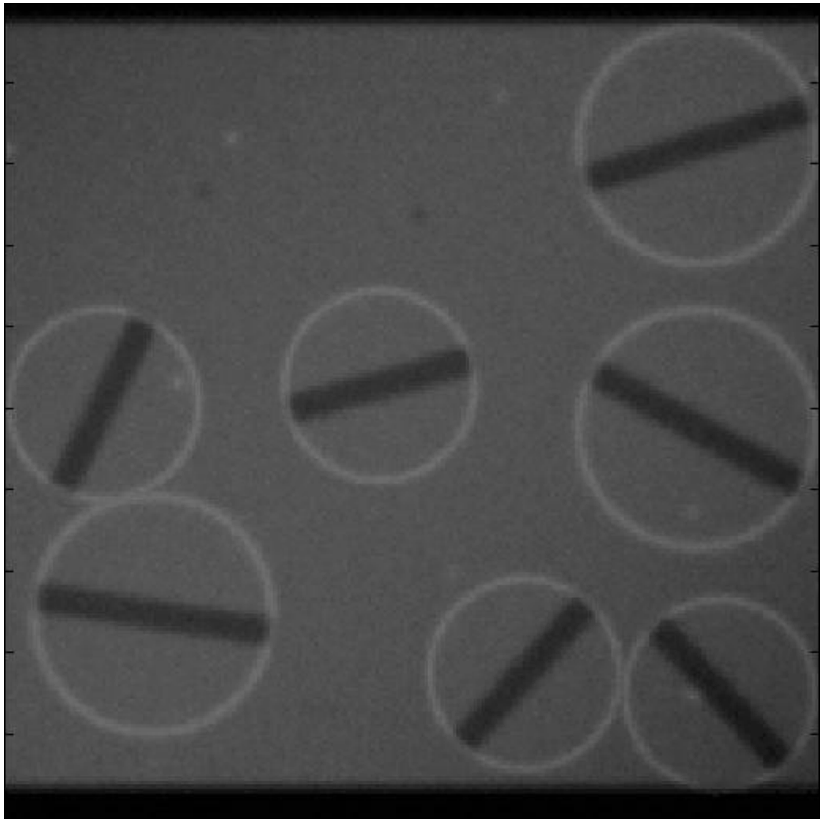
Frictional Families



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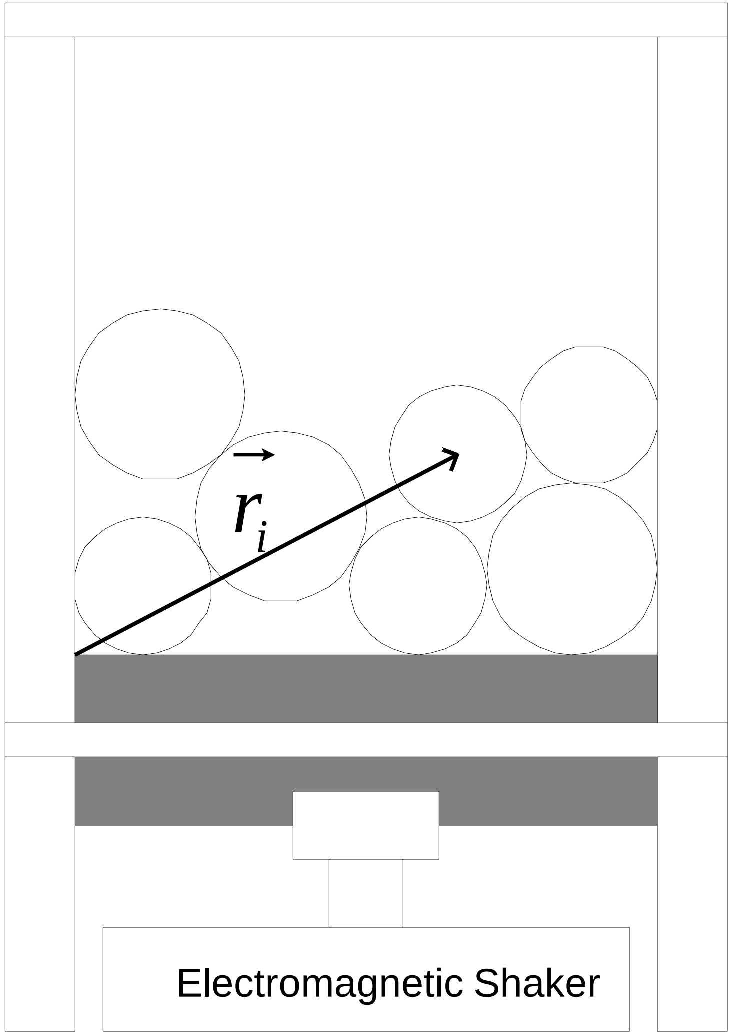
The City College of New York
OF THE CITY UNIVERSITY OF NEW YORK

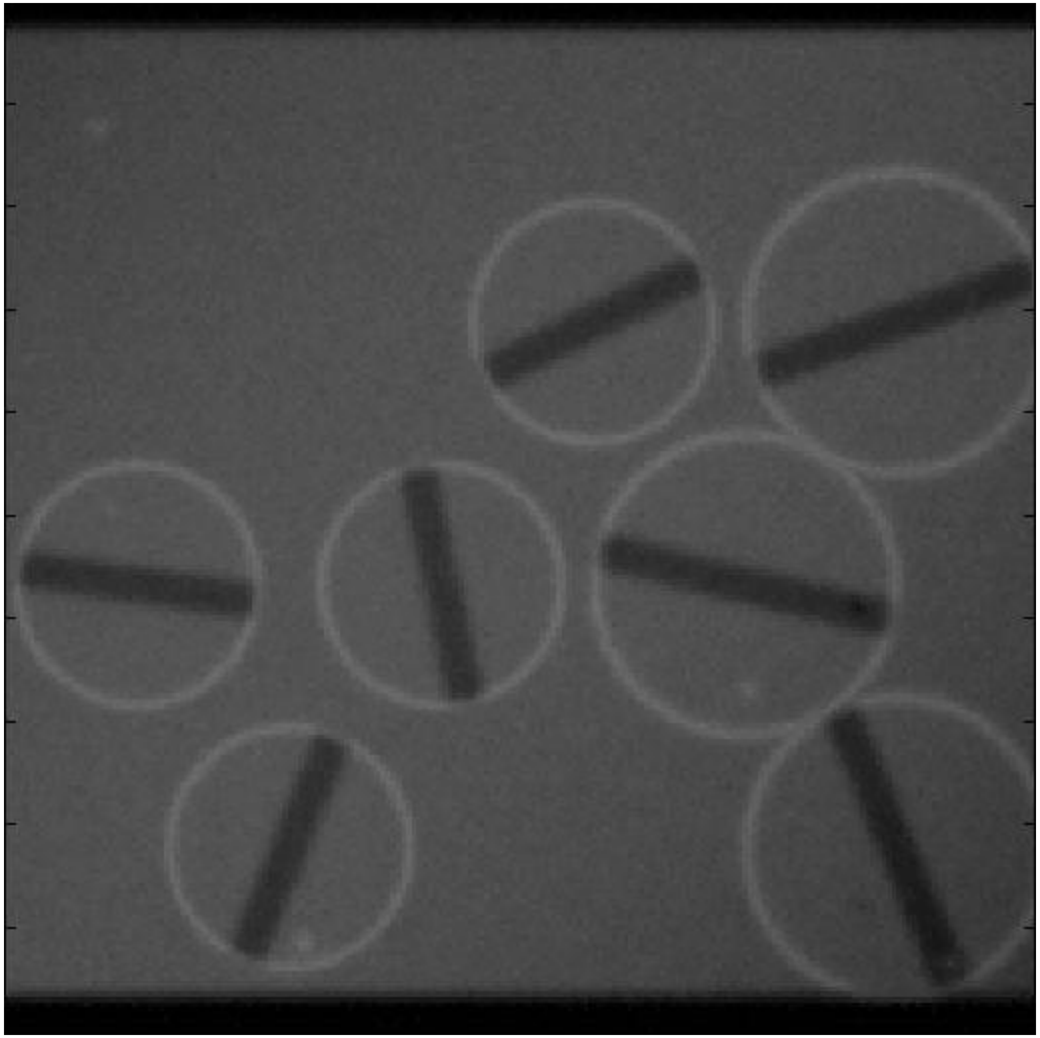
KITP 2014

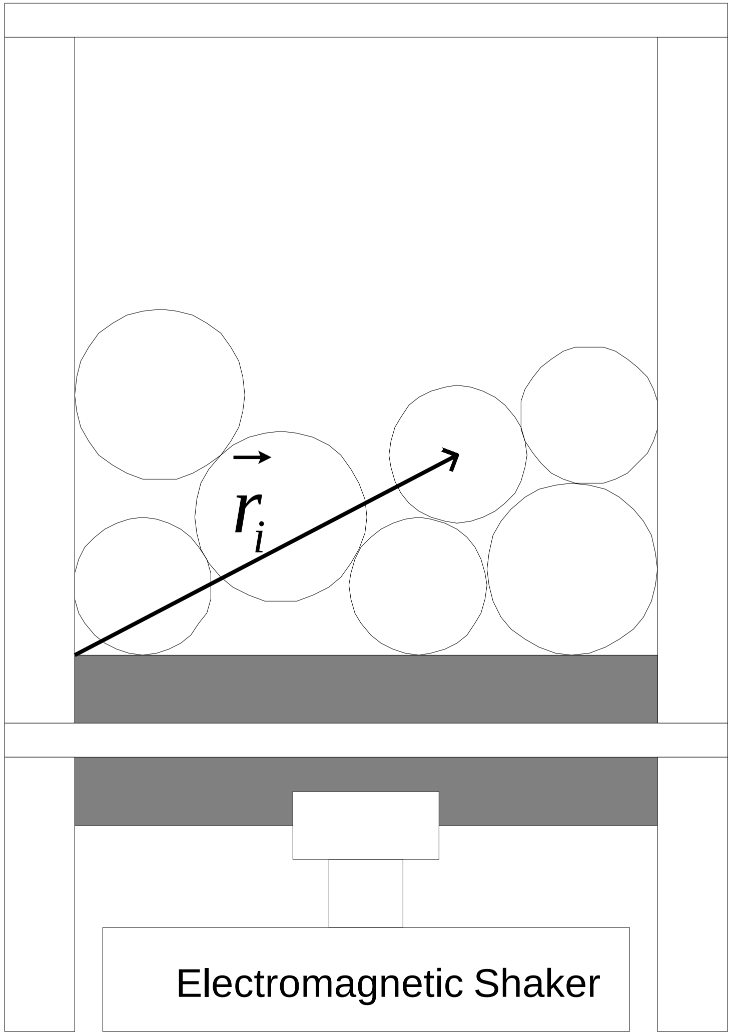
Experimental setup

- 7 1/8" thick (3.1mm) Photoelastic or stainless steel disks
  - 4 D_S = 1/2" (12.685mm)
  - 3 D_L = 5/8" (15.881mm)
  - D_L/D_S=1.25
- Thin (2D) container dimensions
  - Width=4.25D_S x Height=4.13D_S
- Driven from below
  - Oscillating sinusoidally (f=440Hz)
  - $y(t)=A\sin(2\pi ft)$
- High intensity monochromatic LED light source
- Crossed polarizers.



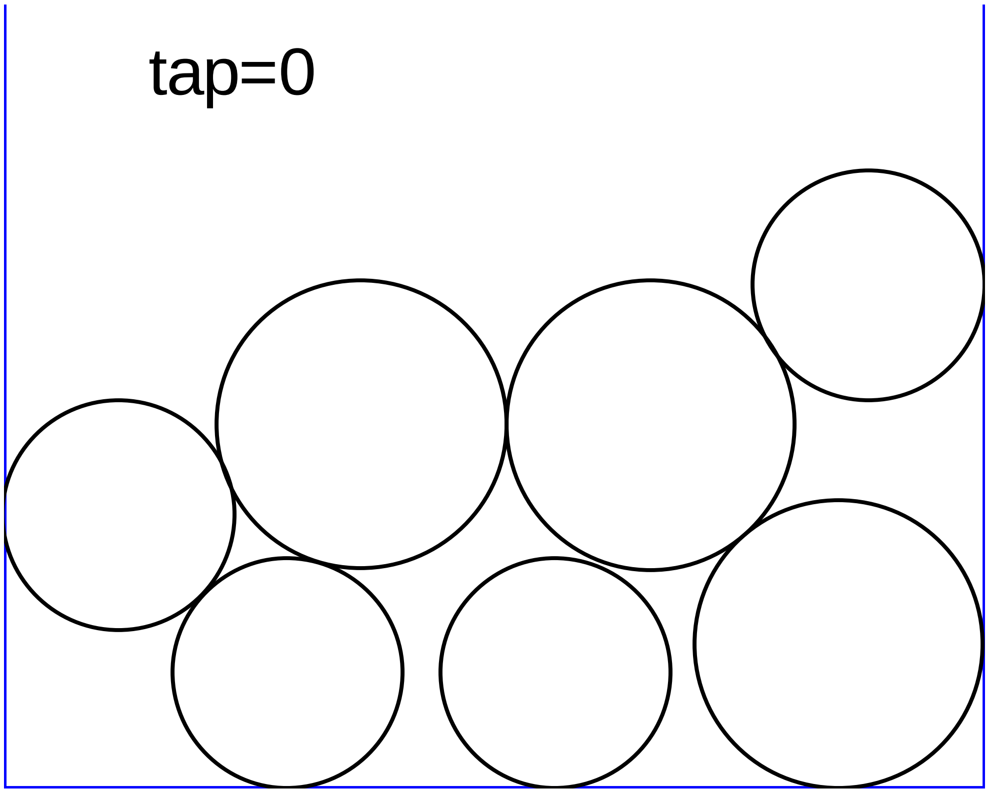
Friction Elimination





Frictional Packings

Friction



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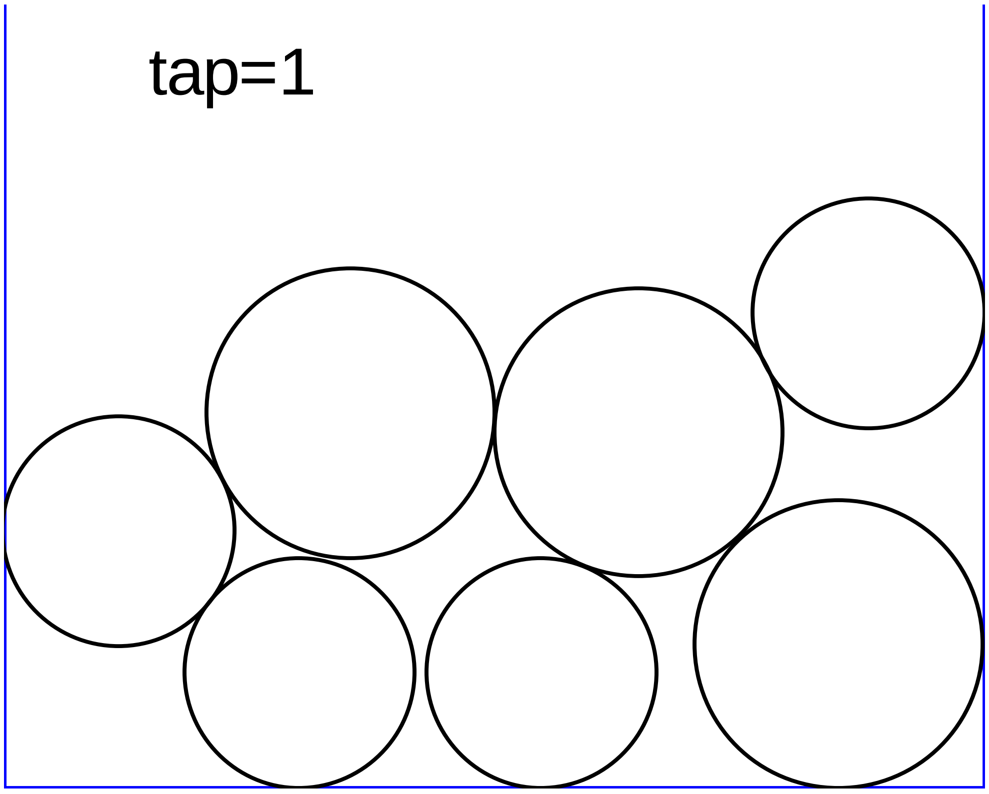

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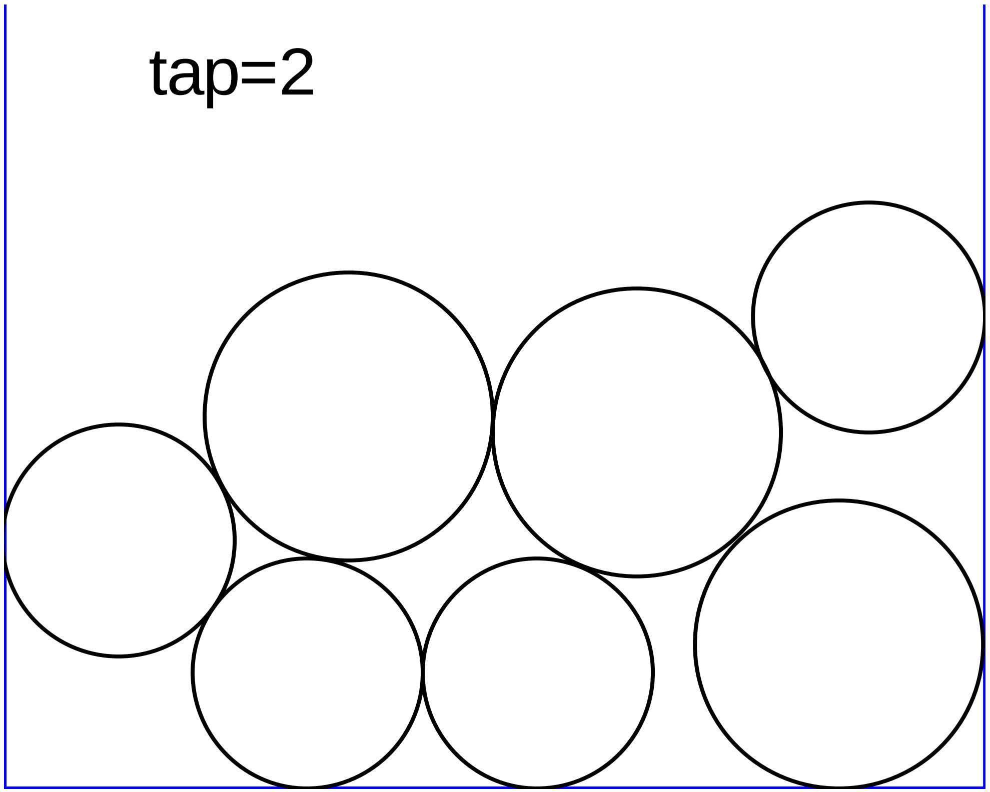

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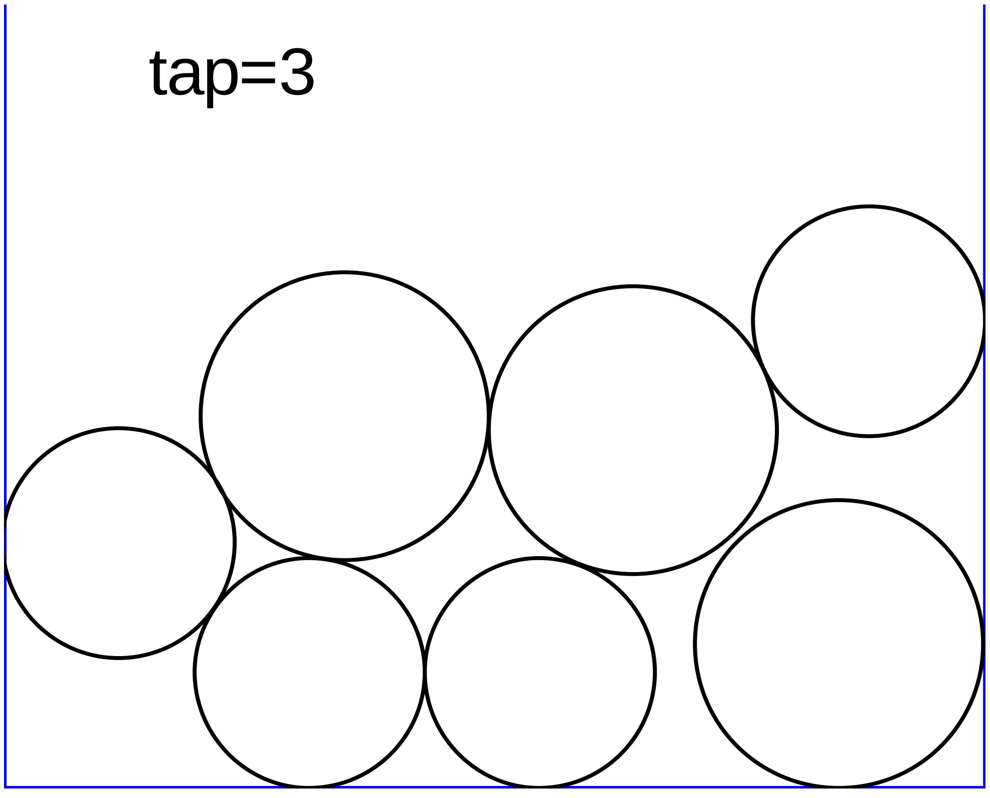

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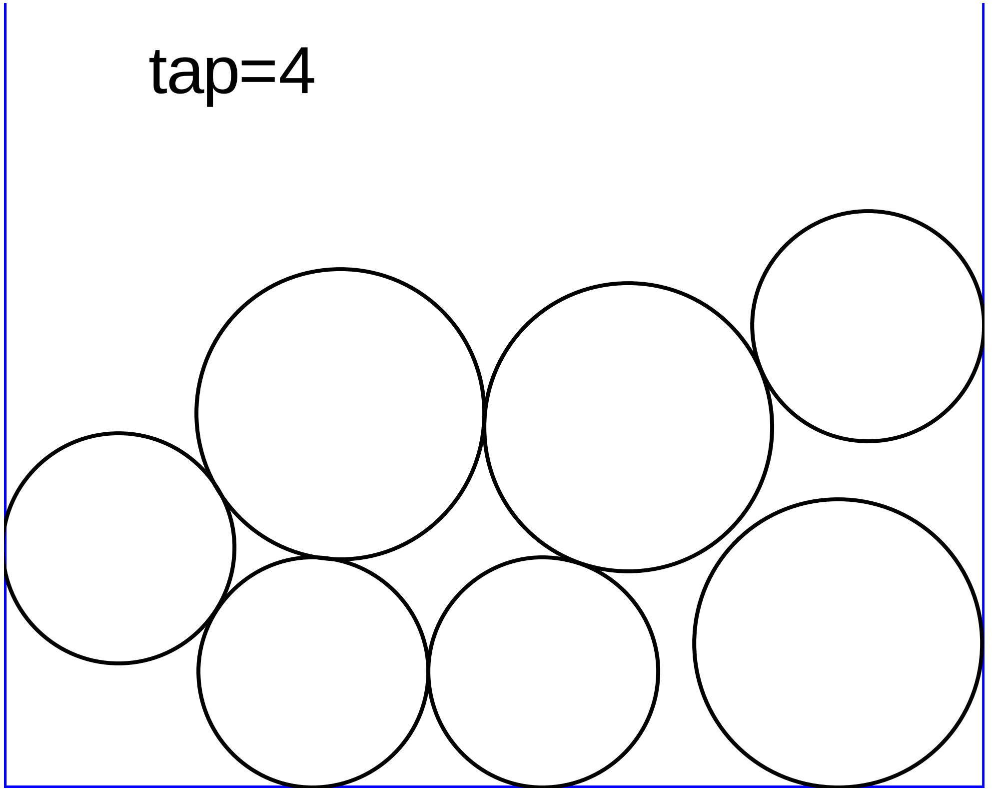

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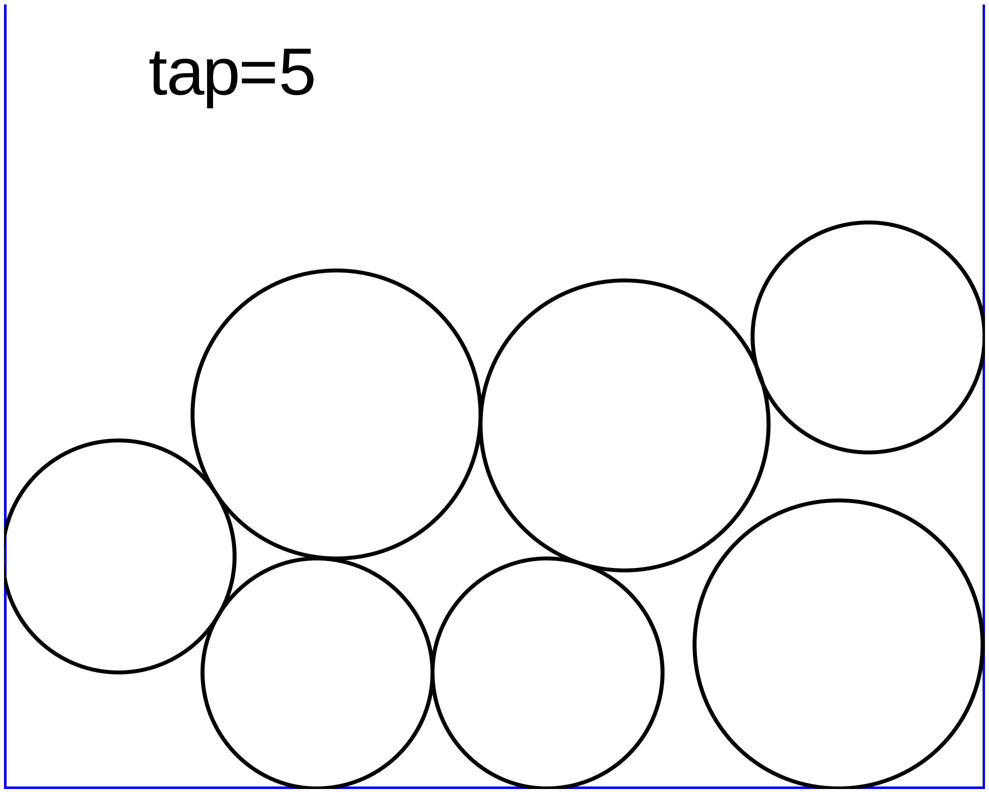

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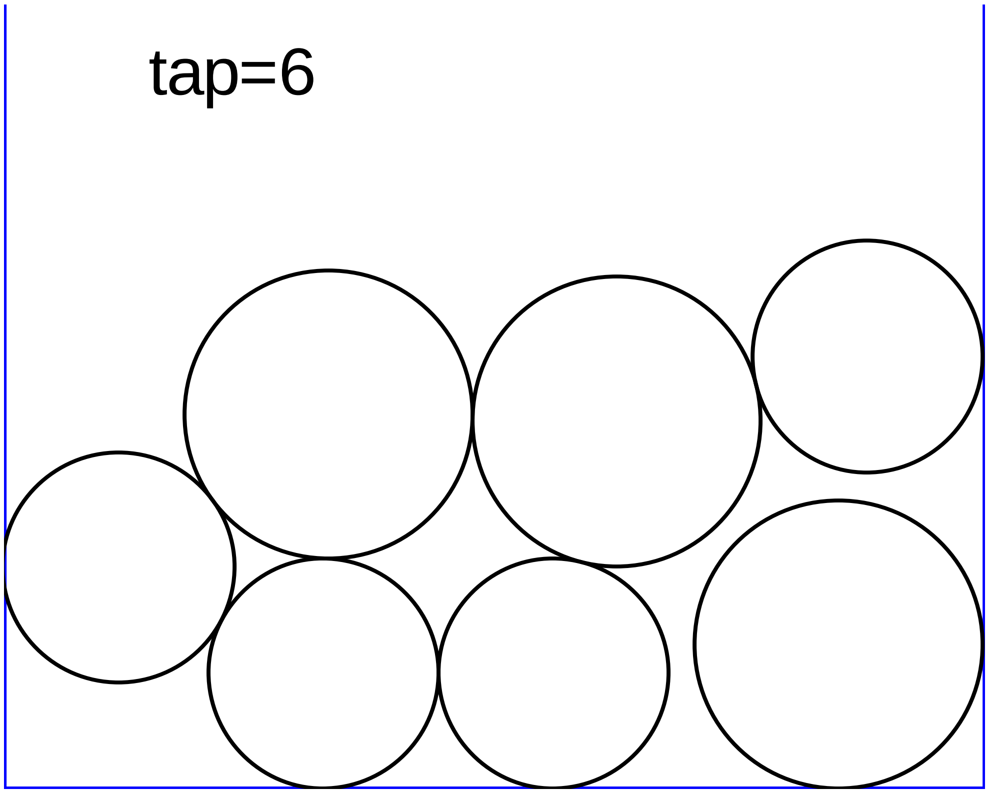

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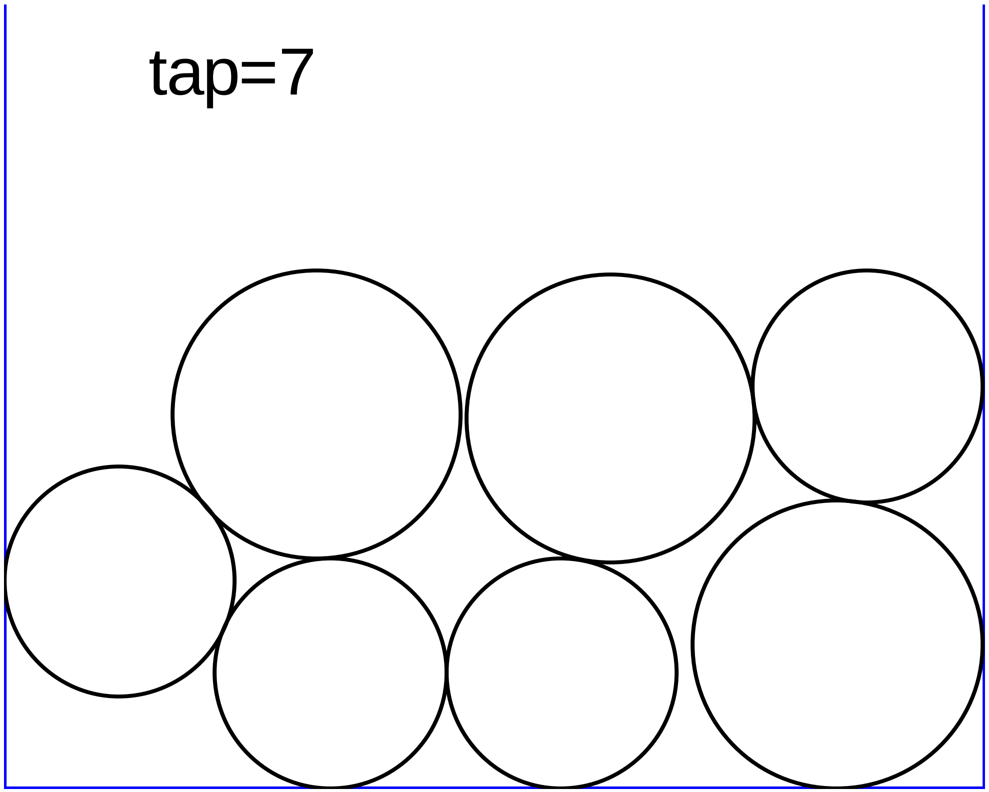

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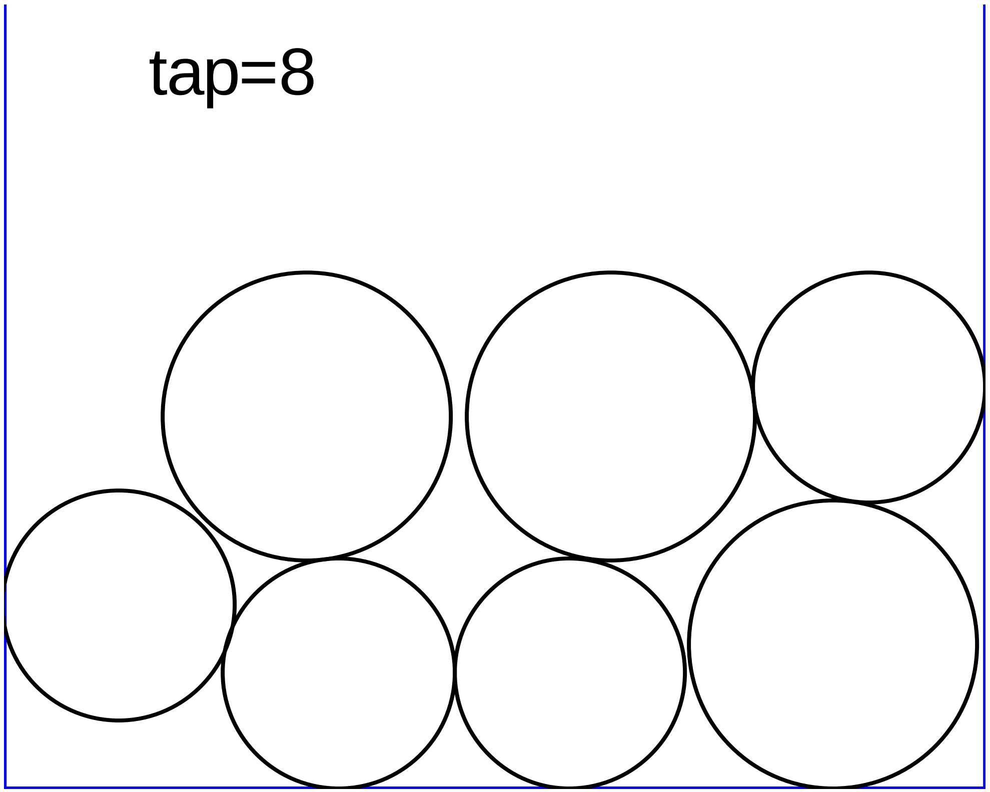

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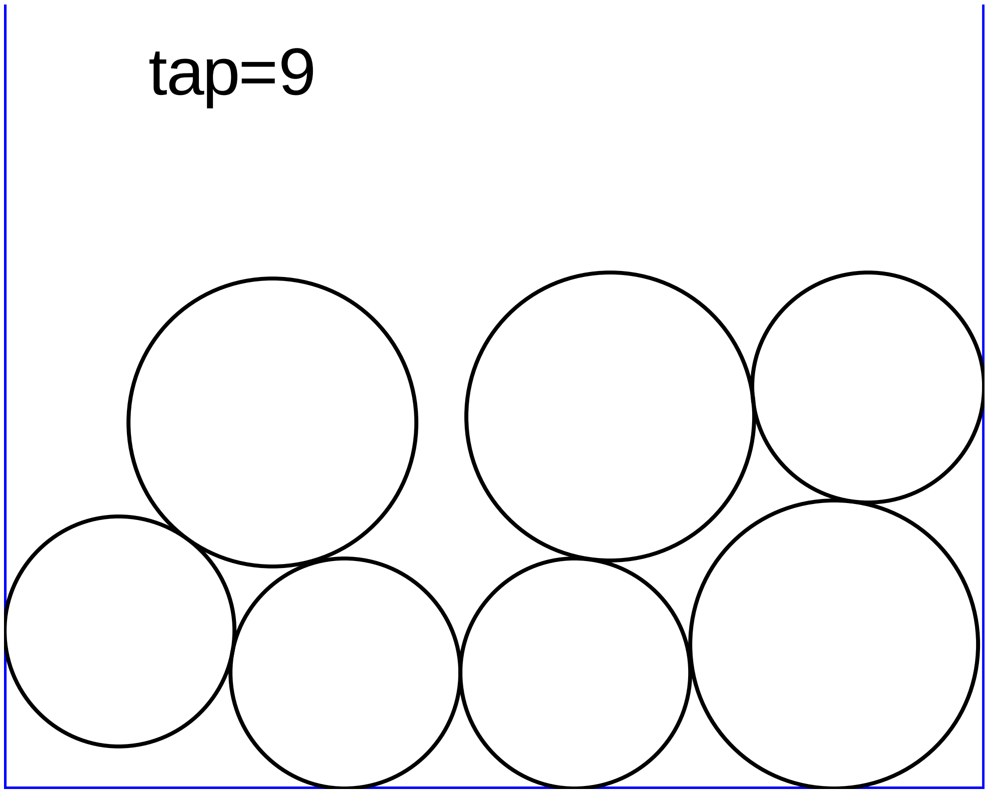

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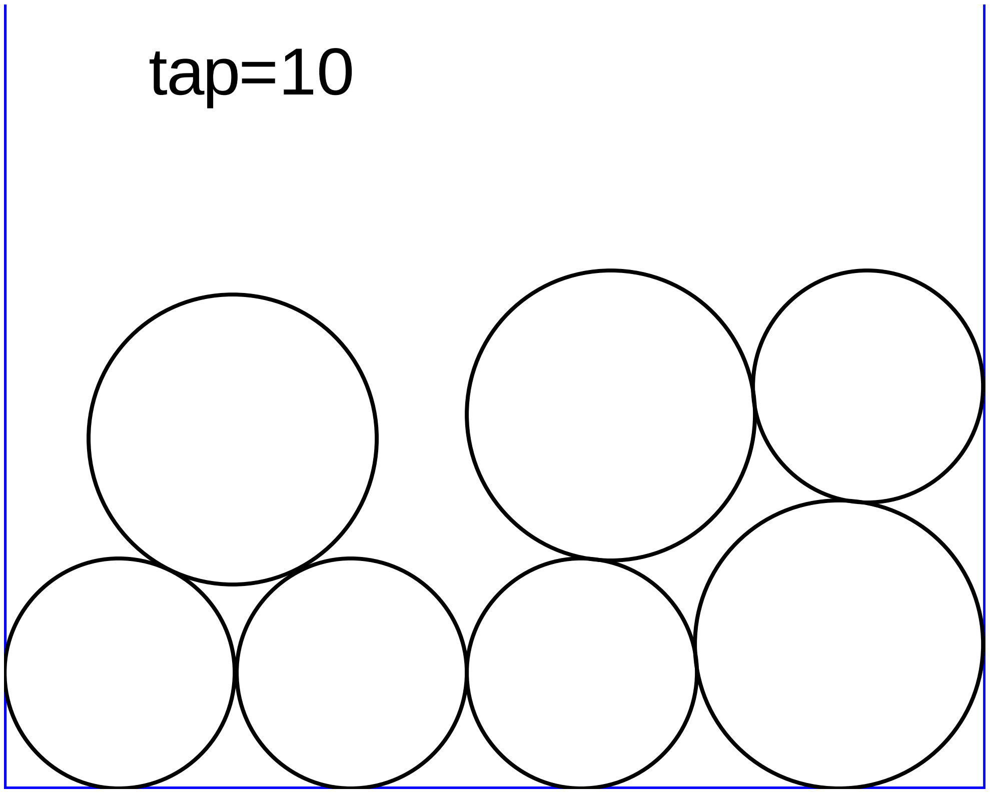

