Local statistics of the abelian sandpile model

David B. Wilson

The looping rate and sandpile density of planar graphs

Joint work with Adrien Kassel

Spanning trees of graphs on surfaces and the intensity of loop-erased random walk on planar graphs

Joint work with Richard Kenyon

[sandpile demo]

Google

abelian sandpile model

Scholar

About 1,240 results (**0.05** sec)

Articles

Case law

My library

Any time

Since 2014

Since 2013

Since 2010

Custom range...

Sort by relevance

Sort by date

include patents

include citations

Create alert

[Equivalence between the **Abelian sandpile model** and the< i> q</i>→ 0 limit of the Potts **model**]

SN Majumdar, D Dhar - Physica A: Statistical Mechanics and its …, 1992 - Elsevier

Abstract We establish an equivalence between the undirected **Abelian sandpile model** and the q→ 0 limit of the q-state Potts **model**. The equivalence is valid for arbitrary finite graphs. Two-dimensional **Abelian sandpile models**, thus, correspond to a conformal field theory **...**

Cited by 236 Related articles All 9 versions Cite Save

[**Abelian sandpile model** on the Bethe lattice]

D Dhar, SN Majumdar - Journal of Physics A: Mathematical and …, 1990 - iopscience.iop.org

Abstract. We study Bak, Tang and Wiesenfeld's **Abelian sandpile model** of selforganised criticality on the Bethe lattice. Exact expressions for various distribution functions including the height distribution at a site and the joint distribution of heights at two sites separated by **...**

Cited by 158 Related articles All 8 versions Cite Save

[Height correlations in the **Abelian sandpile model**]

SN Majumdar, D Dhar - Journal of Physics A: Mathematical and …, 1991 - iopscience.iop.org

Abstract. We study the distribution of heights in the self-organired critical state of the **Abelian sandpile model** an a d-dimensional hypercubic lattice. We calculate analytically the concentration of sites having minimum alluwed value in! he critical stafe. We al~ u **...**

Cited by 136 Related articles All 6 versions Cite Save

[**Abelian sandpile model**]

HF Chau - Physical Review E, 1993 - adsabs.harvard.edu

Abstract A systematic and simple method to find the correlation function of the **Abelian sandpile model** up to any finite order is developed. In addition, an algorithm for evaluating the distribution function of the avalanche size P (s) exactly is also discovered along the **...**

Cited by 1 Related articles All 6 versions Cite Save

[Rare events and breakdown of simple scaling in the **Abelian sandpile model**]

M De Menech, AL Stella, C Tebaldi - Physical Review E, 1998 - APS

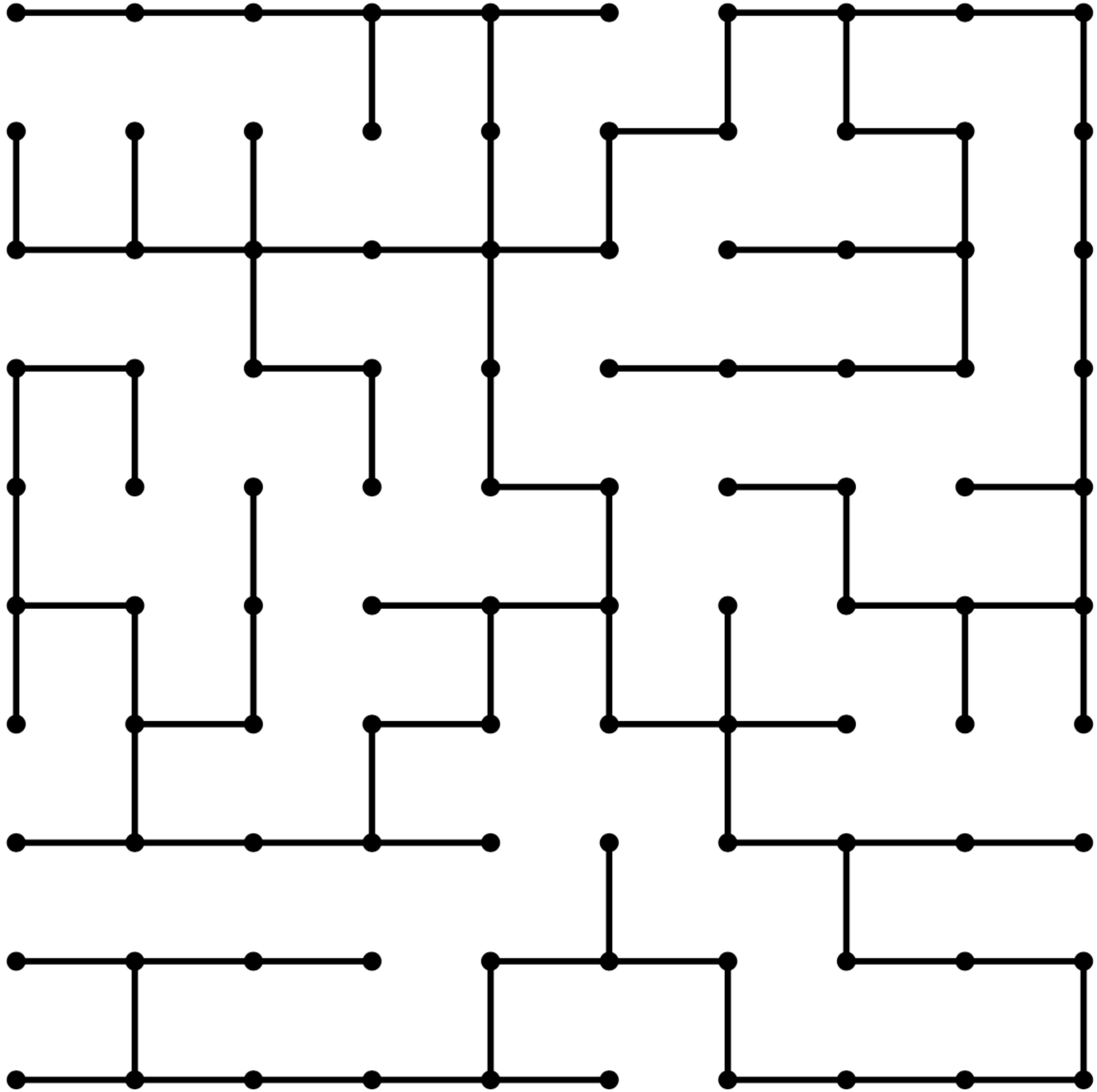
Due to intermittency and conservation, the **Abelian sandpile** in two dimensions obeys multifractal, rather than finite size scaling. In the thermodynamic limit, a vanishingly small fraction of large avalanches dominates the statistics and a constant gap scaling is **...**

Cited by 113 Related articles All 8 versions Cite Save

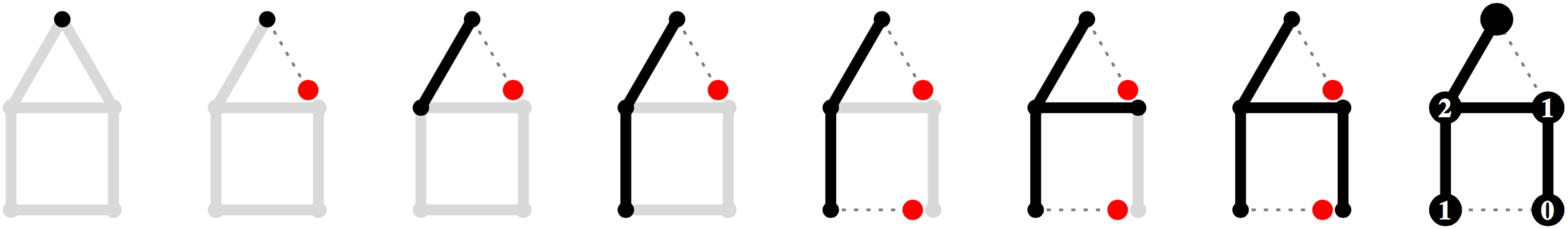
[Formation of avalanches and critical exponents in an **Abelian sandpile model**]

