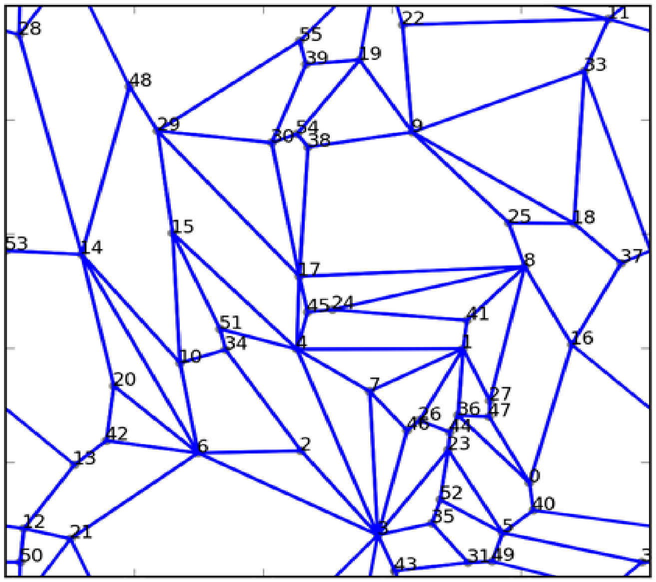
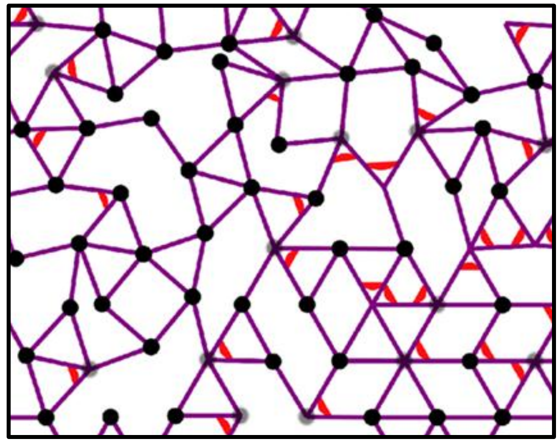
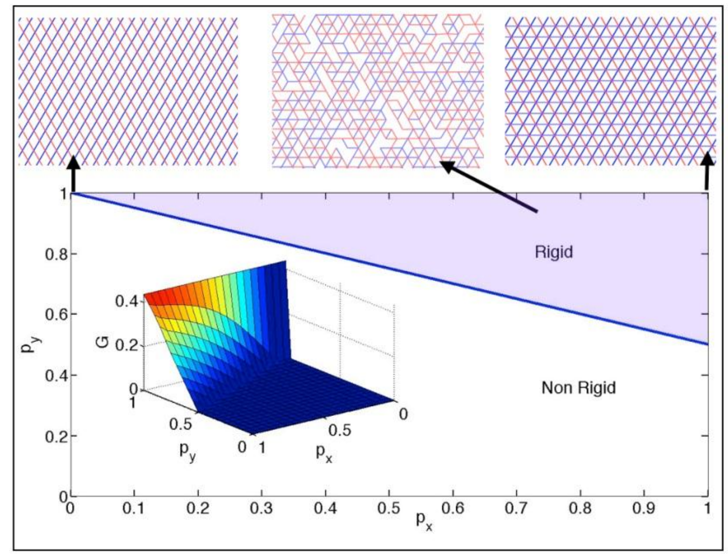
Disordered mechanical networks: From jamming graphs to anisotropic rigidity percolation







Papers:

- *Redundancy and cooperativity in the mechanics of compositely crosslinked filamentous networks*, M. Das, D. A. Quint, and J. M. Schwarz, PLoS ONE **7**(5): e35939 (2012)
- *Jamming graphs: A local approach to global mechanical rigidity*, J. H. Lopez, L. Cao, and J. M. Schwarz, Phys. Rev. E **88**:062130 (2013)
- *The mechanics of anisotropic disordered spring networks***,** T. Zhang, J. M. Schwarz, and M. Das: arXiv:1408.5910.

Maxwell counting argument and rigidity

L. *On the Calculation of the Equilibrium and Stiffness of Frames.* By J. CLERK MAXWELL, F.R.S., *Professor of Natural Philosophy in King's College, London**.

THE theory of the equilibrium and deflections of frameworks subjected to the action of forces is sometimes considered as more complicated than it really is, especially in cases in which

Let us consider a bar-joint framework made of N vertices with N_f degrees of freedom, and N_c bonds. We call $< z >$ the average coordination number.

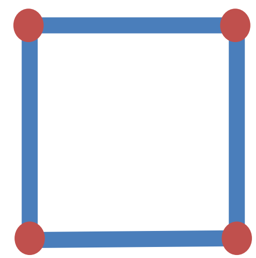
Flexible framework: $N_c < N_f$

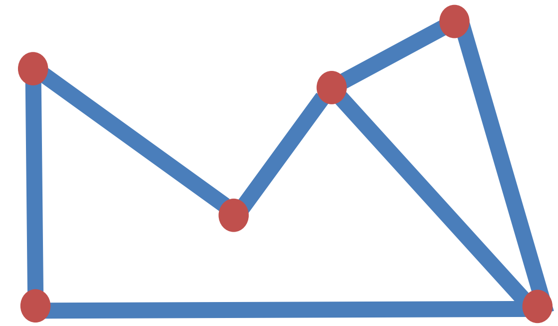
Minimally rigid framework: $N_f = N_c \quad \Rightarrow \quad 2N - 3 =< z_c > N/2 \Rightarrow < z_c >= 4 - 6/\mathrm{N}\ (2d)$

Rigid framework: $N_f < N_c$ (overconstrained)

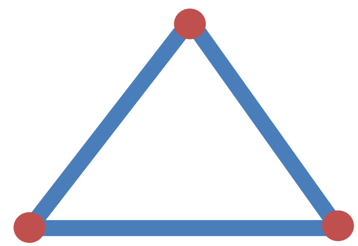
Examples of flexible and rigid graphs

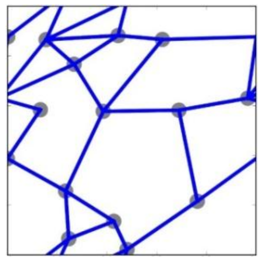
Flexible



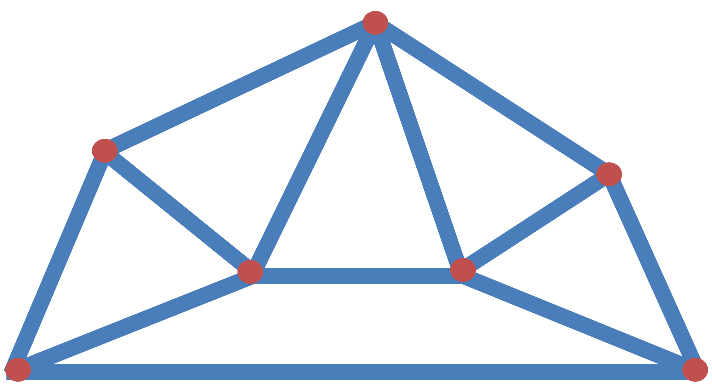


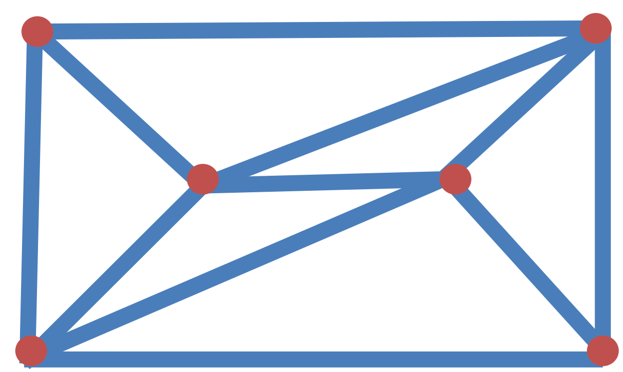
Minimally rigid





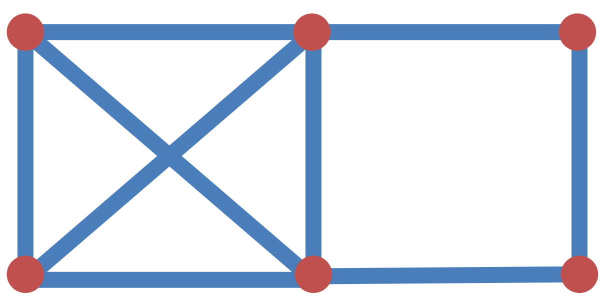
Overconstrained



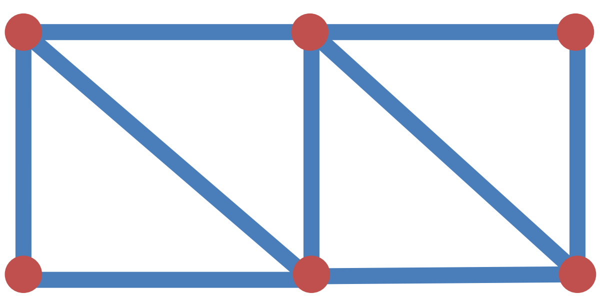


Beyond Maxwell counting

Laman's Theorem: *A 2 dimensional graph G that has N vertices is minimally rigid iff it has 2N-3 edges, and every subgraph with m vertices contains no more than 2m-3 edges.*



