z-w)^{2\lambda}}$$

$$\delta \left\langle D(z)\bar{\zeta}(w)\right\rangle = \left\langle \zeta(z)\bar{\zeta}(w)\right\rangle - \langle D(z) C(w)\rangle = 0$$

So ζ are just the usual primary fields

$$\left\langle \zeta(z)\bar{\zeta}(w)\right\rangle = \frac{1}{(z-w)^{2\lambda}}$$

Finally:

$$\langle D(z)\, D(w)\rangle = -\frac{2\ln(z-w)}{(z-w)^{2\lambda}}$$

because why not??

Stress-energy tensor at c=0: CFT perspective

VG, 1999

Any primary operator with a nonvanishing norm in a CFT satisfies

$$A(z)A(0) = \frac{1}{z^{2\lambda}}\left(1 + \frac{2\lambda}{c}T(z) + \dots\right)$$

Thus the direct limit $c \to 0$ is problematic.

Stress-energy tensor at c=0: CFT perspective

VG, 1999

Any primary operator with a nonvanishing norm in a CFT satisfies

$$A(z)A(0) = \frac{1}{z^{2\lambda}}\left(1 + \frac{2\lambda}{c}T(z) + \dots\right)$$

Thus the direct limit $c \to 0$ is problematic.

Any $c=0$ CFT must contain operators with dimension 2 distinct from the stress-energy tensor. At least one of them, called t, must satisfy

$$T(z)t(0) = \frac{b}{z^4} + \dots$$

Stress-energy tensor at c=0: CFT perspective

VG, 1999

Any primary operator with a nonvanishing norm in a CFT satisfies

$$A(z)A(0) = \frac{1}{z^{2\lambda}}\left(1 + \frac{2\lambda}{c}T(z) + \dots\right)$$

Thus the direct limit *c*→*0* is problematic.

Any *c=0* CFT must contain operators with dimension 2 distinct from the stress-energy tensor. At least one of them, called *t*, must satisfy

$$T(z)t(0) = \frac{b}{z^4} + \dots$$

Then $$A(z)A(0) = \frac{1}{z^{2\lambda}}\left(1 + \frac{\lambda}{b}t(z) + CT(0) + \dots\right)$$

Stress-energy tensor at c=0: supersymmetry perspective

$$\begin{tikzcd}
& t \arrow[dl] \arrow[dr] & \\
\xi \arrow[dr] & & \bar{\xi} \arrow[dl] \\
& T &
\end{tikzcd}$$

Stress-energy tensor is always a part of a reducible but indecomposable multiplet

Stress-energy tensor at c=0: supersymmetry perspective

$$\begin{tikzcd}
& t \arrow[dl] \arrow[dr] & \\
\xi \arrow[dr] & & \bar{\xi} \arrow[dl] \\
& T &
\end{tikzcd}$$

Stress-energy tensor is always a part of a reducible but indecomposable multiplet

Possible consistent OPE:

$$T(z)T(0) = \frac{2T(0)}{z^2} + \ldots$$

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0)}{z^2} + \ldots$$

$$t(z)t(0) = \frac{2t(0)}{z^2} + \ldots$$

Realized in supergroup-based WZW models.

Stress-energy tensor at c=0: supersymmetry perspective

$$\begin{tikzcd} & t \arrow[dl] \arrow[dr] & \\ \xi \arrow[dr] & & \bar{\xi} \arrow[dl] \\ & T & \end{tikzcd}$$

Stress-energy tensor is always a part of a reducible but indecomposable multiplet

Possible consistent OPE:

$$T(z)T(0) = \frac{2T(0)}{z^2} + \ldots$$

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0)}{z^2} + \ldots$$

$$t(z)t(0) = \frac{2t(0)}{z^2} + \ldots$$

Realized in supergroup-based WZW models.

But these are also possible consistent OPE:

$$T(z)T(0) = \frac{2T(0)}{z^2} + \ldots$$

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0) + T(0)}{z^2} + \ldots$$

$$t(z)t(0) = \frac{-2b \ln z}{z^4} + \ldots$$

Makes *t* logarithmic. Realized in *c=0* minimal model.