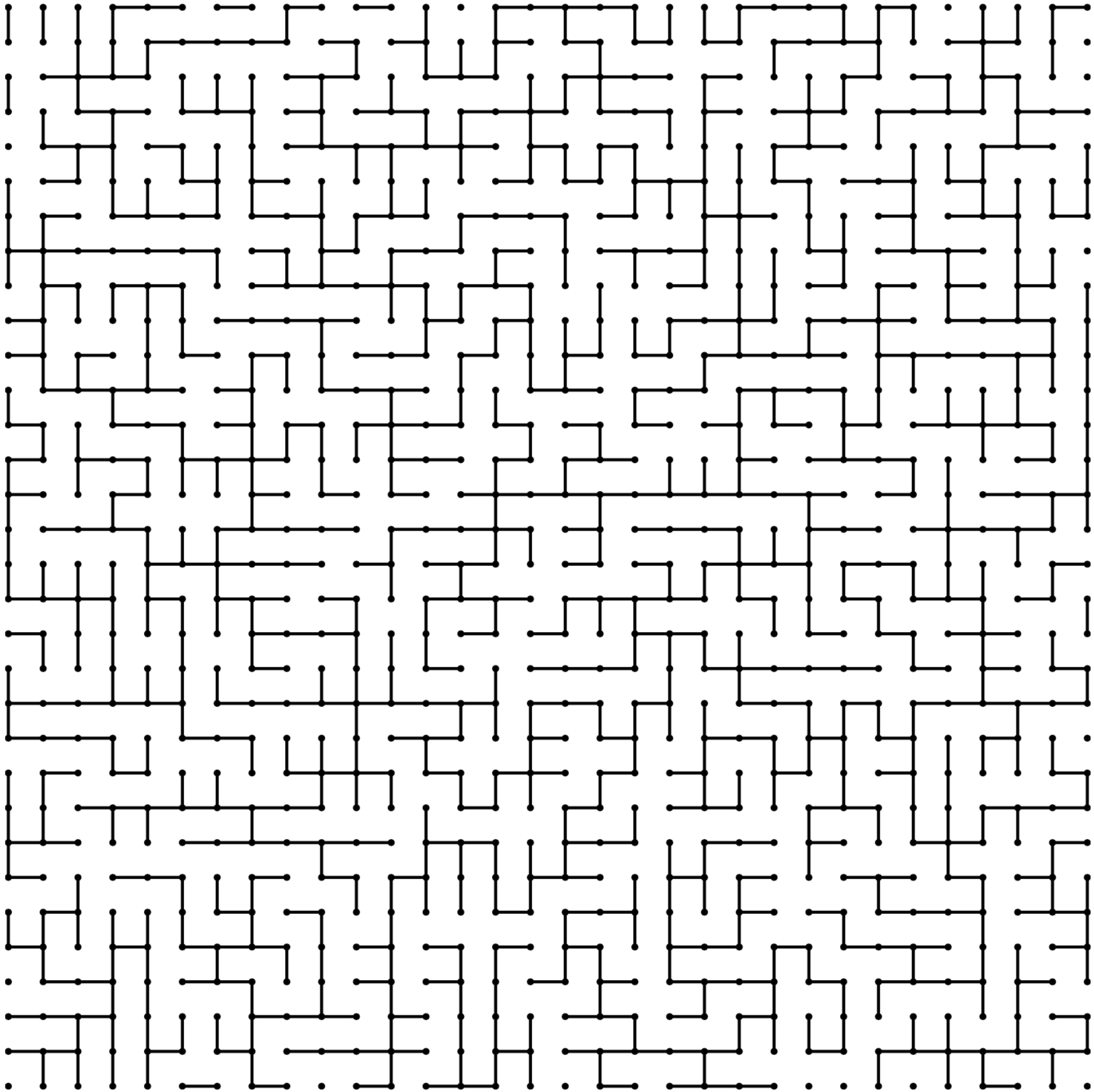
VB Priezzhev, DV Ktitarev, EV Ivashkevich - Physical review letters, 1996 - APS

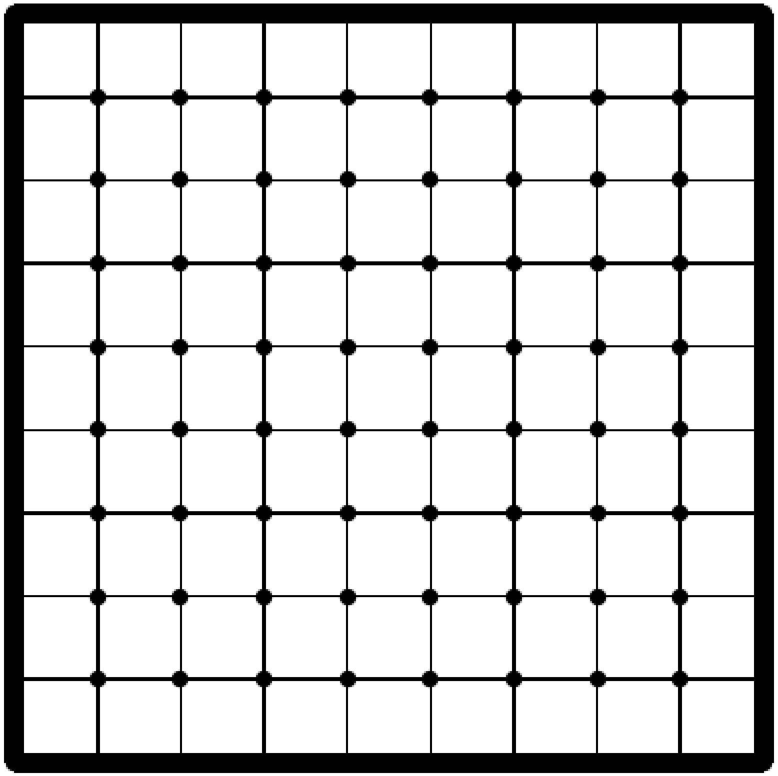
The structure of avalanches in the **Abelian sandpile model** on a square lattice is analyzed. It is shown that an avalanche can be considered as a sequence of waves of decreasing sizes. Being more simple objects, waves admit a representation in terms of spanning trees **...**



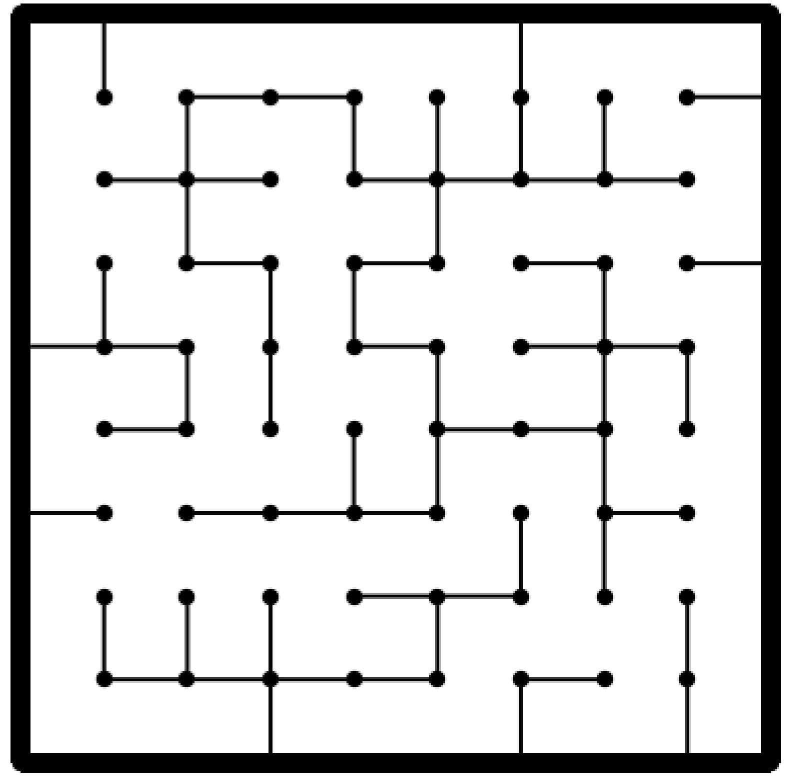
0	2	0	1	2
1	3	1	2	2
1	3	1	3	3
1	2	3	3	0
2	1	2	3	2