Satisfies Maxwell counting
but it is not minimally rigid

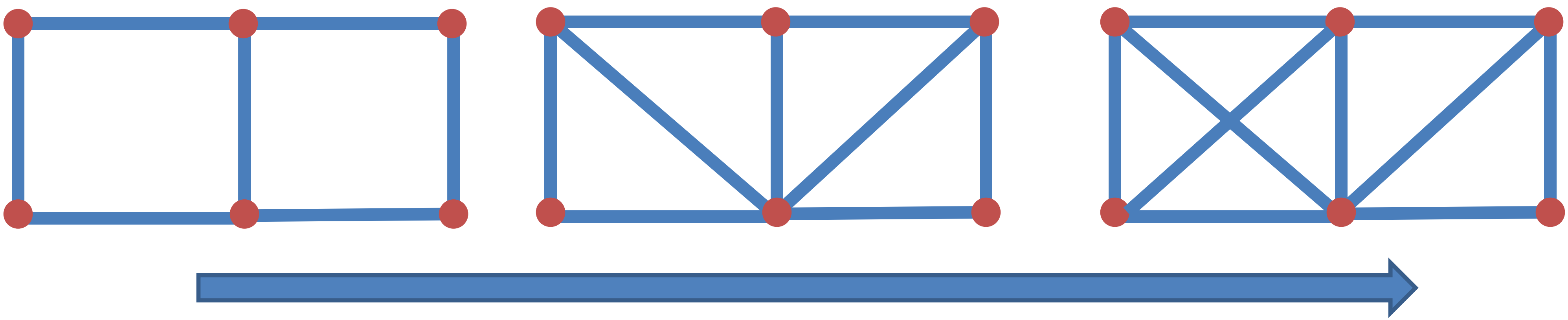


A minimally rigid (Laman) graph

J. Engineering Mathematics **4**: 331 (1970)

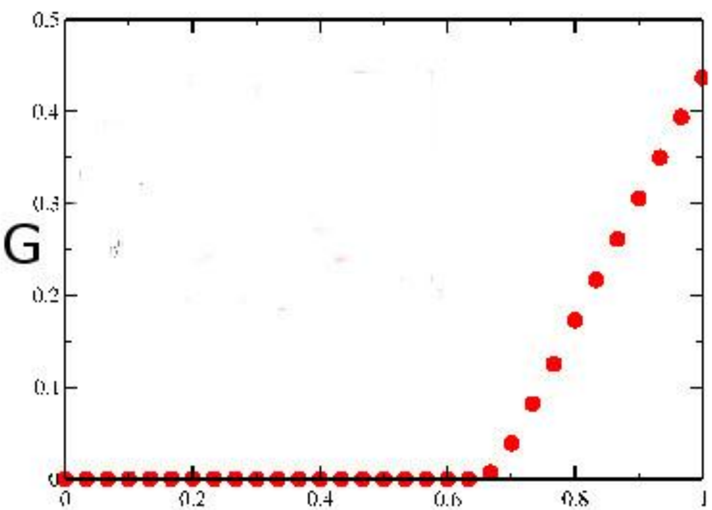
How does a bar-joint system becomes rigid?

Flexible | Minimally rigid | Rigid



Phase Transition

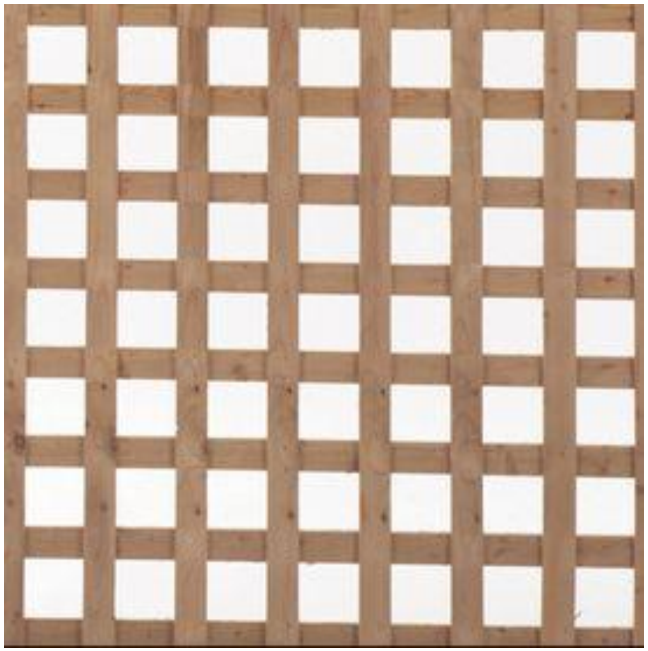
Shear modulus G



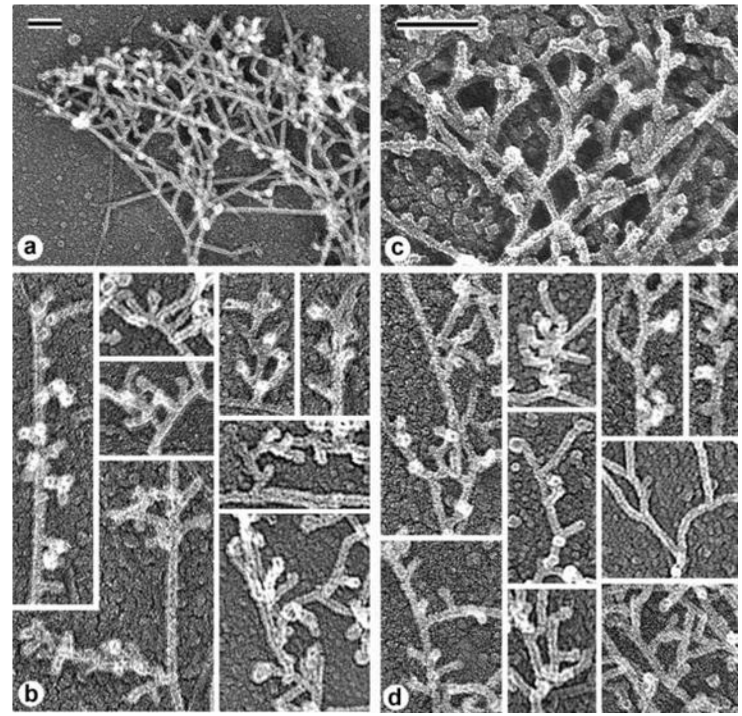
$\langle z \rangle / 6$

S. Feng, M. F. Thorpe, E. Garboczi, *Phys. Rev. B* **31**, 1 (1985)

Phase transitions: So what?



0.1 micron

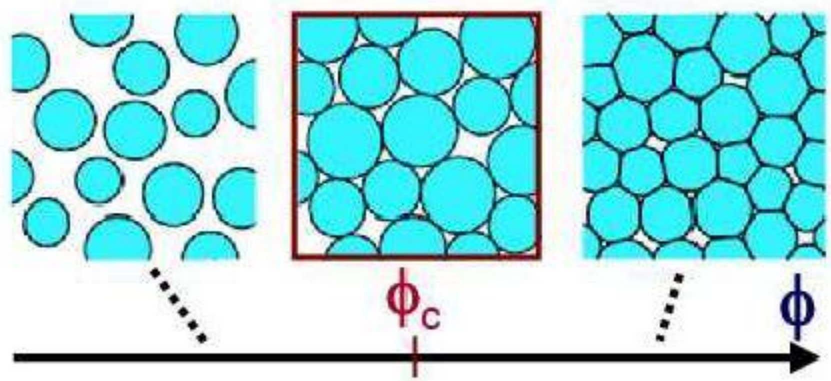


Svitkina/Borisy *J. Cell Biol.* (1999)

What is the most “efficient” way to make these mechanical networks rigid?

From frameworks to sand (coffee, actually)

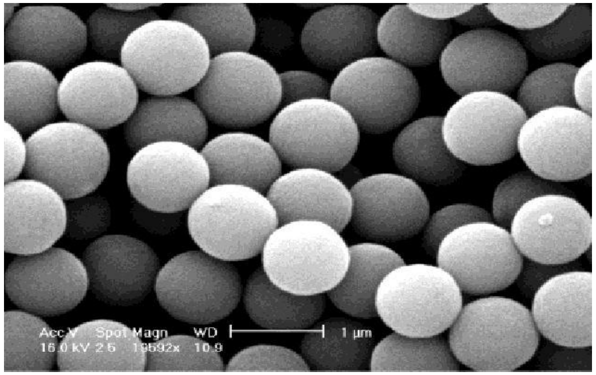
E. Brown, N. Rodenberg,…,H. Jaeger, *PNAS* (2010)



Jamming Transition

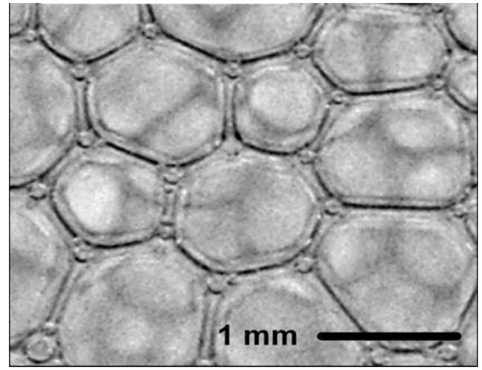
Further experiments on Jamming

Colloids (TiO_2)



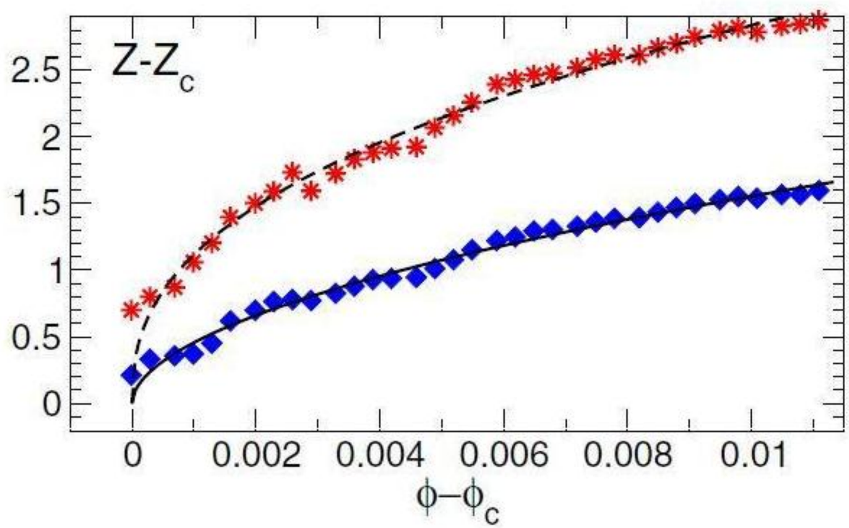
S. Eiden-Assmann, *et al. Chem. Mater*. (2004)