Logarithmic t: supersymmetry emerges

VG, A. Ludwig, 2002

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0) + T(0)}{z^2} + \frac{t'(0)}{z} + \ldots$$

Logarithmic *t*: supersymmetry emerges

VG, A. Ludwig, 2002

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0) + T(0)}{z^2} + \frac{t'(0)}{z} + \ldots$$

$$t(z)t(0) = -\frac{2b\log z}{z^4} + \frac{t(0)\left[1 - 4\log z\right] - T(0)\left[\log z + 2\log^2 z\right]}{z^2}$$

$$\xi(z)\bar{\xi}(0) = \frac{1}{8}T(z)T(0) + \frac{b}{2z^4} + \frac{t(0) + T(0)\log z}{z^2} + \ldots$$

$$t(z)\xi(0) = \frac{1}{4}T(z)\xi(0) - T(z)\xi(0)\log z + \frac{\xi'(0)}{2z} + \ldots$$

These follow from the assumption of logarithmic t by conformal invariance only

Logarithmic *t*: supersymmetry emerges

VG, A. Ludwig, 2002

$$T(z)t(0) = \frac{b}{z^4} + \frac{2t(0) + T(0)}{z^2} + \frac{t'(0)}{z} + \dots$$

$$t(z)t(0) = -\frac{2b\log z}{z^4} + \frac{t(0)\left[1 - 4\log z\right] - T(0)\left[\log z + 2\log^2 z\right]}{z^2}$$

$$\xi(z)\bar{\xi}(0) = \frac{1}{8}T(z)T(0) + \frac{b}{2z^4} + \frac{t(0) + T(0)\log z}{z^2} + \dots$$

$$t(z)\xi(0) = \frac{1}{4}T(z)\xi(0) - T(z)\xi(0)\log z + \frac{\xi'(0)}{2z} + \dots$$

These follow from the assumption of logarithmic t by conformal invariance only

$$\begin{tikzcd} & t \arrow[dl] \arrow[dr] & \\ \xi \arrow[dr] & & \bar{\xi} \arrow[dl] \\ & T & \end{tikzcd}$$

Yet they automatically form the indecomposable representation shown on the left

Logarithmic *t*: minimal model at c=0

n \ m	1	2	3	4	5
5	2	$\frac{5}{8}$	0		
4	1	$\frac{1}{8}$	0		
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$		
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7

$$\lambda_{m,n} = \frac{(2n-3m)^2-1}{24}$$

Differential equations give

$$\langle A(z_1)A(z_2)A(z_3)A(z_4)\rangle = \frac{1}{(z_1-z_2)^{2\lambda}(z_3-z_4)^{2\lambda}}\left(1+\alpha x^2\ln(x)+\dots\right)$$

$$\alpha = \frac{\lambda}{b} \qquad\qquad x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

Logarithmic *t*: minimal model at c=0

n \ m	1	2	3	4	5
5	2	$\frac{5}{8}$	0		
4	1	$\frac{1}{8}$	0		
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$		
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7

$$b = \frac{5}{6}$$

$$\lambda_{m,n} = \frac{(2n-3m)^2-1}{24}$$

Differential equations give

$$\langle A(z_1)A(z_2)A(z_3)A(z_4)\rangle = \frac{1}{(z_1-z_2)^{2\lambda}(z_3-z_4)^{2\lambda}}\left(1+\alpha x^2\ln(x)+\dots\right)$$

$$\alpha = \frac{\lambda}{b} \qquad\qquad x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