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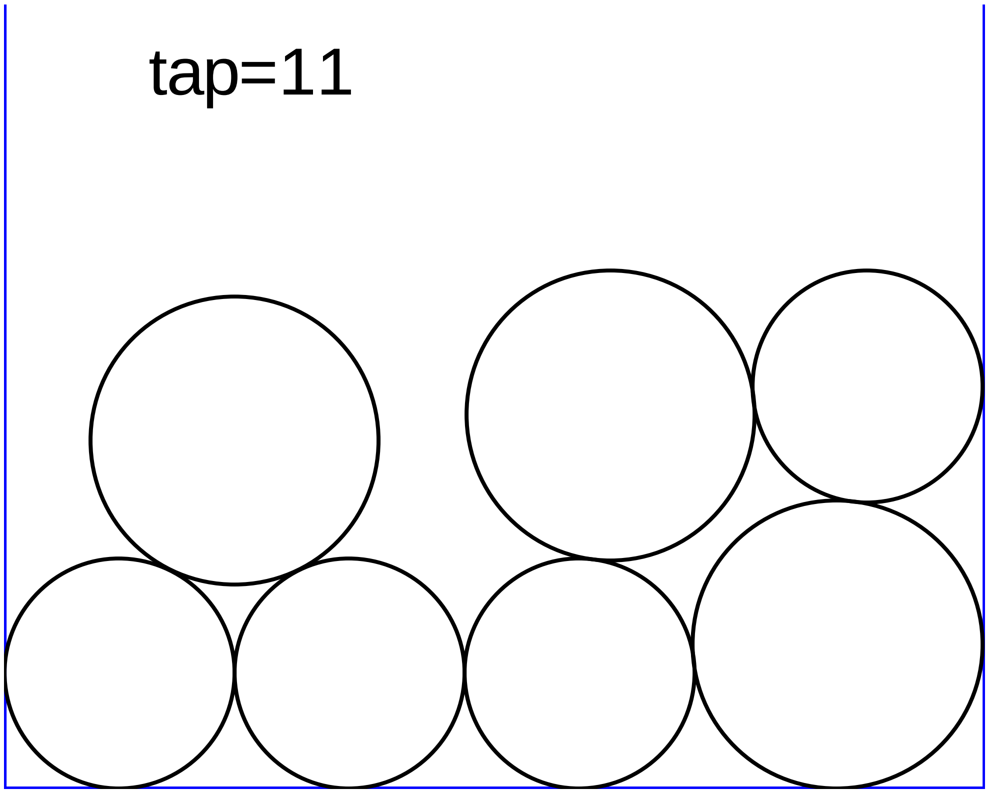

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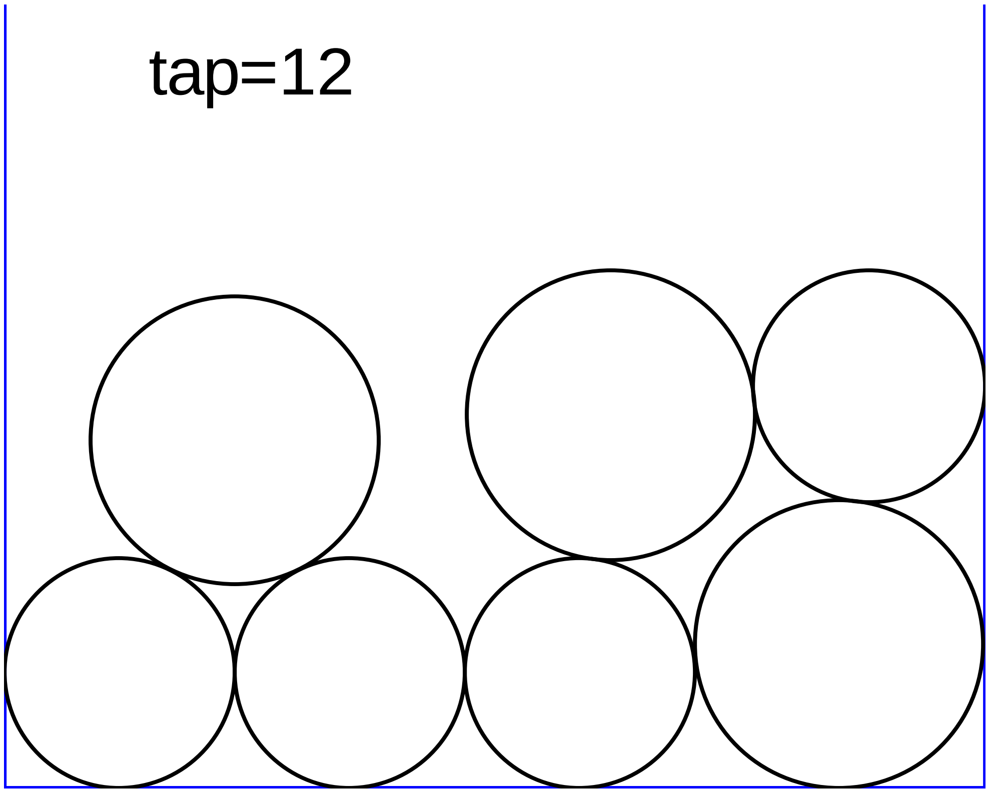

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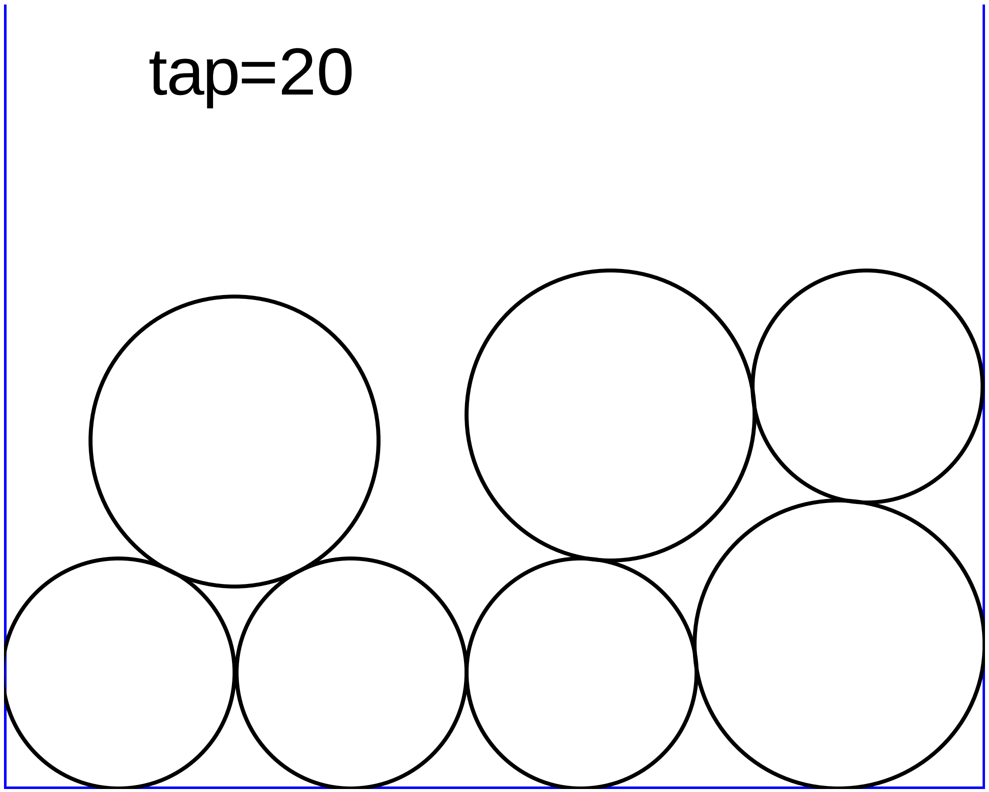
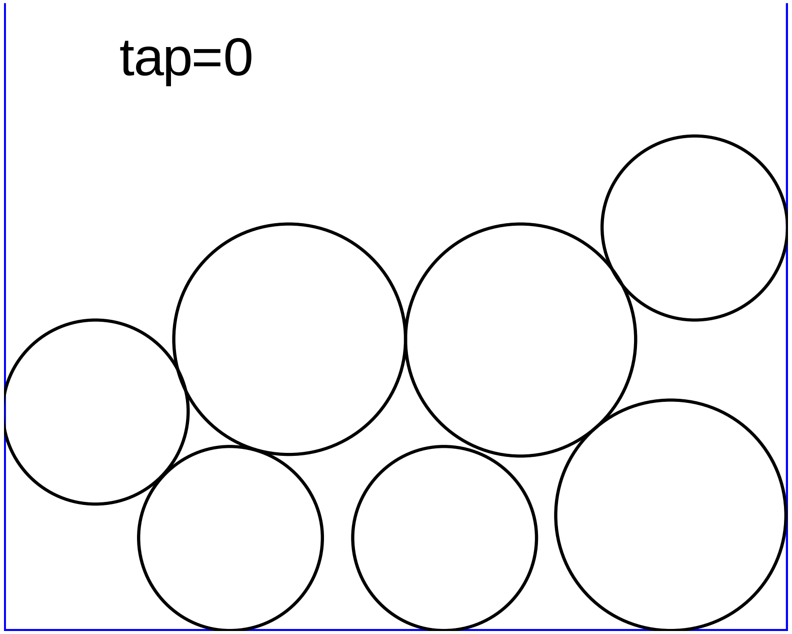
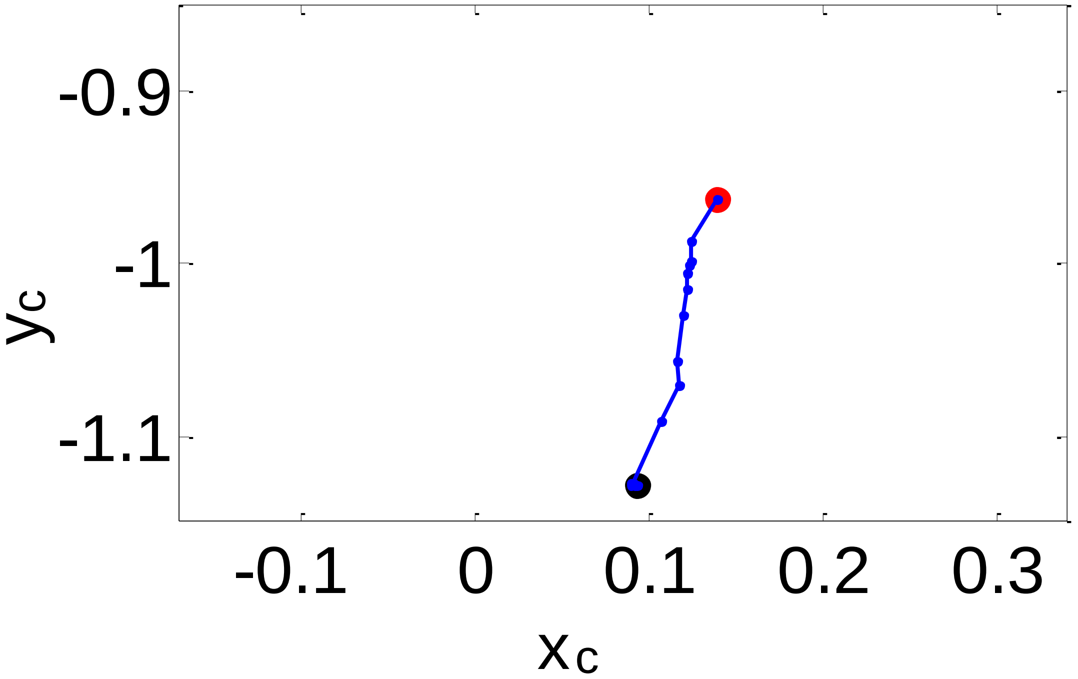


Friction Center of Mass Trajectory





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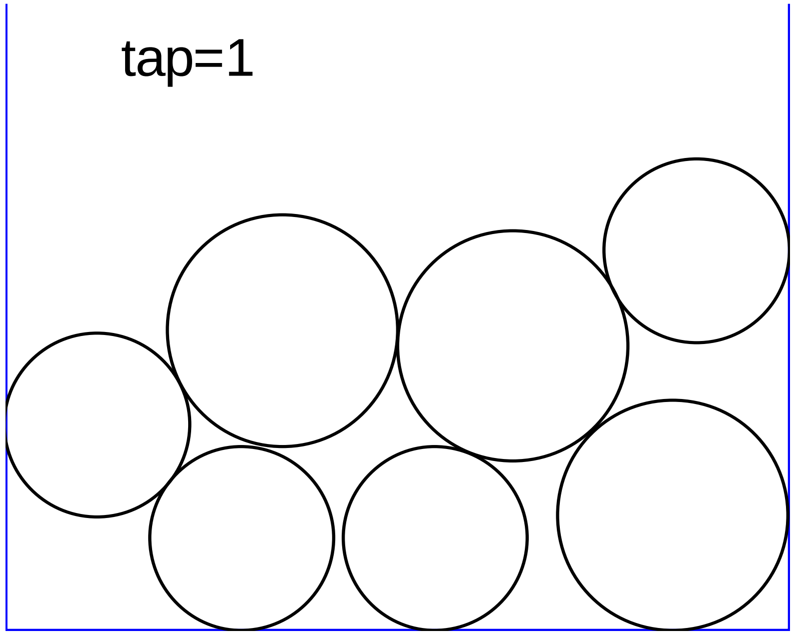

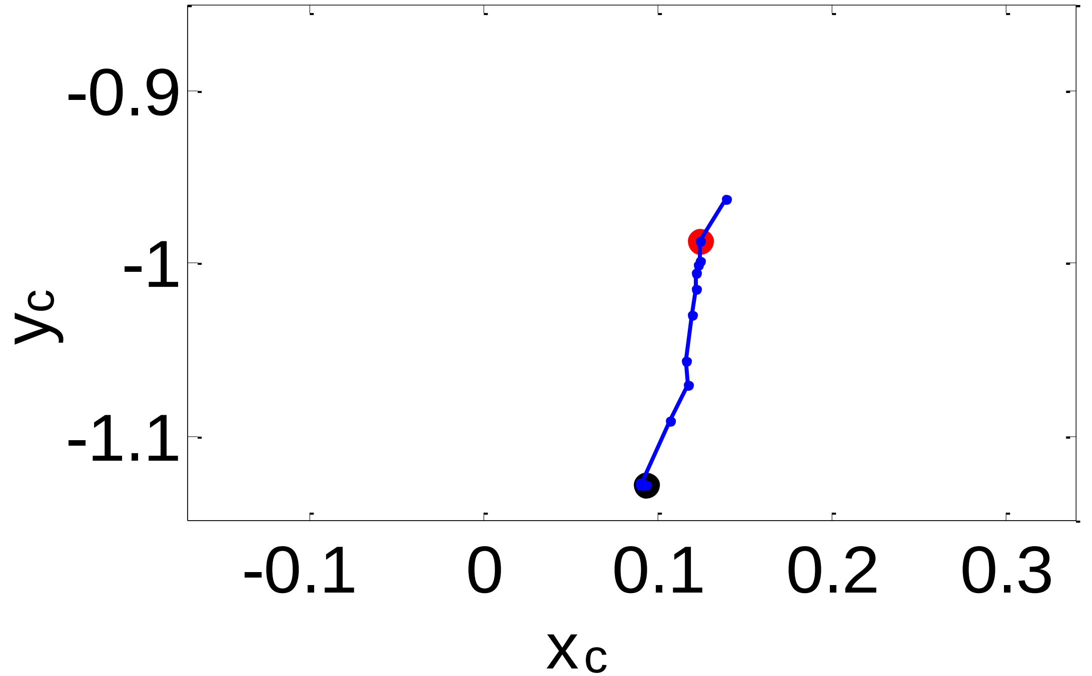

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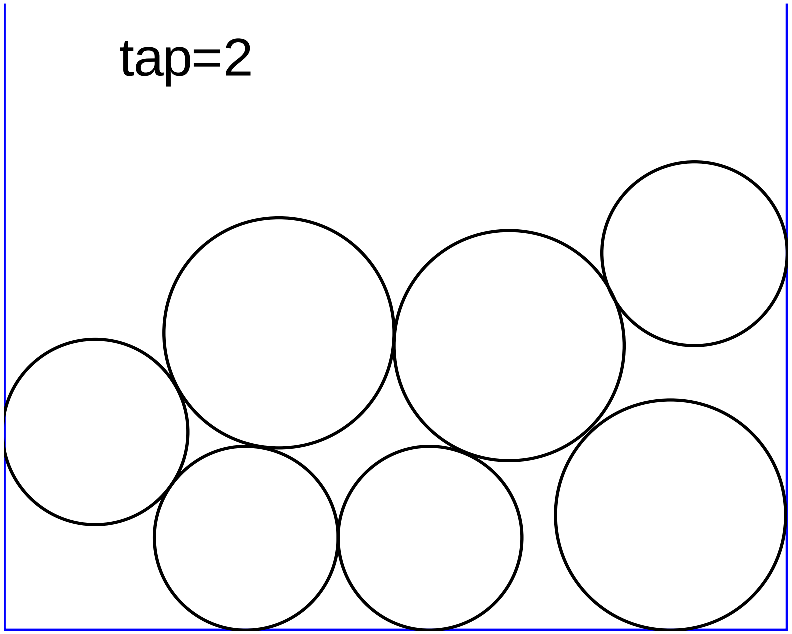

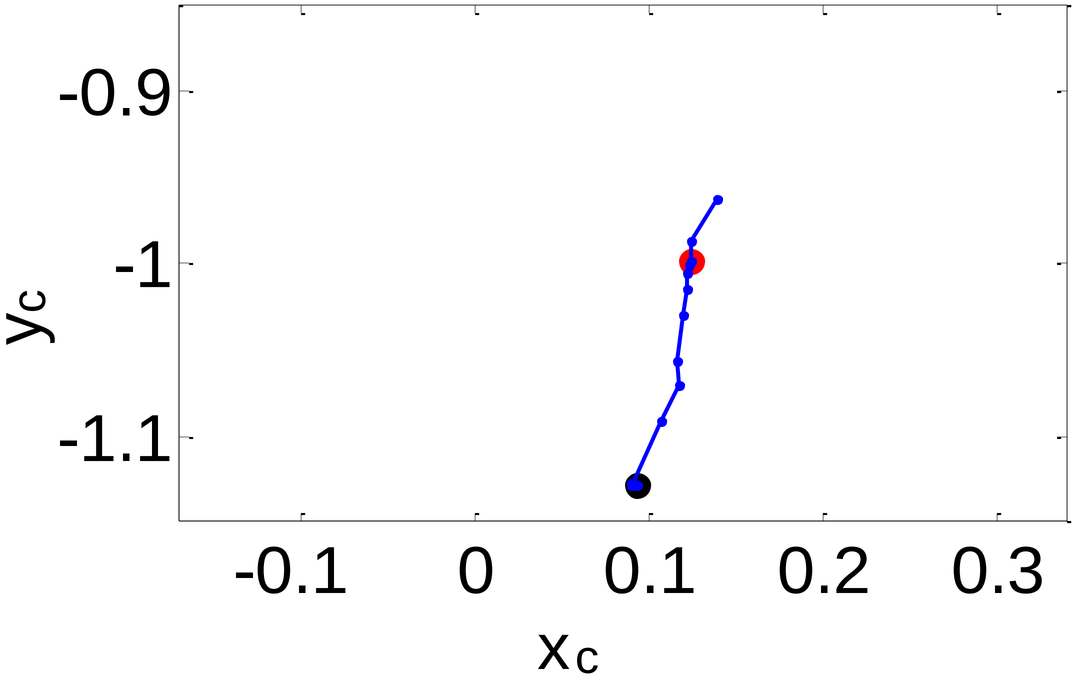

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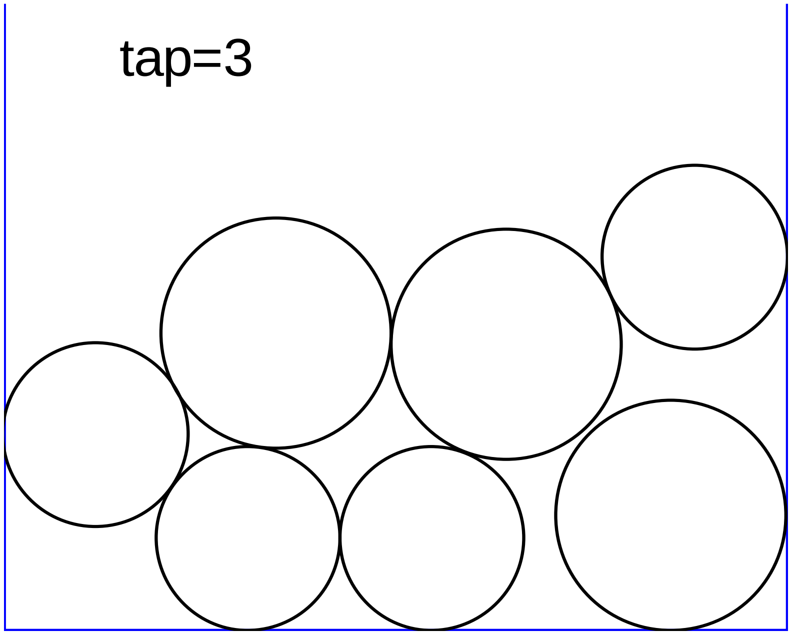

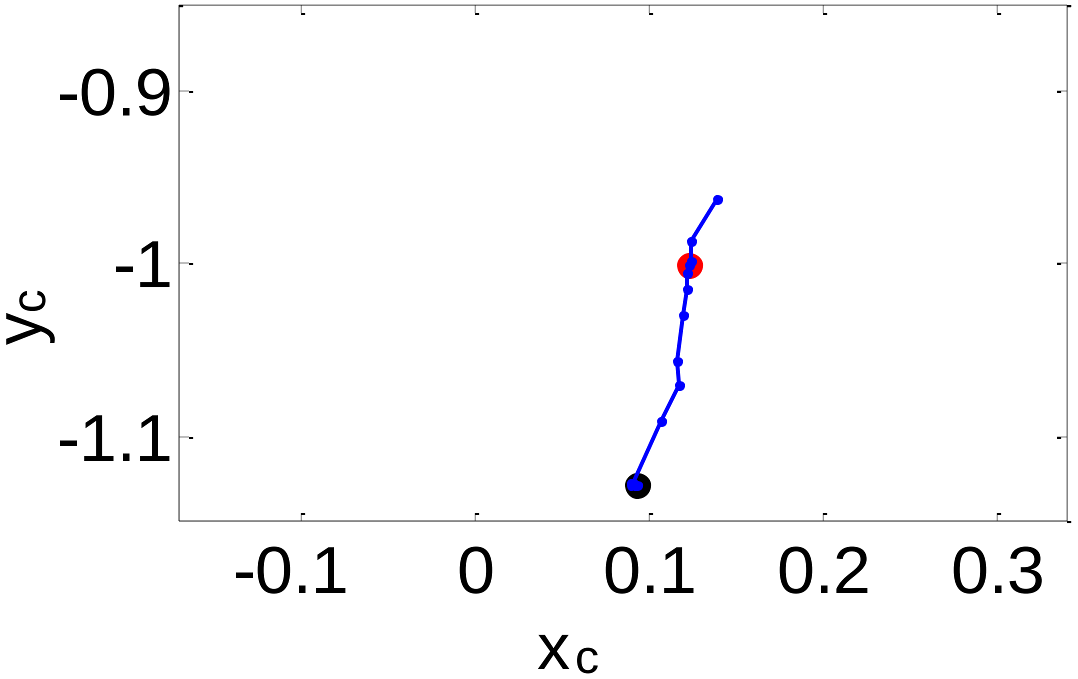

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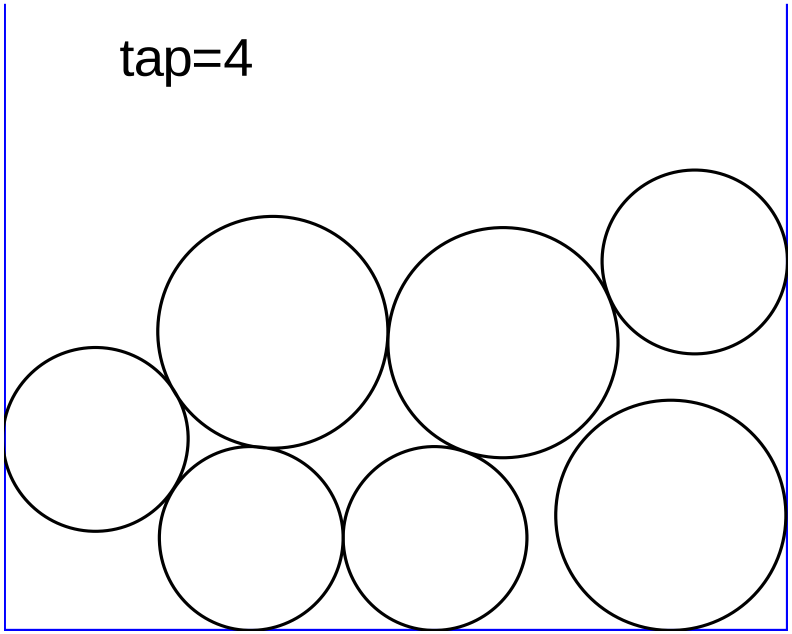

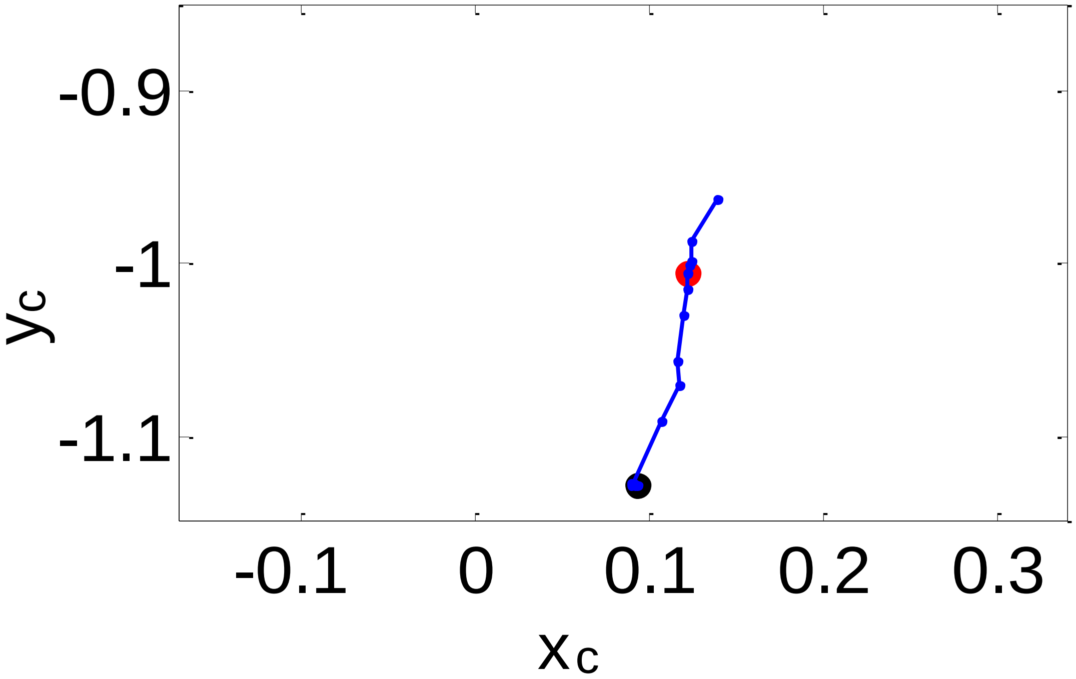

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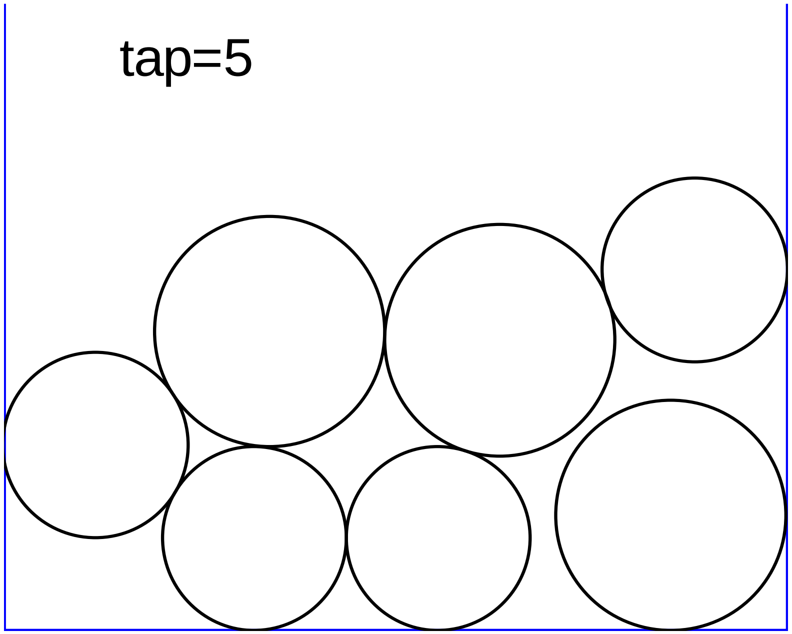

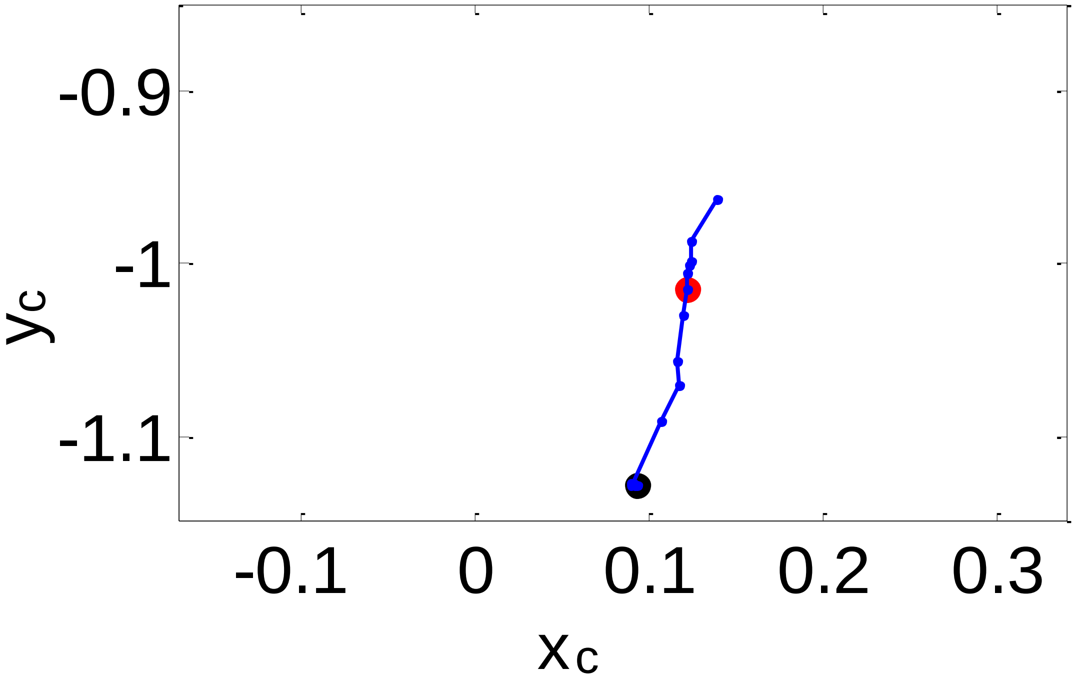

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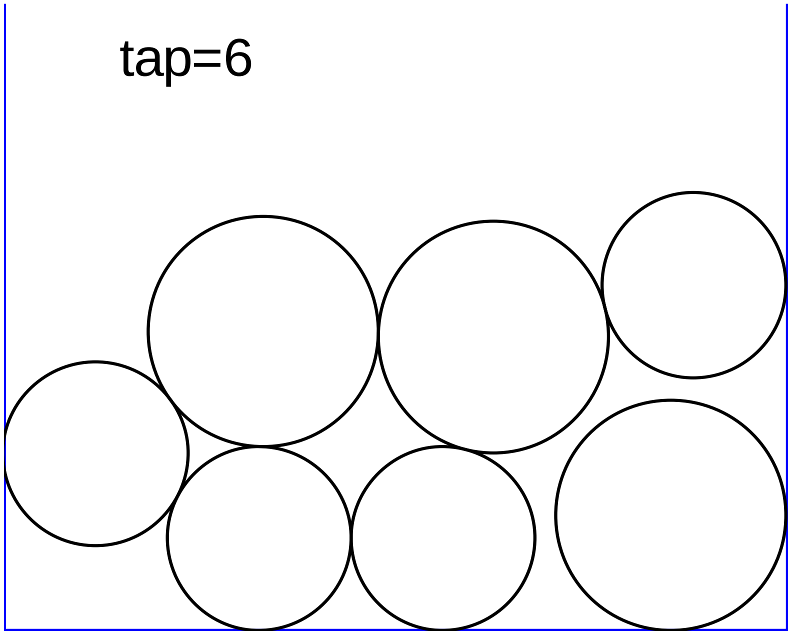

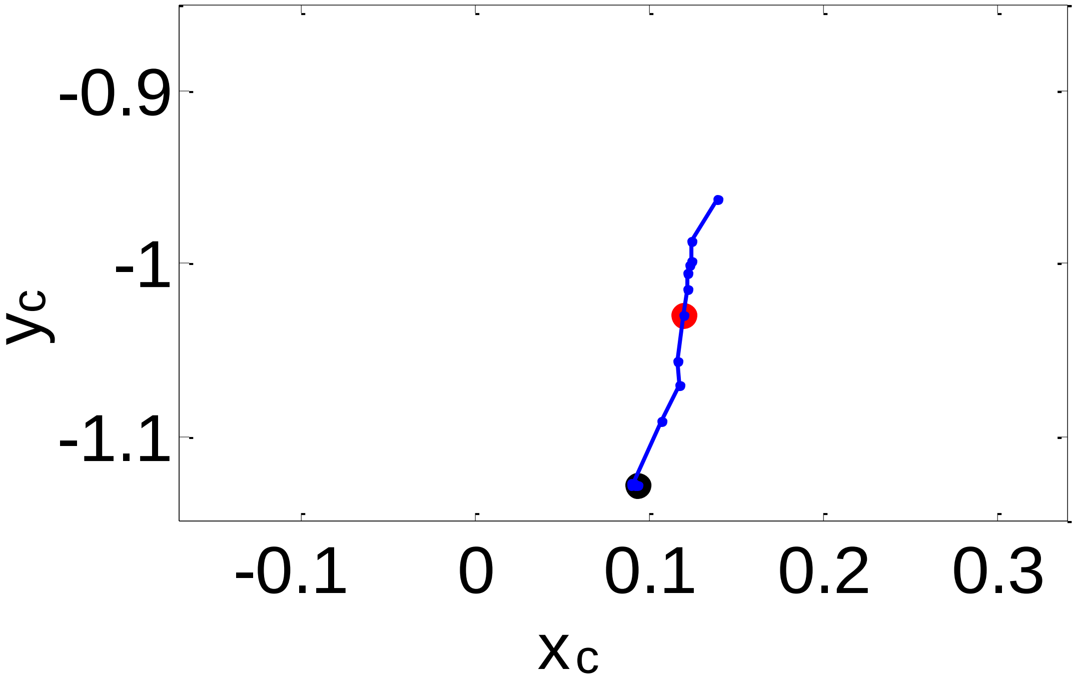

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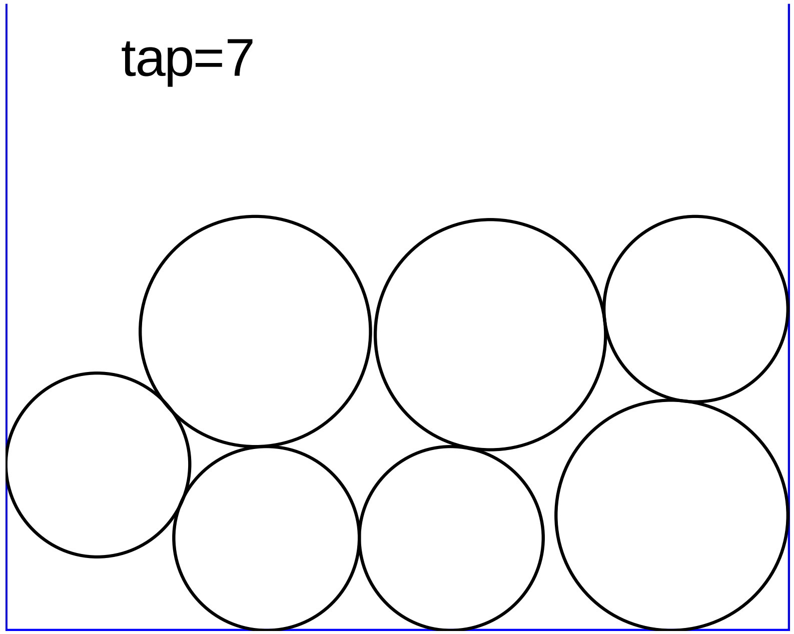