2
1
1
0





Underlying graph

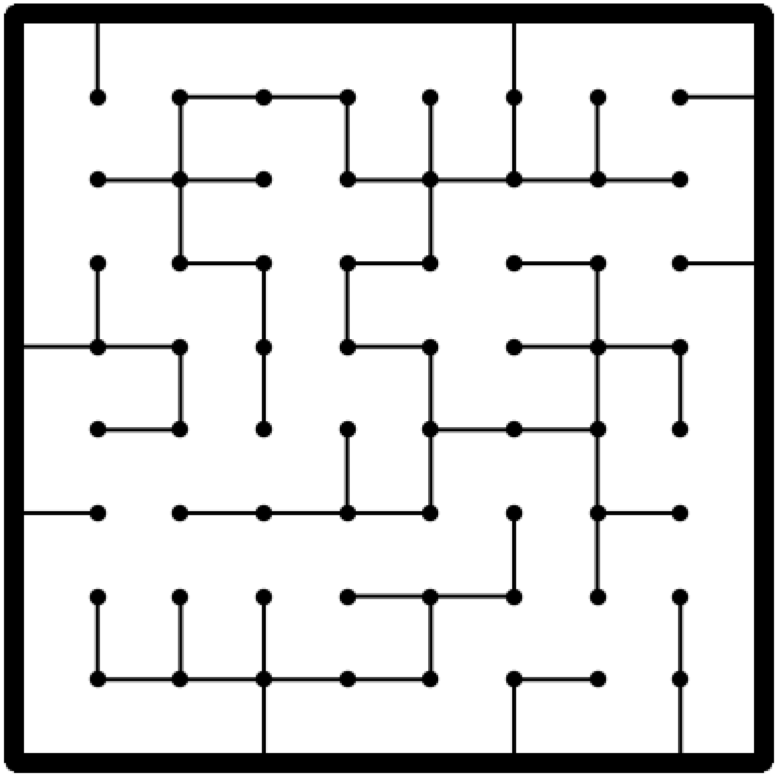


Uniform spanning tree

Uniform spanning tree on infinite grid

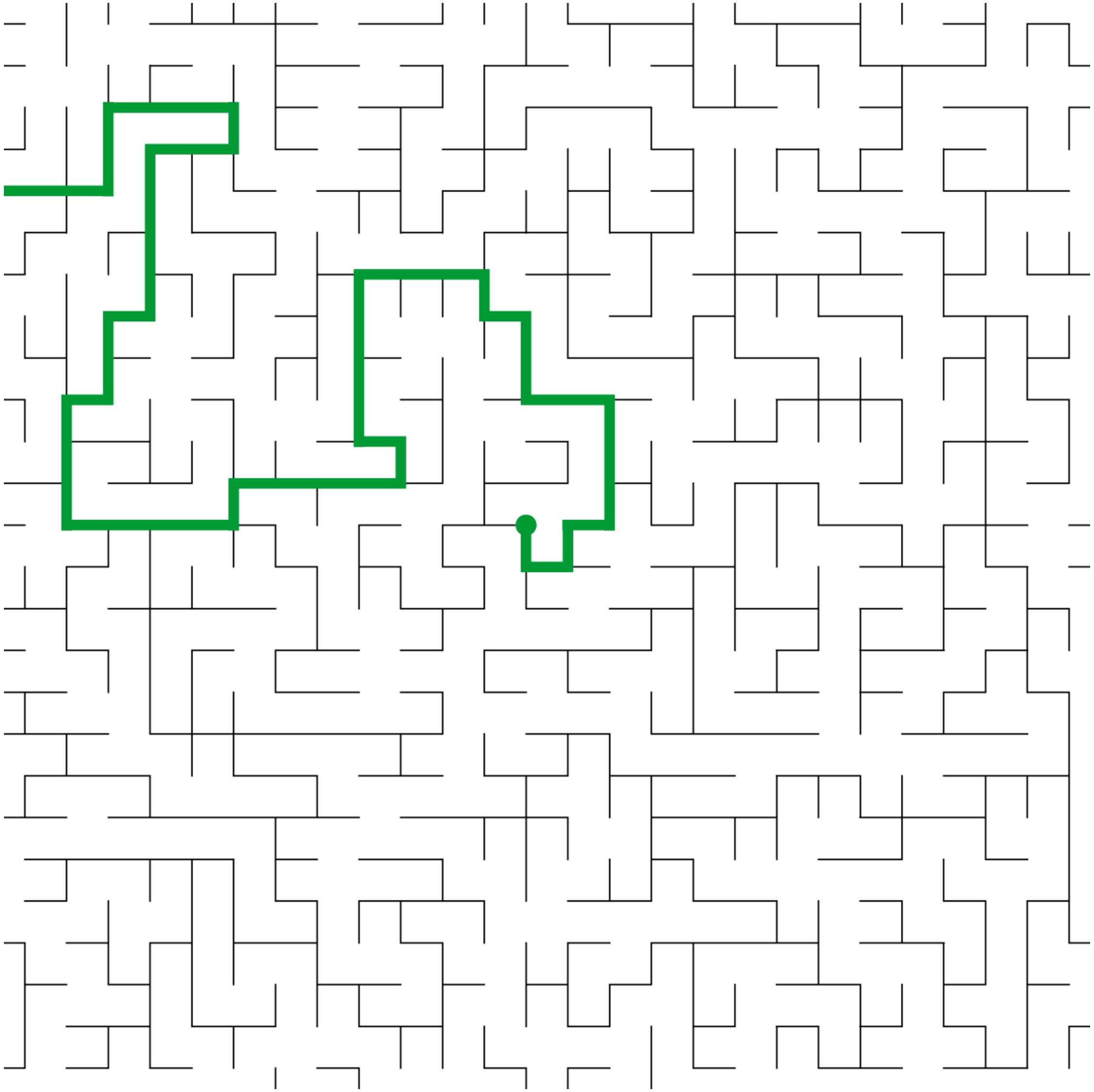
Pemantle: limit of UST on large boxes
converges as boxes tend to Z^d