Foams



Doug Durian group

M&M's



Experiments on *frictional*, photoelastic disks

$$Z_c = 3.04, \phi_c = 0.8422(5)$$

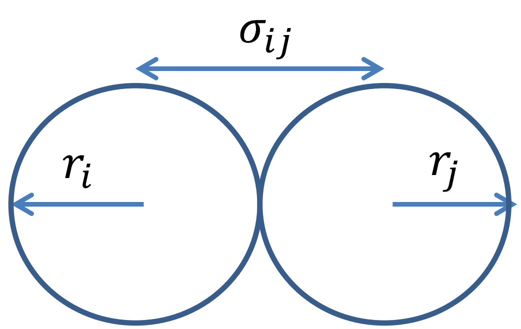
R.P. Behringer, D. Bi, B. Chakraborty, *et al. J. Stat. Mech.* (2014)

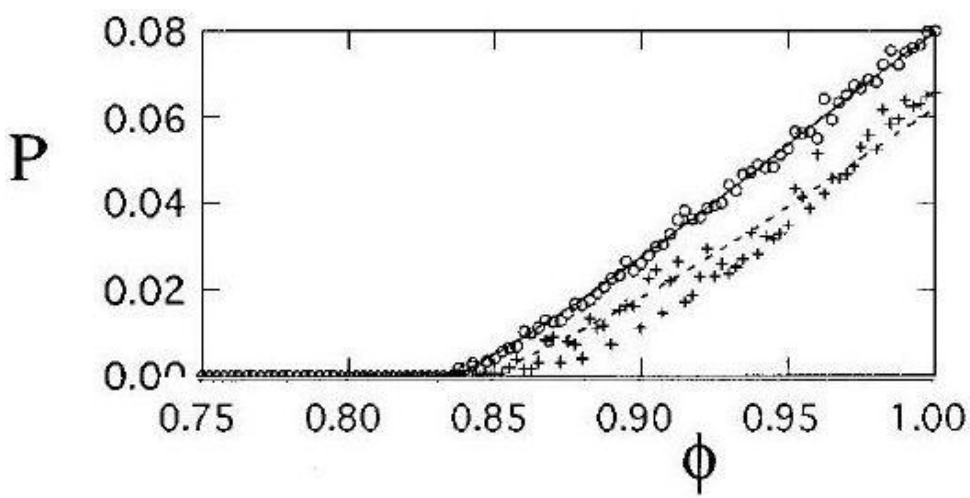
Numerical Studies of jamming

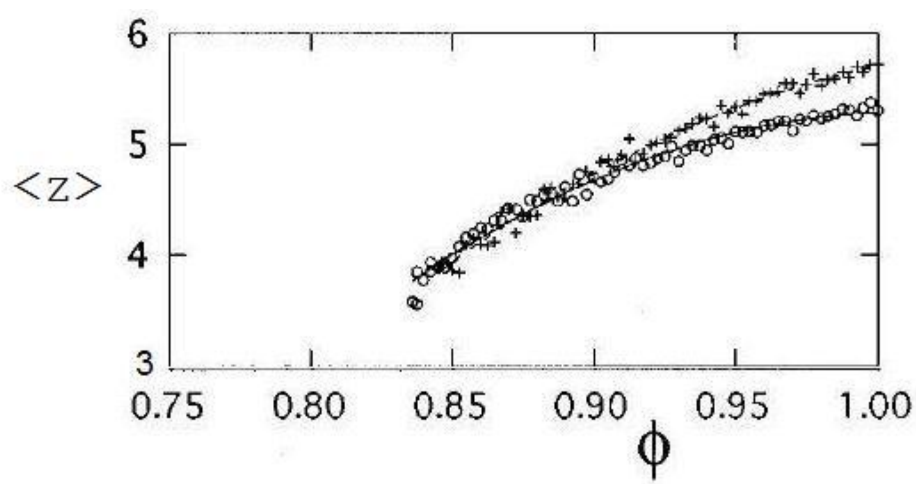
Repulsive soft particles

$$V(r_{ij}) = \begin{cases} \frac{\varepsilon}{\alpha}\left(1 - \frac{r_{ij}}{\sigma_{ij}}\right)^{\alpha} & r_{ij} < \sigma_{ij} \\ 0 & r_{ij} \geq \sigma_{ij}, \end{cases}$$

$\alpha = 2$: Harmonic
$\alpha = 5/2$: Hertzian
$\alpha = 0$: Hard-sphere

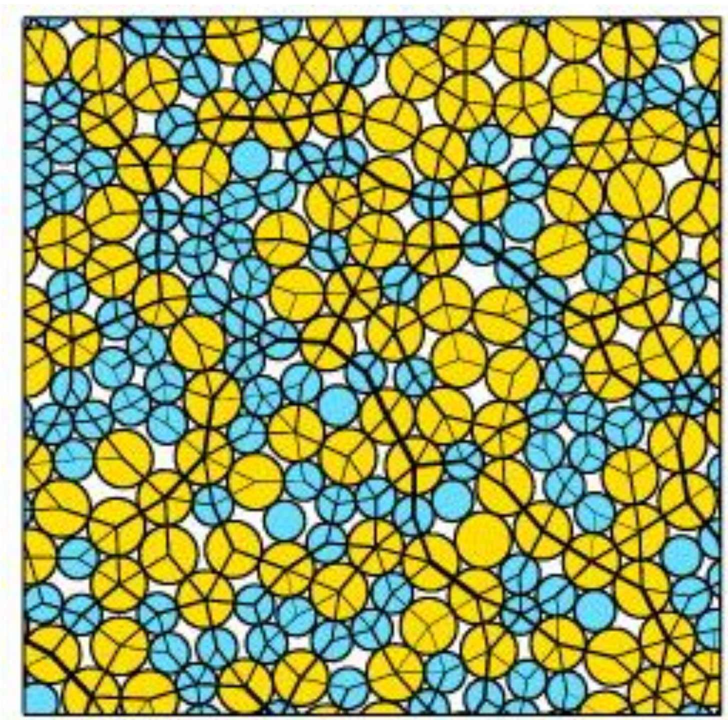






D. J. Durian. *Phys. Rev. E* **55**, 1739 (1996)

Rigidity aspects of Jamming



- Maxwell counting: $< z_c >= 4$

- There is a characteristic length l^* above which the packing is rigid and below it is not

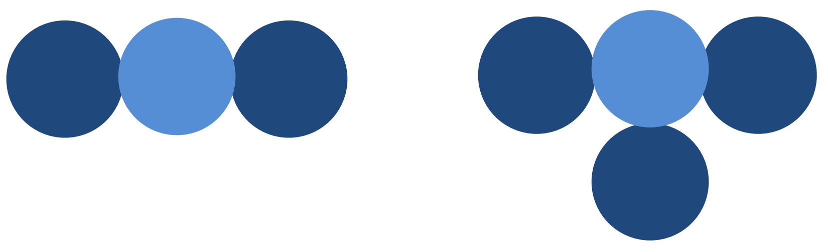
$$l^* \propto \frac{1}{< z > - < z_c >}$$

C. P. Goodrich, *et al. Soft Matter* (2013)

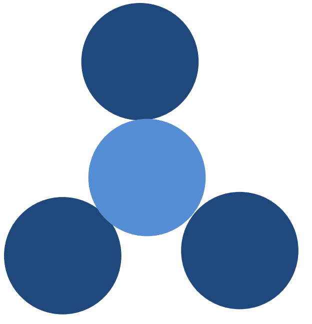
How to extend *rigidity theory* to more closely resemble *jamming*?

Jamming systems do not have fixed connectivity!

For purely repulsive particles, to obtain mechanically stable particle packings, we must impose local mechanical stability.



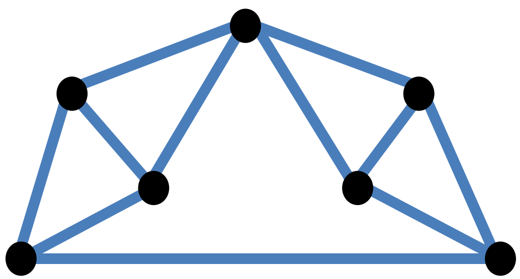
Not mechanically stable



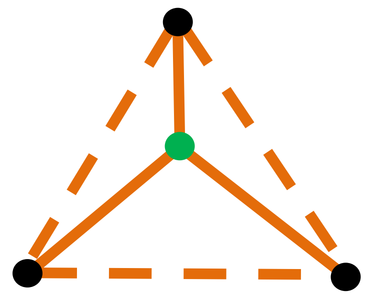
Mechanically stable

Jamming graphs

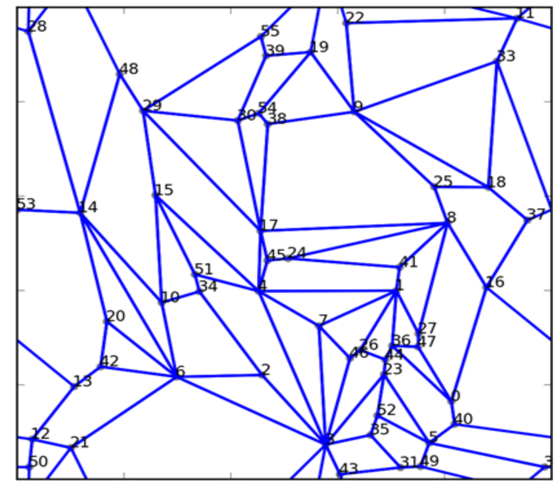
Minimal Rigidity + *Local Stability* = *Jamming Graph*