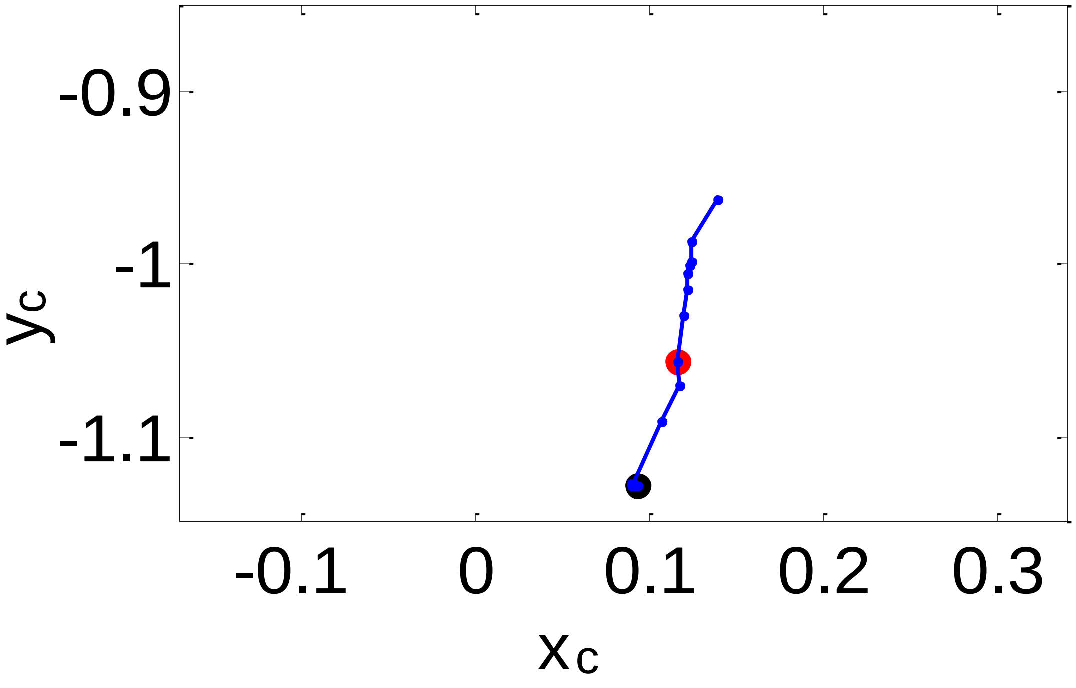

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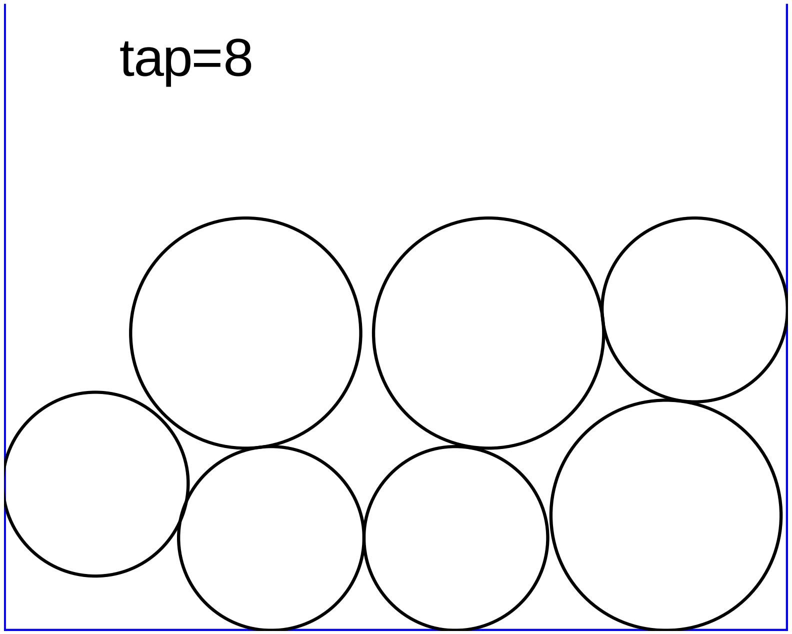

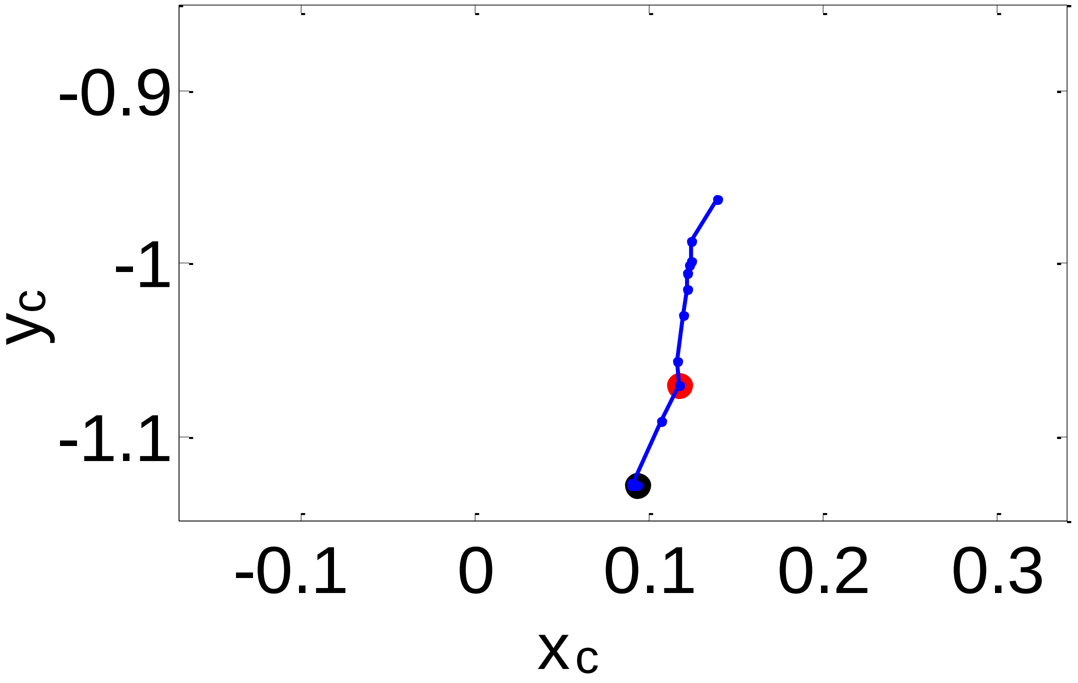

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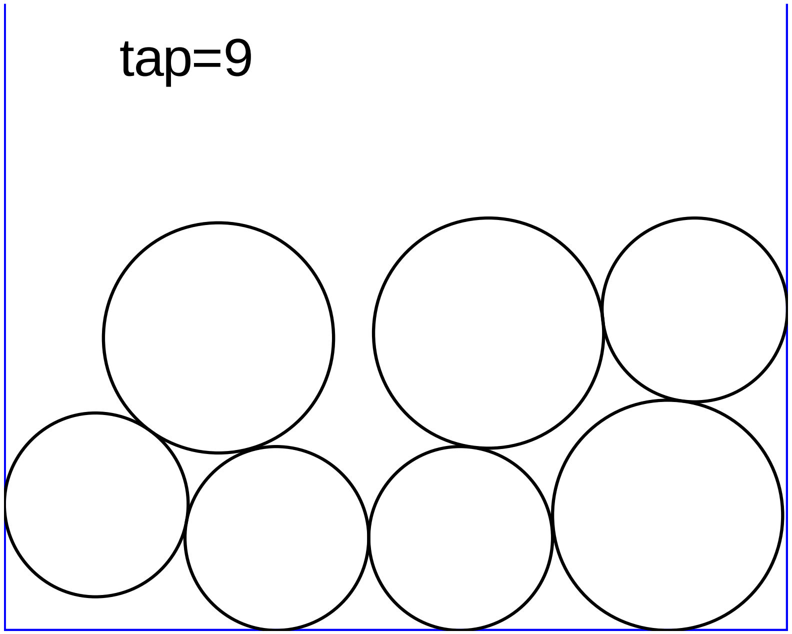

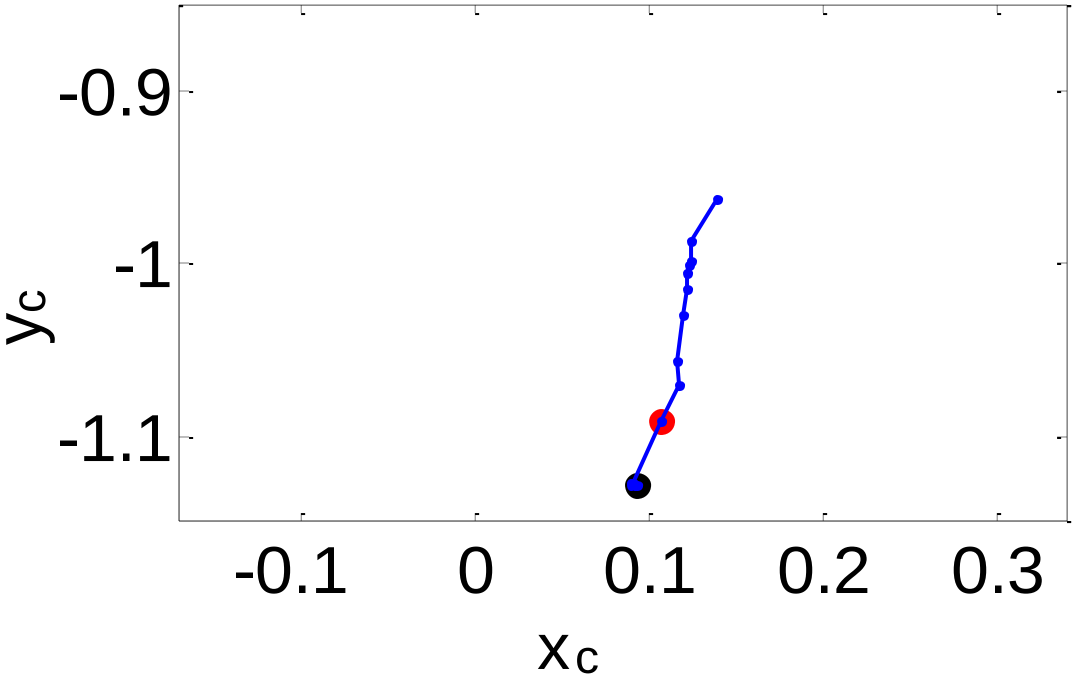

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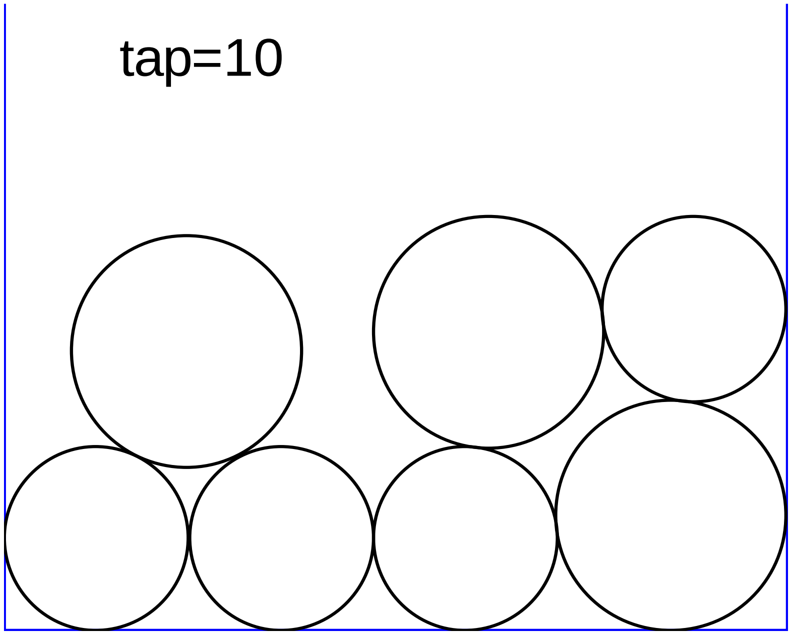

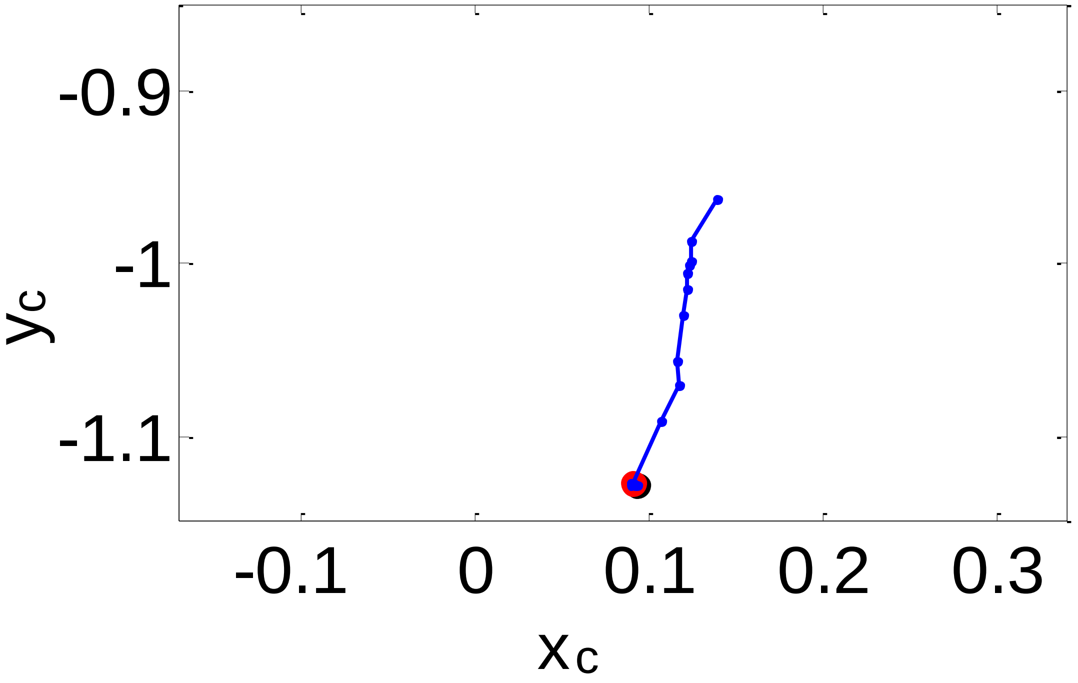

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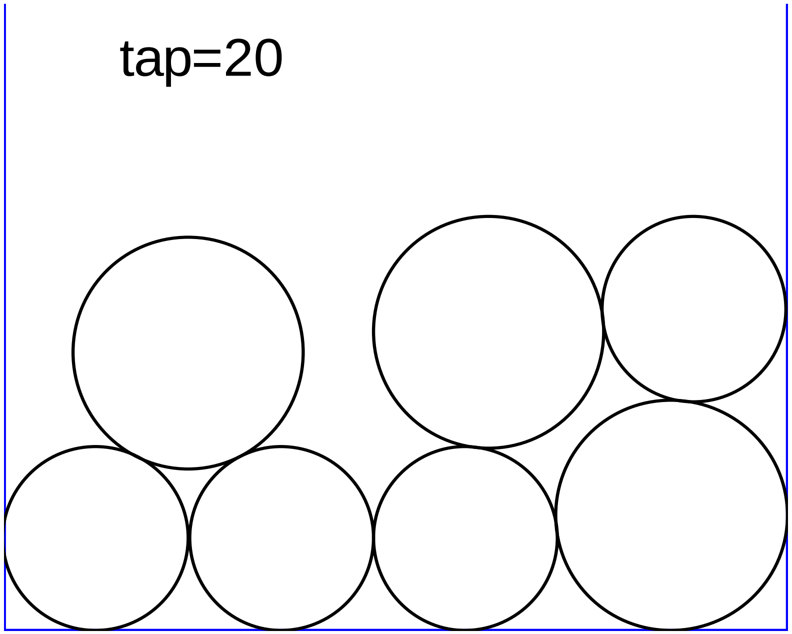
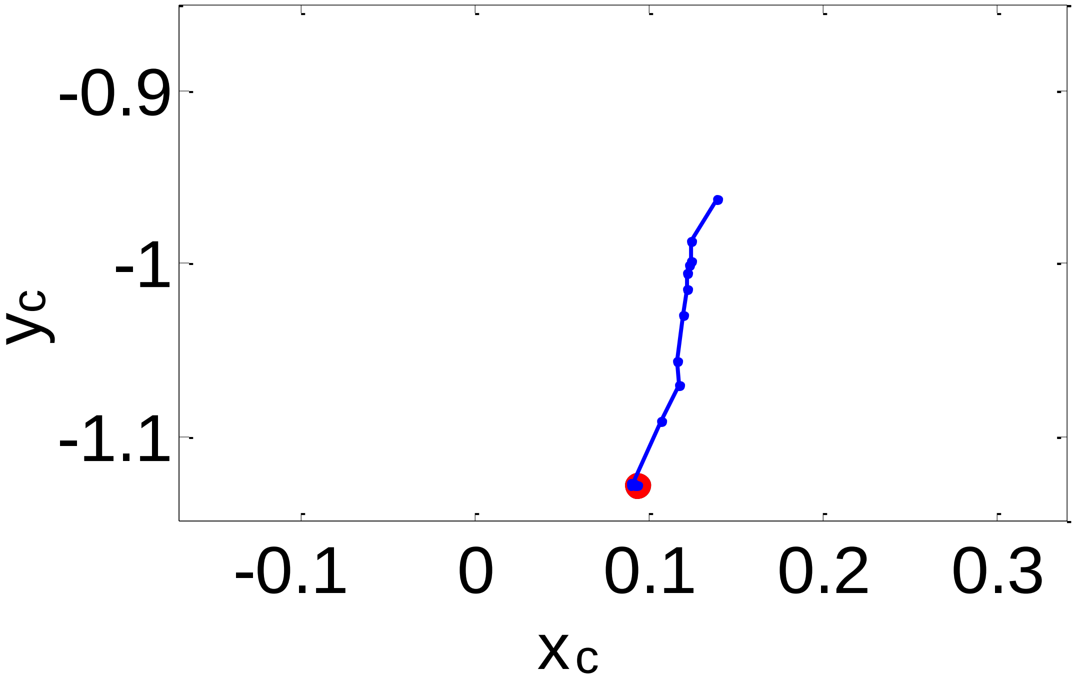
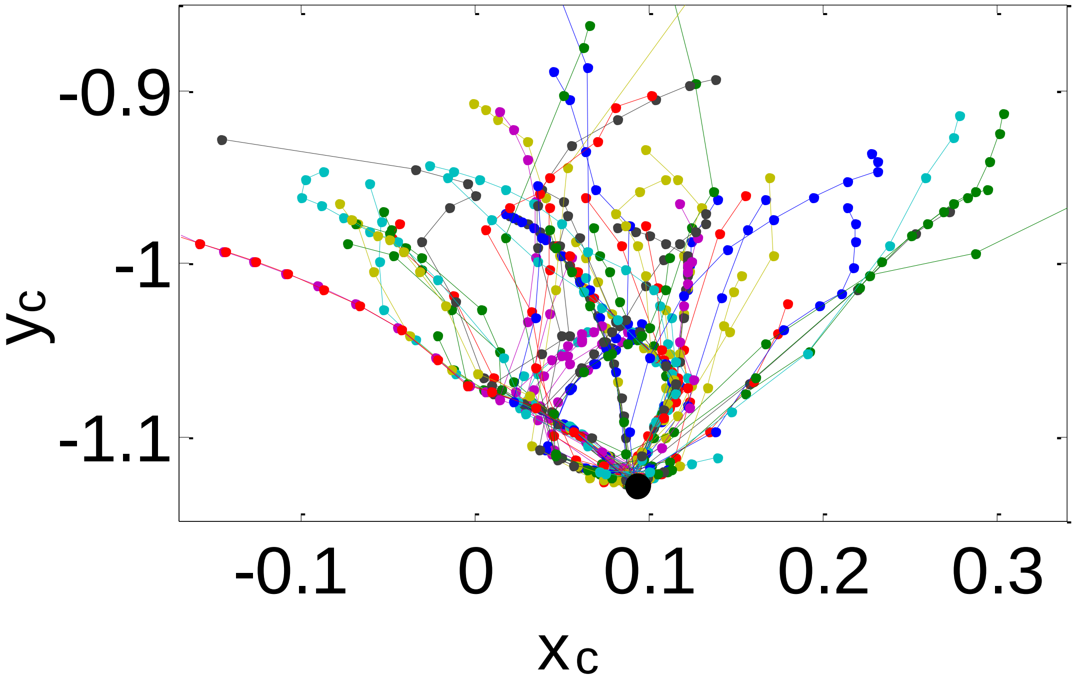




Frictional Families



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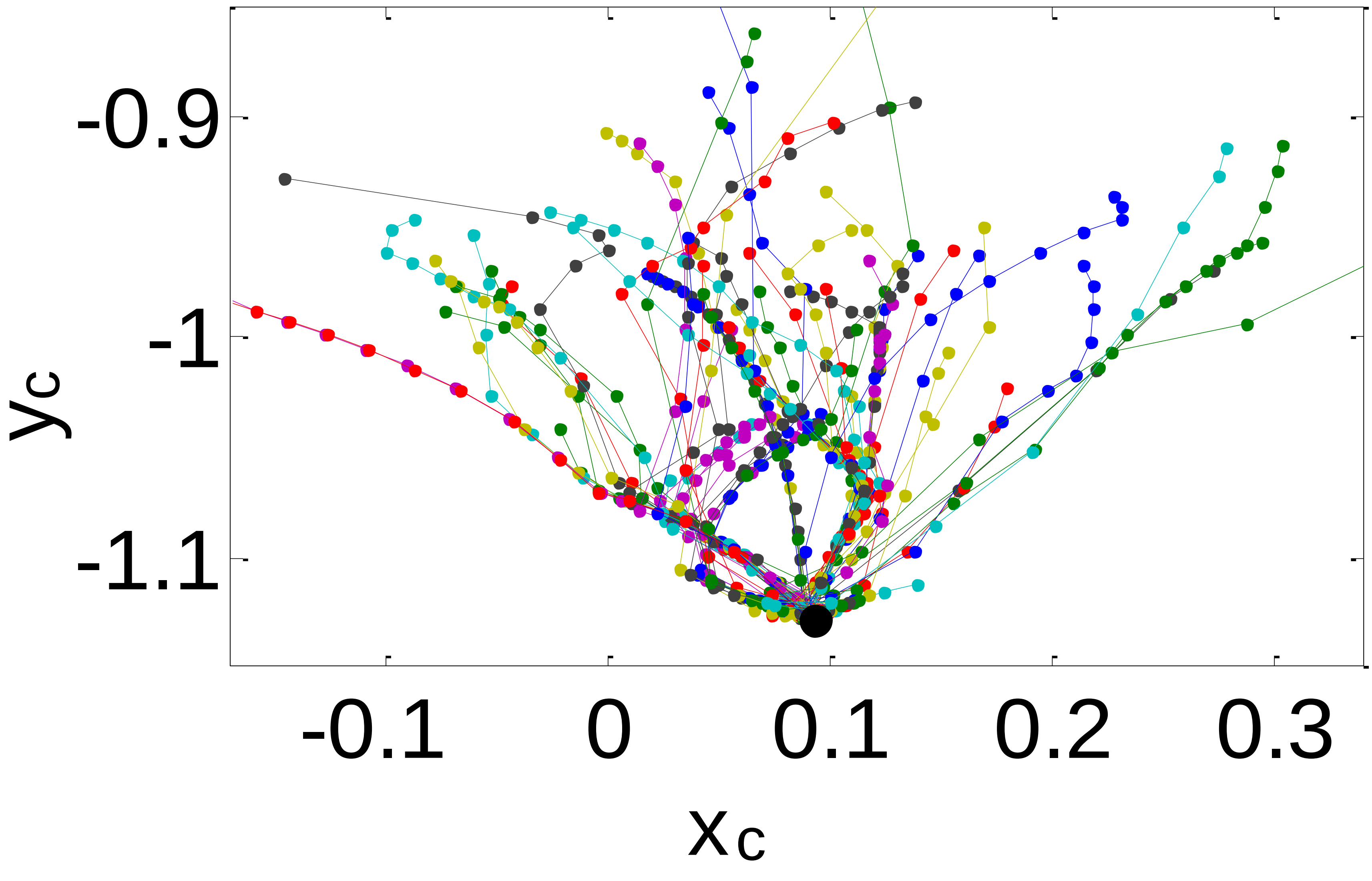

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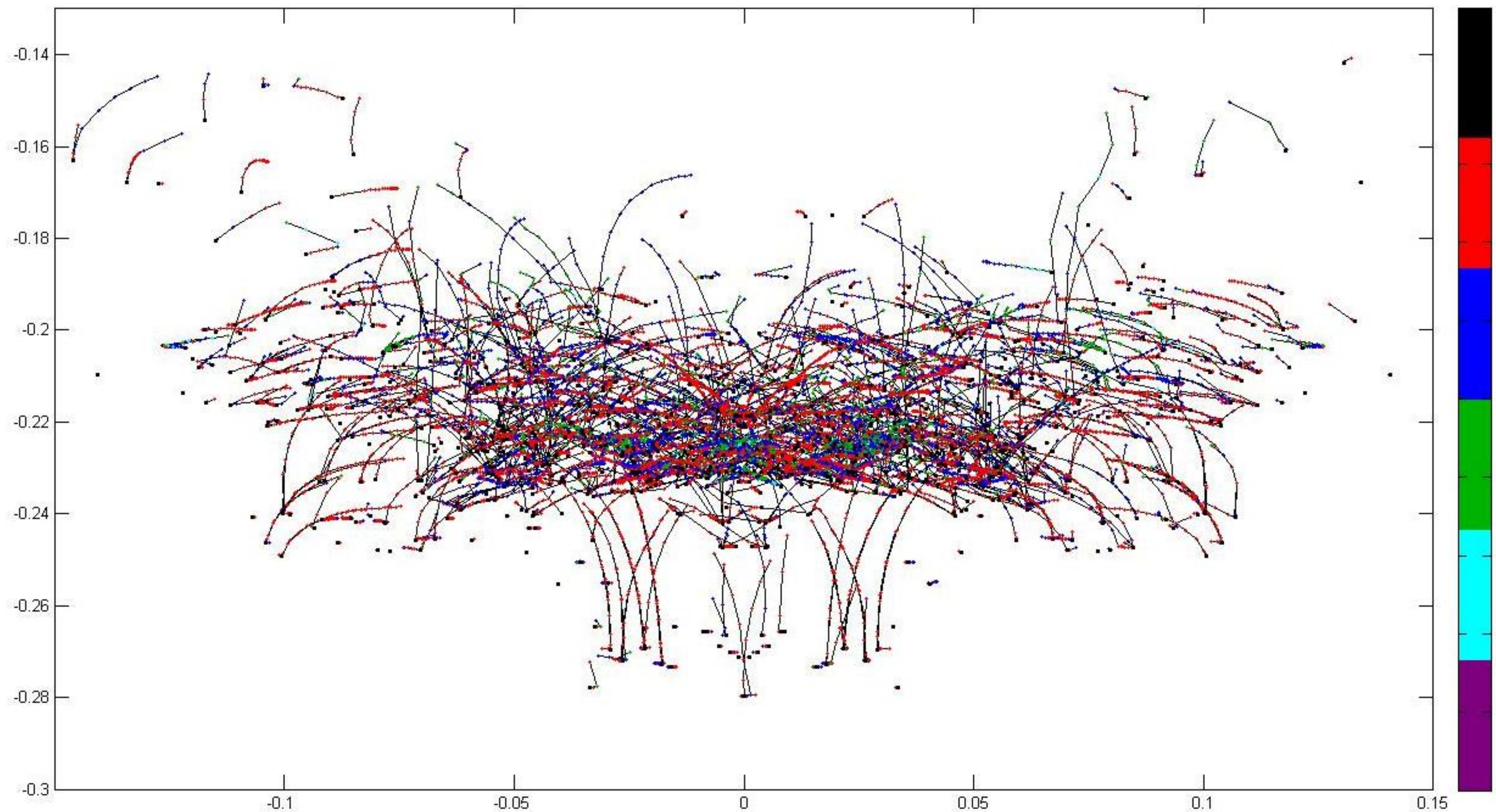
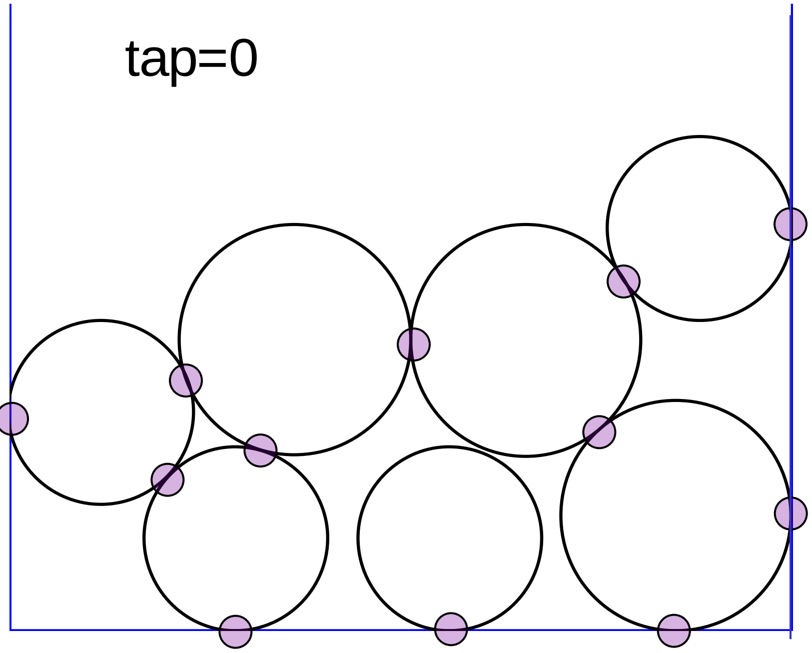
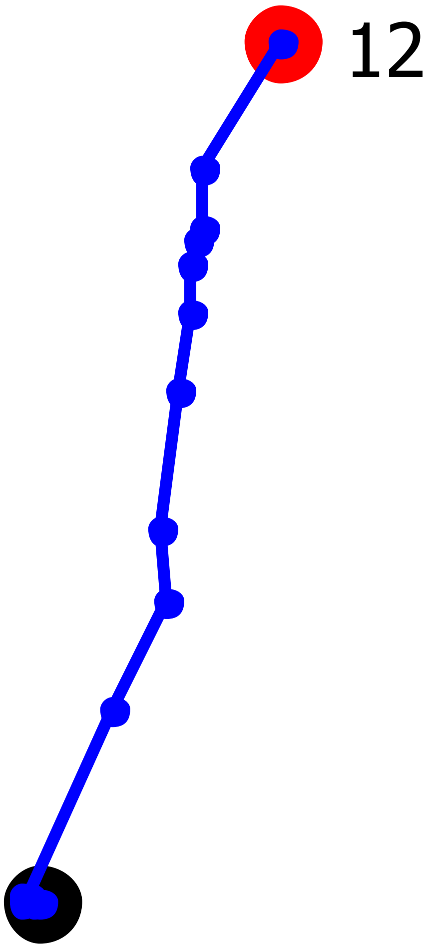


Contact Evolution





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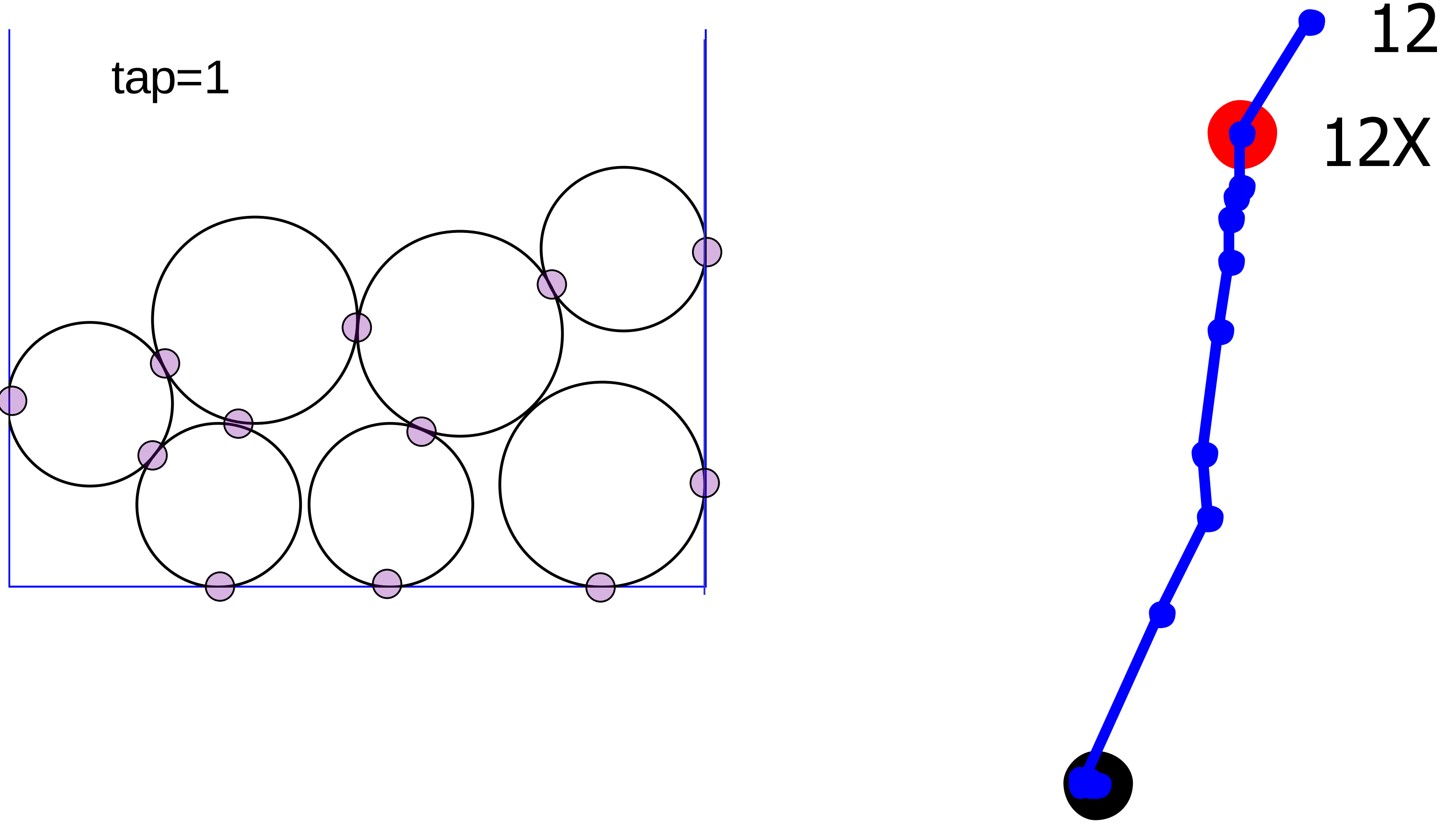

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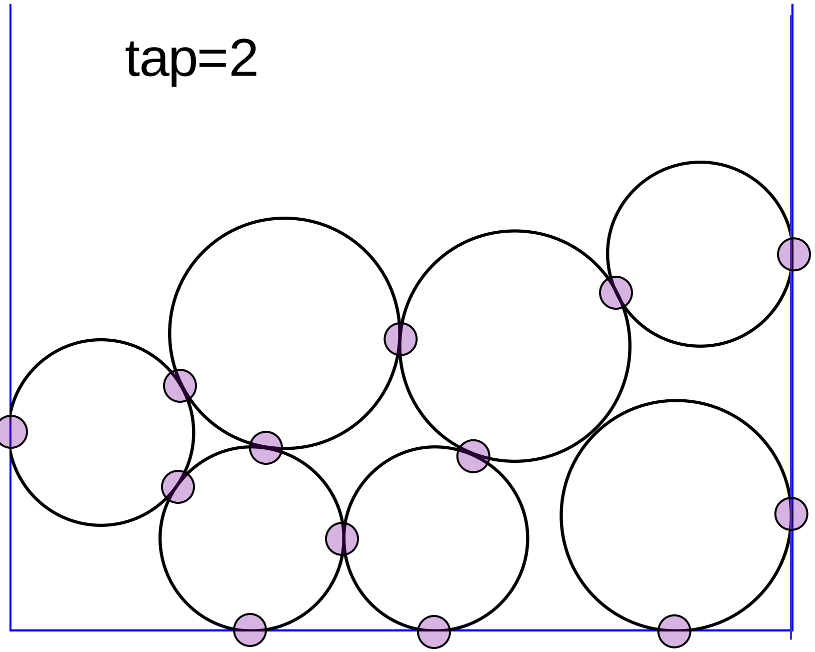