Logarithmic *t*: minimal model at c=0

$$b = -\frac{5}{8}$$

n \ m	1	2	3	4	5
5	2	$\frac{5}{8}$	0		
4	1	$\frac{1}{8}$	0		
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$		
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7

$$b = \frac{5}{6}$$

$$\lambda_{m,n} = \frac{(2n-3m)^2 - 1}{24}$$

Differential equations give

$$\langle A(z_1)A(z_2)A(z_3)A(z_4)\rangle = \frac{1}{(z_1 - z_2)^{2\lambda}(z_3 - z_4)^{2\lambda}}\left(1 + \alpha x^2 \ln(x) + \dots\right)$$

$$\alpha = \frac{\lambda}{b} \qquad x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

Determination of b

PRL **108,** 161602 (2012) PHYSICAL REVIEW LETTERS week ending 20 APRIL 2012

Puzzle of Bulk Conformal Field Theories at Central Charge $c = 0$

Romain Vasseur,[1,2,5] Azat Gainutdinov,[1,5] Jesper Lykke Jacobsen,[2,3,5] and Hubert Saleur[1,4,5]

[1]*Institut de Physique Théorique, CEA Saclay, 91191 Gif Sur Yvette, France*
[2]*LPTENS, 24 rue Lhomond, 75231 Paris, France*
[3]*Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris, France*
[4]*Department of Physics, University of Southern California, Los Angeles, California 90089-0484, USA*
[5]*Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75231 Paris Cedex 05, France*

(Received 12 October 2011; published 20 April 2012)

Nontrivial critical models in 2D with a central charge $c = 0$ are described by logarithmic conformal field theories (LCFTs), and exhibit, in particular, mixing of the stress-energy tensor with a "logarithmic" partner under a conformal transformation. This mixing is quantified by a parameter (usually denoted b), introduced in Gurarie [Nucl. Phys. **B546**, 765 (1999)]. The value of b has been determined over the last few years for the boundary versions of these models: $b_{\text{perco}} = -\frac{5}{8}$ for percolation and $b_{\text{poly}} = \frac{5}{6}$ for dilute polymers. Meanwhile, the existence and value of b for the bulk theory has remained an open problem. Using lattice regularization techniques we provide here an "experimental study" of this question. We show that, while the chiral stress tensor has indeed a single logarithmic partner in the chiral sector of the theory, the value of b is not the expected one; instead, $b = -5$ for *both* theories. We suggest a theoretical explanation of this result using operator product expansions and Coulomb gas arguments, and discuss the physical consequences on correlation functions. Our results imply that the relation between bulk LCFTs of physical interest and their boundary counterparts is considerably more involved than in the non-logarithmic case.

DOI: 10.1103/PhysRevLett.108.161602 PACS numbers: 11.25.Hf, 02.30.Ik, 64.60.ah

Determination of b

PRL **108,** 161602 (2012) PHYSICAL REVIEW LETTERS week ending 20 APRIL 2012

Puzzle of Bulk Conformal Field Theories at Central Charge $c = 0$

Romain Vasseur,[1,2,5] Azat Gainutdinov,[1,5] Jesper Lykke Jacobsen,[2,3,5] and Hubert Saleur[1,4,5]

[1]*Institut de Physique Théorique, CEA Saclay, 91191 Gif Sur Yvette, France*
[2]*LPTENS, 24 rue Lhomond, 75231 Paris, France*
[3]*Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris, France*
[4]*Department of Physics, University of Southern California, Los Angeles, California 90089-0484, USA*
[5]*Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75231 Paris Cedex 05, France*

(Received 12 October 2011; published 20 April 2012)

Nontrivial critical models in 2D with a central charge $c = 0$ are described by logarithmic conformal field theories (LCFTs), and exhibit, in particular, mixing of the stress-energy tensor with a "logarithmic" partner under a conformal transformation. This mixing is quantified by a parameter (usually denoted b), introduced in Gurarie [Nucl. Phys. **B546**, 765 (1999)]. The value of b has been determined over the last few years for the boundary versions of these models: $\boxed{b_{\text{perco}} = -\frac{5}{8}}$ for percolation and $\boxed{b_{\text{poly}} = \frac{5}{6}}$ for dilute polymers. Meanwhile, the existence and value of b for the bulk theory has remained an open problem. Using lattice regularization techniques we provide here an "experimental study" of this question. We show that, while the chiral stress tensor has indeed a single logarithmic partner in the chiral sector of the theory, the value of b is not the expected one; instead, $\boxed{b = -5}$ for *both* theories. We suggest a theoretical explanation of this result using operator product expansions and Coulomb gas arguments, and discuss the physical consequences on correlation functions. Our results imply that the relation between bulk LCFTs of physical interest and their boundary counterparts is considerably more involved than in the non-logarithmic case.