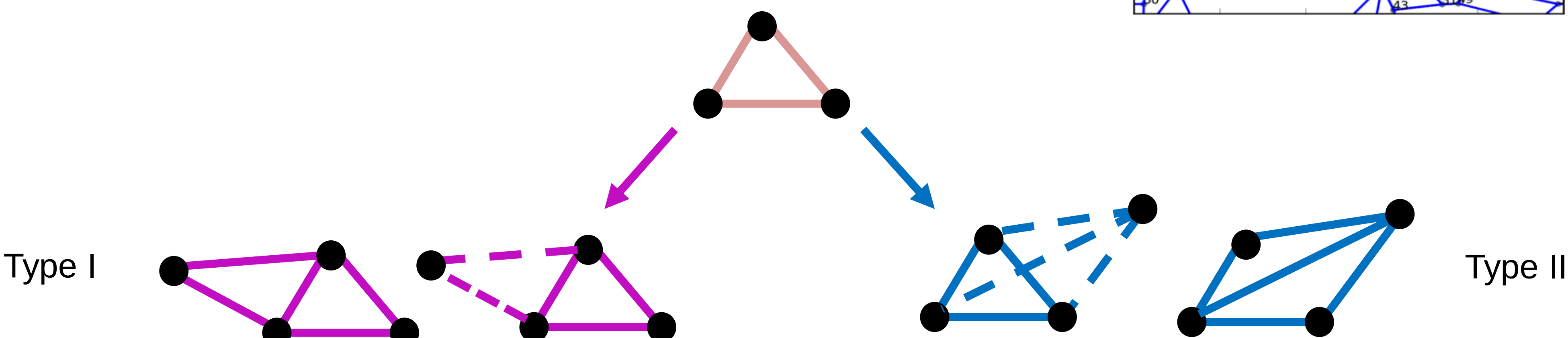


Laman Graph



Counter Balancing



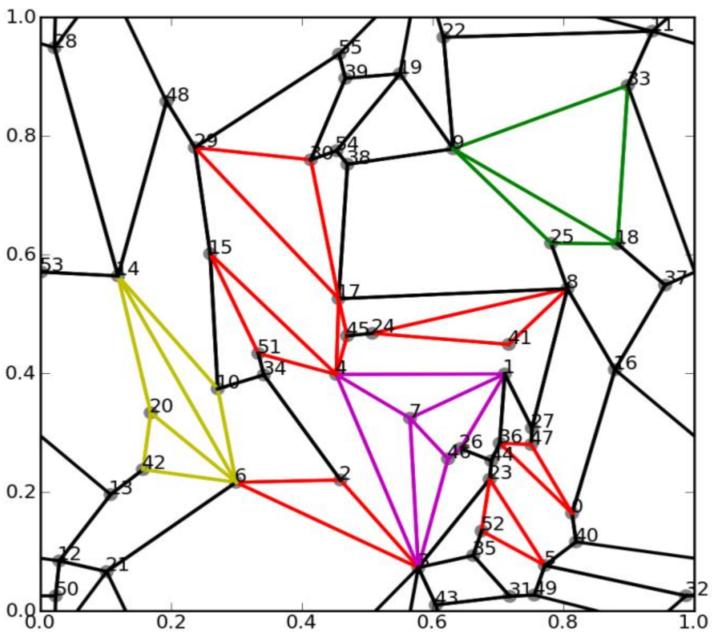


Henneberg steps: *Local rules*

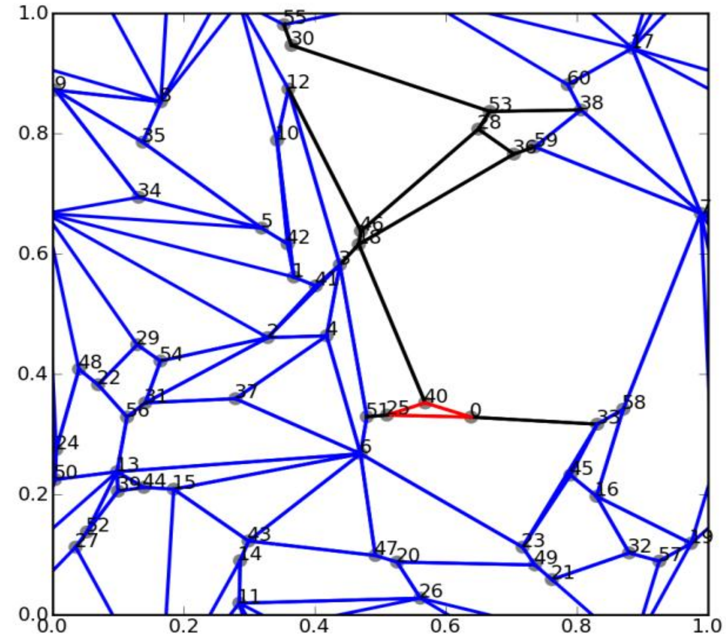
Henneberg, manuscript (1911)

Perturbing jamming graphs

Many small rigid clusters



Big rigid cluster

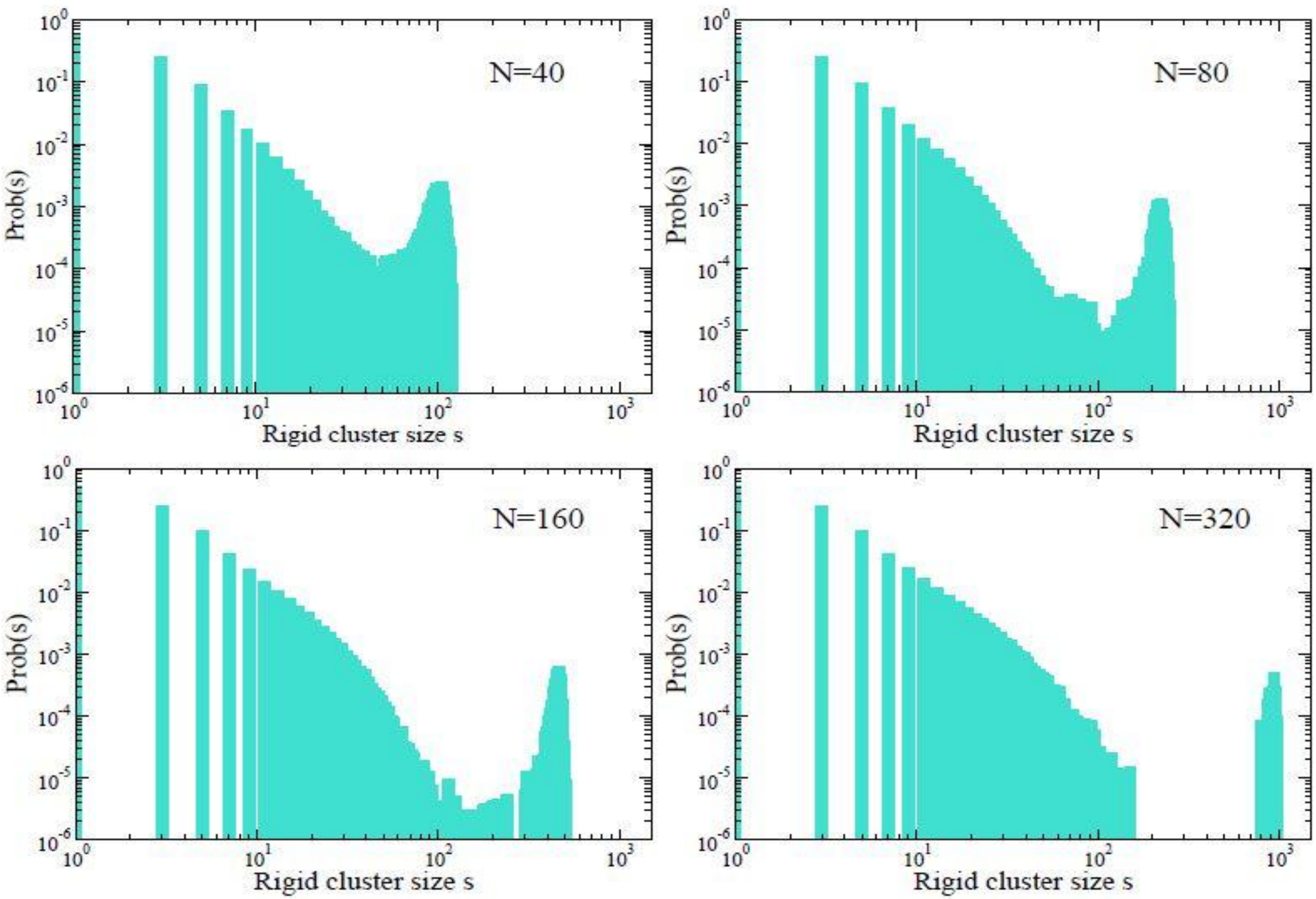


Cluster Sizes: 1 3 5 7 9 101

Let us do some statistics.