Pemantle: limiting process has one tree
if d<=4, infinitely many trees if d>4



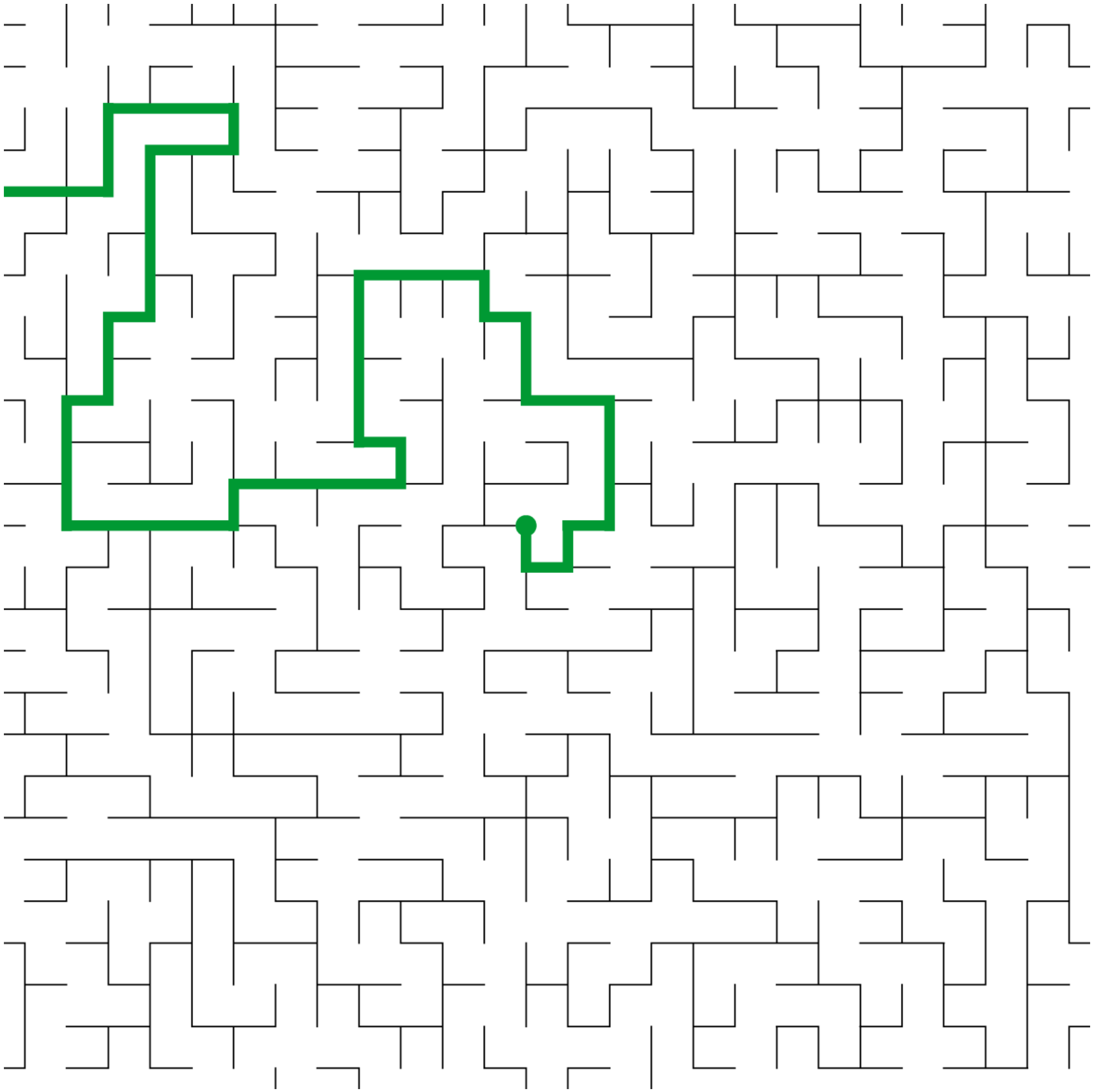
Uniform spanning tree

UST and LERW on Z^2



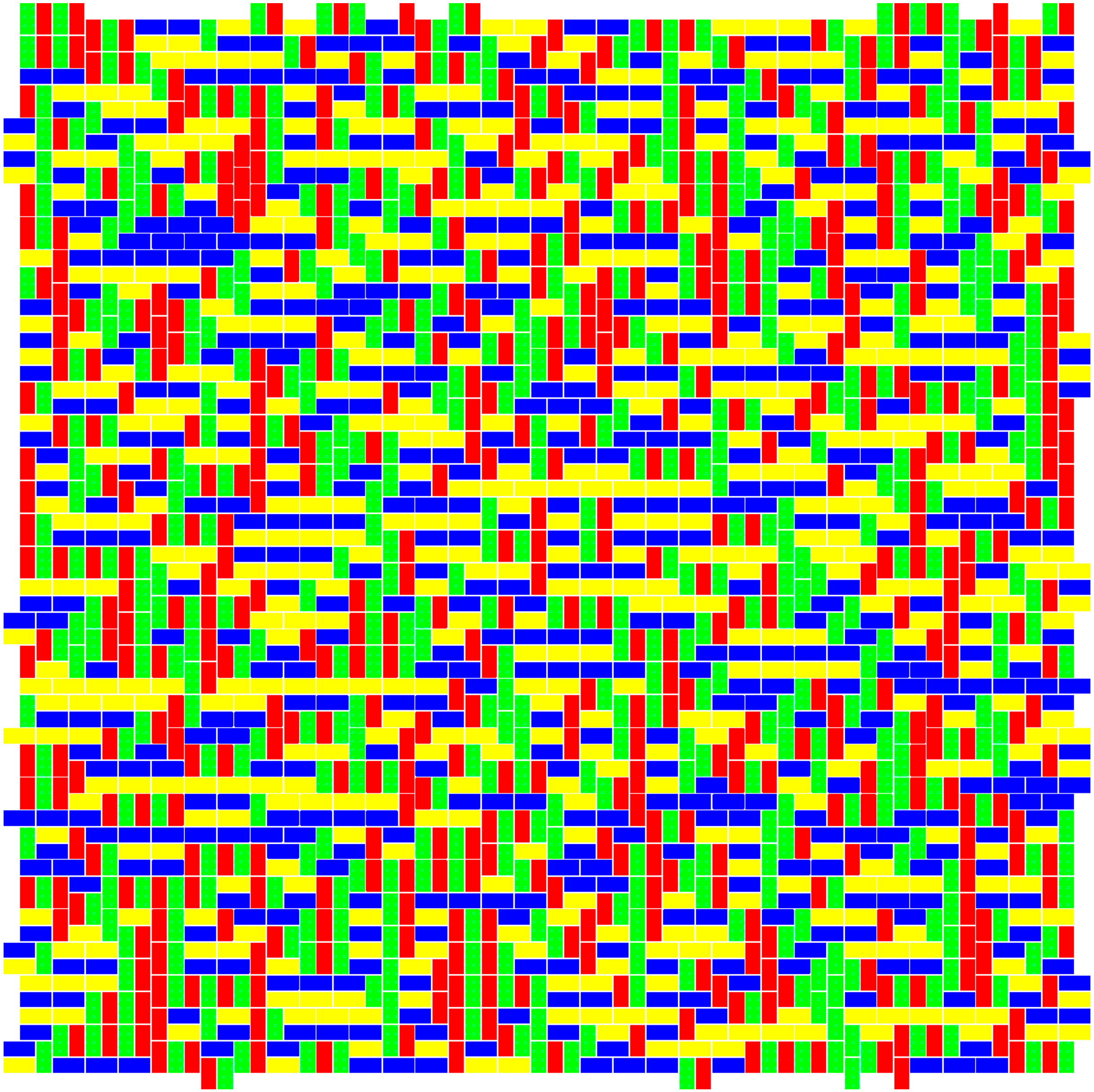
Benjamini-Lyons-Peres-Schramm:
UST on Z^d has one end if d>1,
i.e., one path to infinity