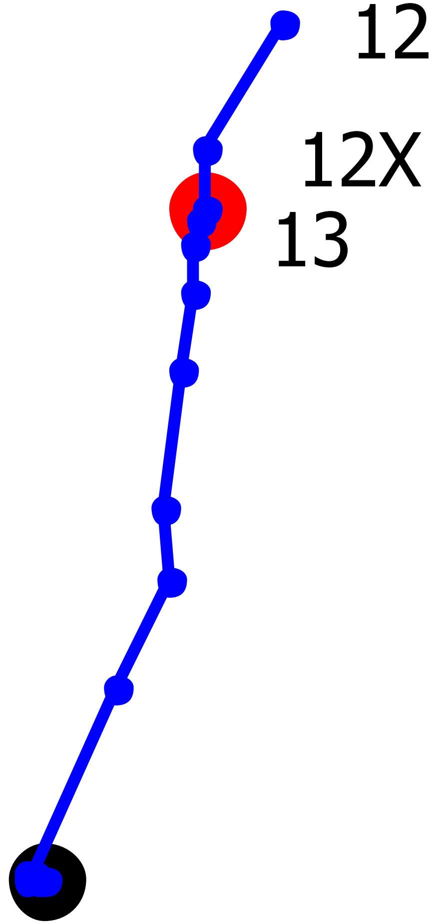

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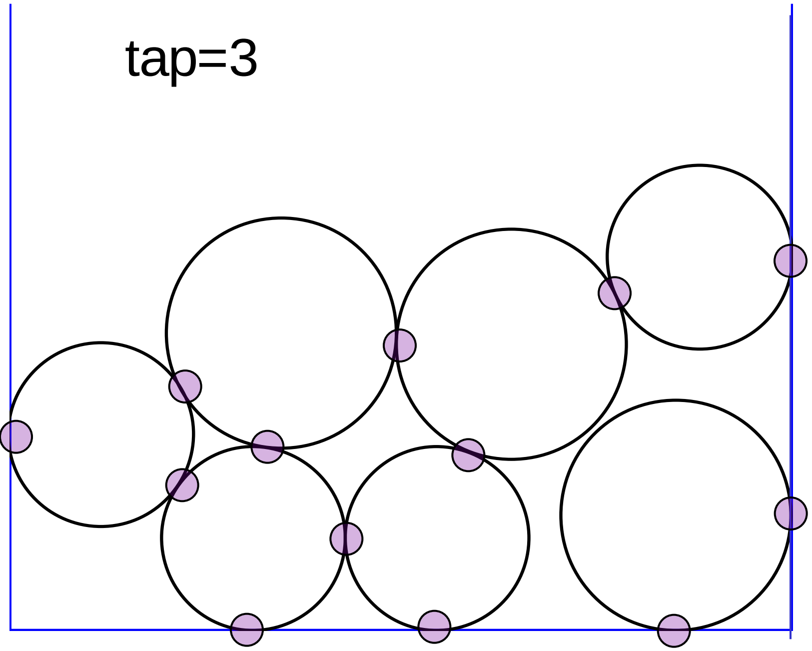

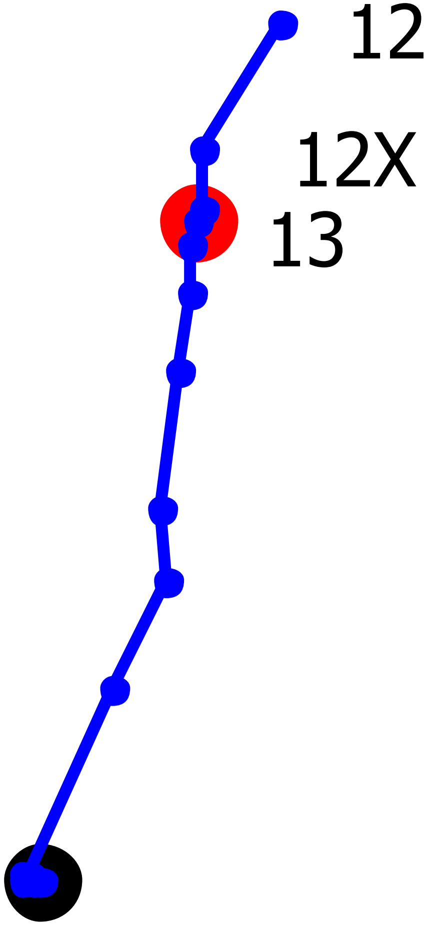

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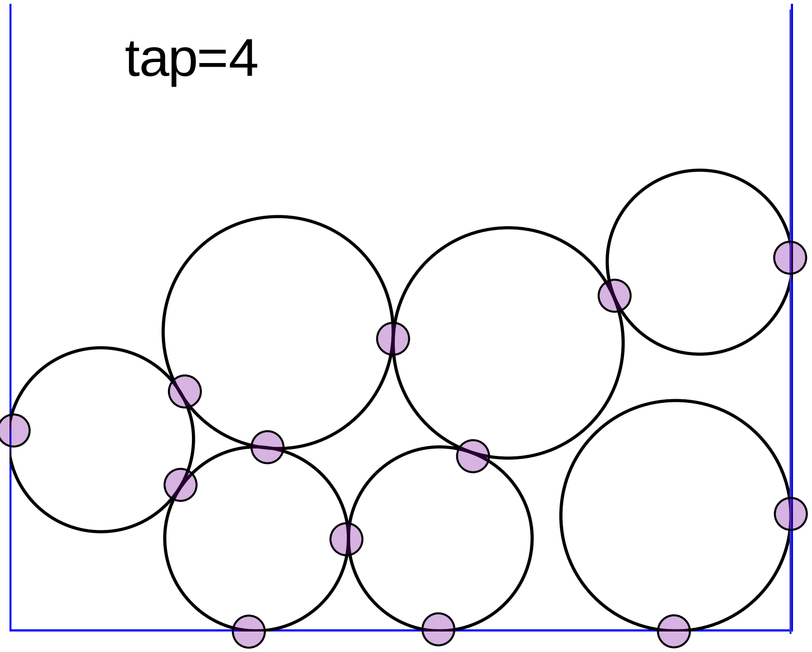

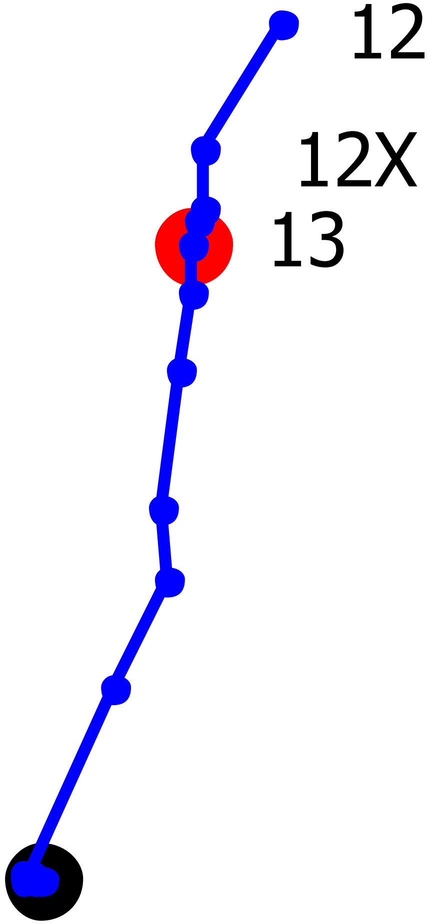

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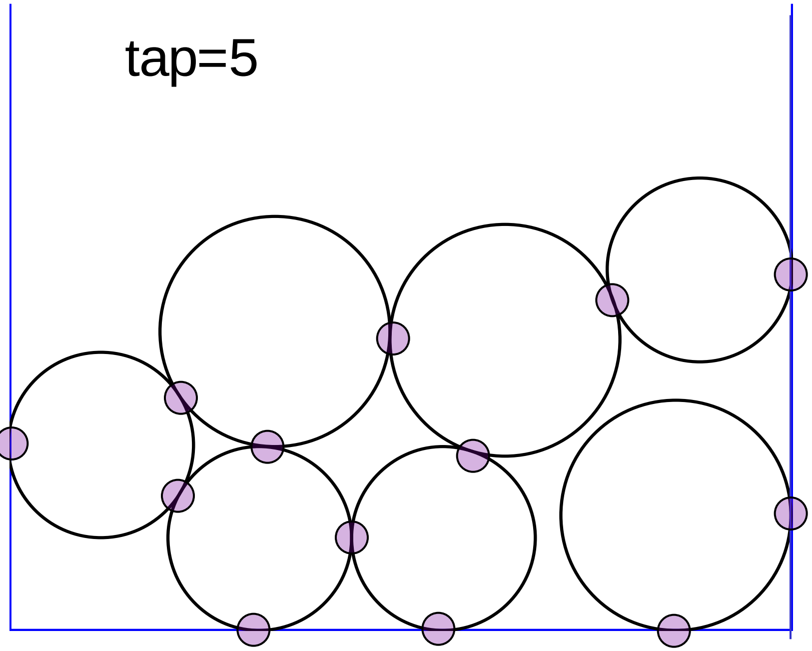

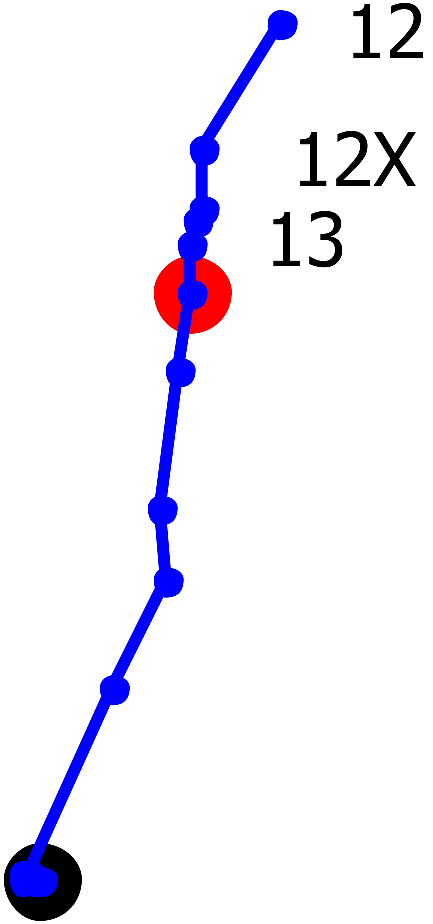

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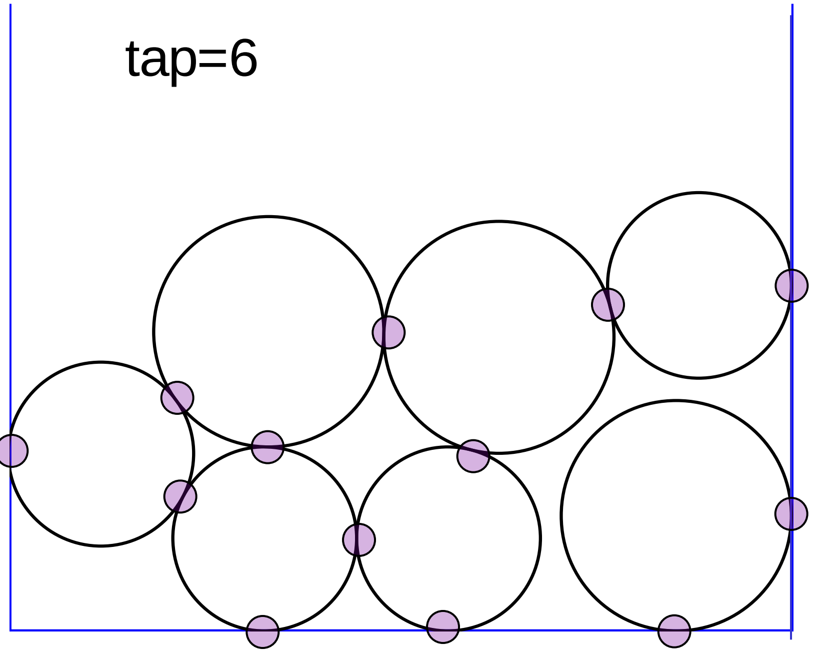

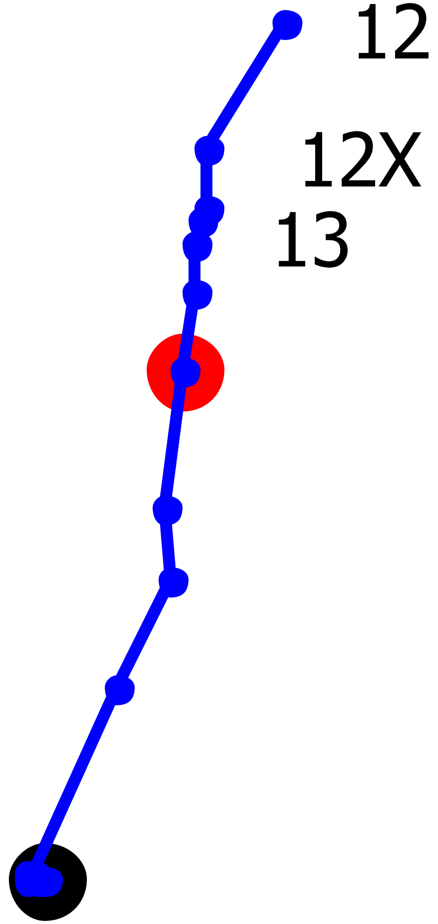

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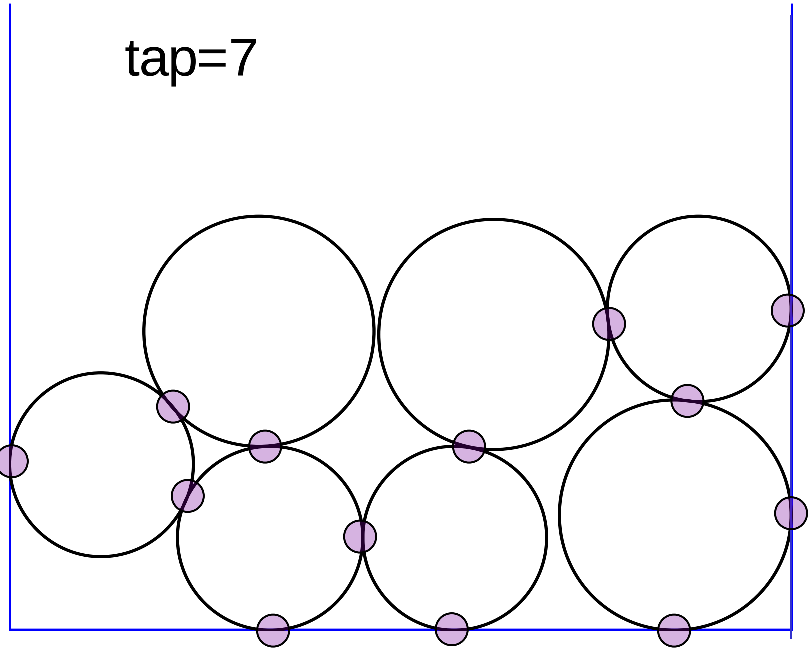

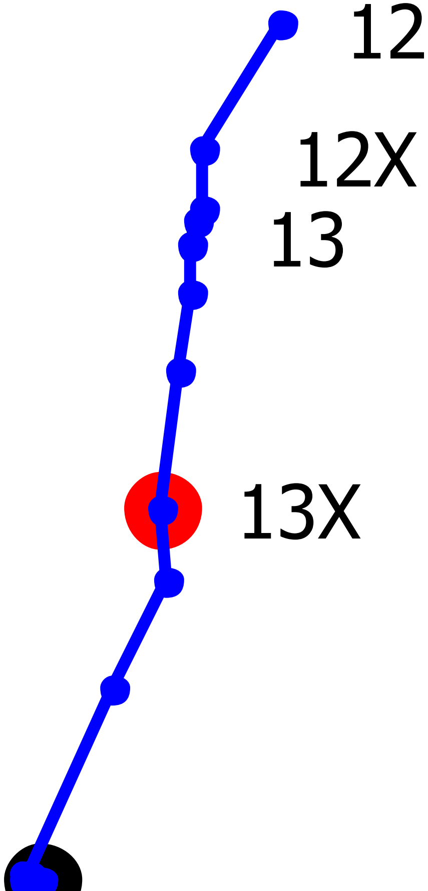

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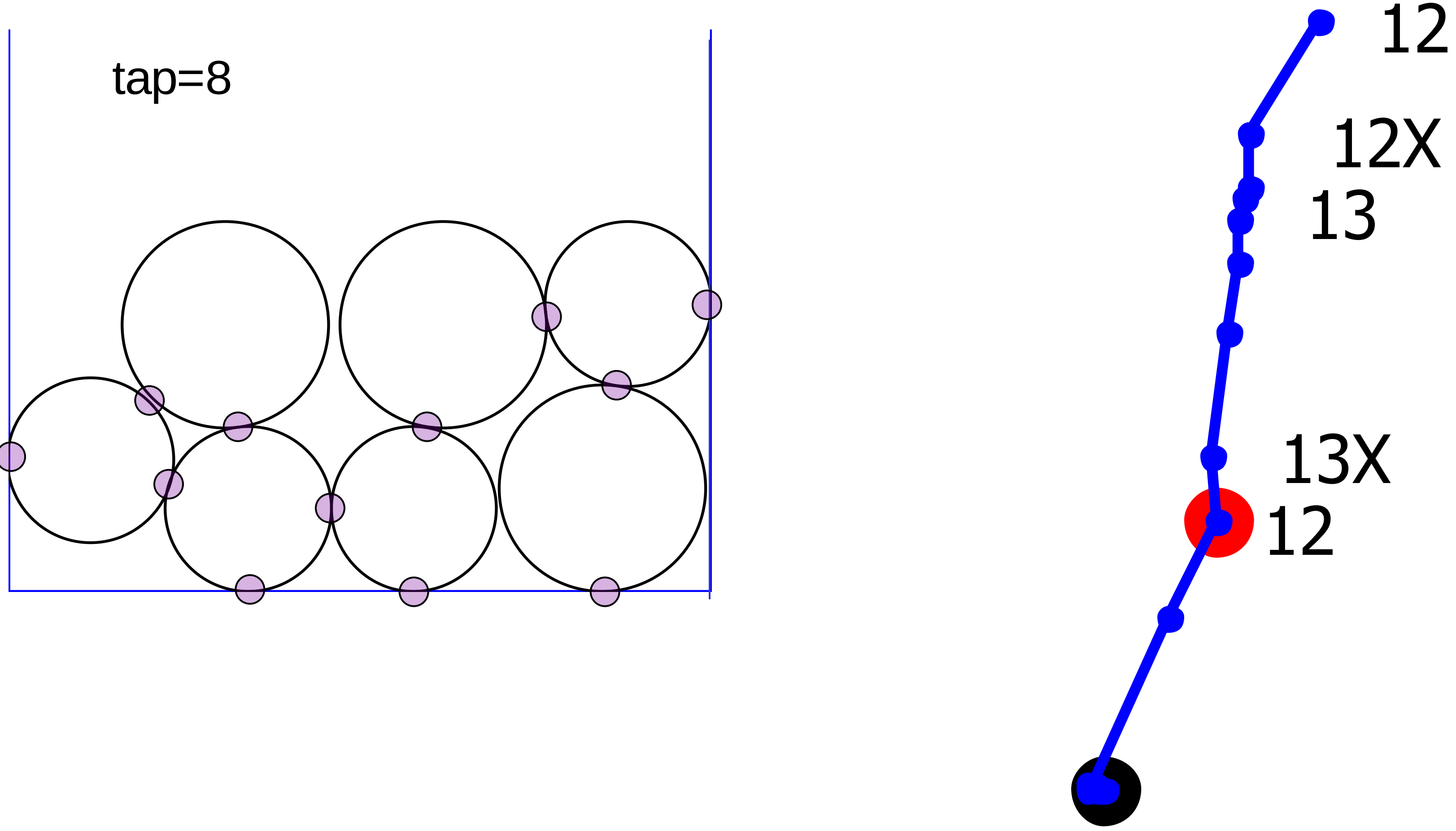

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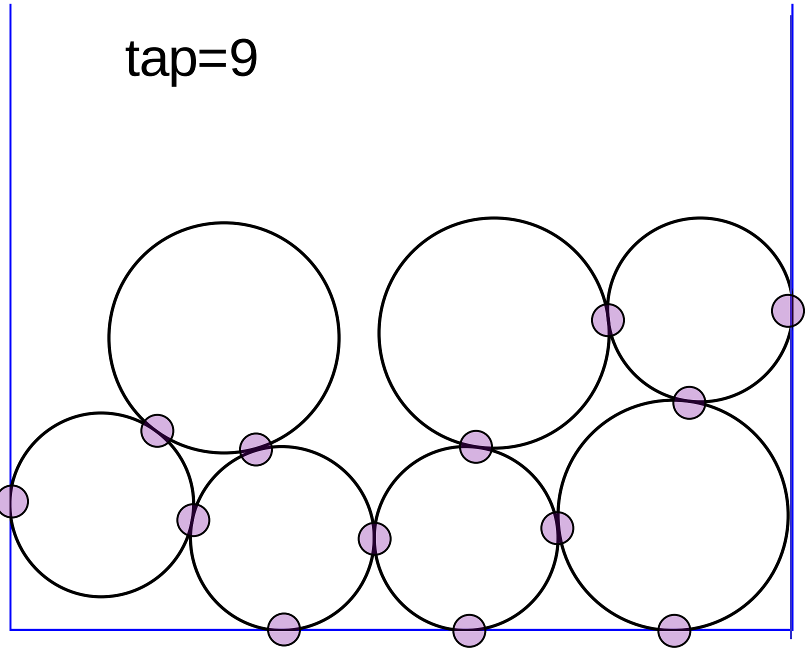

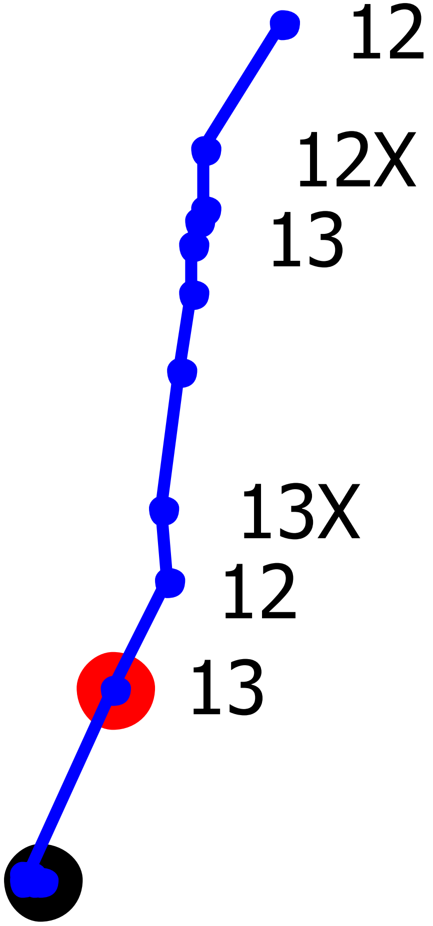

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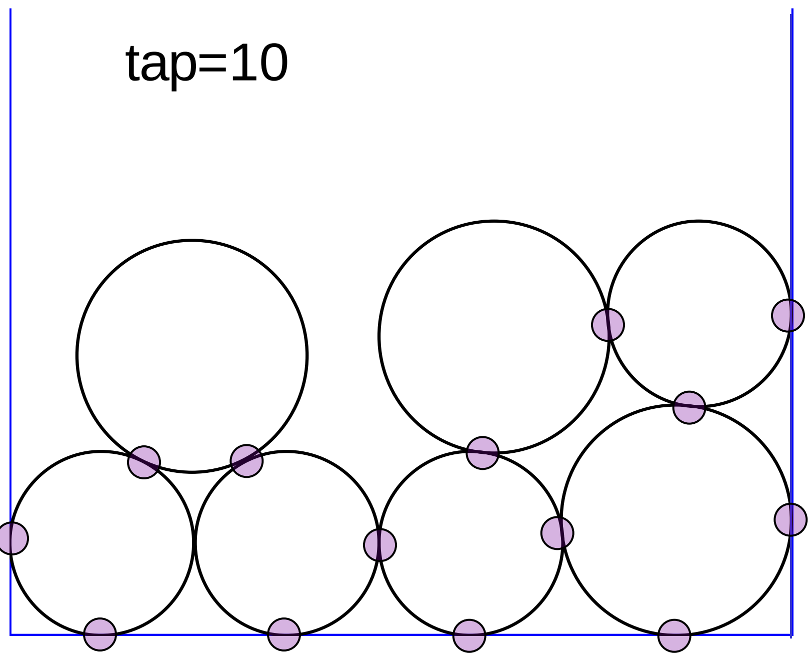

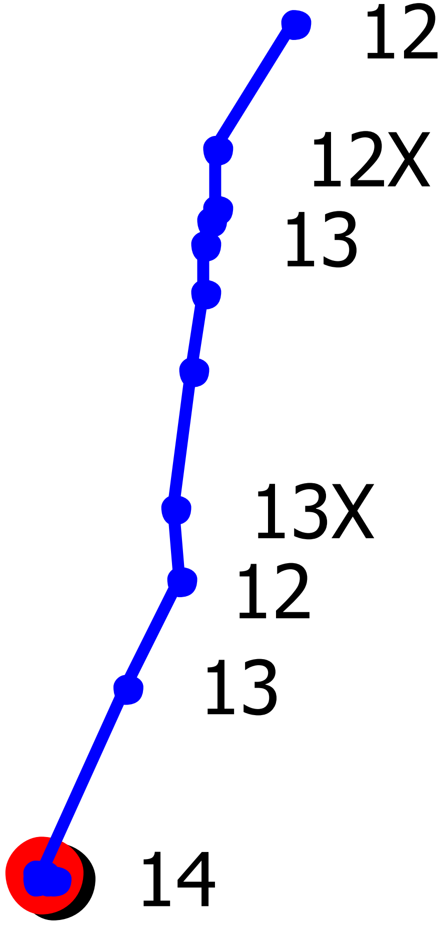

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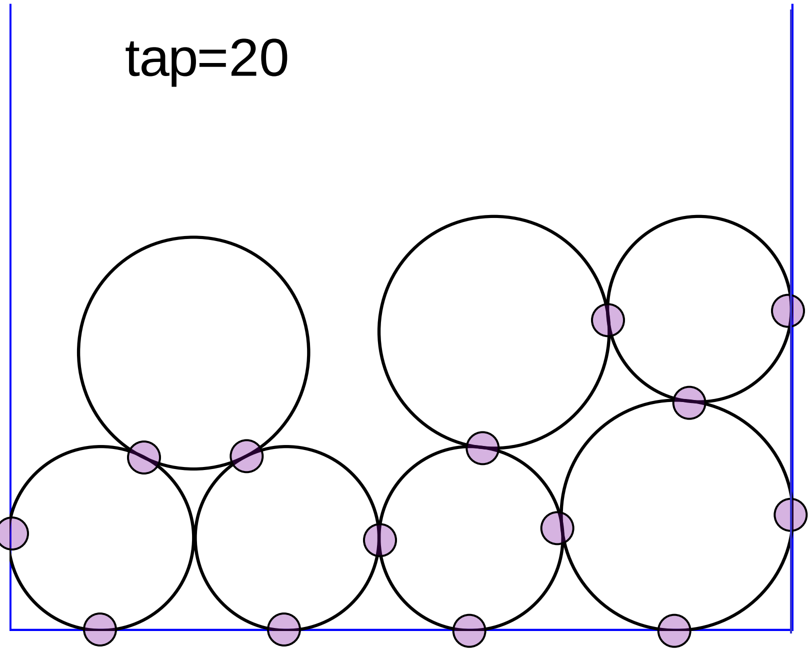

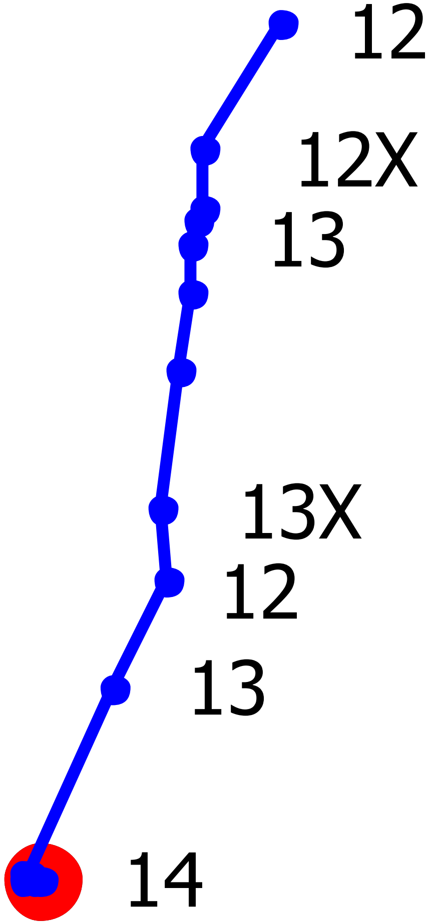

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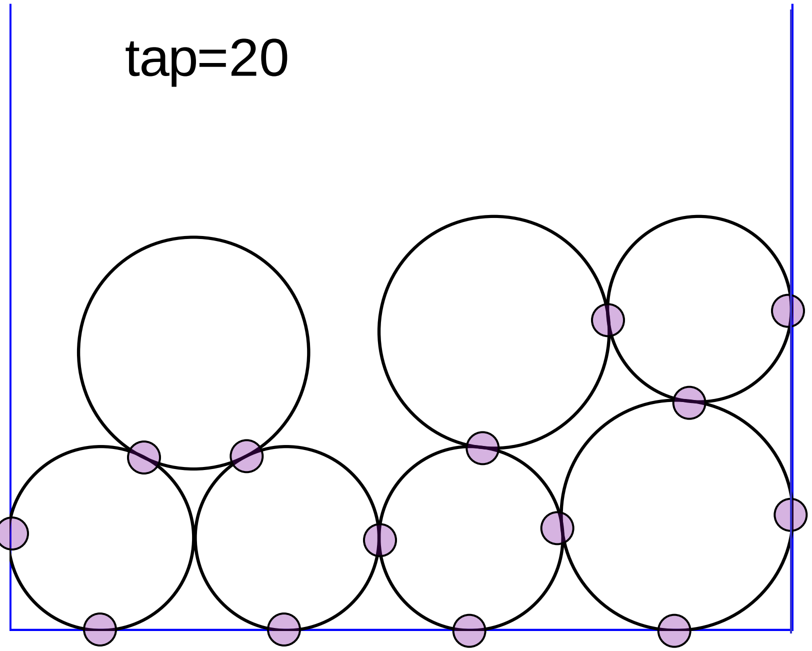
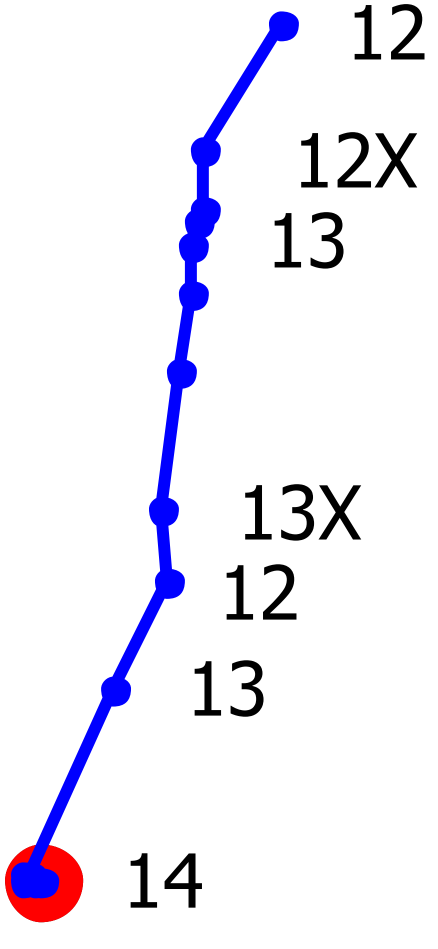
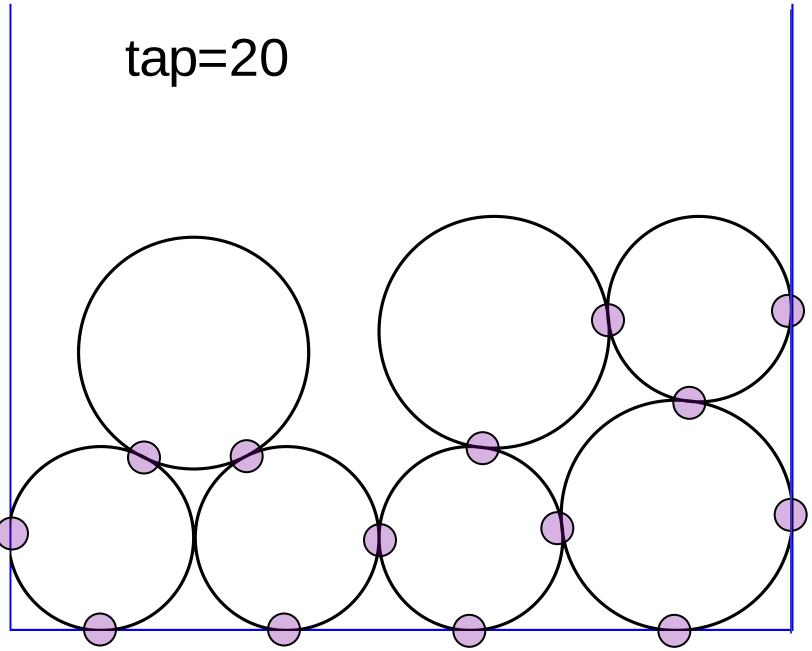
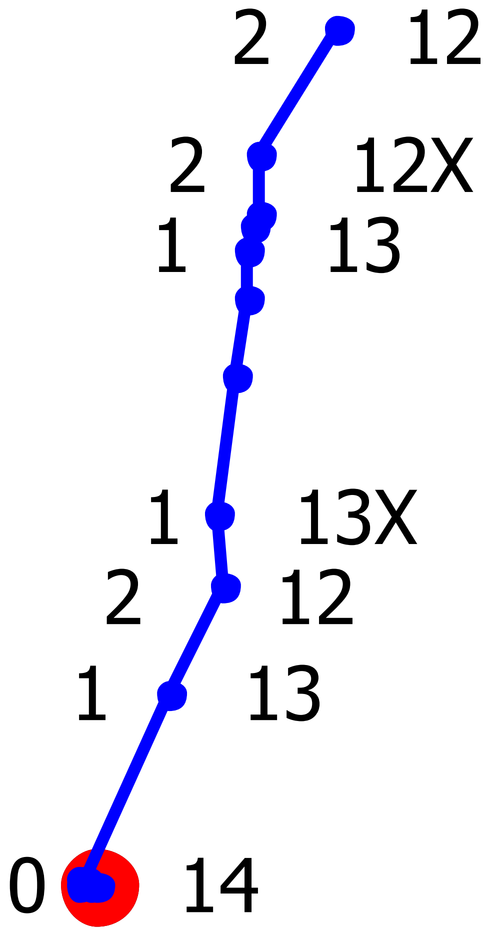