J. H. Lopez, L. Cao, JMS, *Phys. Rev. E* (2013)

Rigid cluster size distribution



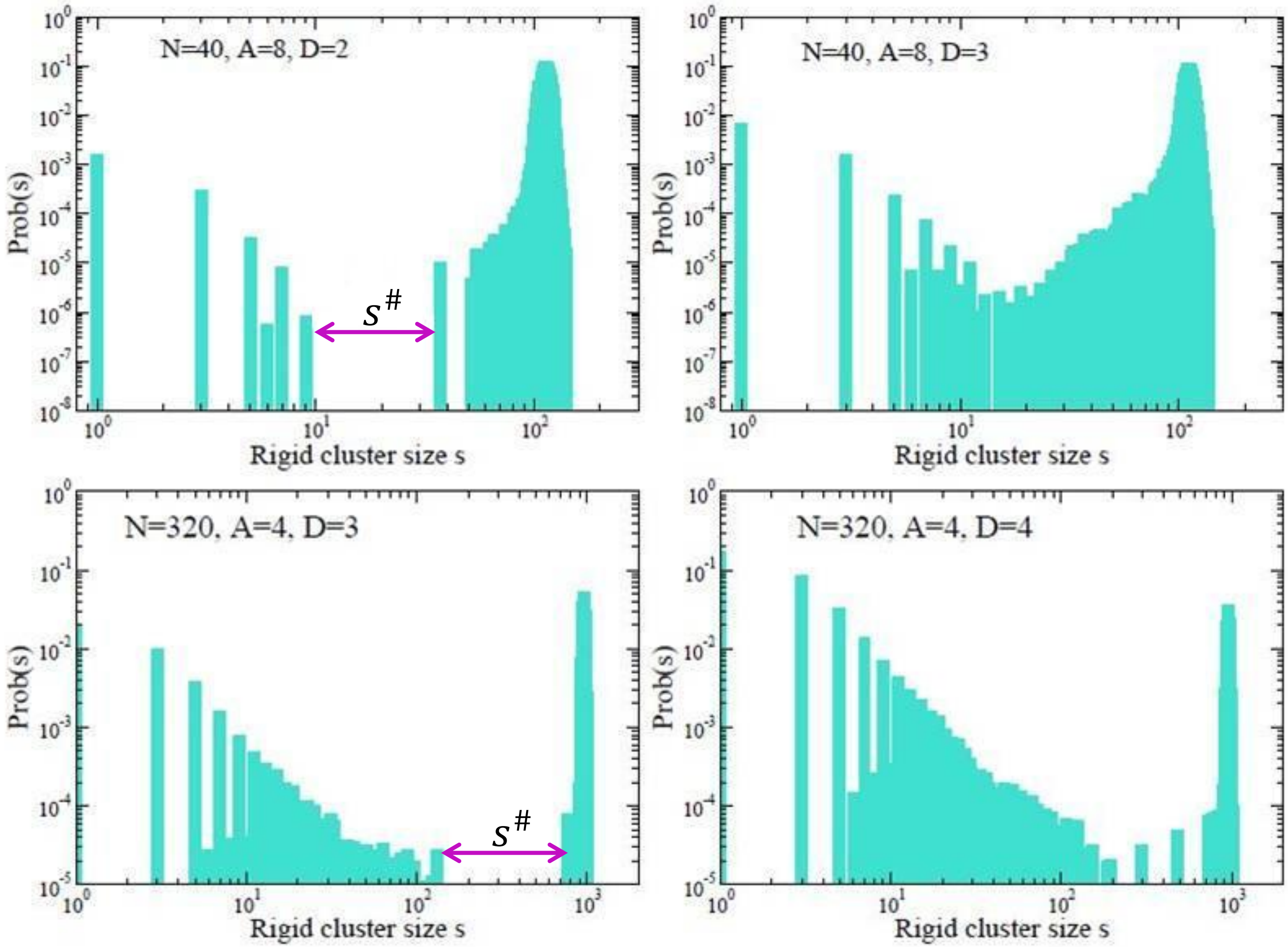
N: Initial number of vertices

Broad distribution (of small clusters) + characteristic peak (big cluster)

Going beyond the jamming transition

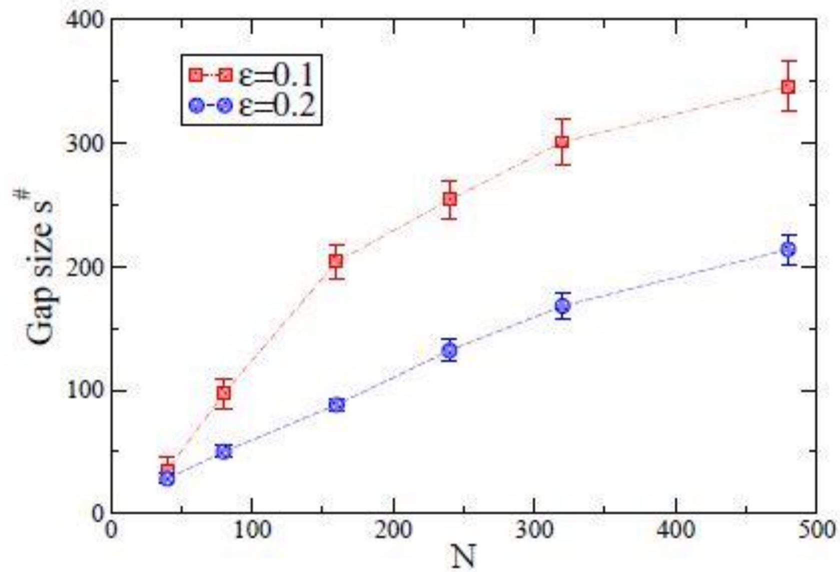
A: Additional bonds beyond Min. Rigid
N: Initiall number of vertices

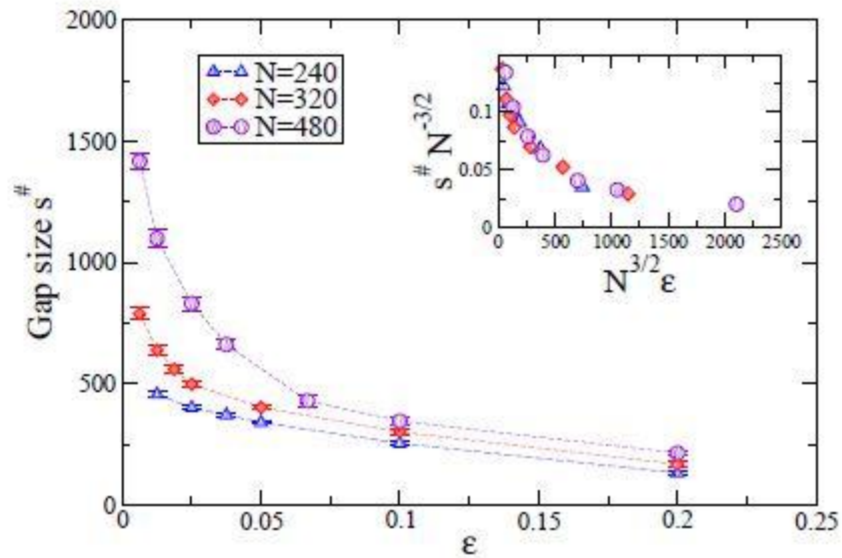
D: Number of deleted bond
$\varepsilon = A/N$



$$s^{\#} \sim \left(l^{\#}\right)^2$$

Extraction of a diverging correlation length





$$\left.\begin{array}{r} l^{\#} \sim \varepsilon^{-\rho} \\ l \sim \varepsilon^{-\nu} \end{array}\right\} \quad l^{\#} \sim l^{\rho/\nu}$$

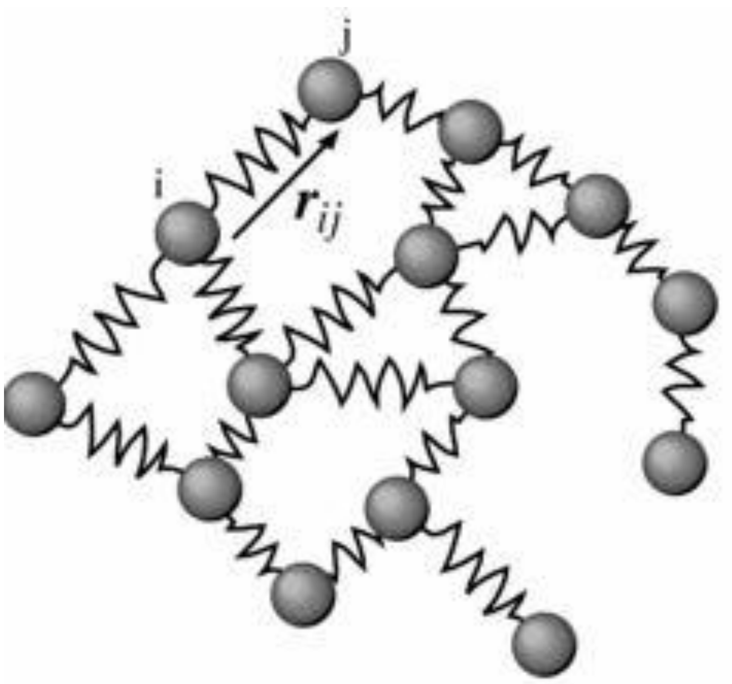
$$l^{\#} = l^{\rho/\nu} \mathrm{f}\left(l^{1/\nu} \varepsilon\right) \longrightarrow \boxed{\nu = 1/3}$$

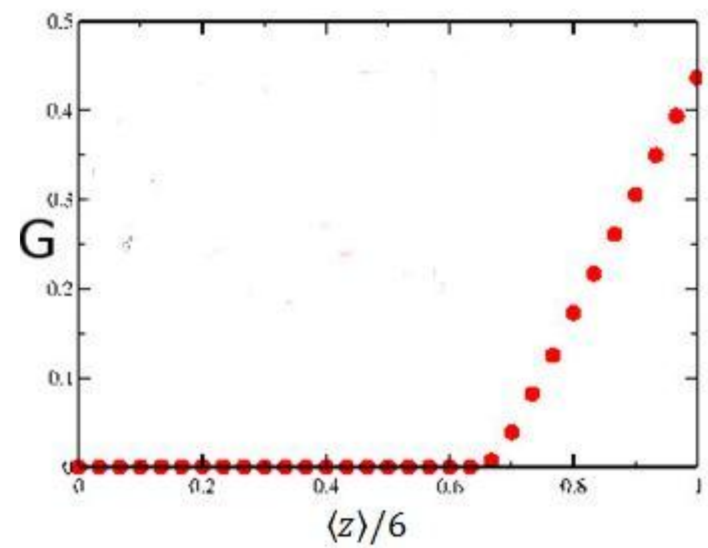
More questions than answers

- Created an algorithm to generate the “invention” of the jamming graph
- Jamming graph fluidizes into either many “small” rigid clusters (catostrophic failure) with a characteristic peak in the rigid cluster size distribution at the order of the system size
- Can this new exponent of 1/3 be observed in jamming systems?
- Given the jamming graph, can we now input particles? The circle packing theorem (Thurston and others) may help us.
- Can we define a sequence of moves (edge additions and deletions) to get back to minimal rigidity (after fluidizing)?