DOI: 10.1103/PhysRevLett.108.161602 PACS numbers: 11.25.Hf, 02.30.Ik, 64.60.ah

Further developments

1. arXiv:1409.0167 [pdf, ps, other]
   The periodic sl(2|1) alternating spin chain and its continuum limit as a bulk Logarithmic Conformal Field Theory at c=0
   A.M. Gainutdinov, N. Read, H. Saleur, R. Vasseur
   Comments: 69pp, 8 figs
   Subjects: **High Energy Physics - Theory (hep-th)**; Statistical Mechanics (cond-mat.stat-mech); Mathematical Physics (math-ph); Quantum Algebra (math.QA); Representation Theory (math.RT)

2. arXiv:1303.2082 [pdf, ps, other]
   Logarithmic Conformal Field Theory: a Lattice Approach
   A.M. Gainutdinov, J.L. Jacobsen, N. Read, H. Saleur, R. Vasseur
   Comments: 44pp, 6 figures, many comments added
   Journal-ref: J. Phys. A: Math. Theor. 46 (2013) 494012
   Subjects: **High Energy Physics - Theory (hep-th)**; Statistical Mechanics (cond-mat.stat-mech); Mathematical Physics (math-ph)

3. arXiv:1207.6334 [pdf, ps, other]
   Associative algebraic approach to logarithmic CFT in the bulk: the continuum limit of the gl(1|1) periodic spin chain, Howe duality and the interchiral algebra
   A. M. Gainutdinov, N. Read, H. Saleur
   Comments: 69 pp., 10 figs, v2: the paper has been substantially modified - new proofs, new refs, new App C with inductive limits construction, etc
   Subjects: **High Energy Physics - Theory (hep-th)**; Statistical Mechanics (cond-mat.stat-mech); Mathematical Physics (math-ph); Quantum Algebra (math.QA)

4. arXiv:1112.3407 [pdf, ps, other]
   Bimodule structure in the periodic gl(1|1) spin chain
   A. M. Gainutdinov, N. Read, H. Saleur
   Comments: latex, 42 pp., 13 figures + 5 figures in color, many comments added
   Journal-ref: Nuclear Physics B 871 [FS] (2013) 289-329
   Subjects: **High Energy Physics - Theory (hep-th)**; Statistical Mechanics (cond-mat.stat-mech); Mathematical Physics (math-ph); Quantum Algebra (math.QA); Representation Theory (math.RT)

5. arXiv:1112.3403 [pdf, ps, other]
   Continuum limit and symmetries of the periodic gl(1|1) spin chain
   A. M. Gainutdinov, N. Read, H. Saleur
   Comments: 43 pp, few comments added
   Journal-ref: Nuclear Physics B 871 [FS] (2013) 245-288
   Subjects: **High Energy Physics - Theory (hep-th)**; Statistical Mechanics (cond-mat.stat-mech); Mathematical Physics (math-ph); Quantum Algebra (math.QA)

Extensive work by H. Saleur and N. Read elucidating the structure of percolation, $Q \to 1$ limit of Potts model, as a logarithmic CFT.

Conclusions and outlook

c=0 theories: modern developments

- Logarithmic scaling at certain fixed points of renormalization group is unavoidable

- In some examples it affects certain correlation functions and is relatively easy to study

- In other examples, it affect the whole structure of the theory and makes it very difficult to understand it.

- Problems with disorder generally have logarithmic correlators. Exact solutions to the critical points in 2D involving disorder are supposed to involve logarithmic structure and are very hard to study.