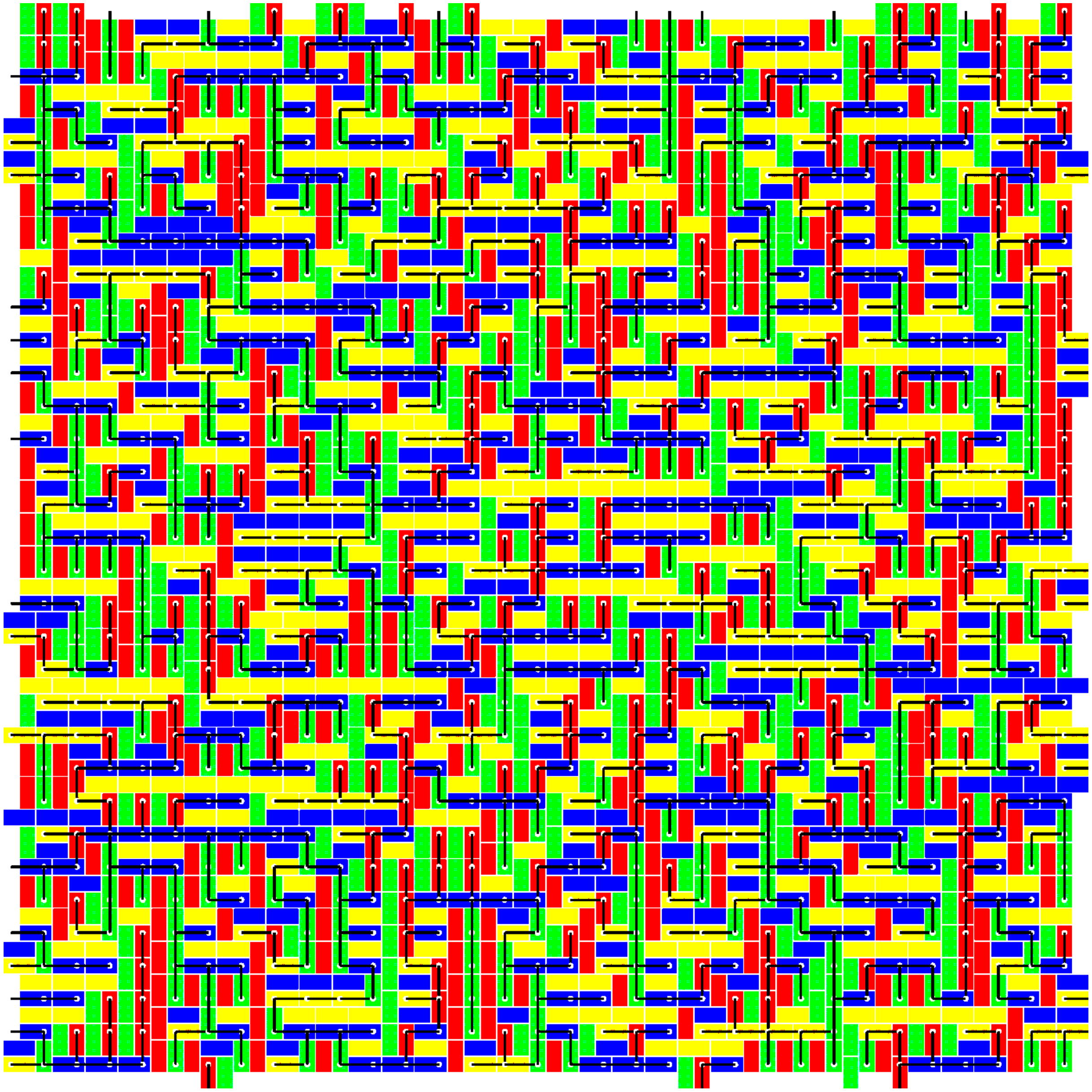
Local statistics of UST

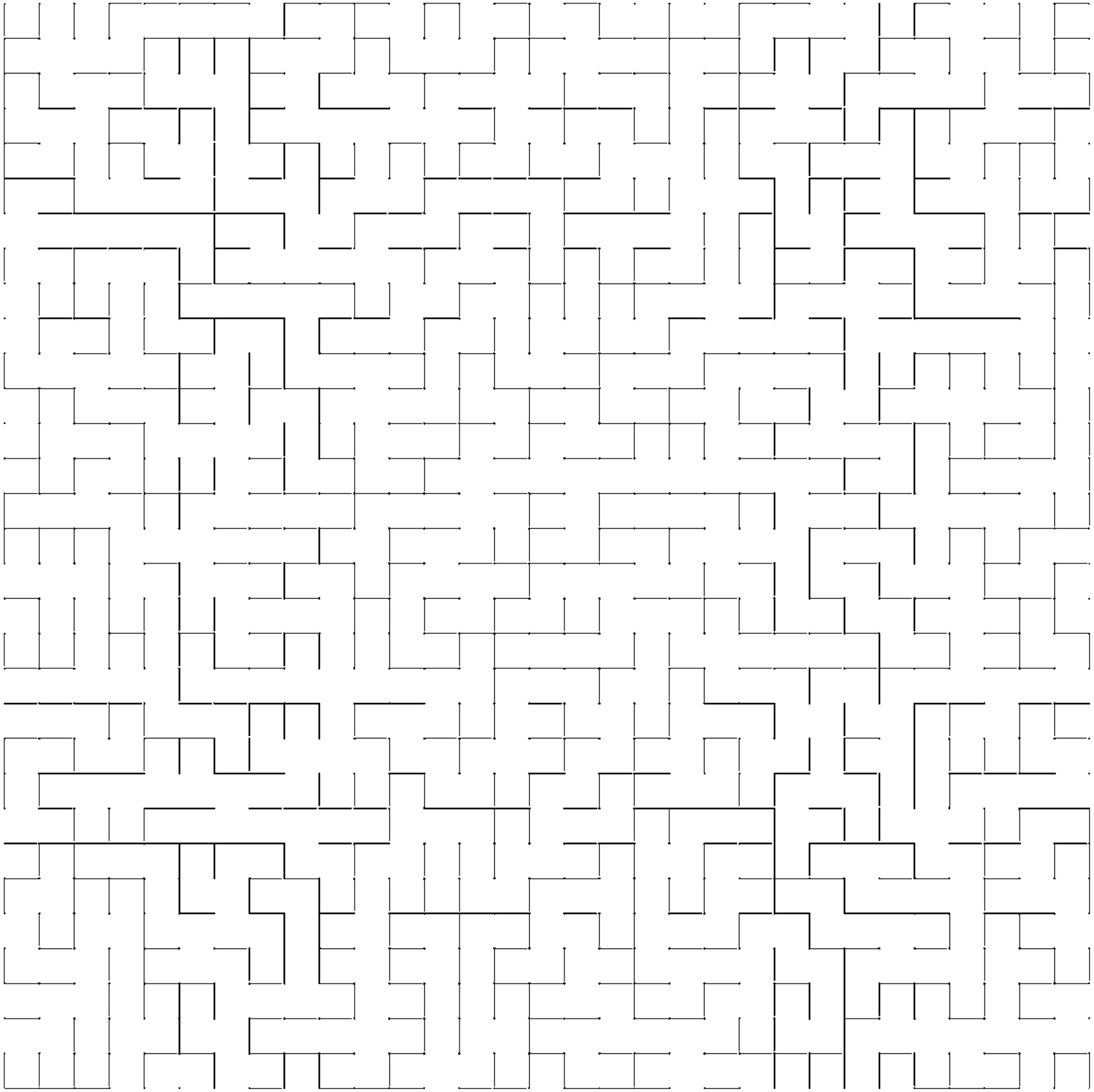


Local statistics of UST can be computed via determinants of transfer impedance matrices
(Burton—Pemantle)

Why doesn't this give local statistics of sandpiles?