Rigidity percolation

Study of the mechanics of a disordered spring networks and the search for phase transitions in the mechanics





$$E_{spring} = \frac{\mu}{2} \sum_{\langle ij \rangle} p_{ij} (\vec{u}_{ij} . \hat{r}_{ij})^2$$

Effective medium theory for effective spring constant yields:

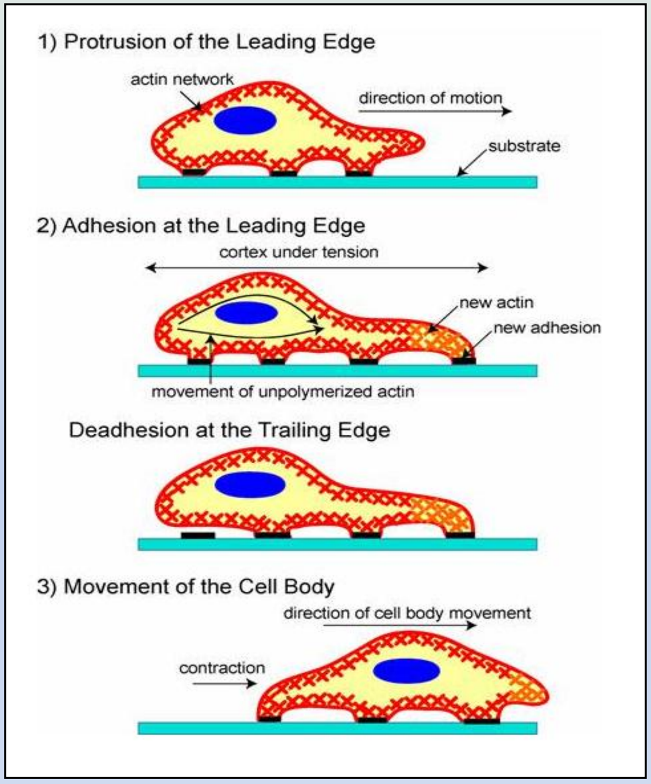
$$\mu_m = 3\mu(p - \frac{2}{3})$$

$$for \quad p \geq \frac{2}{3}$$

How can we go beyond a mean field theory for the mechanics of disordered linear spring networks to include angular spring?

Why would we want to include angular springs?

How does a cell crawl?

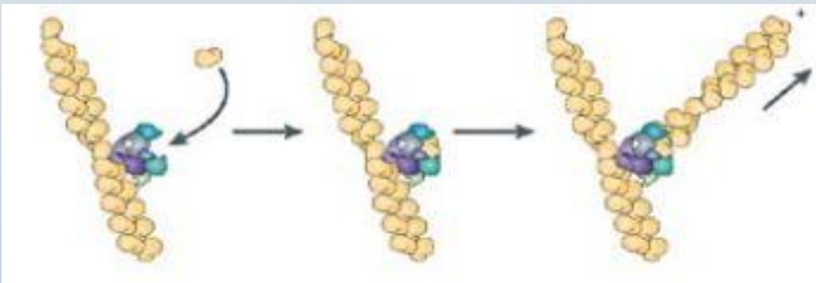


David Rogers of Vanderbilt University in the 1950s

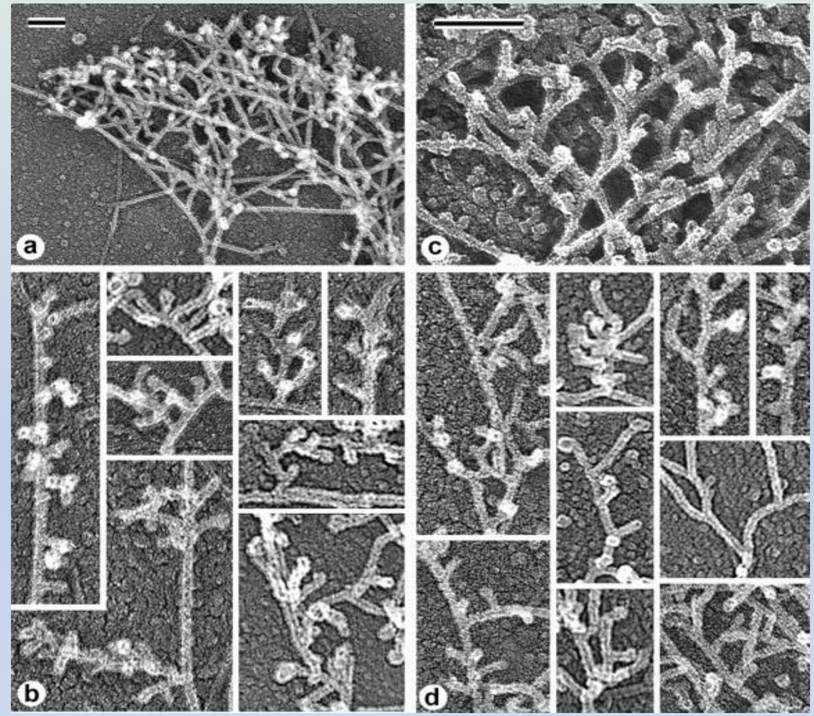
The protrusion/extension at the leading edge of a crawling cell is driven by monomeric actin polymerizing into *biopolymers.*
The polymerization is directional due to actin's inherent *polarity.*

Branched, anisotropic actin networks

New actin biopolymers are generated via branching driven by the Arp2/3 complex:

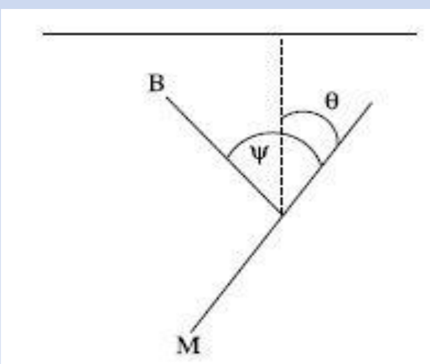


0.1 μm   0.1 μm

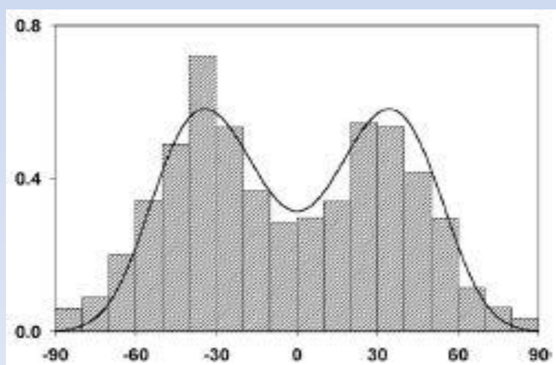


Svitkina/Borisy *J. Cell Biol.* (1999)

The branch angle (with respect to the mother filament) is rather regular:



D(θ)

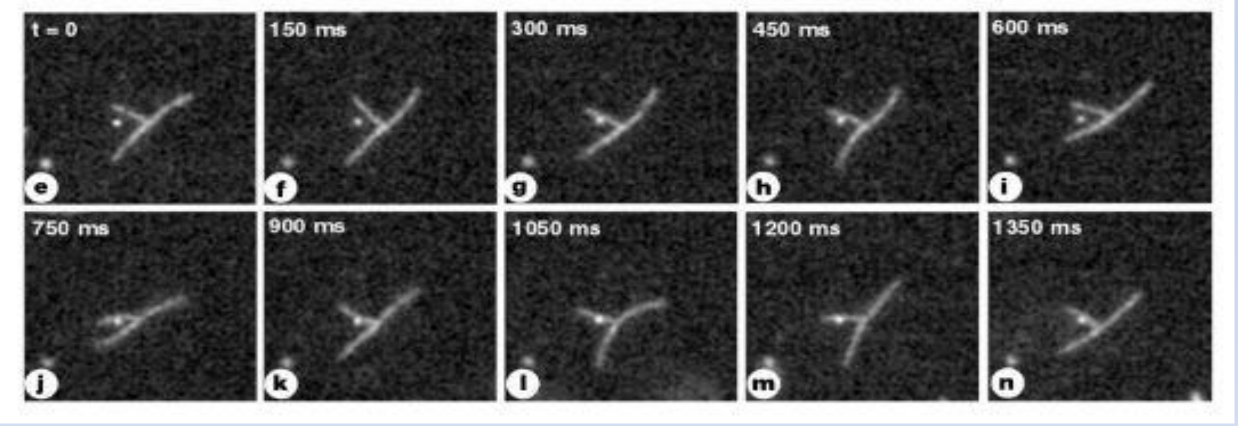


θ

Maly/Borisy *PNAS* (2001); Quint/JMS *J. Math. Biol.* (2011)

Question

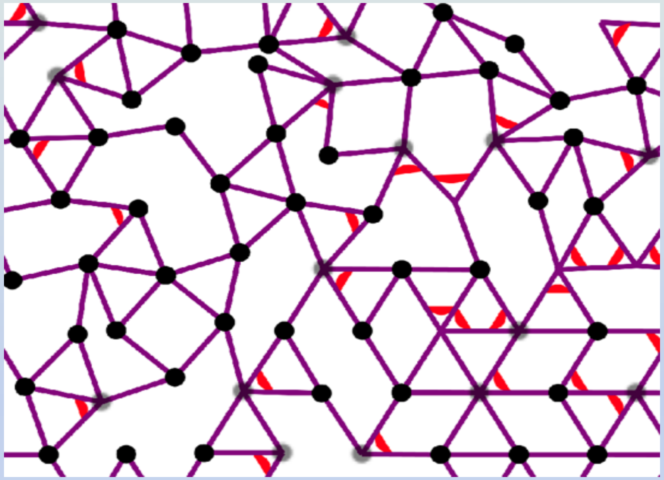
Arp2/3 as a generator of actin biopolymers has long been appreciated by scientists. What about its contribution to the mechanics of actin networks as an *angle-constraining y-linker*?