Isostatic: Nc = 2N = 14

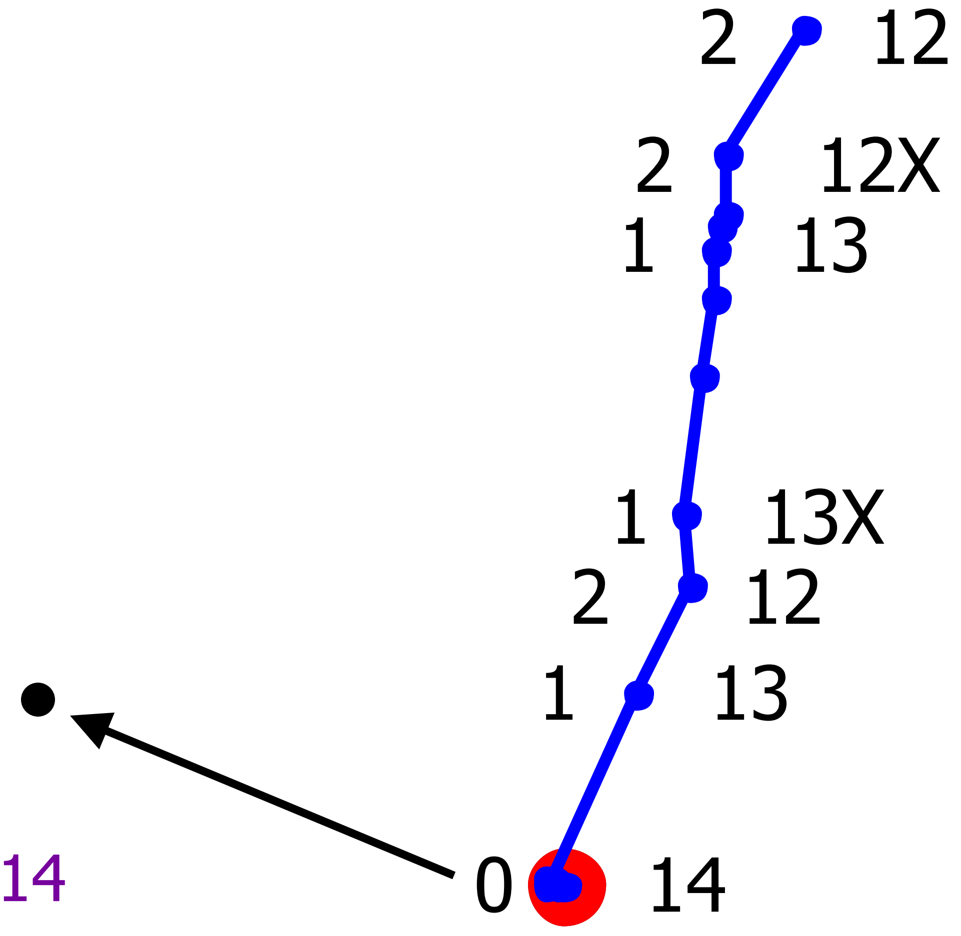
Contact Evolution





Isostatic: Nc = 2N = 14

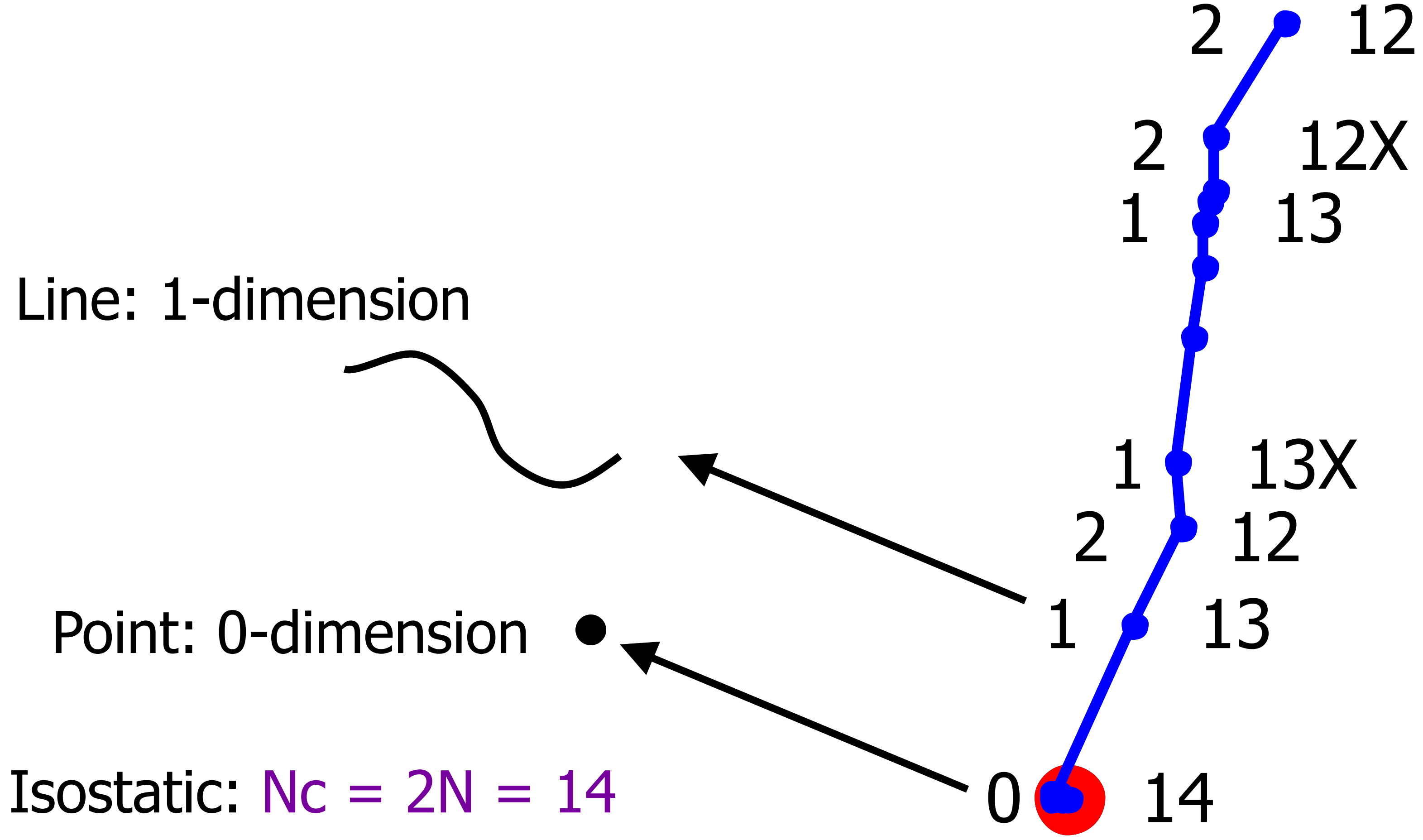
Phase Space Evolution



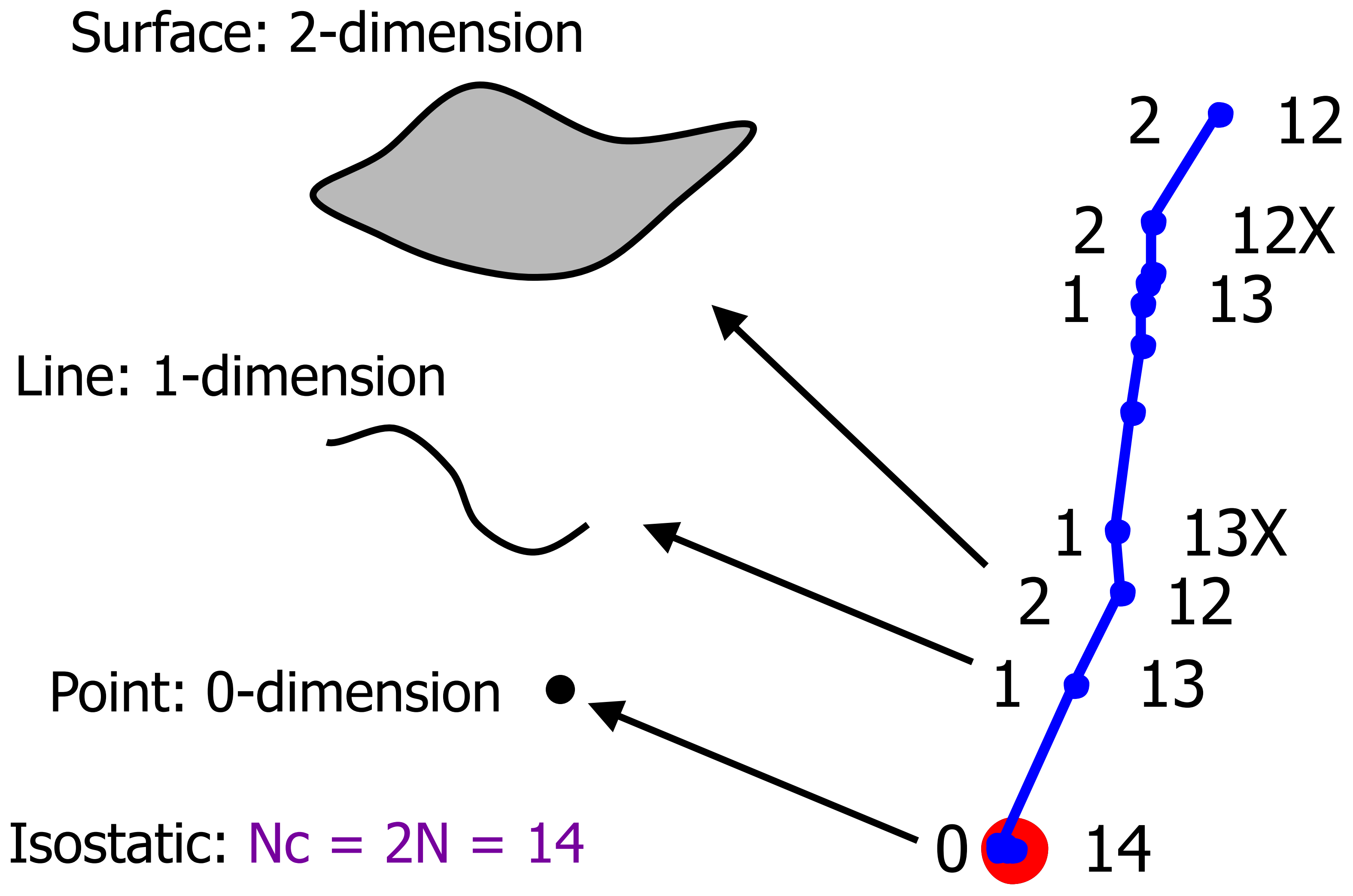
Point: 0-dimension

Isostatic: Nc = 2N = 14

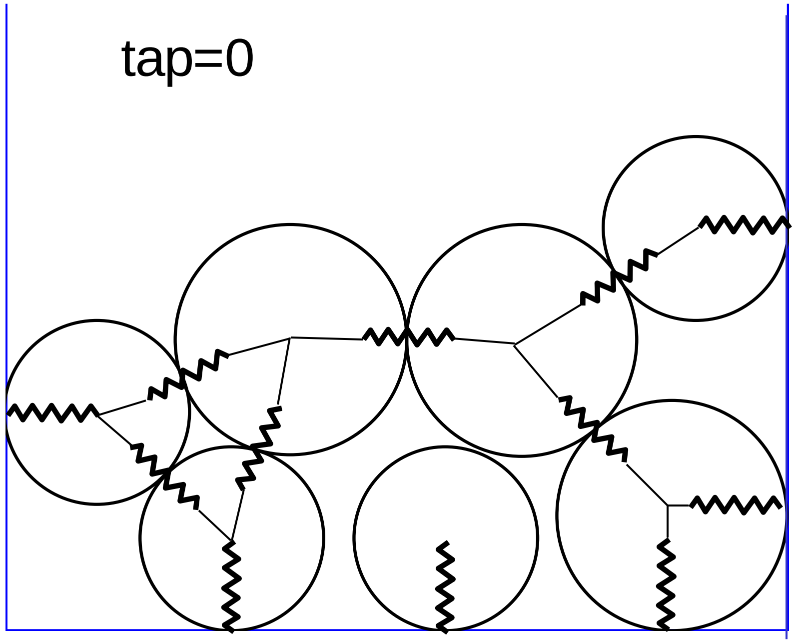
Phase Space Evolution

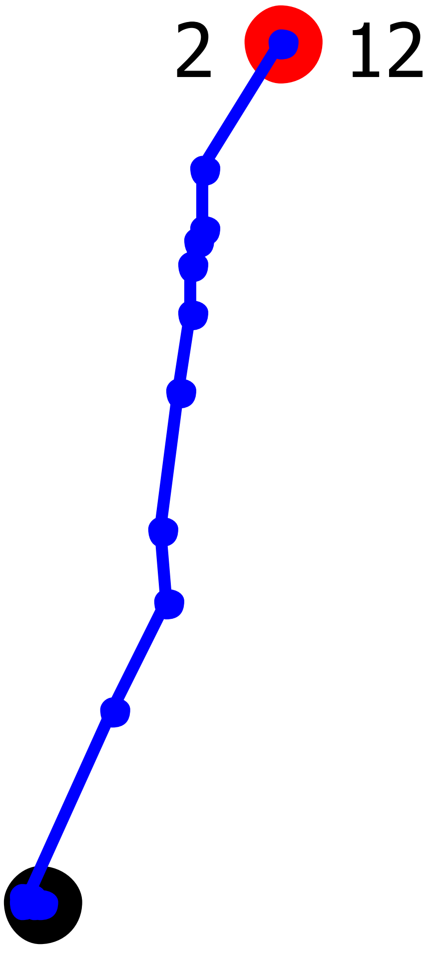


Phase Space Evolution



Spring Network





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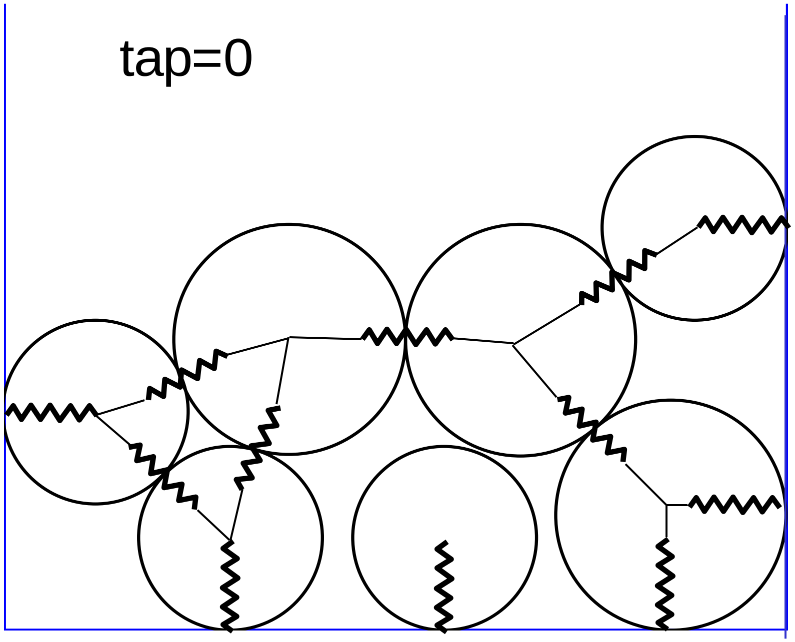
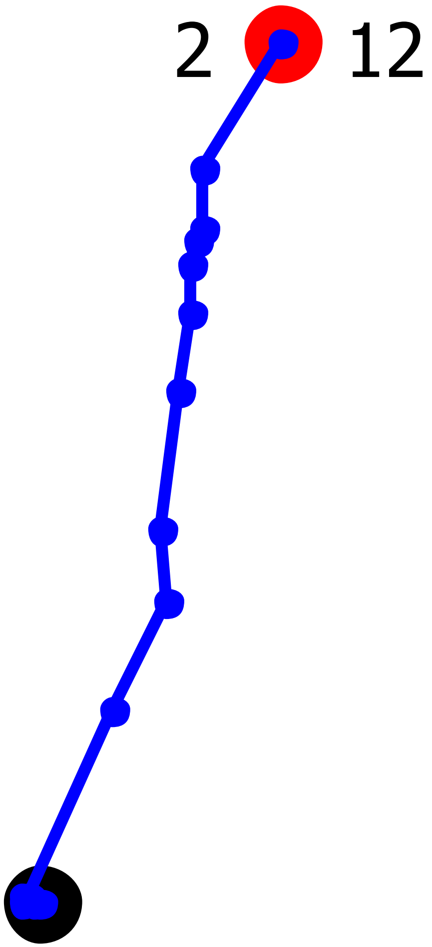
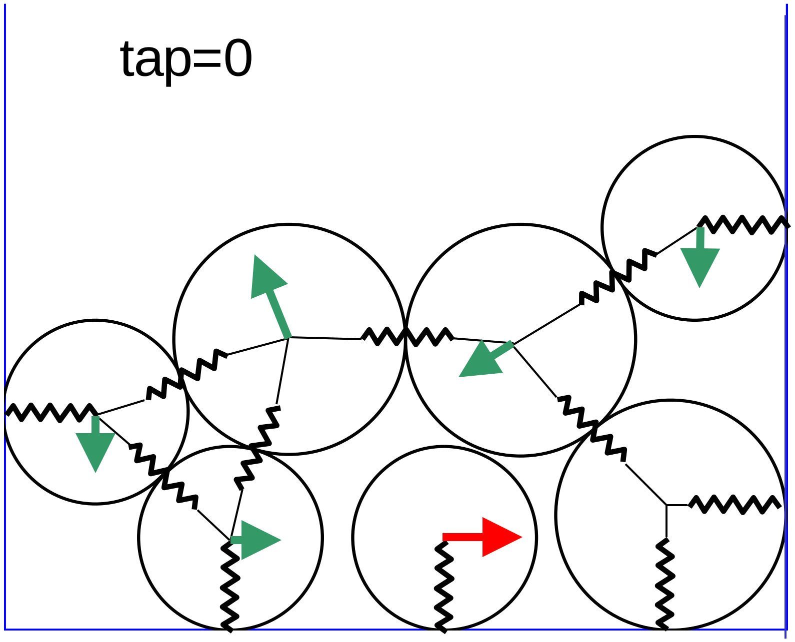


Dynamical Matrix:

$$\kappa_{nk} = \nabla_n \nabla_k V(\mathbf{X}_n^0)$$



Spring Network



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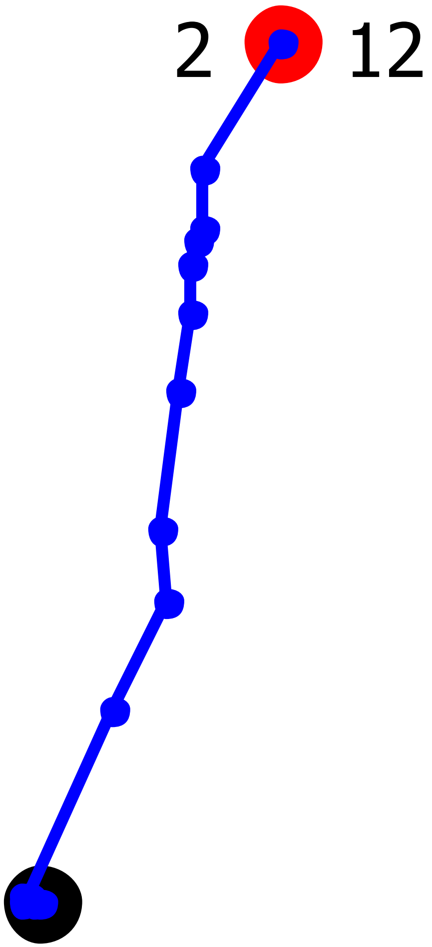
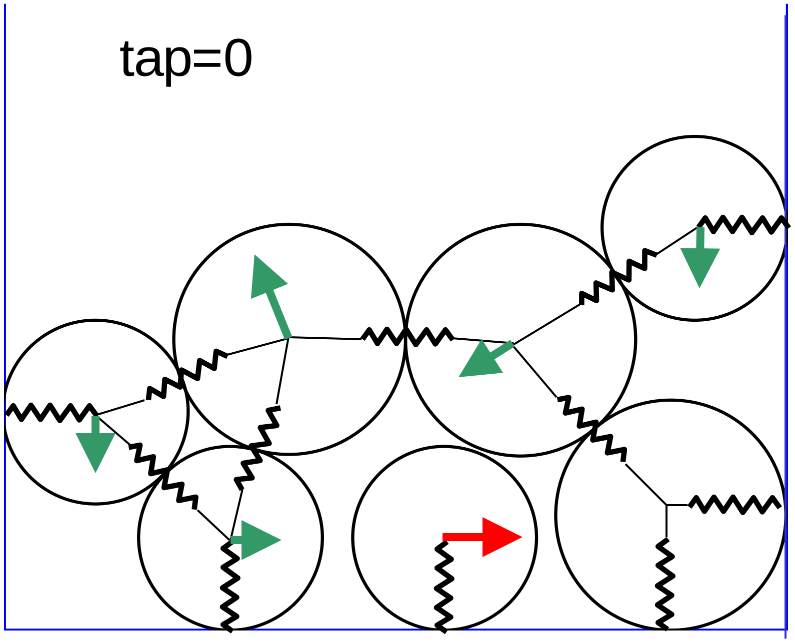
$$\kappa_{nk} = \nabla_n \nabla_k V(\mathbf{X}_n^0)$$

Null Space: 2D



Particles Move in Null Space



Dynamical Matrix:

$$\kappa_{nk} = \nabla_n \nabla_k V(\mathbf{X}_n^0)$$

Null Space: 2D

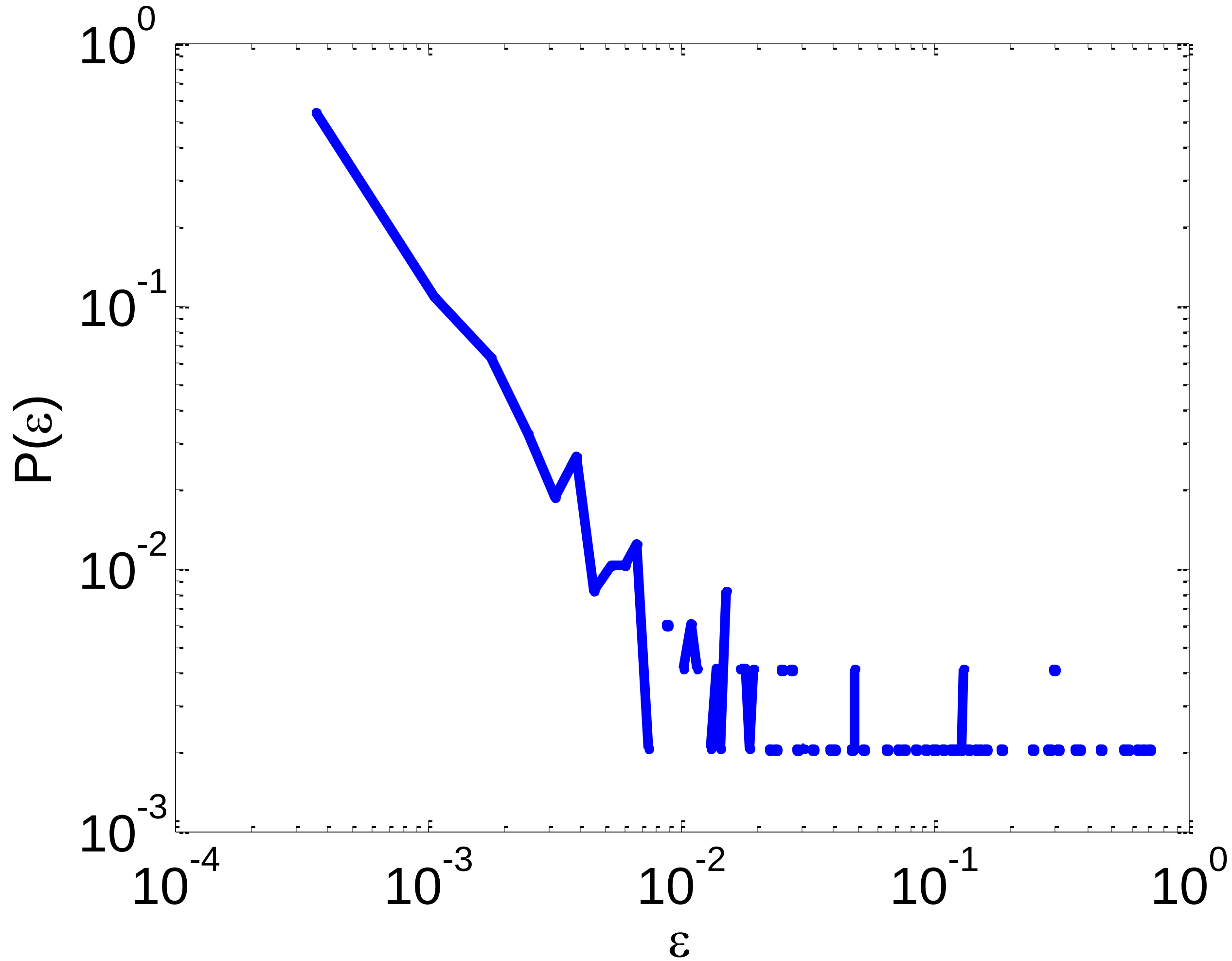
Project Displacement:

$$A_n = \frac{1}{|\mathbf{\Delta X}|} \sum_{k=1}^{2N} \Delta X_k \hat{\varepsilon}_{kn}^{-1}$$

= [-0.002, 0.996, 0.00, ...]

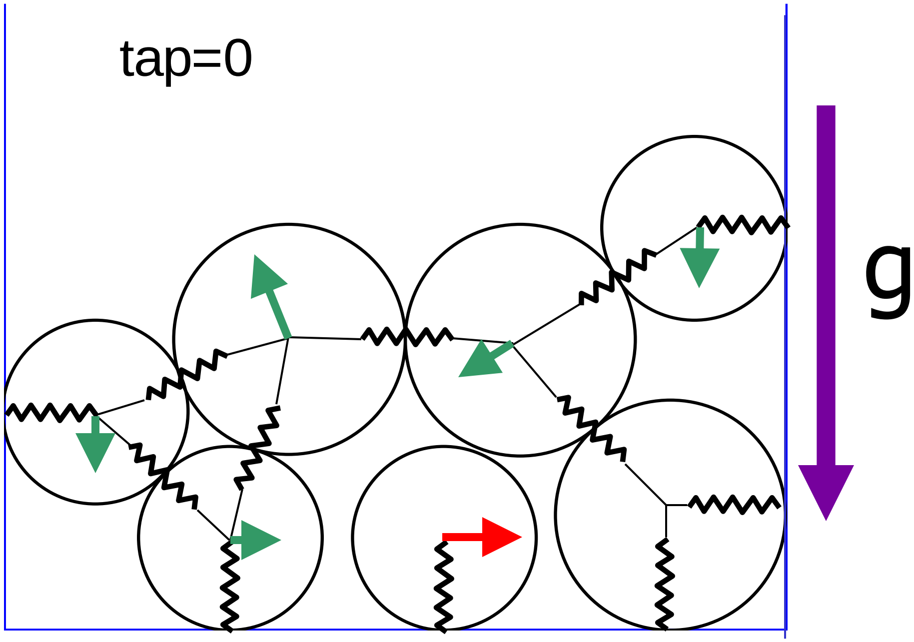
$$\varepsilon = 1 - \sqrt{\sum_{k=1}^{m} A_n^2} = 0.004$$

Particles Move in Null Space



Fraction of Movement Not in Null Space

Particles Move *Down* in Null Space



Dynamical Matrix:

$$\kappa_{nk} = \nabla_n \nabla_k V(\mathbf{X}_n^0)$$

Null Space: 2D

Project Downward Displacement:

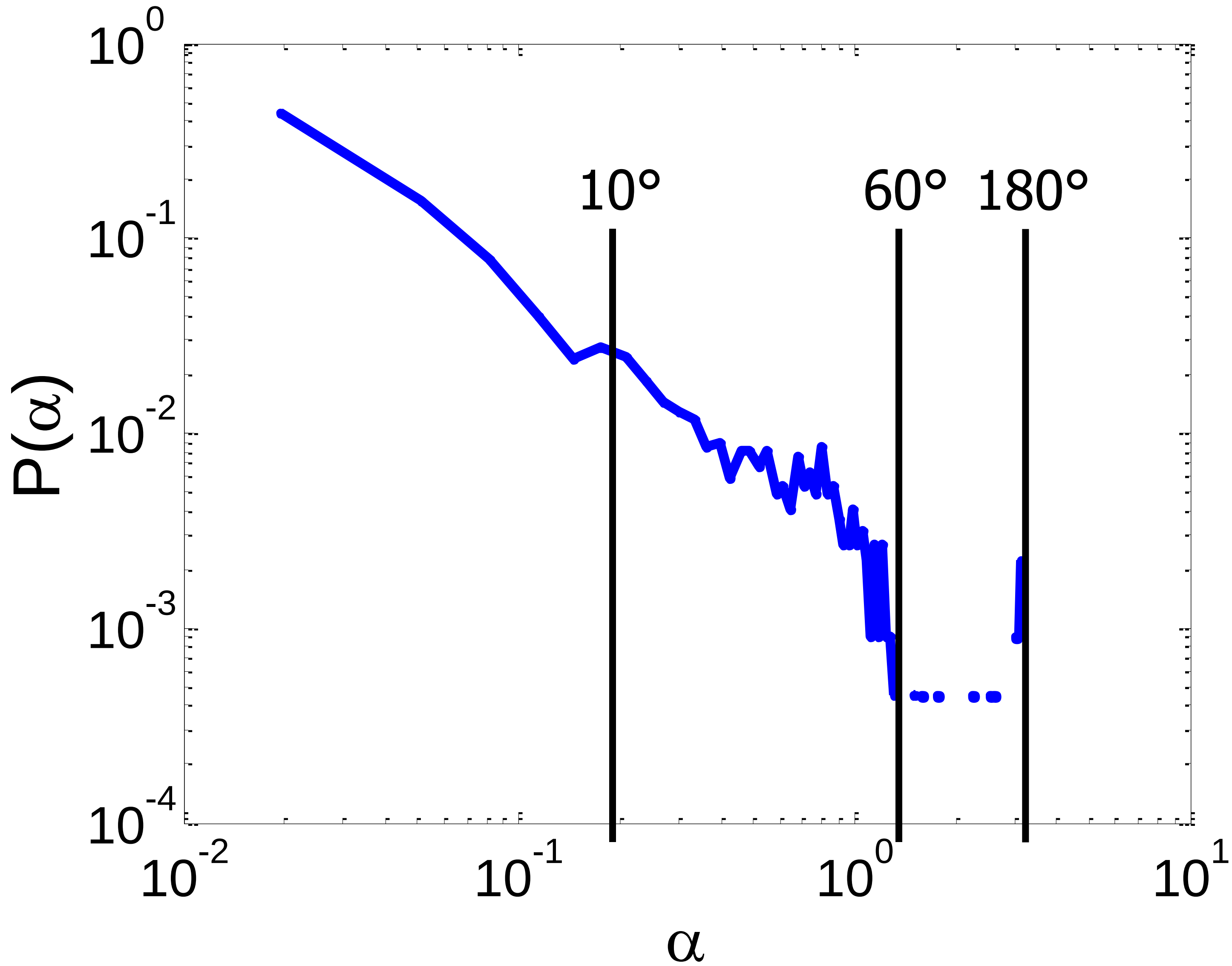
$$A_n = \sum_{k=1}^{2N} g_k \hat{\varepsilon}_{kn}^{-1}$$

Project-Back *only* in Null Space:

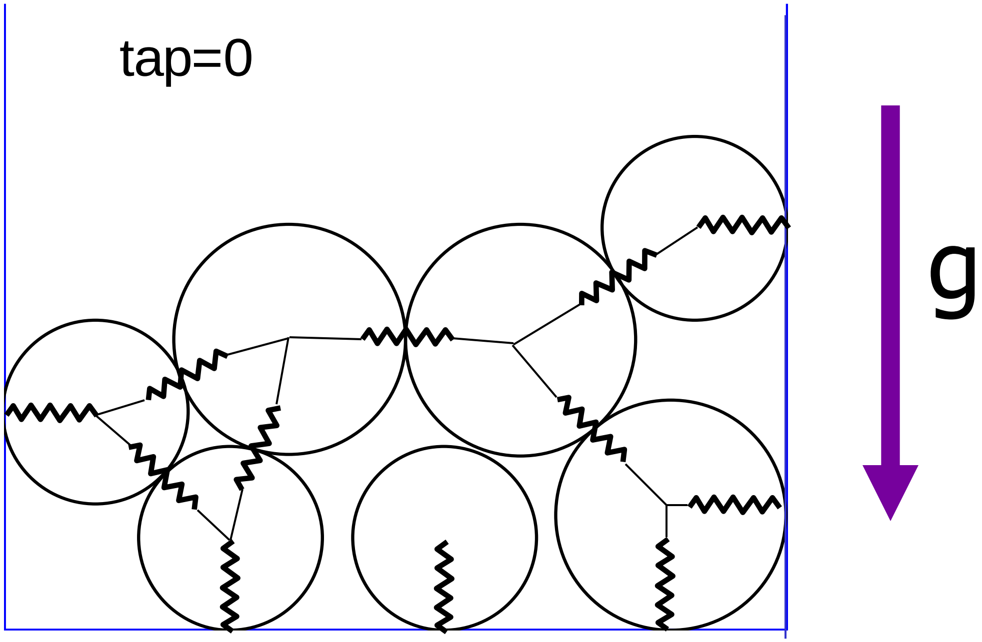
$$\Delta X_k^{\text{predict}} = \sum_{n=1}^{m} A_n \hat{\varepsilon}_{kn}$$

$$\cos(\alpha) = \frac{\Delta \mathbf{X}^{\text{predict}} \cdot \Delta \mathbf{X}}{\left|\Delta \mathbf{X}^{\text{predict}}\right| \left|\Delta \mathbf{X}\right|}$$

Distribution of Angle Between Experiment and Predicted Displacement



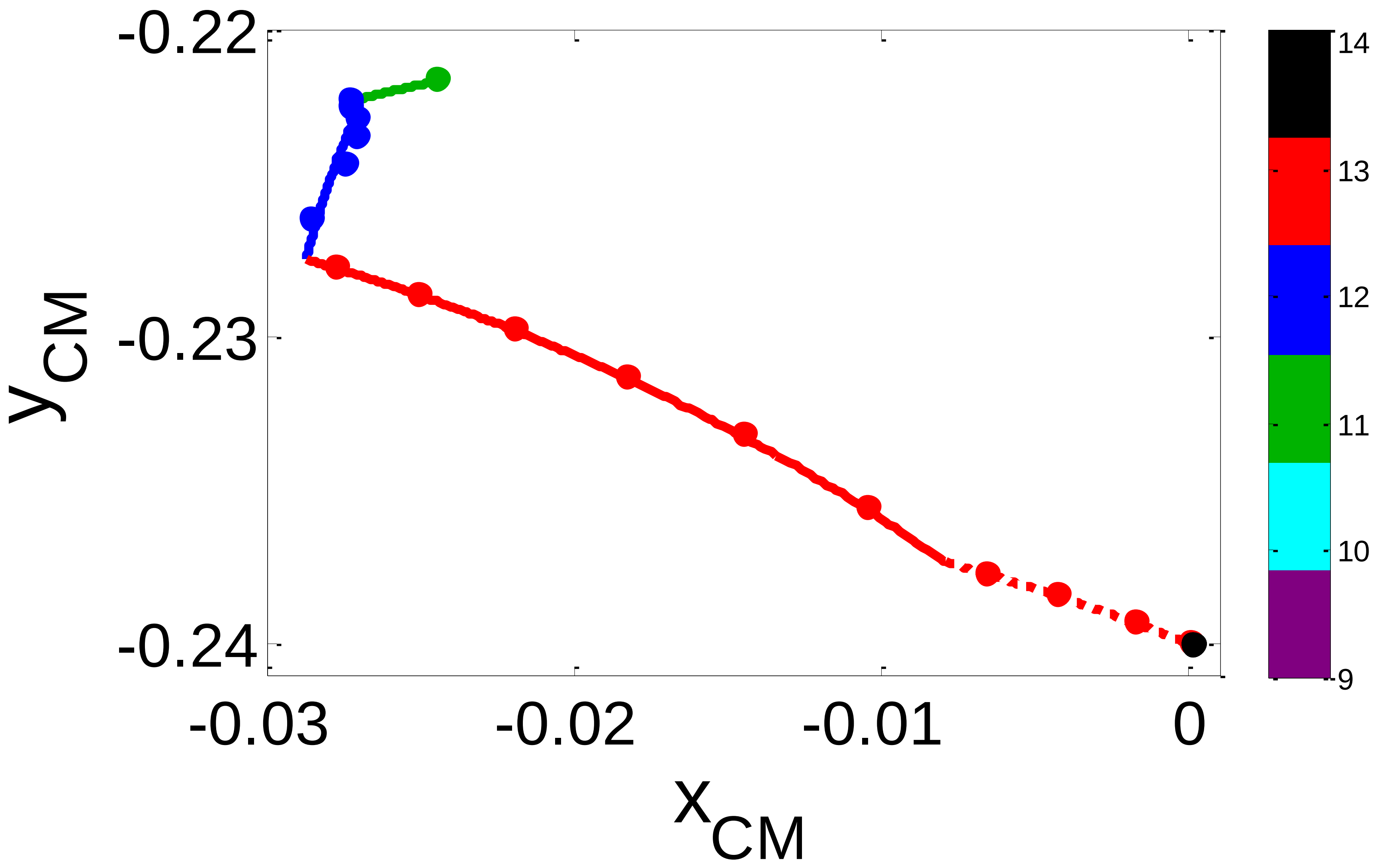
Spring Network with Gravity



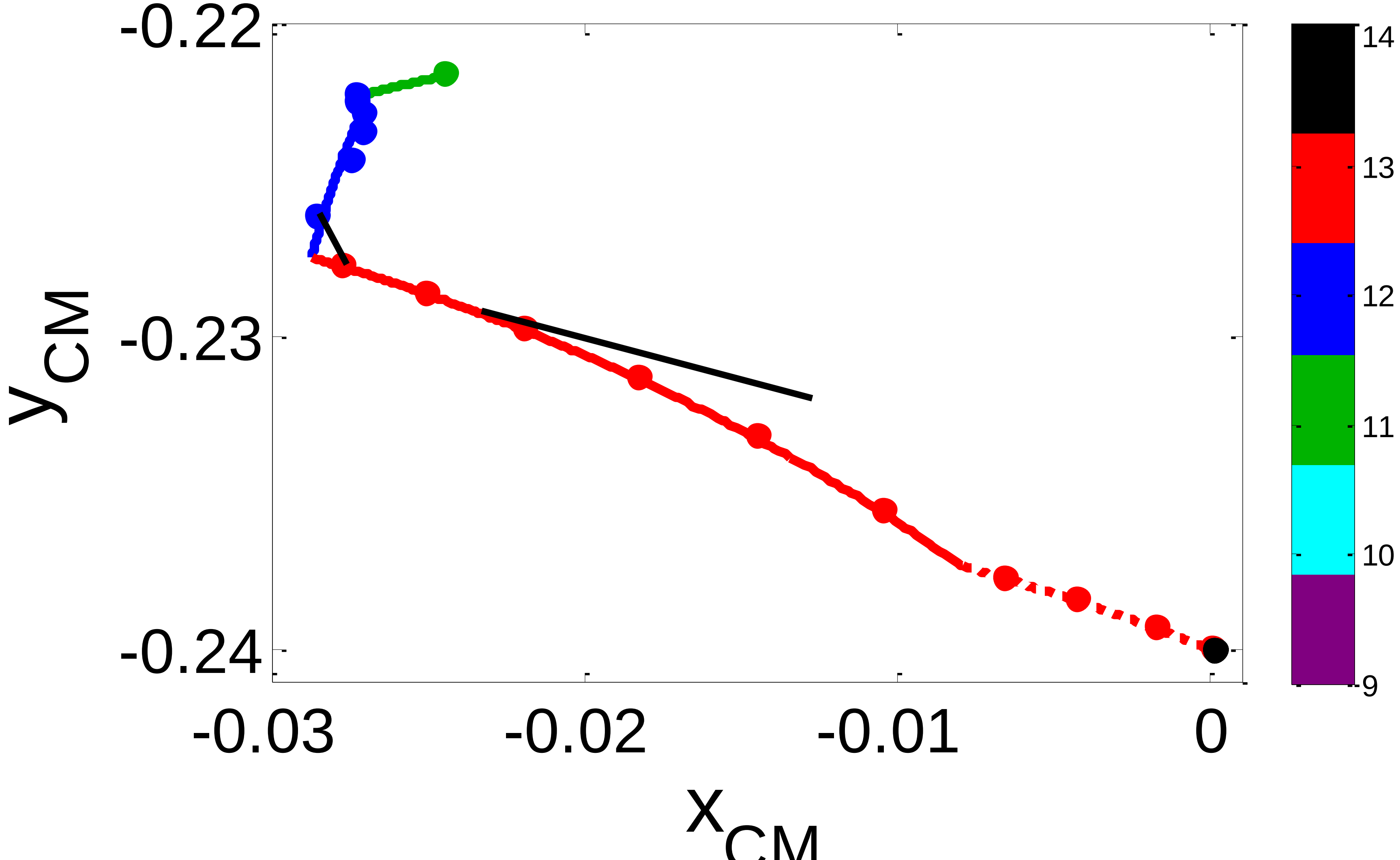
Spring Model

1) Apply gravity force to spring network for a short time.
2) Relax network.
3) Repeat (1-2) until new contact forms or reach steady-state.
4) If new contact add spring and goto 1).
5) If steady and isostatic quit, else break weakest contact and goto 1).