Infinite volume limit

- Infinite volume limit exists (Athreya—Jarai ’04)
- Pr[h=0]= $\frac{2}{\pi^2} - \frac{4}{\pi^3}$ (Majumdar—Dhar ’91)
- Other one-site probabilities computed by Priezzhev (’93)

$$P(2) = \frac{1}{2} - \frac{3}{2\pi} - \frac{2}{\pi^2} + \frac{12}{\pi^3} + \frac{I_0}{4}$$

with

$$I_0 = \frac{1}{(2\pi)^4} \iiiint_0^{2\pi} \frac{i \sin(\beta_1) \det(\mathbf{M})}{D(\alpha_1, \beta_1) D(\alpha_2, \beta_2) D(\alpha_1 + \alpha_2, \beta_1 + \beta_2)} \times d\alpha_1\, d\alpha_2\, d\beta_1\, d\beta_2 \qquad (2)$$

where

$$D(\alpha, \beta) = 2 - \cos(\alpha) - \cos(\beta) \qquad (3)$$

and $\mathbf{M}$ is a 4×4 matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & e^{i\alpha_2} & 1 \\ 3 & e^{i(\beta_1 + \beta_2)} & e^{i(\alpha_2 - \beta_2)} & e^{i\beta_1} \\ (4/\pi) - 1 & e^{i(\alpha_1 + \alpha_2)} & 1 & e^{-i\alpha_1} \\ (4/\pi) - 1 & e^{-i(\alpha_1 + \alpha_2)} & e^{2i\alpha_2} & e^{i\alpha_1} \end{pmatrix} \qquad (4)$$

The numerical evaluation of the integral (2) leads to $P(2) = 0.1739 \ldots$. The solution is based on an analogy

Priezzhev ('94)

$$P(2)=\frac{1}{2}-\frac{3}{2\pi}-\frac{2}{\pi^2}+\frac{12}{\pi^3}+\frac{I_1}{4} \tag{3}$$

$$P(3)=\frac{1}{4}+\frac{3}{2\pi}+\frac{1}{\pi^2}-\frac{12}{\pi^3}-\frac{I_1}{2}-\frac{3I_2}{32} \tag{4}$$

$$P(4)=\frac{1}{4}-\frac{1}{\pi^2}+\frac{4}{\pi^3}+\frac{I_1}{4}+\frac{3I_2}{32} \tag{5}$$

Here I_ν, $\nu=1, 2$, are integrals:

$$I_\nu=\frac{1}{(2\pi)^4}\iiiint_0^{2\pi}\frac{i\sin(\beta_1)\det(M_\nu)}{D(\alpha_1,\beta_1)\,D(\alpha_2,\beta_2)\,D(\alpha_1+\alpha_2,\beta_1+\beta_2)}\,d\alpha_1\,d\alpha_2\,d\beta_1\,d\beta_2 \tag{6}$$

where

$$D(\alpha,\beta)=2-\cos(\alpha)-\cos(\beta) \tag{7}$$

and M_1, M_2 are matrices,

$$M_1=\begin{pmatrix} 1 & 1 & e^{i\alpha_2} & 1 \\ 3 & e^{i(\beta_1+\beta_2)} & e^{i(\alpha_2-\beta_2)} & e^{-i\beta_1} \\ 4/\pi-1 & e^{i(\alpha_1+\alpha_2)} & 1 & e^{-i\alpha_1} \\ 4/\pi-1 & e^{-i(\alpha_1+\alpha_2)} & e^{2i\alpha_2} & e^{i\alpha_1} \end{pmatrix} \tag{8}$$

and

$$M_2=\begin{pmatrix} e^{i\beta_2} & e^{-i(\alpha_1+\alpha_2)-i(\beta_1+\beta_2)} & e^{i\beta_1} \\ e^{-i\alpha_2} & 1 & e^{-i\alpha_1} \\ e^{i\alpha_2} & e^{-2i(\alpha_1+\alpha_2)} & e^{i\alpha_1} \end{pmatrix} \tag{9}$$

The numerical evaluation of integrals in Eq. (6) leads to $P(2)=0.1739...$, $P(3)=0.3063...$, $P(4)=0.4461...$, in good agreement with the high-statistics data.

Jeng—Piroux—Ruelle ('06)

$$P_2 = \frac{1}{2} - \frac{1}{\pi} - \frac{3}{\pi^2} + \frac{12}{\pi^3} - \frac{\pi - 2}{2\pi} J_2 \simeq 0.1739, \qquad (4.10)$$

$$P_3 = \frac{1}{4} + \frac{2}{\pi} - \frac{12}{\pi^3} - \frac{8 - \pi}{4\pi} J_2 \simeq 0.3063. \qquad (4.11)$$

$$J_2 = \frac{4}{\pi^2} - \frac{14}{\pi} - 8 - \frac{4\sqrt{2}}{\pi^2} \int_0^\pi \frac{\mathrm{d}\beta_1}{\sqrt{3 - \cos\beta_1}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\beta_2}{1 - t_1 t_2 t_3} \sin\frac{\beta_1 - \beta_2}{2} \left[\cos\frac{\beta_1 - \beta_2}{2} - 2\cos\frac{\beta_1 + \beta_2}{2}\right]$$
$$\times \left[(3 - \cos\beta_1 + \cos\beta_2)\cos\frac{\beta_1}{2} - 2\sin\beta_2 \sin\frac{\beta_1}{2}\right], \qquad (4.16)$$

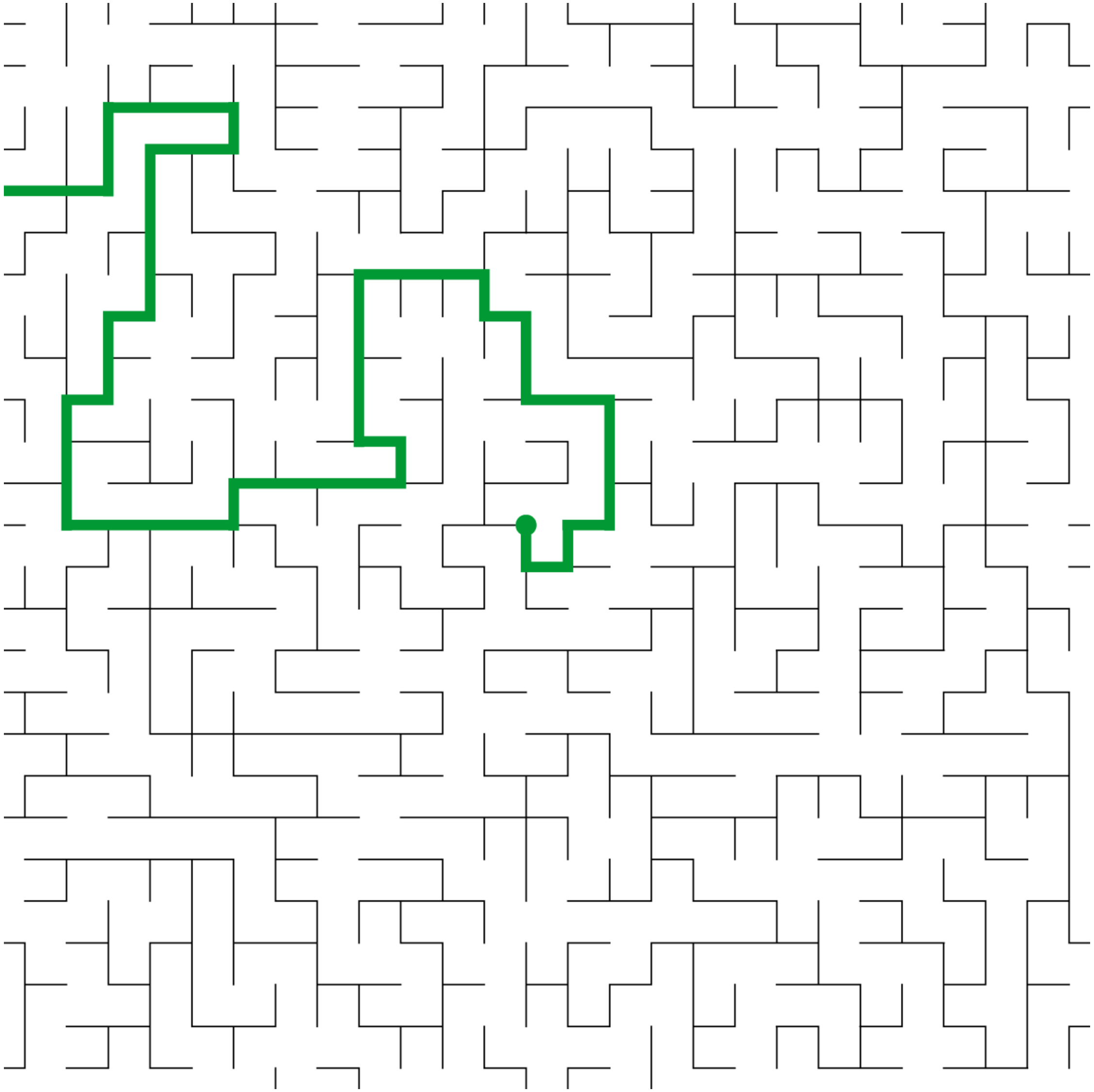
where $t_i = y_i - \sqrt{y_i^2 - 1}$, $y_i = 2 - \cos\beta_i$ and $\beta_3 = -(\beta_1 + \beta_2)$. This integral expression has been used for the numerical evaluation of J_2, yielding $J_2 = 0.5 + o(10^{-12})$.