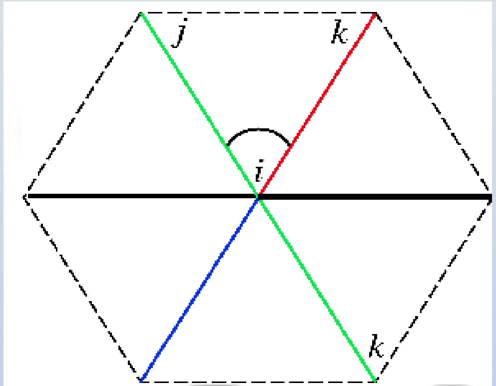


Blanchoin et al. *Nature* (2000)

Back up: What is the mechanical role of *angle-constraining* crosslinkers in *isotropic* actin networks?

Consider a model network:





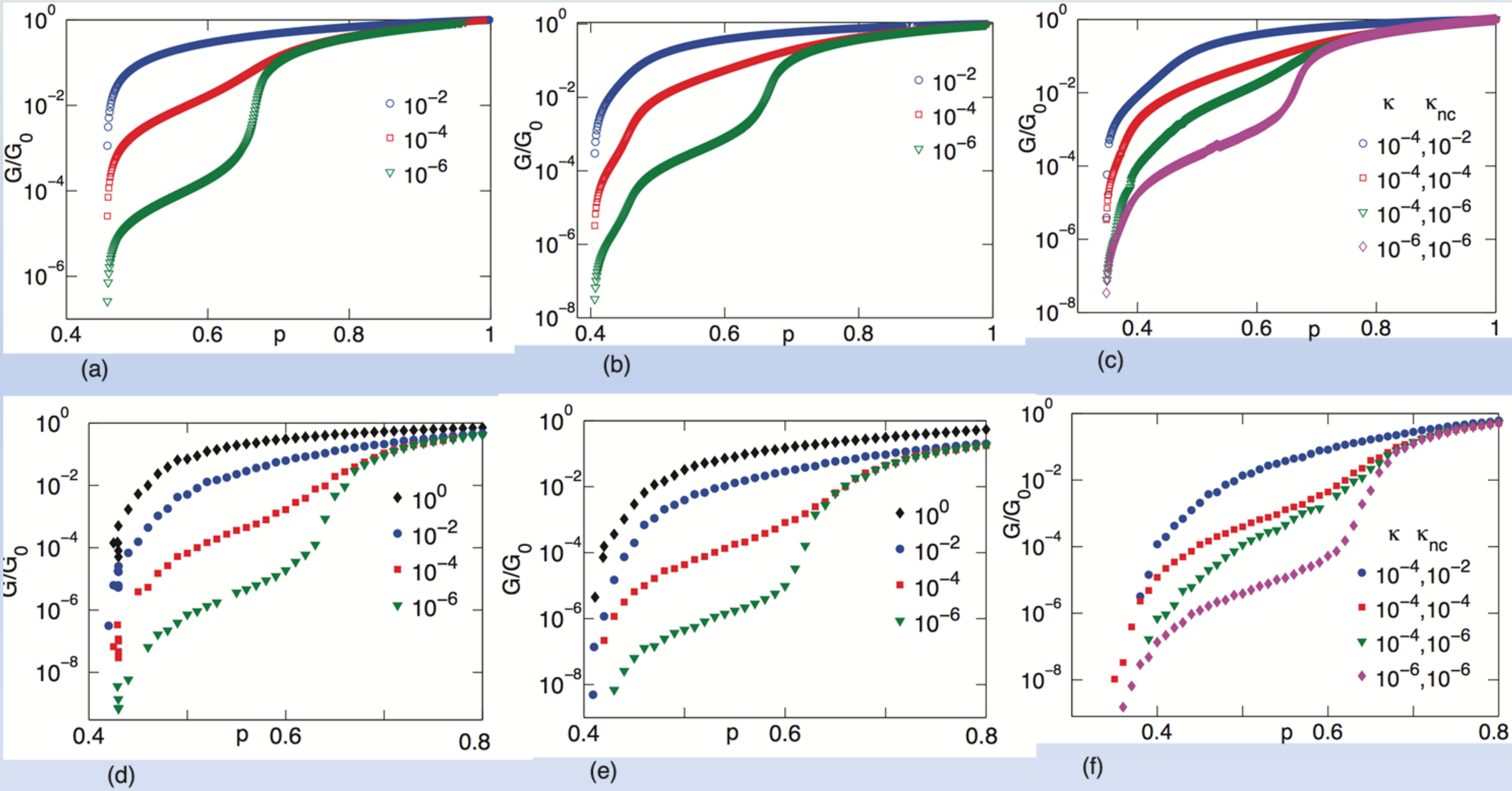
$$E = E_{spring} + E_{filament\ bending} + E_{non-collinear\ bending}$$

$$E_{spring} = \frac{\mu}{2} \sum_{\langle ij \rangle} p_{ij} (\vec{u}_{ij} . \hat{r}_{ij})^2$$

$$E_{filament\ bending} = \frac{\kappa}{2} \sum_{\langle jik \rangle} p_{ji} p_{ik} (\theta_{jik})^2$$

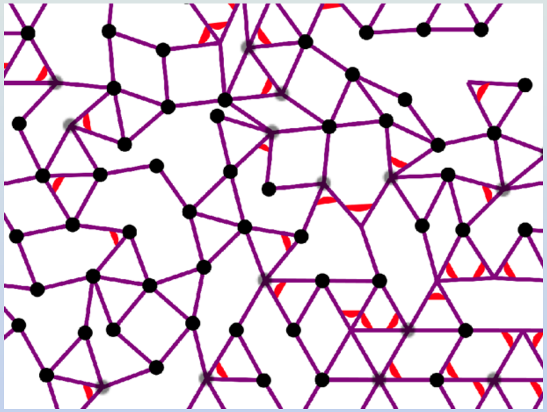
$$E_{non-collinear\ bending} = \frac{\kappa_{nc}}{2} \sum_{\langle jik' \rangle} p_{ji} p_{ik'} p_{nc} (\Delta\theta_{jik'})^2$$

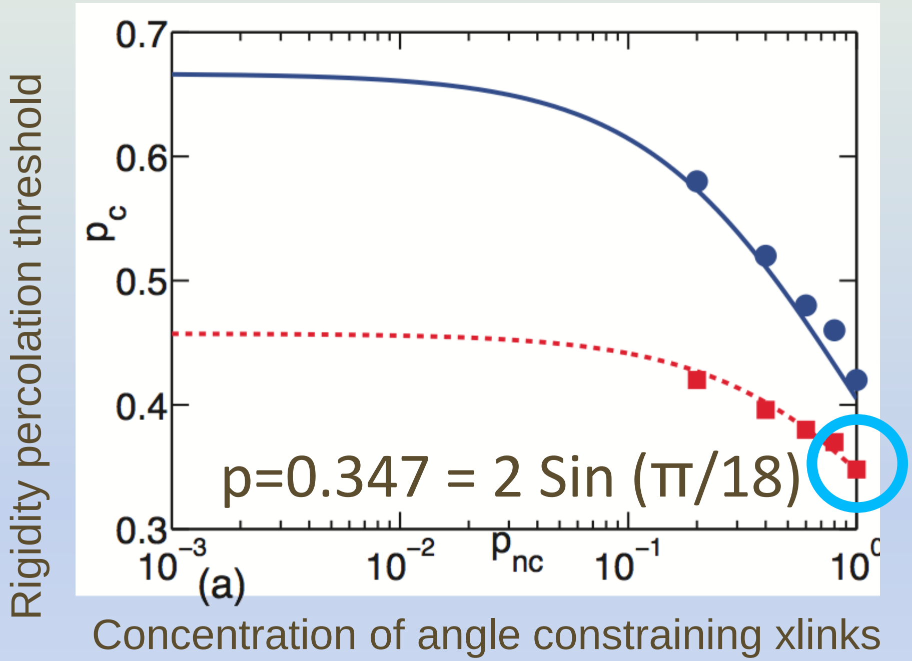
Comparison of EMT and numerics

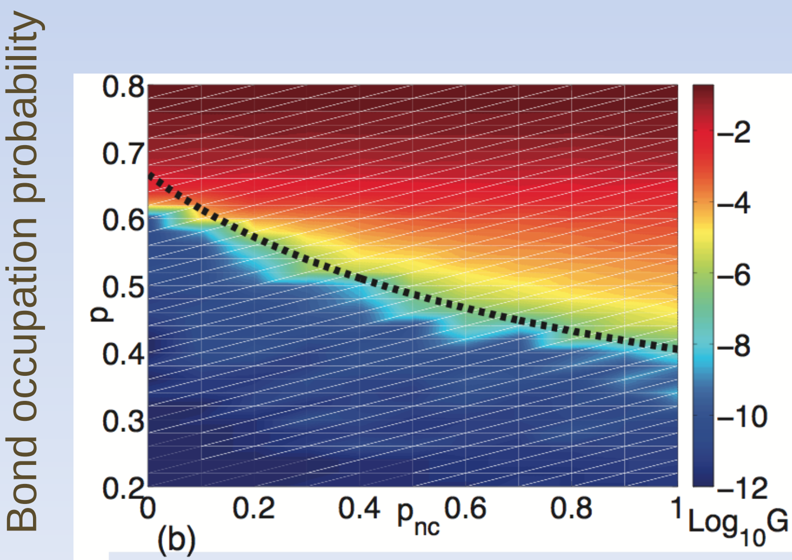


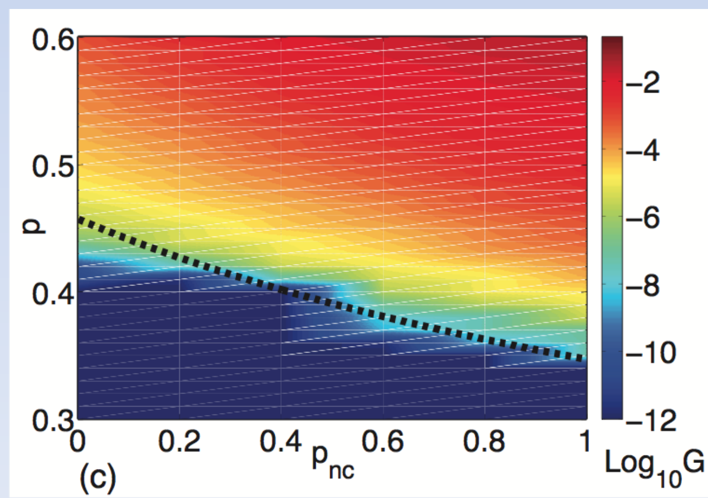
Das, Quint, JMS *PLoS ONE* (2012)