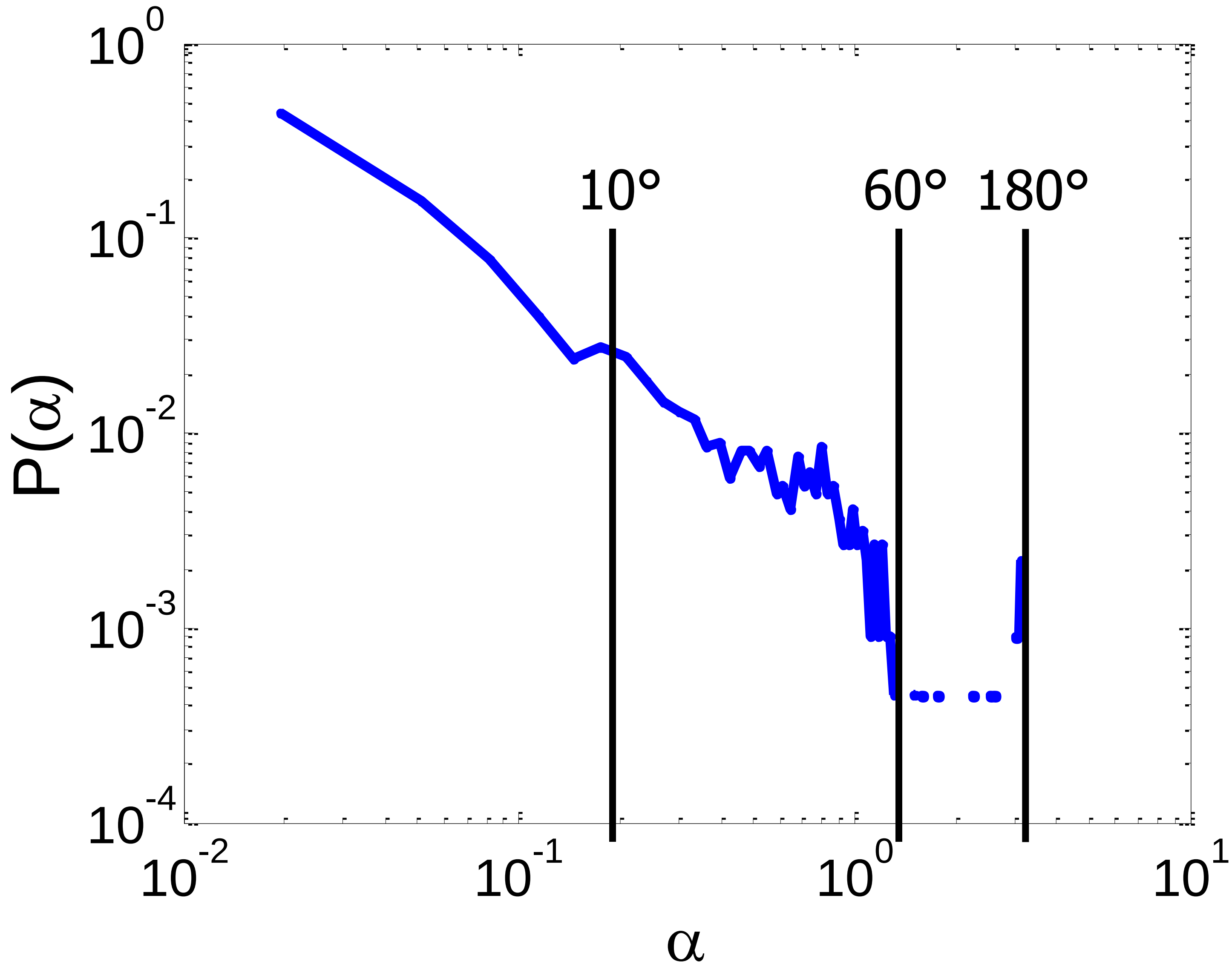
Comparison to Springs with Gravity



Some Experimental Errors



Distribution of Angle Between Experiment and Predicted Displacement



Experimental Results

Frictional Families

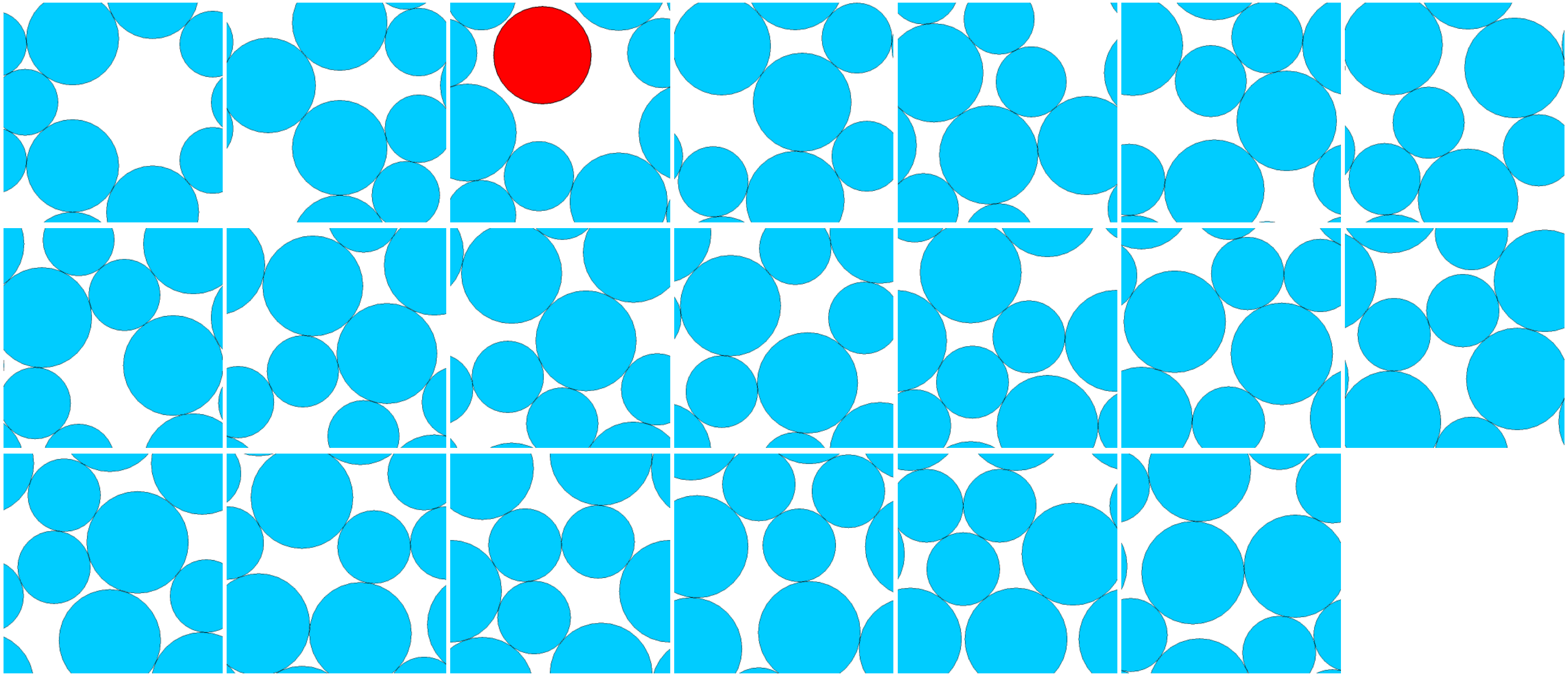
- Frictional packings form 1-D families under periodic gravitational compaction.
  - The states evolve along the families.
  - The evolution is described by the dynamical matrix of a normal spring network formed from the contacts of the frictional state.
  - The system evolves in the direction of gravity projected on the null space.
- Frictional families are not equally probable.
  - Most/Least probable similar to frictionless.

Frictional Families for Packings Creation by Compression

N=6 Periodic Packings

Isostatic: $N_c = 2N - 1$

N = 6 (5) => **N_c = 11 (9)**



diameter ratio σ_L/σ_S=1.4

Invariant map: a 2-D representation

- Use distance matrix **D**
  D_{ij} = the distance between particle i and j (D_{ij} =0 in case i = j)

- Solve for the invariant of distant matrix D:

$$q_1 = tr(\hat{D}) \quad \mathbf{= 0}$$

$$q_2 = \frac{1}{2}[(tr(\hat{D}))^2 - tr(\hat{D}^2)]$$

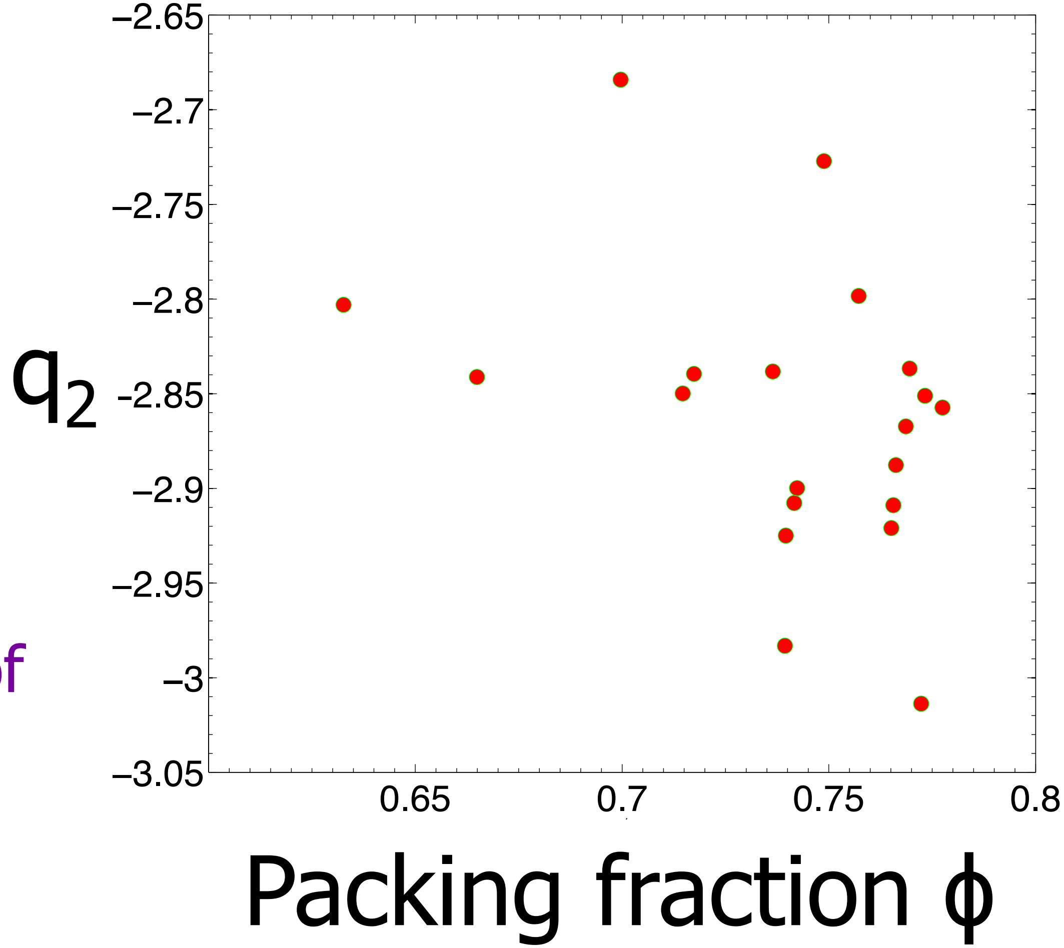
$$q_3 = \ldots$$

.

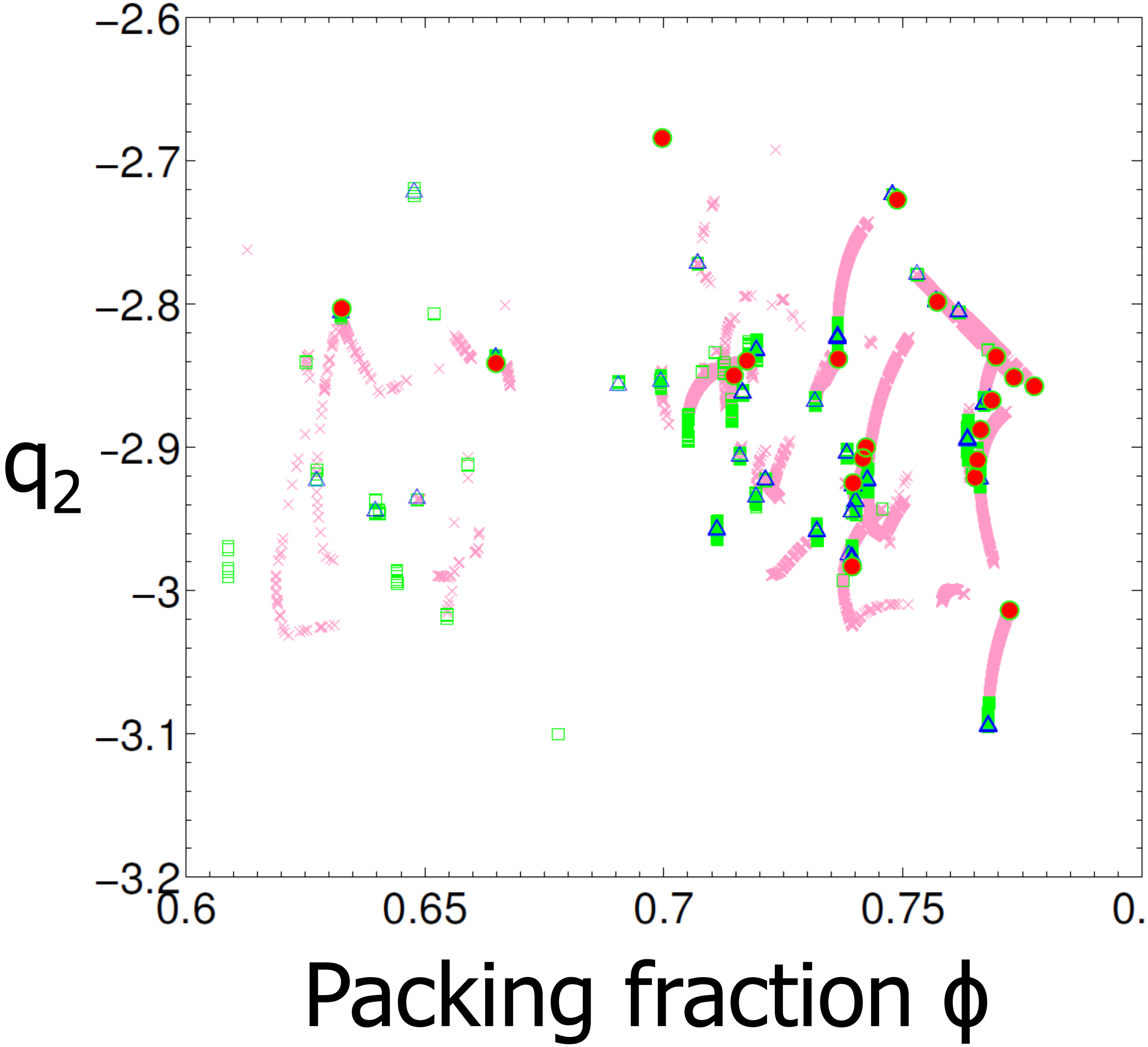
.

.

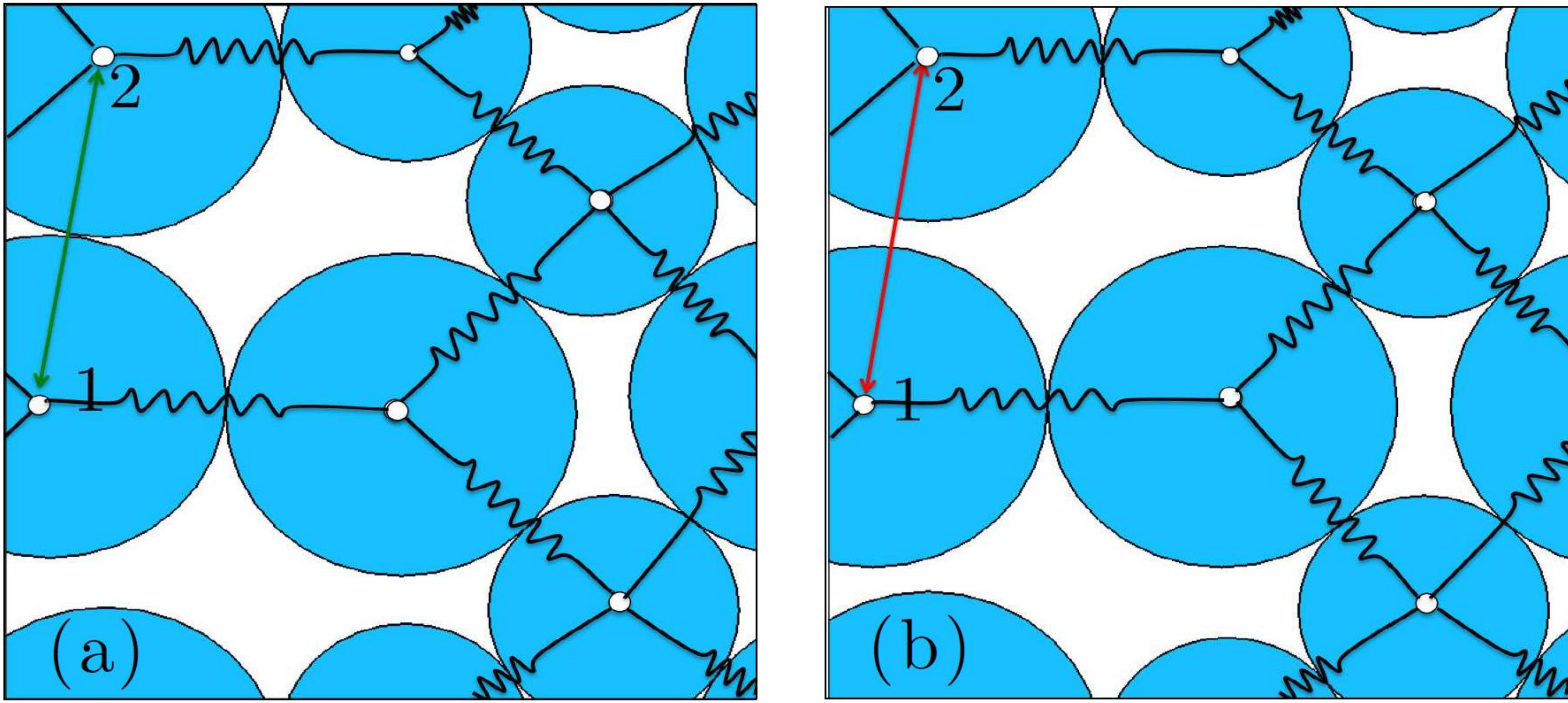
Packings are minima of the density landscape or 0th order saddles.



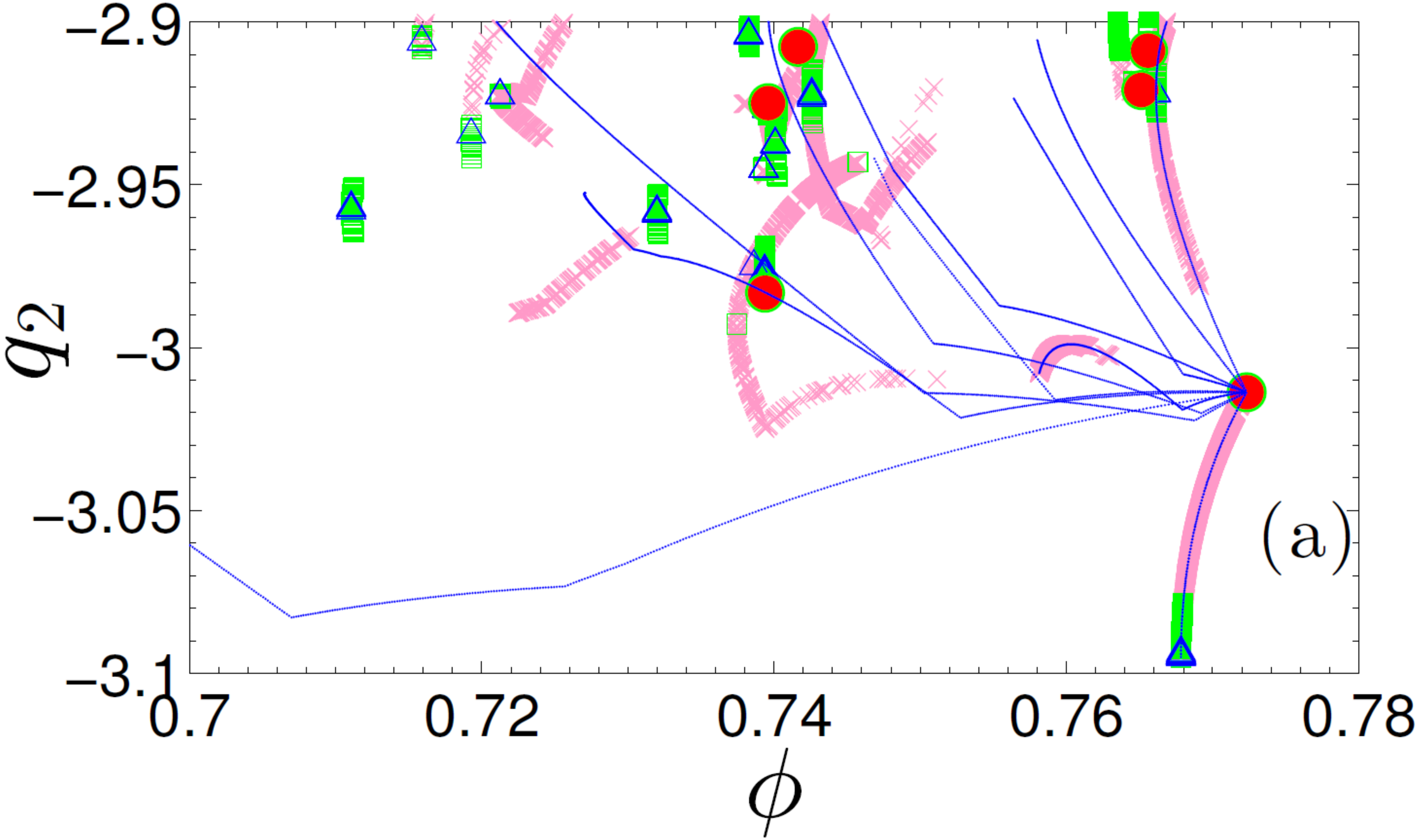
Cundall-Strack Frictional Packings
1^{st} Order Saddles (N_{iso}-1)



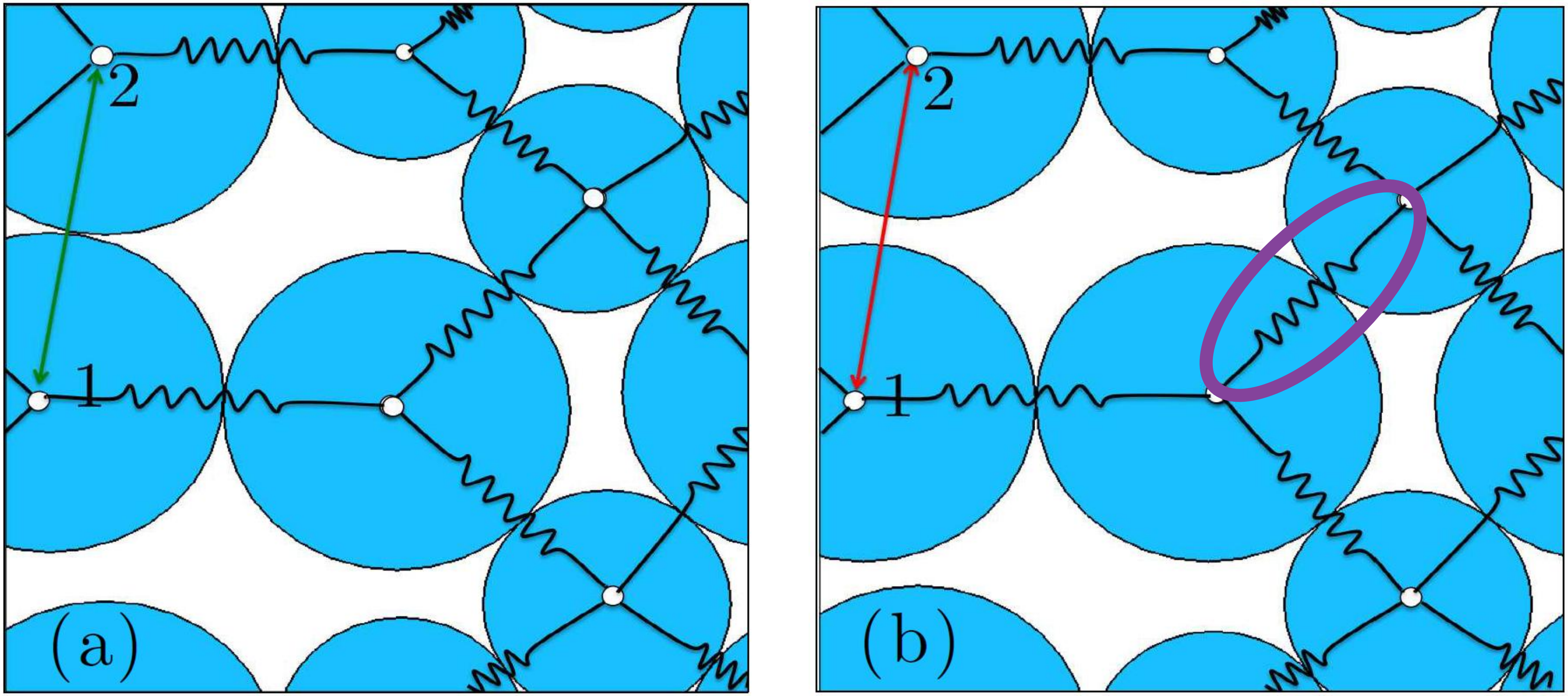
Spring Model



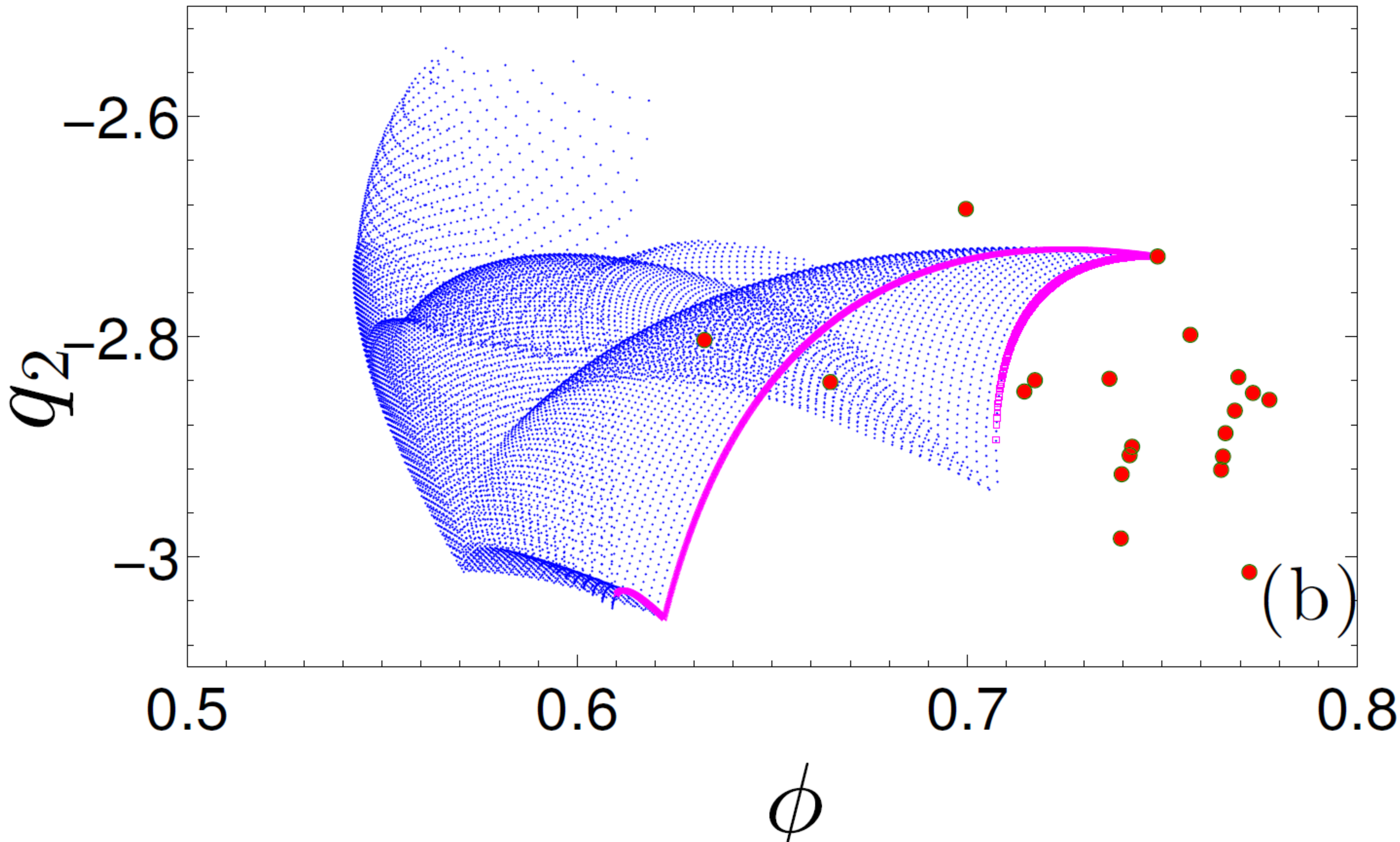
Cundall-Strack Frictional Packings
1st Order Saddles enumerated



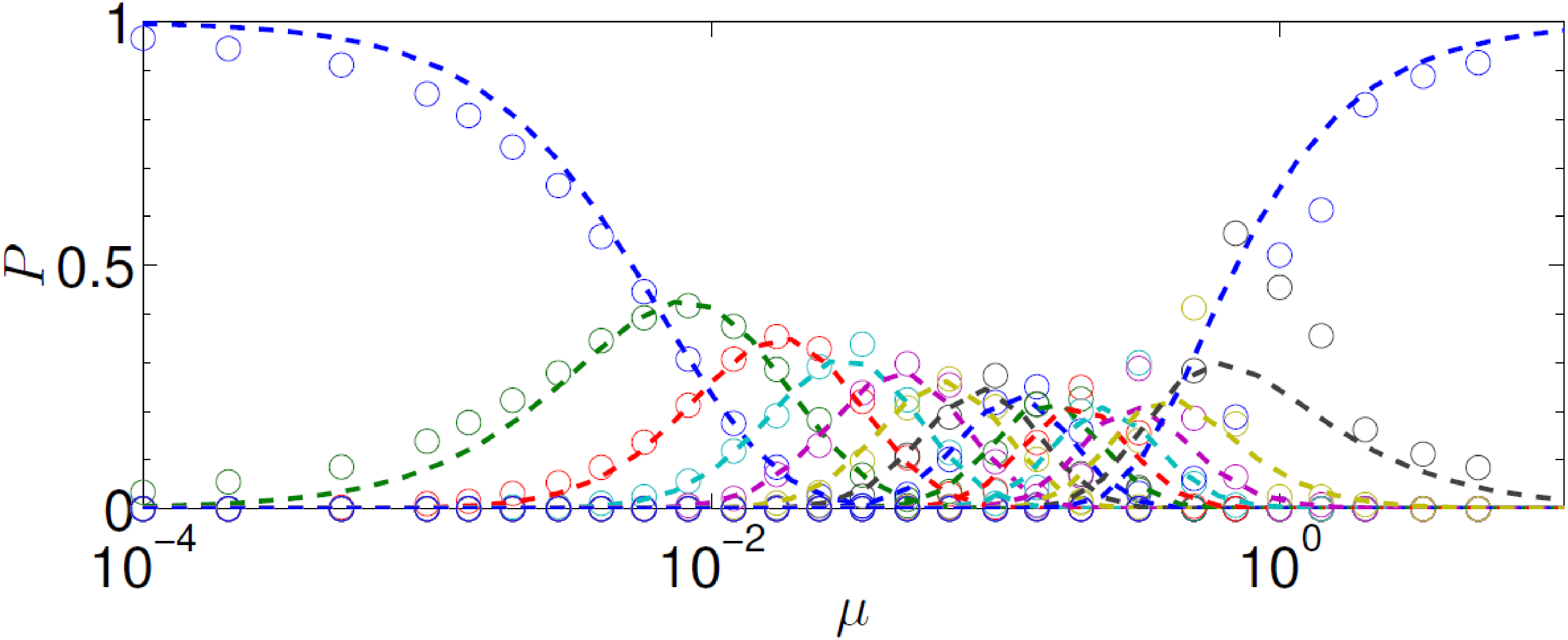
Spring Model



Cundall-Strack Frictional Packings
2nd Order Saddles enumerated



Probability of m^{th} Order Saddles For N=30 Frictional Packings

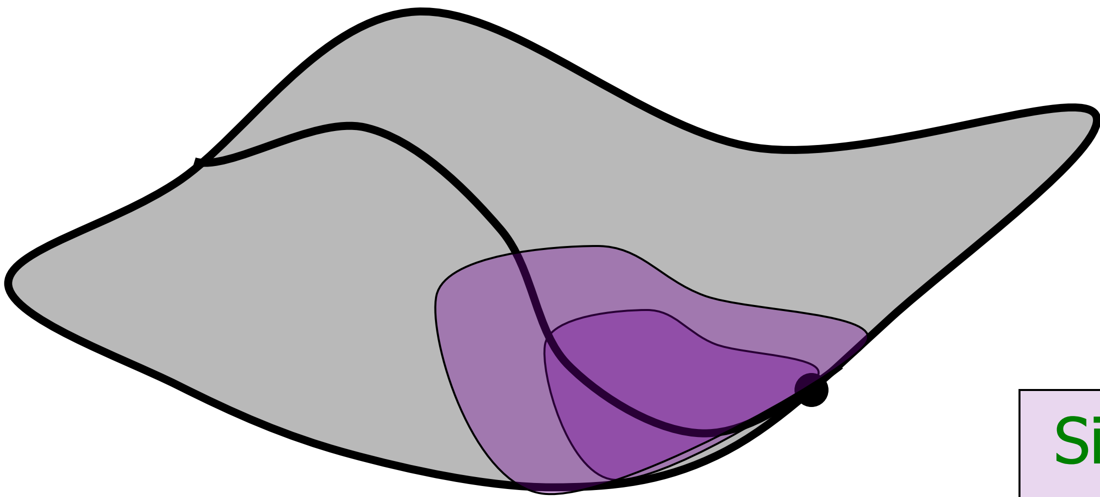


Static Friction Coefficient

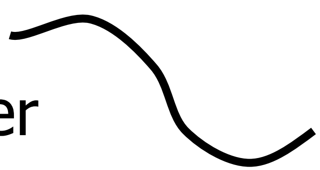
Dependence on Saddle Order and Friction

Real Space

Surface: 2nd order



Line: 1st order



Point: 0th order

Force Space

Force Balance:

$$\mathbf{CF} = 0$$

Solution Null Space of C

Size of C Null Space *shrinks* as Real Space *grows*

Configurational Entropy:

$$V_m^R(\mu) = A_m \mu^m$$

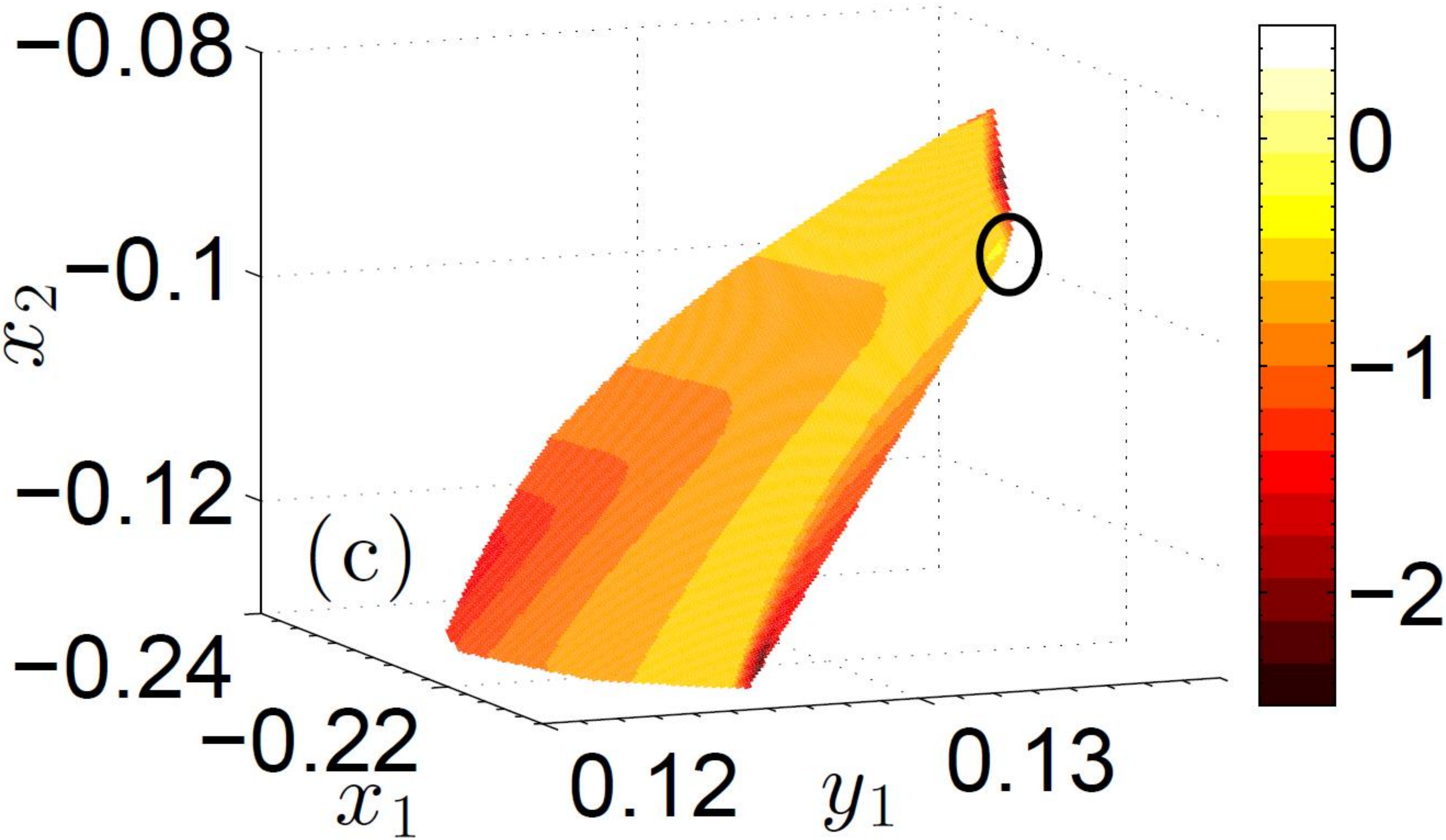
Probability of m^th Order Saddles

$$Z_m(\mu) \propto V_m(\mu)\delta^{2N-1-m}$$

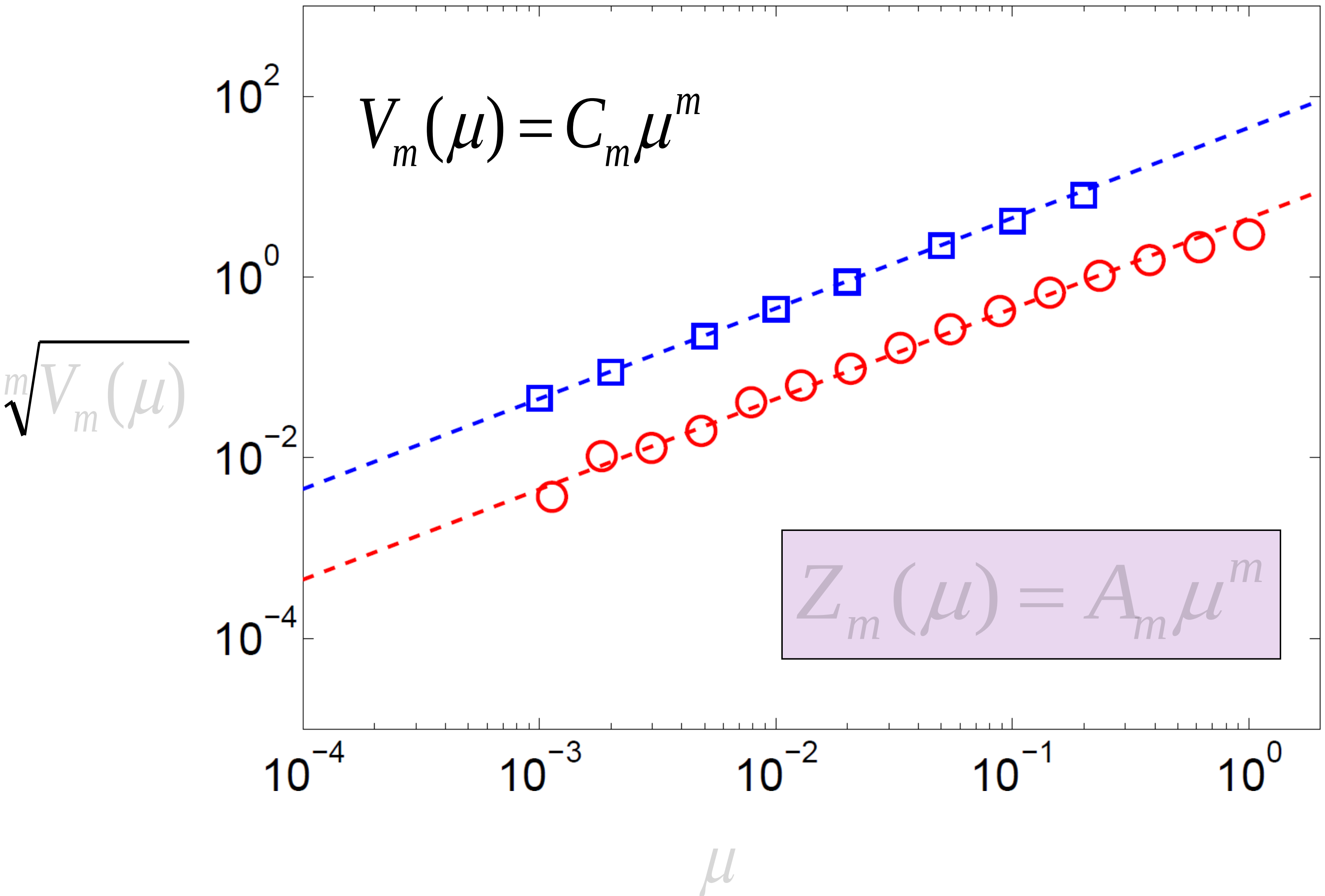
$$V_m(\mu) \sim [l(\mu)]^m$$

$$\frac{Z_m(\mu)}{Z_0(\mu)} \propto \left(\frac{l(\mu)}{\delta}\right)^m$$

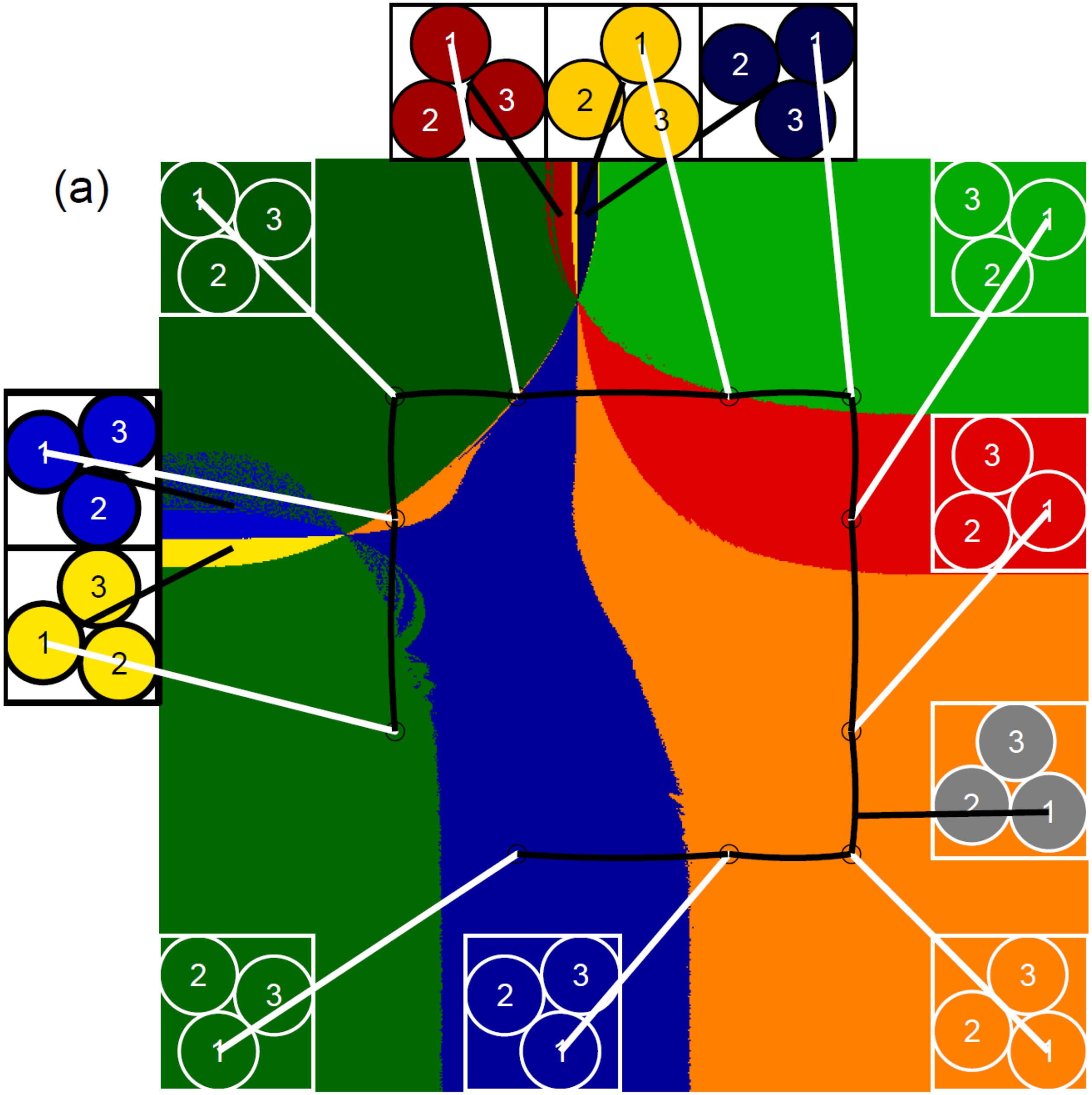
Real Space



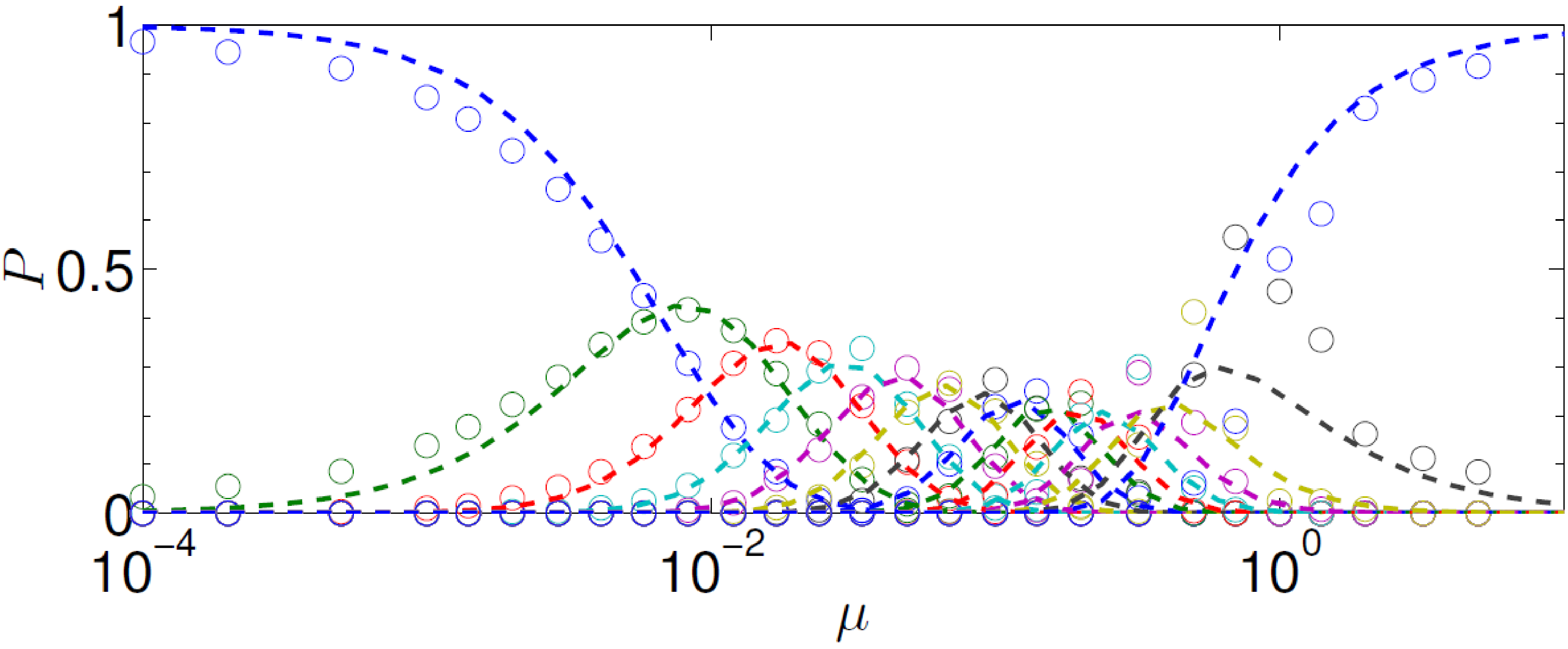
Partition Function



Basin Diagram



Probability of mth Order Saddles For N=32 Frictional Packings



$$P_m(\mu) = Z_m(\mu) / \sum_m Z_m(\mu)$$

Probability of mth Order Saddles

$$P_m(\mu) = \frac{A_m \mu^m}{\sum_{m=0}^{m_{\max}} A_m \mu^m}$$

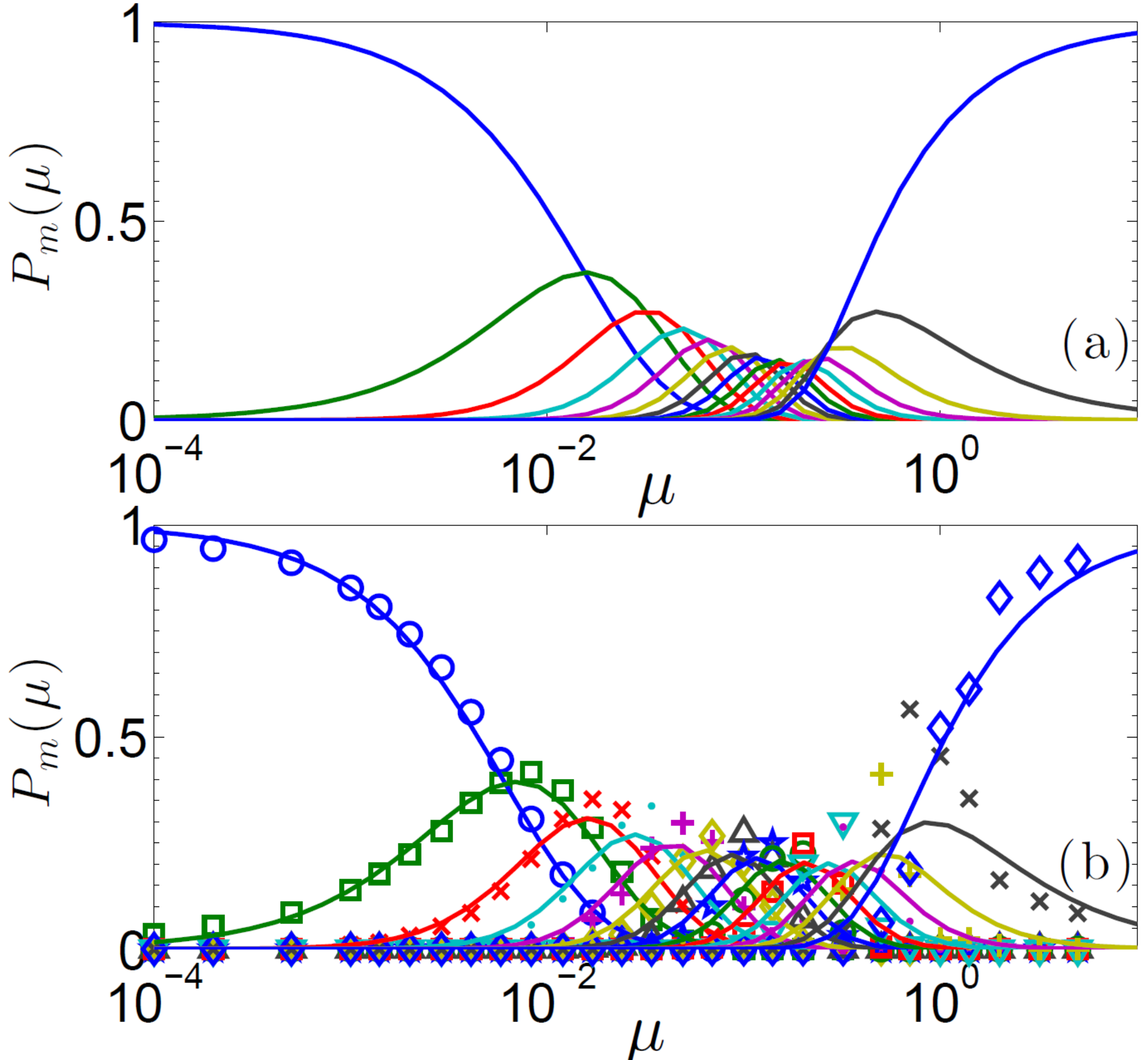
$$= \frac{a_m \mu^m}{1 + \sum_{m=1}^{m_{\max}} a_m \mu^m}$$

$$A_m = N_0(N) N_B(N,m)$$

$$a_m = \frac{A_m}{A_0} = N_B(N,m) = C^m_{Nc=2N-1}$$

Probability of mth Order Saddles N=30

$$P_m(\mu) = Z_m(\mu) / \sum_m Z_m(\mu)$$

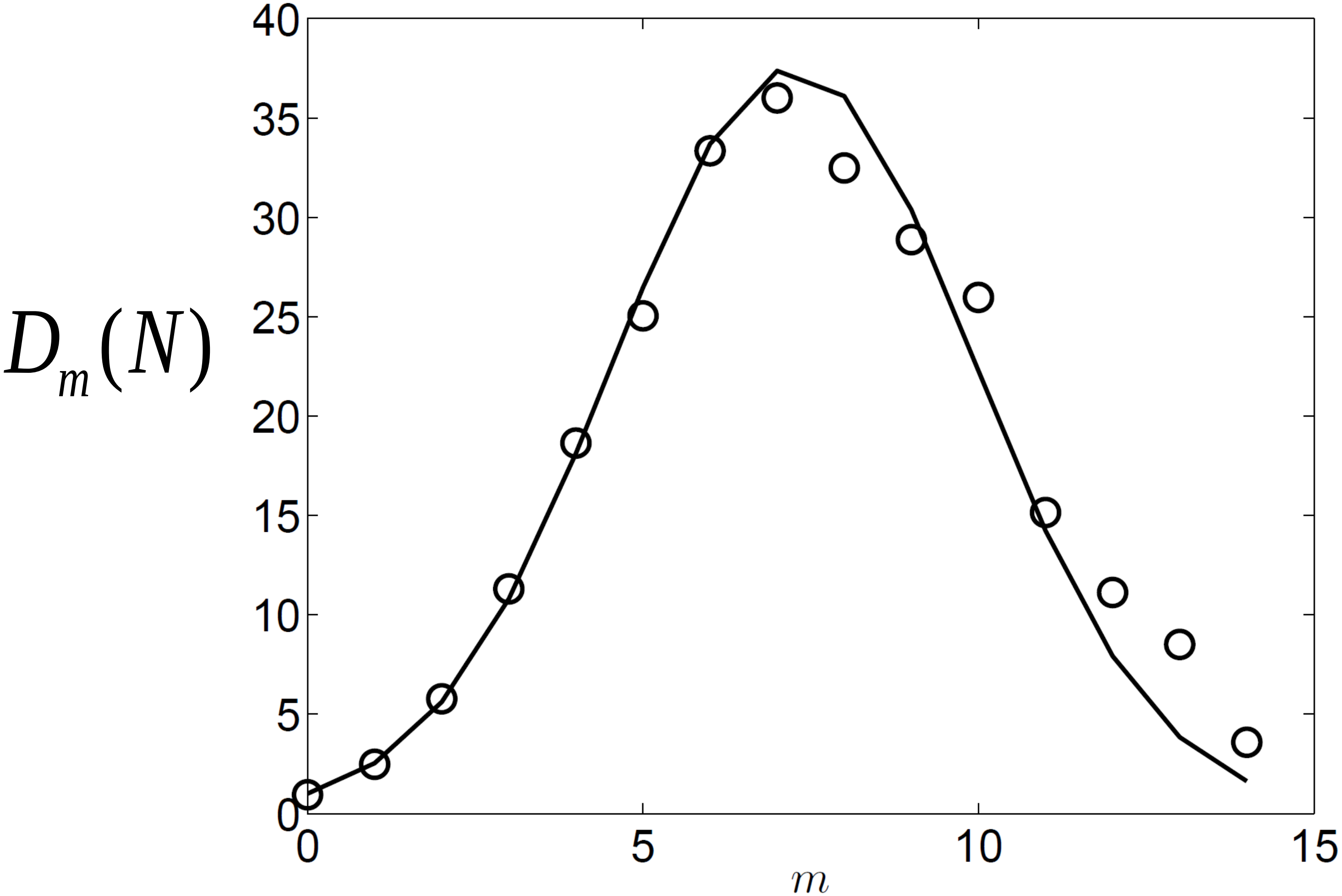


Probability of mth Order Saddles

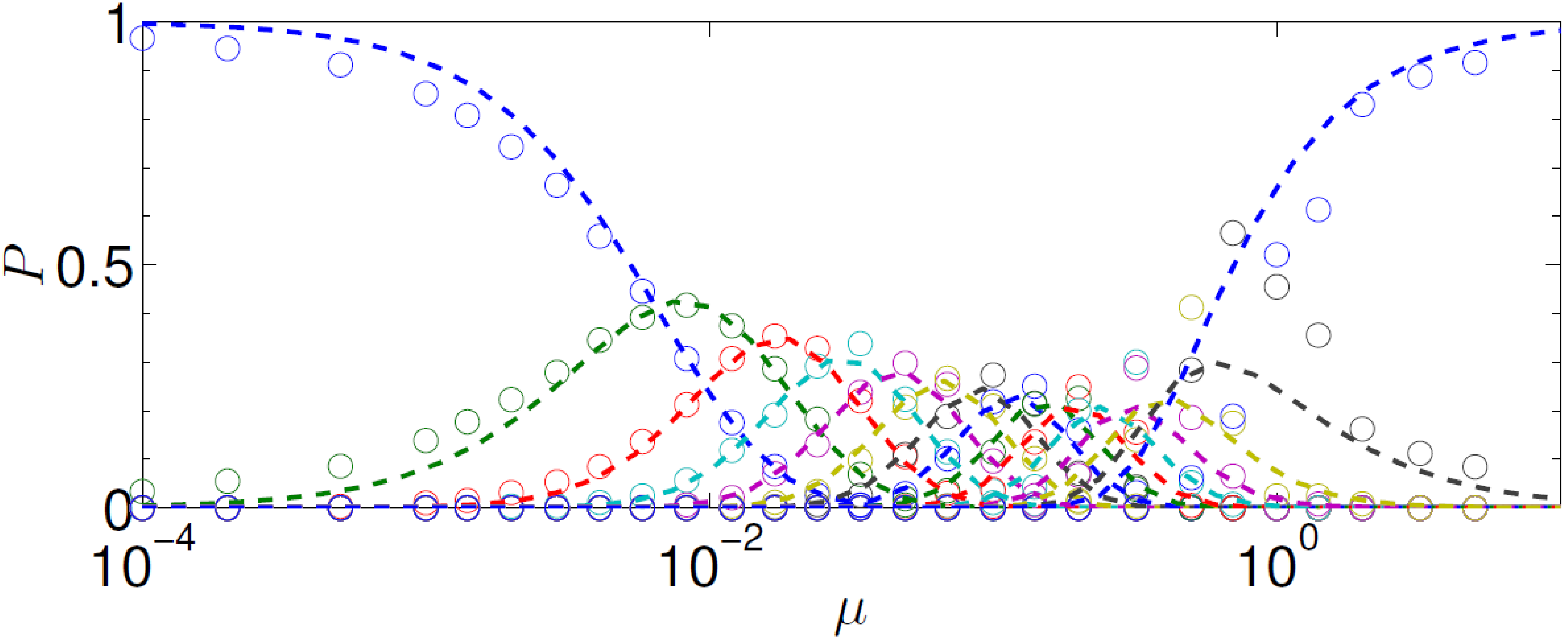
$$a_m = \frac{A_m}{A_0} = N_B(N,m) = C^m_{Nc=2N-1}$$

$$a_m = D_m(N) C^m_{Nc}$$

Probability of mth Order Saddles

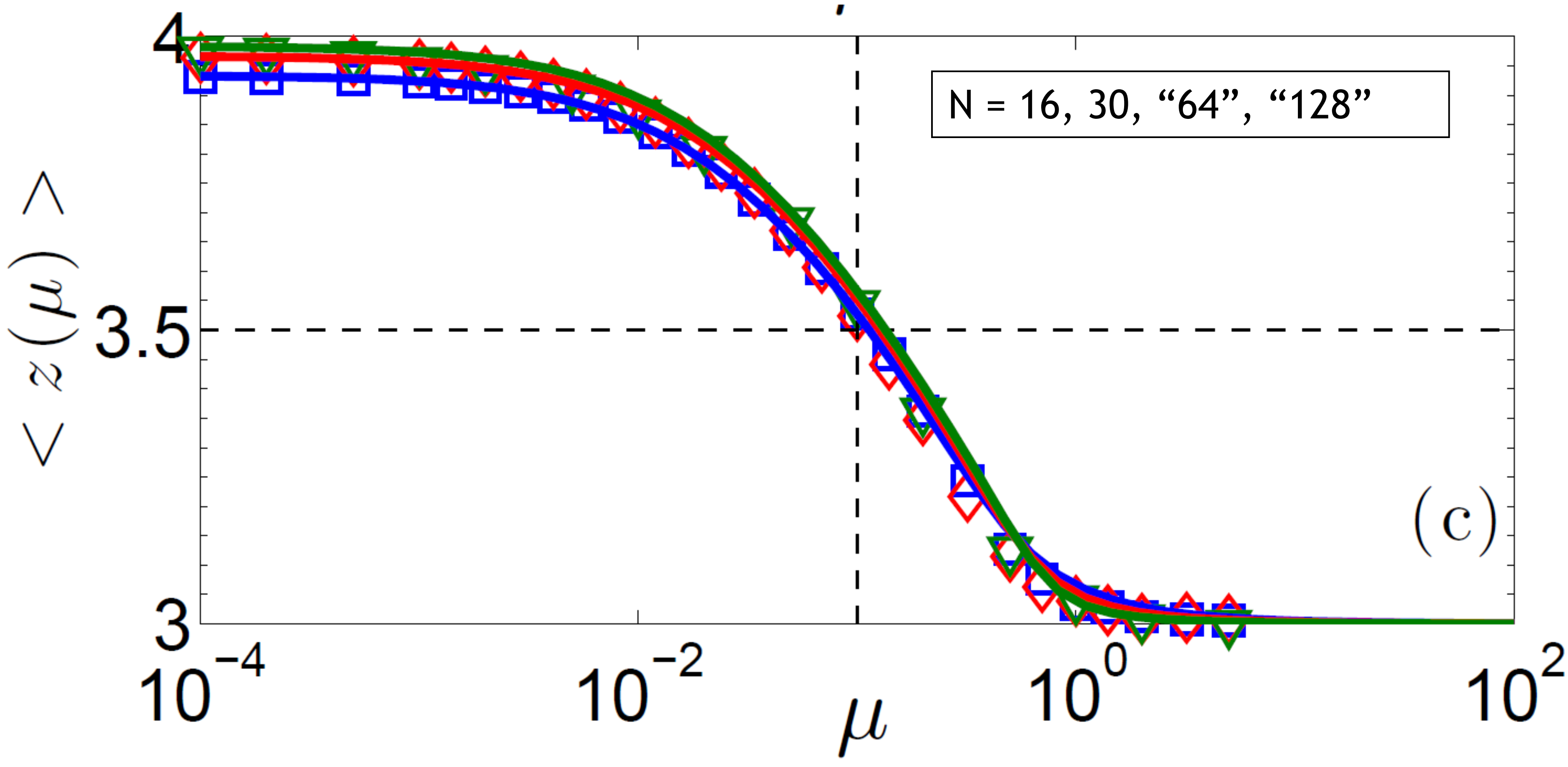


Probability of mth Order Saddles For N=32 Frictional Packings



$$P_m(\mu) = Z_m(\mu) / \sum_m Z_m(\mu)$$

Contact Number vs Friction



Simulation/Theory Results

Frictional Families

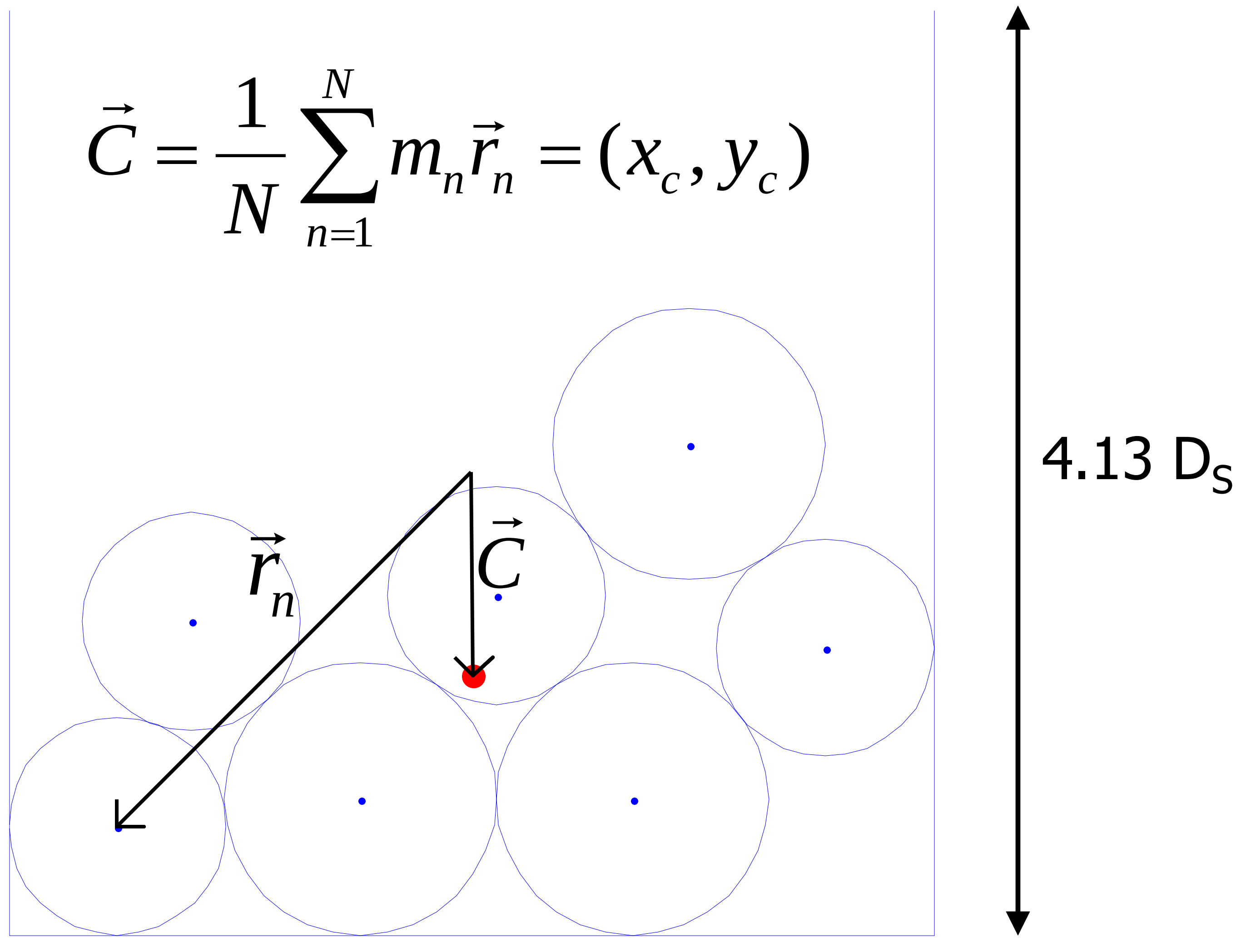
- Frictional states lie on reduced dimension manifolds in the full configuration space.
- The partition Z_m function for each family m is determined by a simple theory.
  - configurational entropy ~ V_R
  - Found that $V_R \sim \mu^m$.
  - Z_m works for all N with 1 fit parameters
  - Predicts the probability of family m as a function of friction and the dependence of z

The End

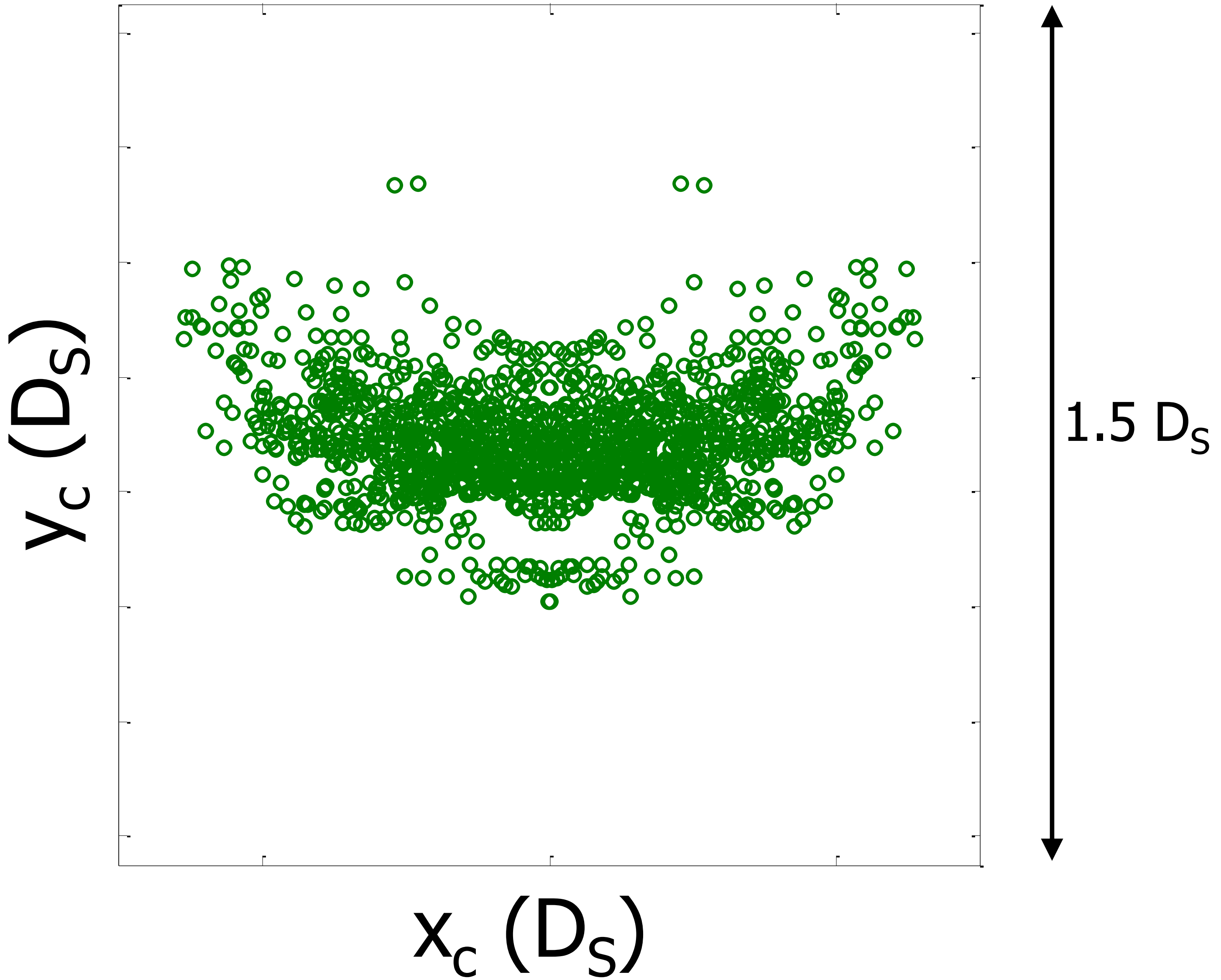
Frictionless Packings

Characterization of Stable Packings (Center of Mass)