Remarkably these values imply an even simpler formula for the mean height in the bulk,

$$\langle h \rangle = P_1 + 2P_2 + 3P_3 + 4P_4 = \frac{25}{8}, \qquad (4.15)$$

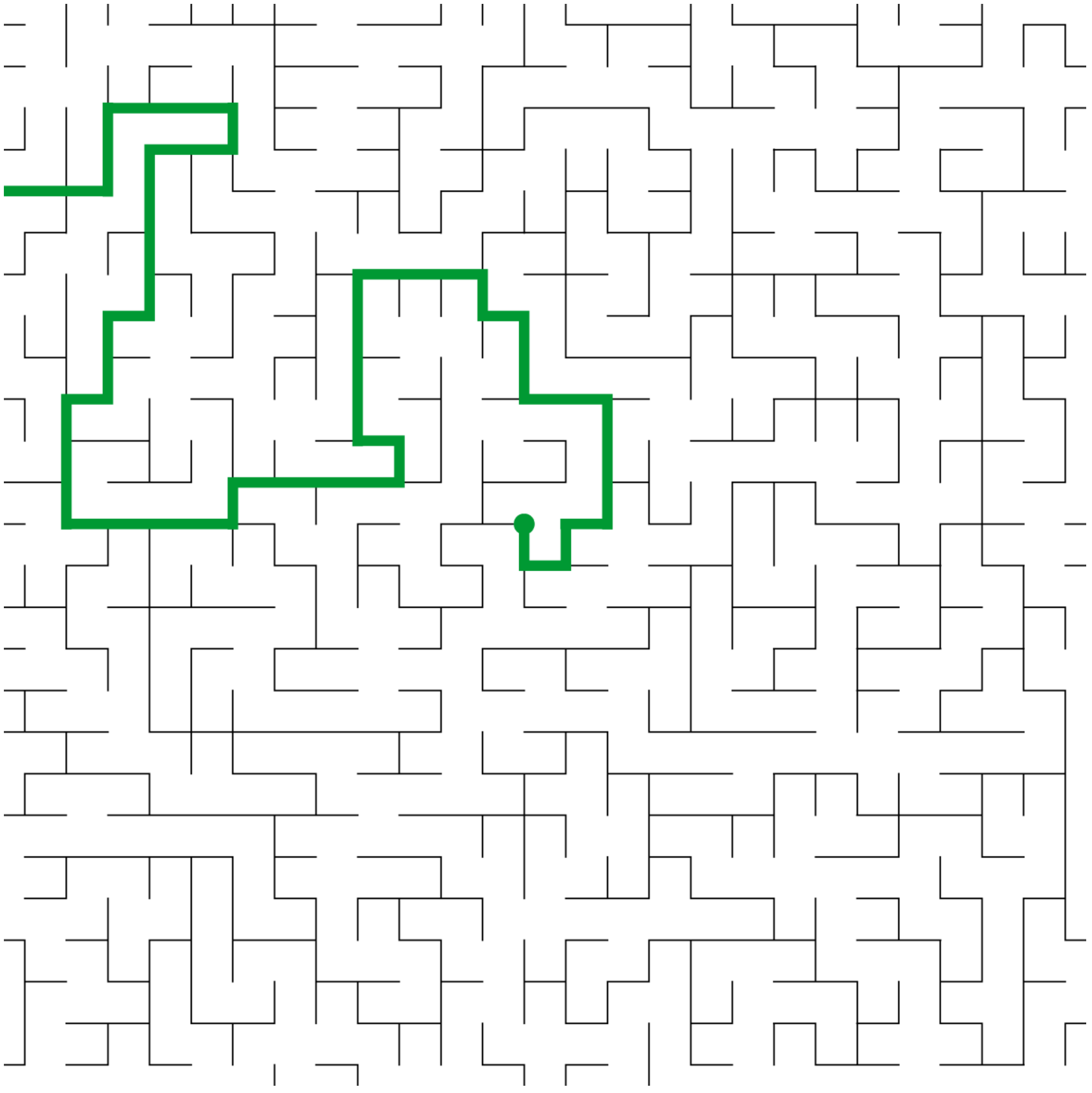
a value conjectured by Grassberger [3]. The striking simplicity of this result clearly calls for a

Sandpile density and LERW



Conjecture: path to infinity visits neighbor to right
with probability 5/16
(Levine—Peres,
Poghosyan—Priezzhev)