Network attains rigidity at minimal actin filament concentration and there exists cooperativity with the addition of a small amount of angle-constraining xlinkers

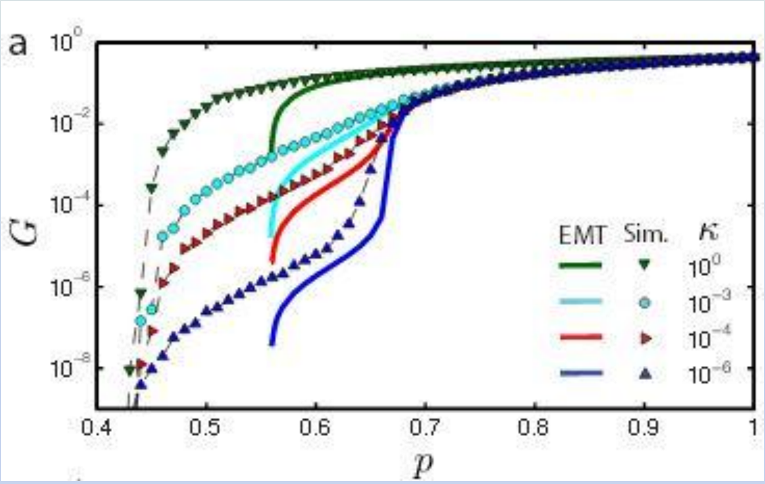




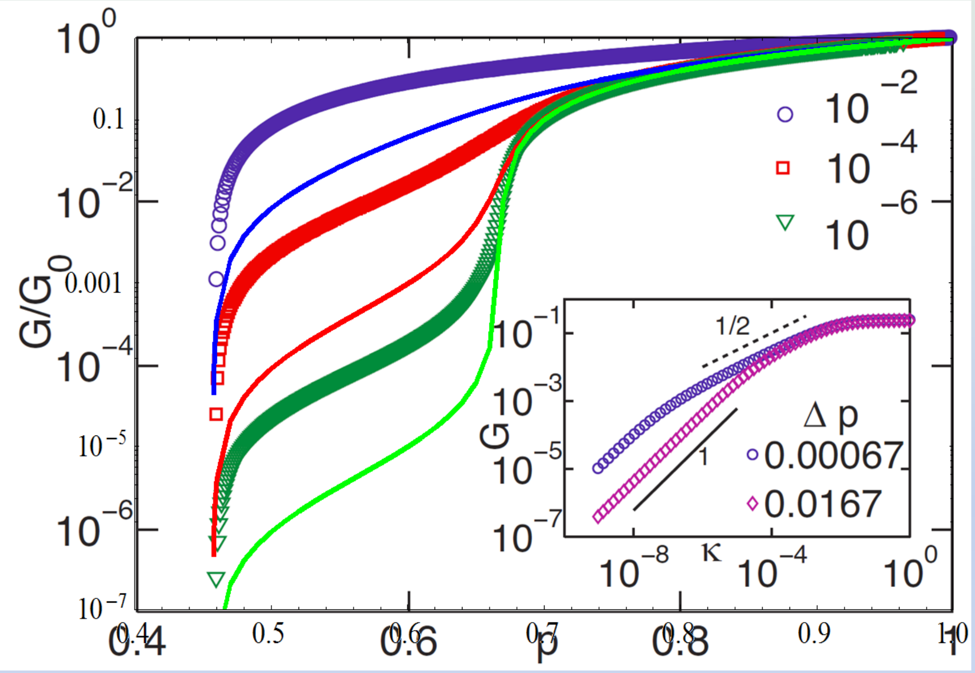




Footnote:



vs.



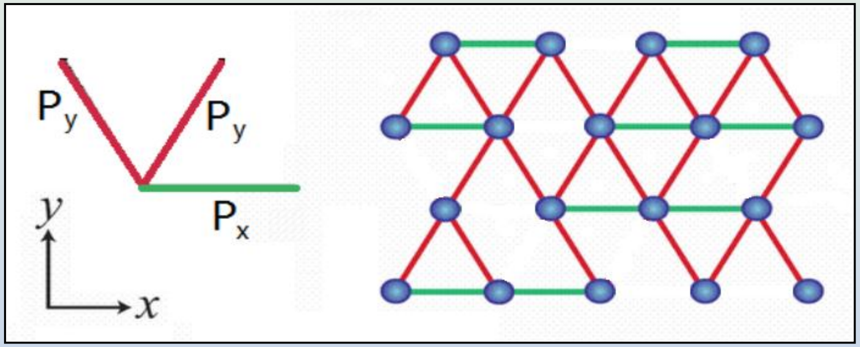
Broedersz, Mao, MacKintosh, Lubensky, *Nature Phys.* (2011)

Das, Quint, JMS, *PLoS ONE* (2012)

72 citations so far….

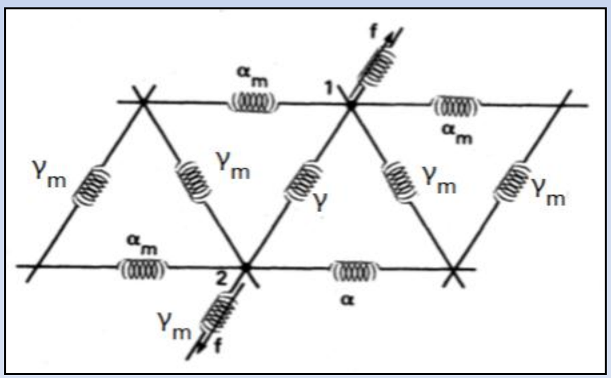
9 citations so far……

Model anistropic spring network



$$E_{spring} = \frac{\alpha}{2}\sum_{<ij>}\left[\vec{u}_{xij}\cdot\hat{r}_{xij}\right]^2 p_{xij} + \frac{\gamma}{2}\sum_{<ij>}\left[\vec{u}_{yij}\cdot\hat{r}_{yij}\right]^2 p_{yij}$$

Effective medium theory:

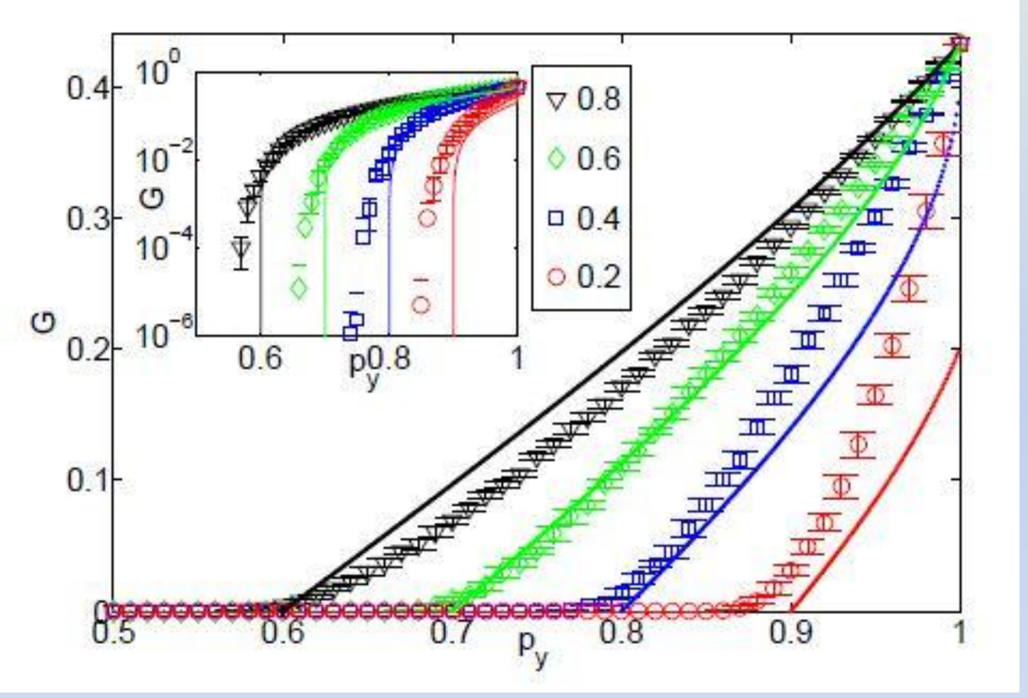


$$\boxed{\begin{aligned}\frac{\alpha_m}{\alpha} &= \frac{p_x - a^*}{1-a^*}\\ \frac{\gamma_m}{\gamma} &= \frac{p_y - b^*}{1-b^*}\end{aligned}}$$

The effective spring constants:

$$\boxed{\alpha_{eff} = \frac{\alpha_m}{a^*},\ \gamma_{eff} = \frac{\gamma_m}{b^*}}$$

Comparison of EMT and numerics



- Curvature in the curves in the solid phase
- Changing the amount of anisotropy changes the location of the transition
- When $p_x = 1$, the bulk modulus jumps from zero to non-zero at the transition, while the shear modulus increases from zero continuously

Zhang, JMS, Das: arXiv:1408.5910

Now on to anisotropic spring networks with linear and angular springs……..