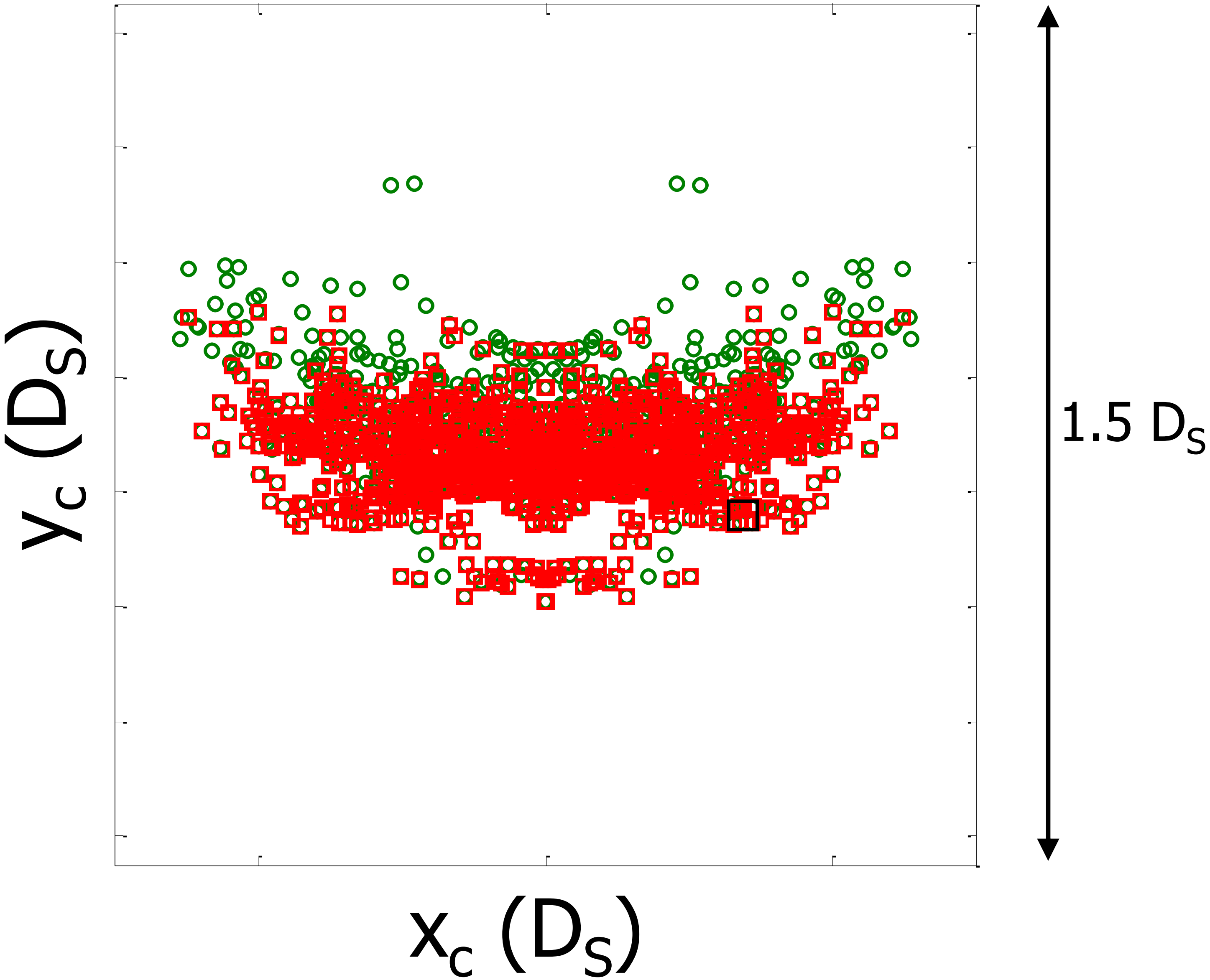
$$\vec{C} = \frac{1}{N}\sum_{n=1}^{N} m_n \vec{r}_n = (x_c, y_c)$$



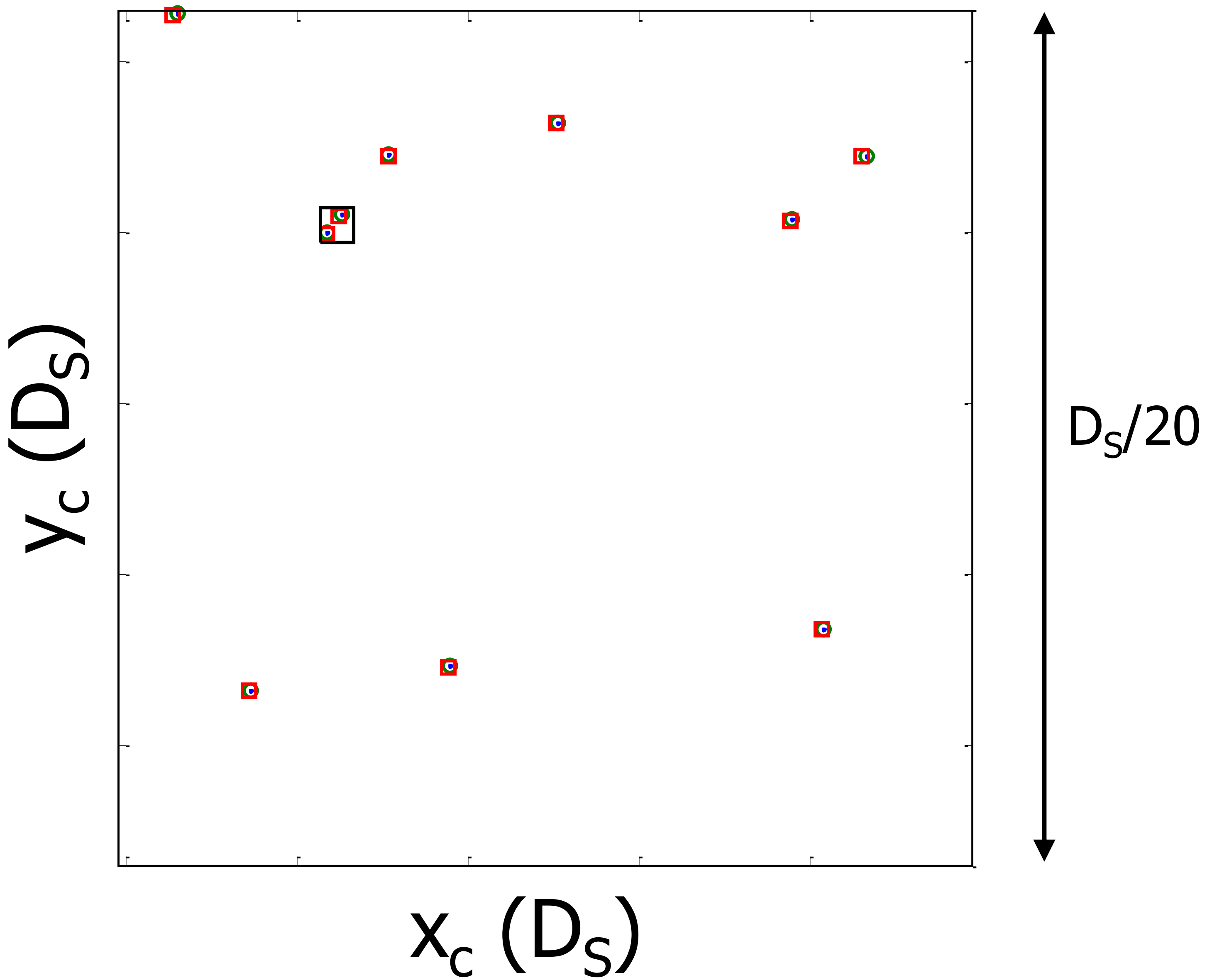
CoM of Stable Granular Packings (Experiments)



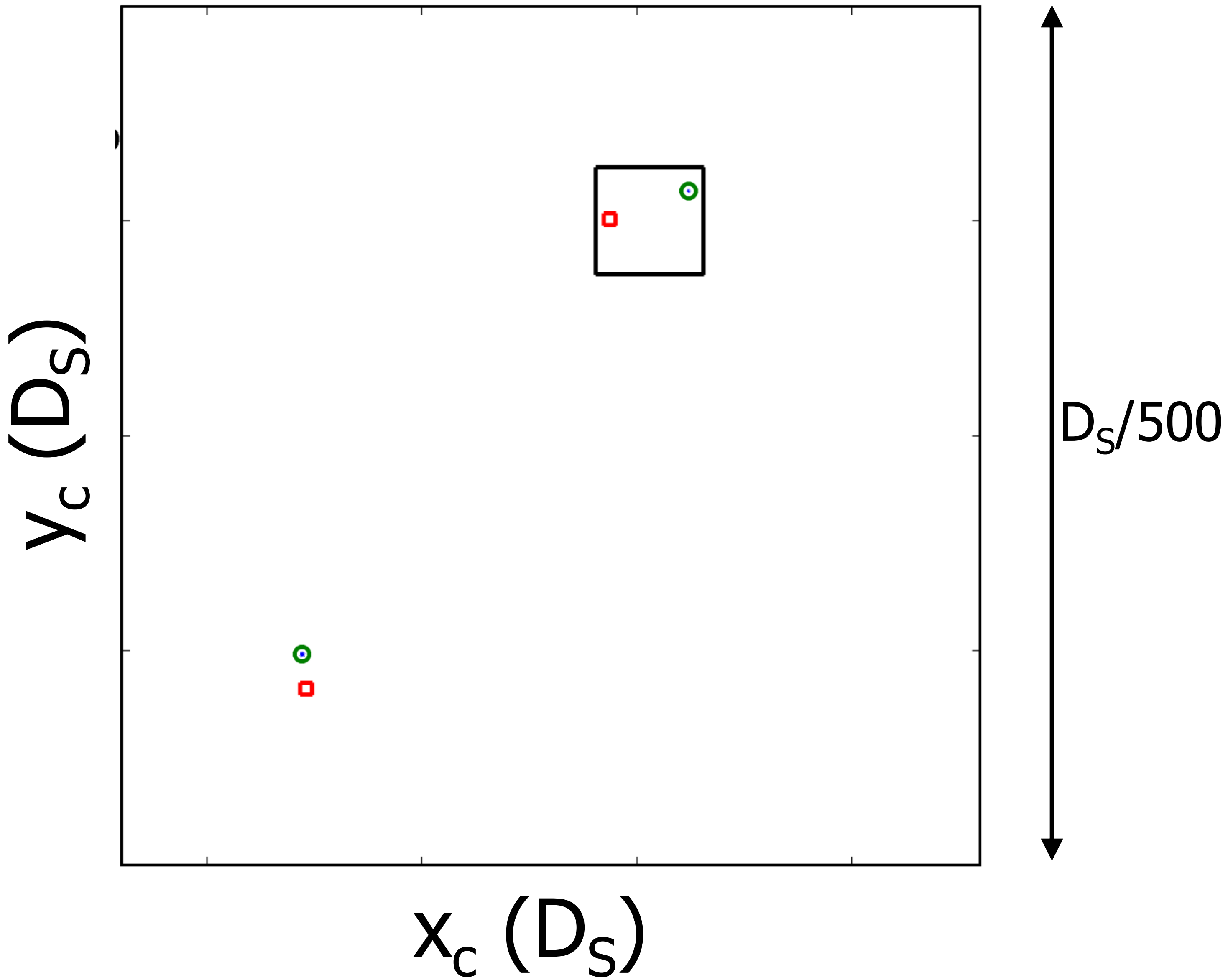
Centroids of Stable Granular Packings (Experiments and Simulations)



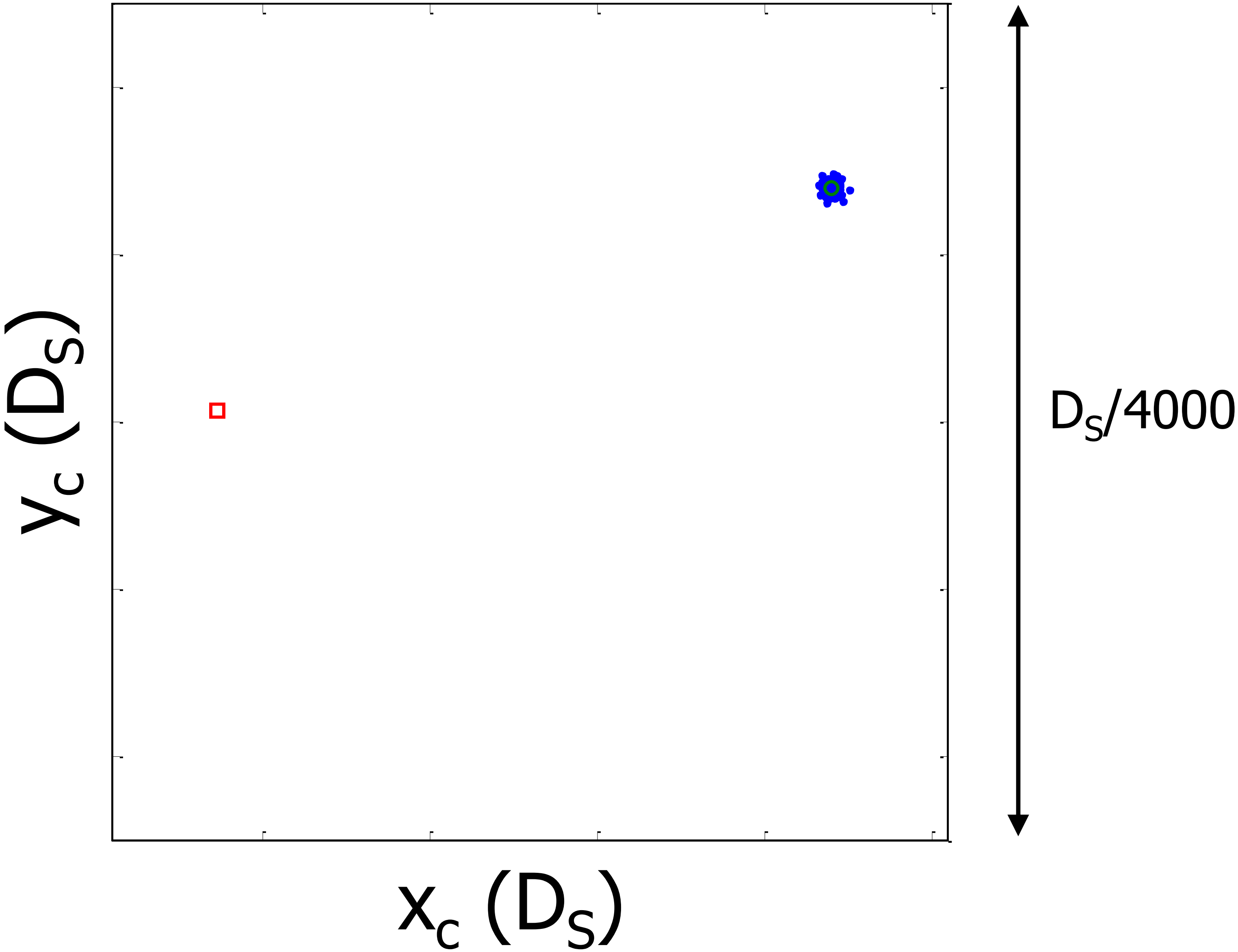
Centroids of Stable Granular Packings (Experiments and Simulations x30)



Centroids of Stable Granular Packings (Experiments and Simulations x750)

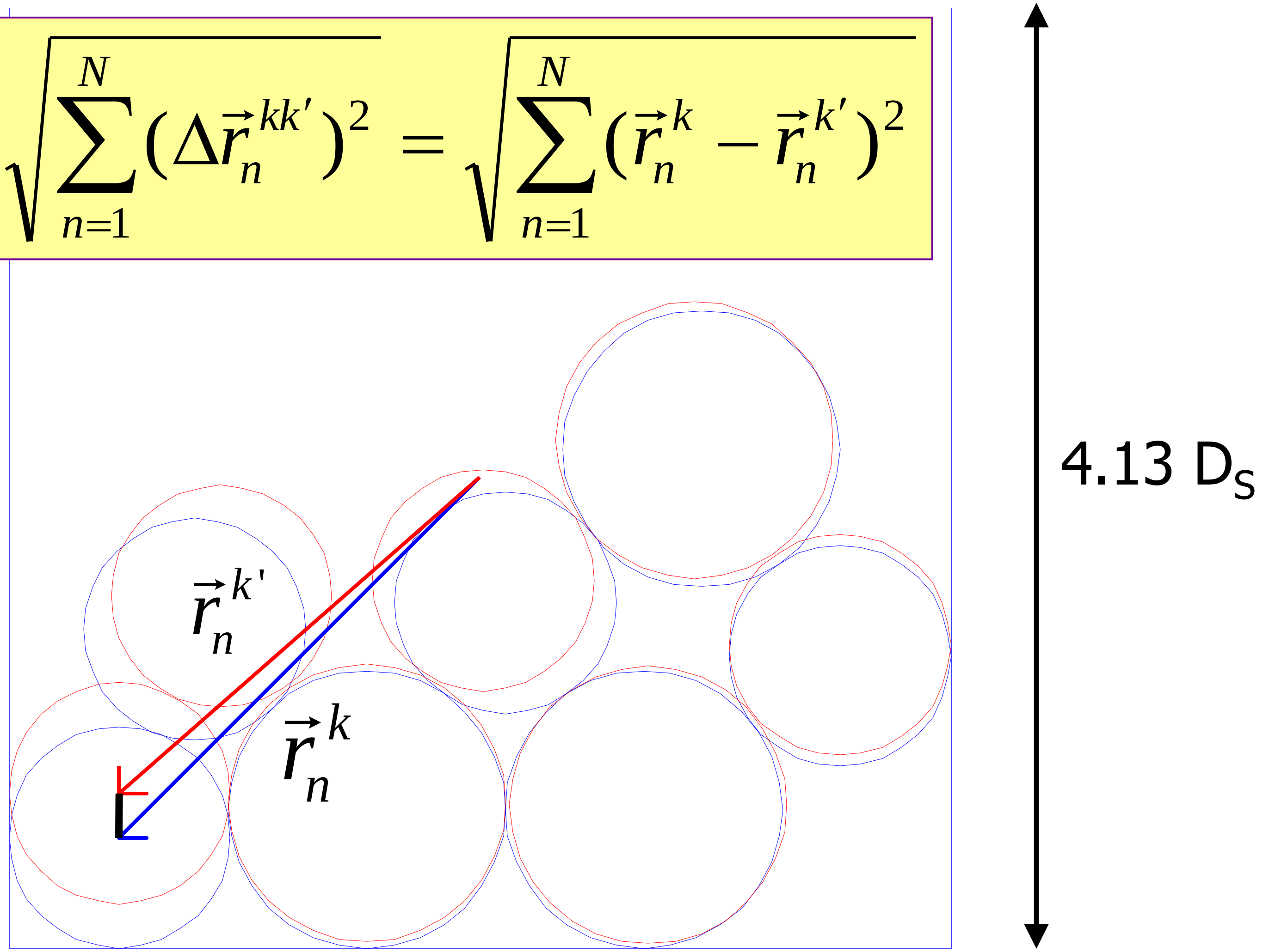


Centroids of Stable Granular Packings (Experiments and Simulations x6000)

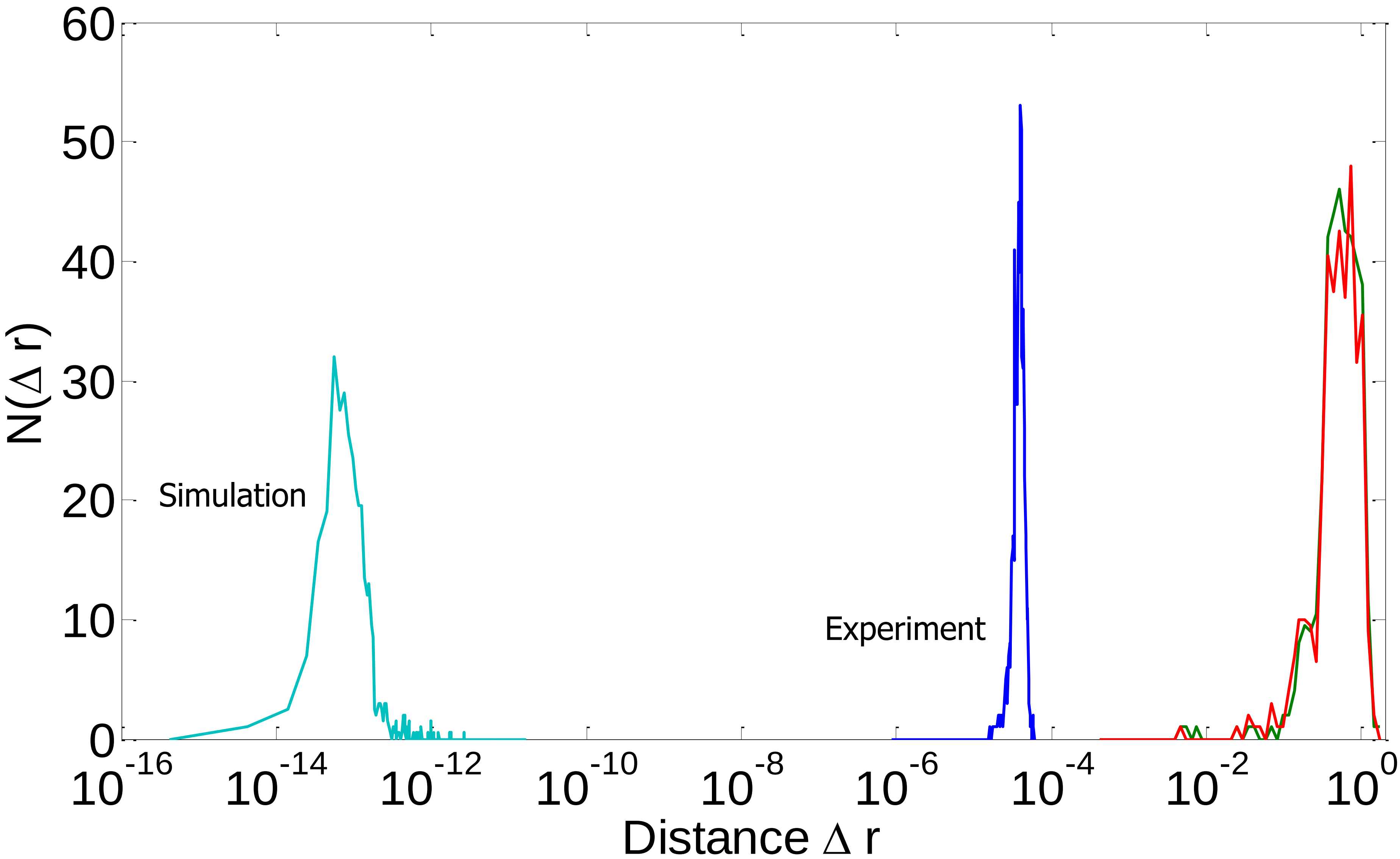


Comparison of Stable Packings
Phase Space Distance

$$\Delta r_{kk'} = \sqrt{\sum_{n=1}^{N} (\Delta \vec{r}_n^{\,kk'})^2} = \sqrt{\sum_{n=1}^{N} (\vec{r}_n^{\,k} - \vec{r}_n^{\,k'})^2}$$



States are Distinct



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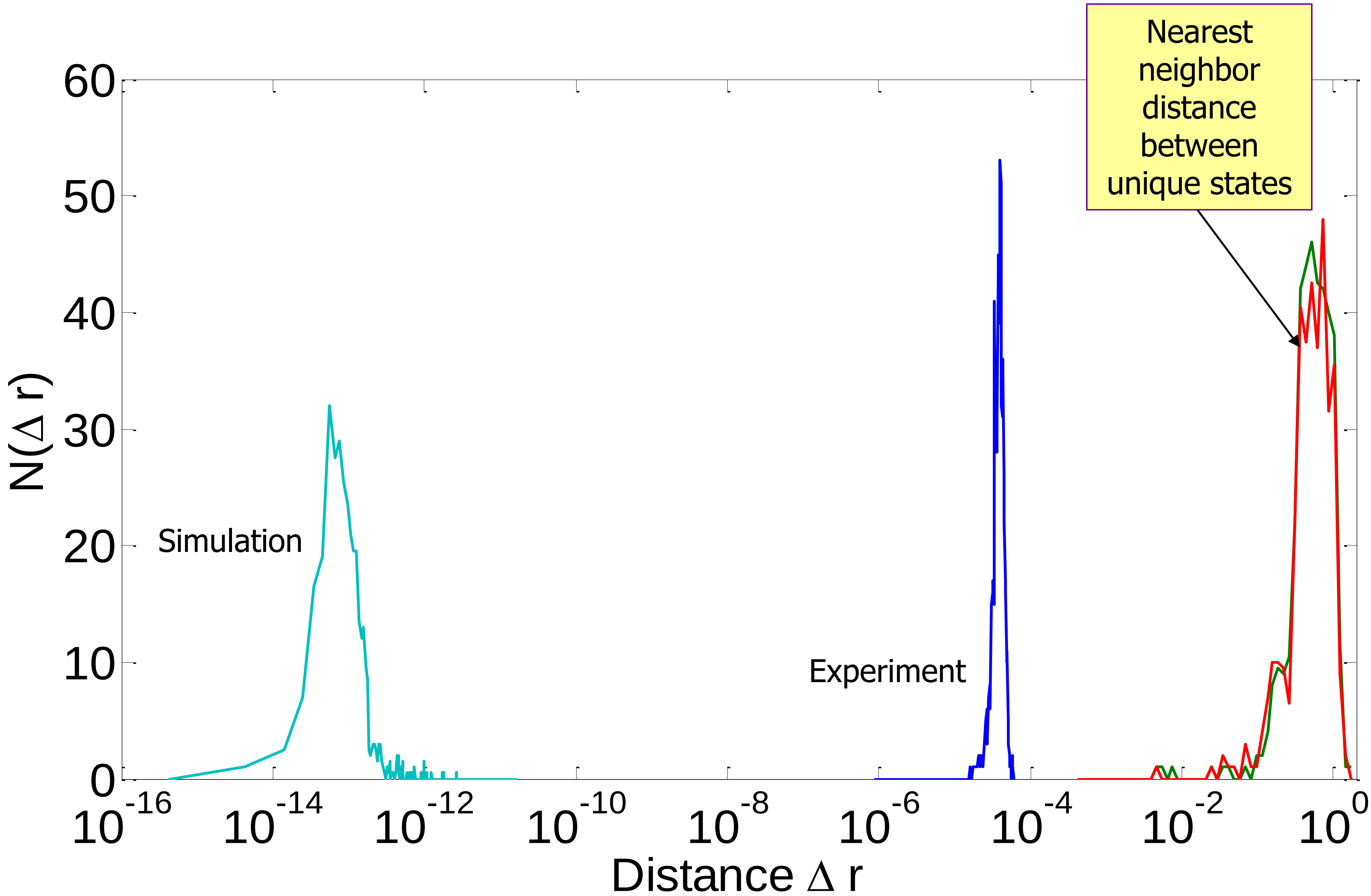

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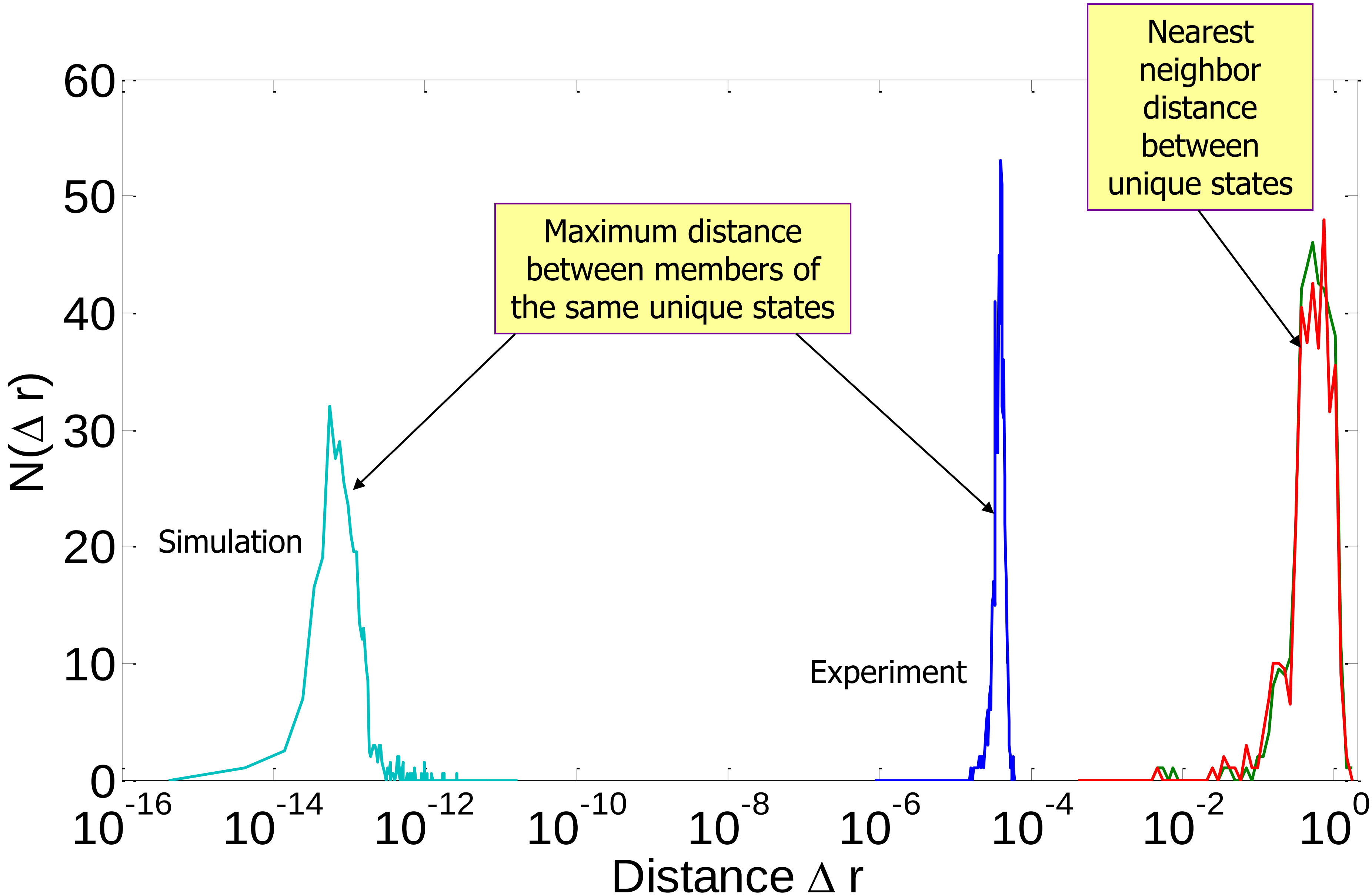
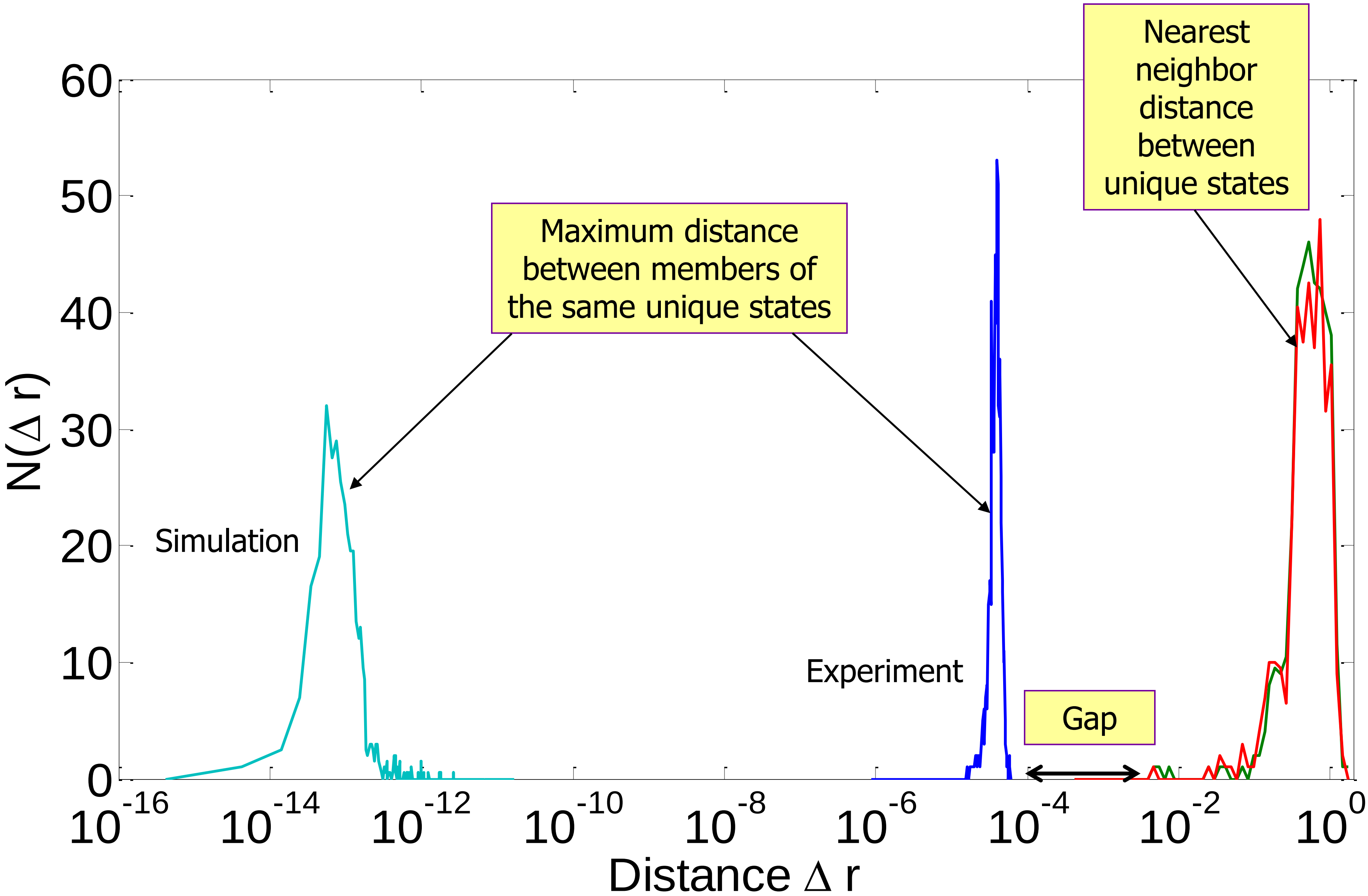
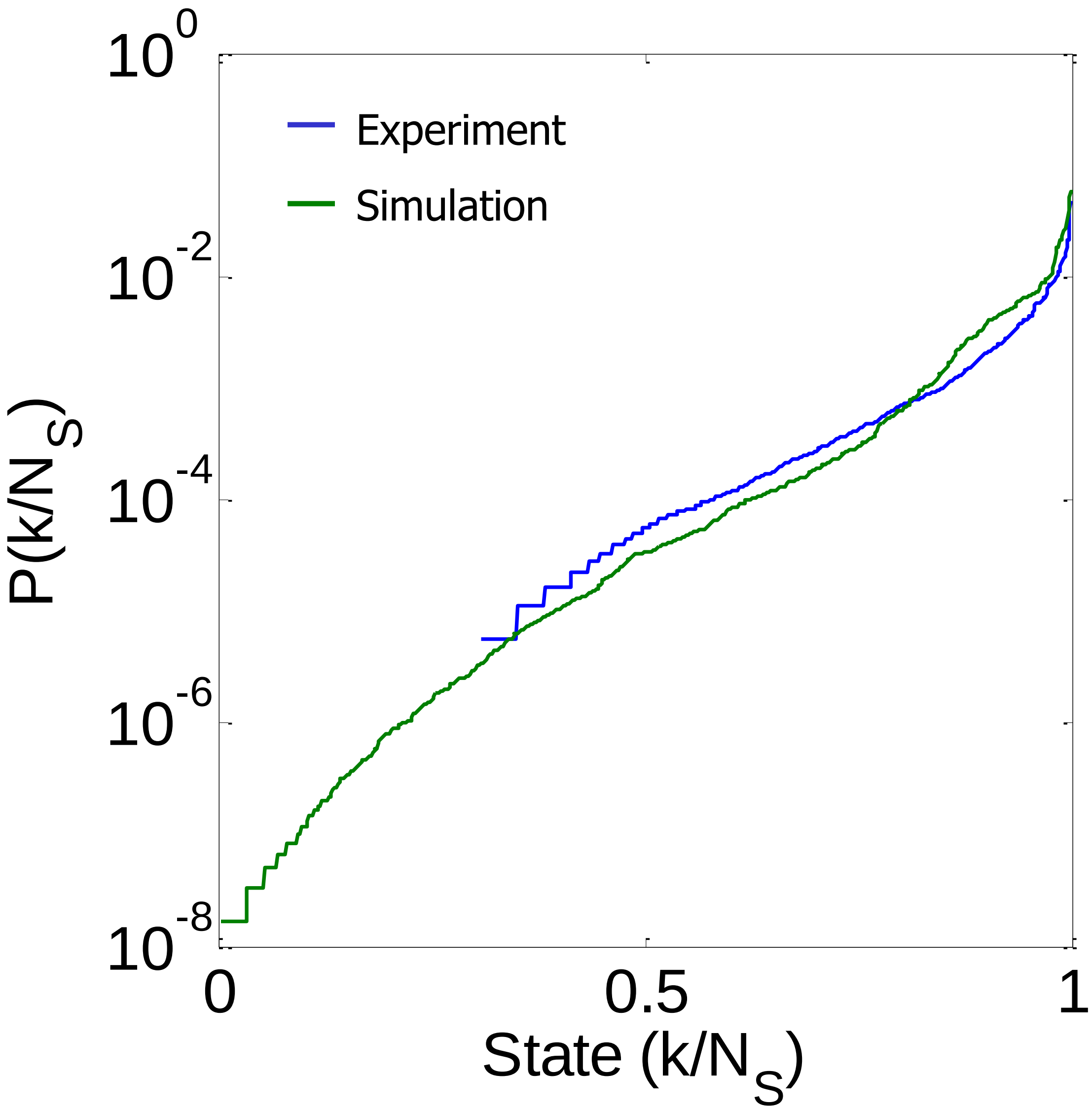


States are Distinct



Probability Distribution of States



Frictionless Results

Experimental Frictionless Packings

- New experimental technique creates stable frictionless packings.
- Experimental packing are distinct.
  - Well separated in phase space.
- Experimental unique states are not equally probable.
  - Most/Least probable > 10^4 (10^7 in simulation).
- Experiments and simulations agree.
  - Most probable states are the same.
  - Probability distribution of states is the same for the highly probable states.
- Properties of most probable states are largely independent of dynamics.

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