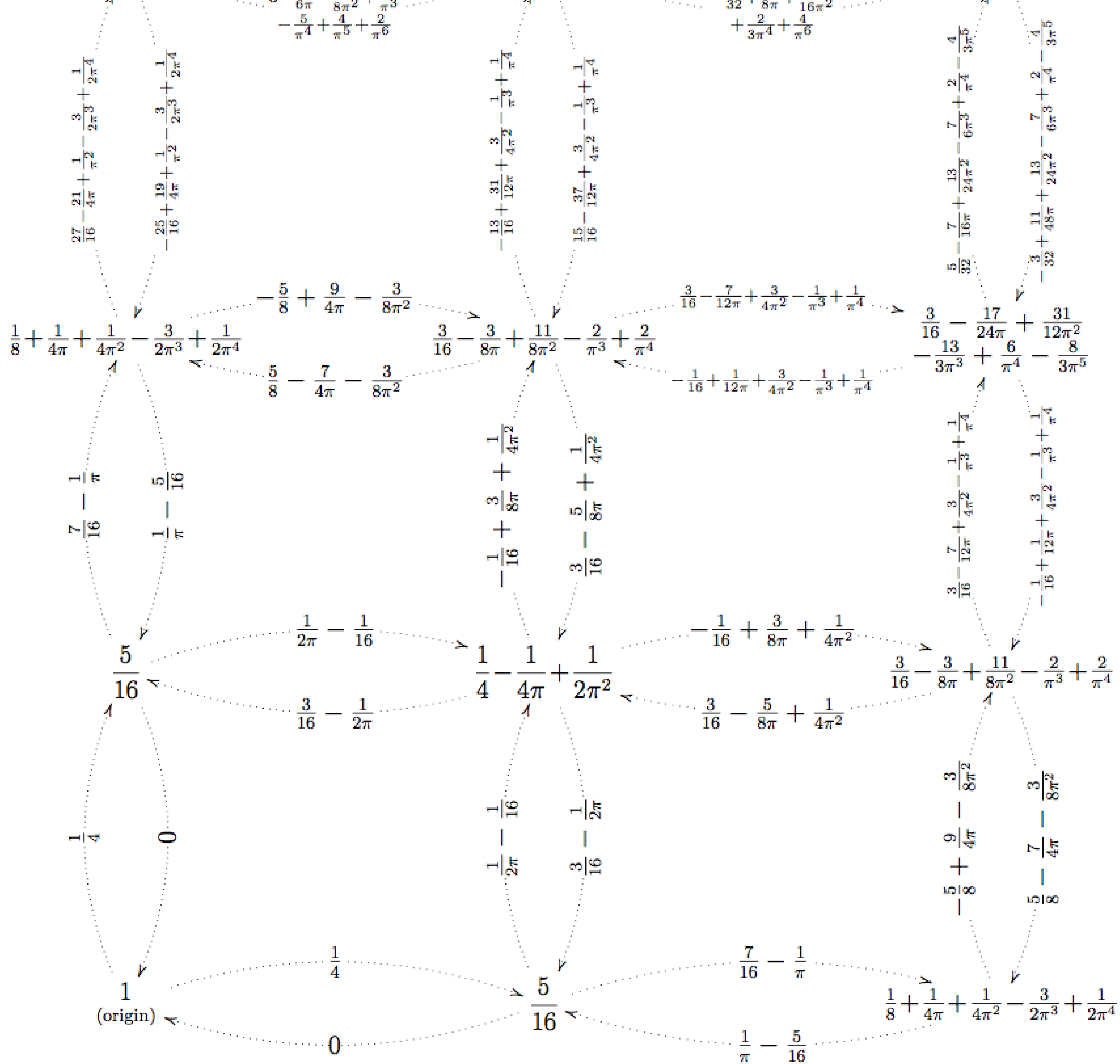
Sandpile density and LERW



Theorem: path to infinity visits neighbor to right with probability 5/16 (Poghosyan-Priezzhev-Ruelle, Kenyon—W)

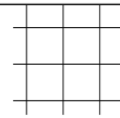
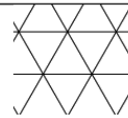
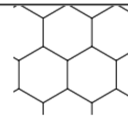
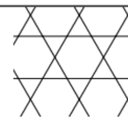
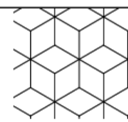
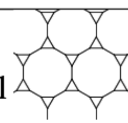
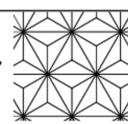
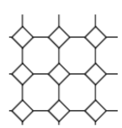
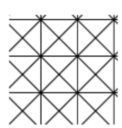
JPR integral evaluates to ½ (Caracciolo—Sportiello)

Proof involving only rationals (Kassel—W)

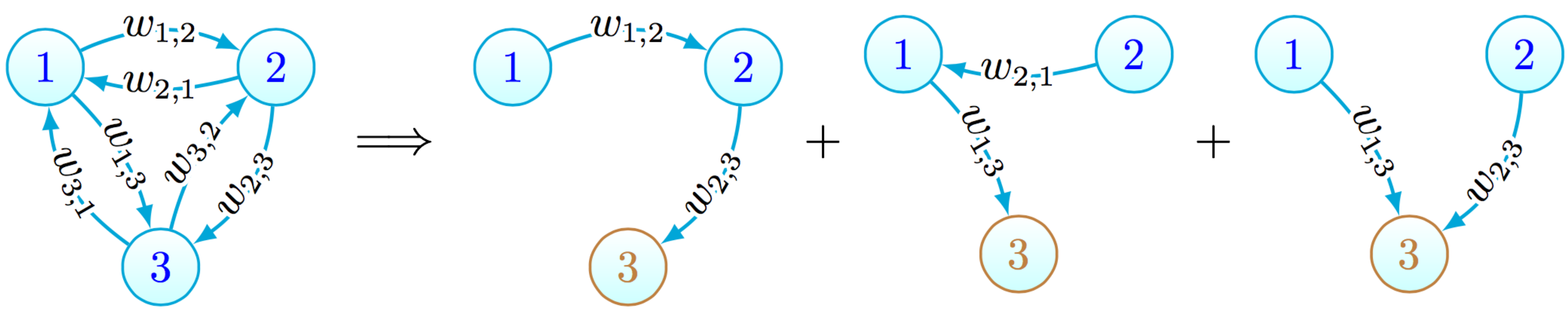


1 (origin)
$\frac{5}{16}$
$\frac{1}{4}$
0
$\frac{1}{4}-\frac{1}{4\pi}+\frac{1}{2\pi^2}$
$\frac{1}{8}+\frac{1}{4\pi}+\frac{1}{4\pi^2}-\frac{3}{2\pi^3}+\frac{1}{2\pi^4}$
$\frac{3}{16}-\frac{3}{8\pi}+\frac{11}{8\pi^2}-\frac{2}{\pi^3}+\frac{2}{\pi^4}$
$\frac{3}{16}-\frac{17}{24\pi}+\frac{31}{12\pi^2}-\frac{13}{3\pi^3}+\frac{6}{\pi^4}-\frac{8}{3\pi^5}$
$\frac{7}{16}-\frac{1}{\pi}$
$\frac{1}{\pi}-\frac{5}{16}$
$\frac{1}{2\pi}-\frac{1}{16}$
$\frac{3}{16}-\frac{1}{2\pi}$
$-\frac{1}{16}+\frac{3}{8\pi}+\frac{1}{4\pi^2}$
$\frac{3}{16}-\frac{5}{8\pi}+\frac{1}{4\pi^2}$
$-\frac{5}{8}+\frac{9}{4\pi}-\frac{3}{8\pi^2}$
$\frac{5}{8}-\frac{7}{4\pi}-\frac{3}{8\pi^2}$
$\frac{3}{16}-\frac{7}{12\pi}+\frac{3}{4\pi^2}-\frac{1}{\pi^3}+\frac{1}{\pi^4}$
$-\frac{1}{16}+\frac{1}{12\pi}+\frac{3}{4\pi^2}-\frac{1}{\pi^3}+\frac{1}{\pi^4}$
$-\frac{1}{16}+\frac{3}{8\pi}+\frac{1}{4\pi^2}$
$\frac{3}{16}-\frac{5}{8\pi}+\frac{1}{4\pi^2}$
$\frac{27}{16}-\frac{21}{4\pi}+\frac{1}{\pi^2}-\frac{3}{2\pi^3}+\frac{1}{2\pi^4}$
$-\frac{25}{16}+\frac{19}{4\pi}+\frac{1}{\pi^2}-\frac{3}{2\pi^3}+\frac{1}{2\pi^4}$
$\frac{13}{16}+\frac{31}{12\pi}+\frac{3}{4\pi^2}-\frac{1}{\pi^3}+\frac{1}{\pi^4}$
$\frac{15}{16}-\frac{37}{12\pi}+\frac{3}{4\pi^2}-\frac{1}{\pi^3}+\frac{1}{\pi^4}$
$\frac{5}{32}-\frac{7}{16\pi}+\frac{13}{24\pi^2}-\frac{7}{6\pi^3}+\frac{2}{\pi^4}-\frac{4}{3\pi^5}$
$-\frac{3}{32}+\frac{11}{48\pi}+\frac{13}{24\pi^2}-\frac{7}{6\pi^3}+\frac{2}{\pi^4}-\frac{4}{3\pi^5}$
$\frac{3}{16}-\frac{7}{12\pi}+\frac{3}{4\pi^2}-\frac{1}{\pi^3}+\frac{1}{\pi^4}$
$-\frac{1}{16}+\frac{1}{12\pi}+\frac{3}{4\pi^2}-\frac{1}{\pi^3}+\frac{1}{\pi^4}$
$-\frac{5}{8}+\frac{9}{4\pi}-\frac{3}{8\pi^2}$
$\frac{5}{8}-\frac{7}{4\pi}-\frac{3}{8\pi^2}$
$\frac{7}{16}-\frac{1}{\pi}$
$\frac{1}{\pi}-\frac{5}{16}$
$\frac{1}{2\pi}-\frac{1}{16}$
$\frac{3}{16}-\frac{1}{2\pi}$
$-\frac{1}{16}+\frac{3}{8\pi}+\frac{1}{4\pi^2}$
$\frac{3}{16}-\frac{5}{8\pi}+\frac{1}{4\pi^2}$
$-\frac{5}{\pi^4}+\frac{4}{\pi^5}+\frac{2}{\pi^6}$
$+\frac{2}{3\pi^4}+\frac{4}{\pi^6}$

Kenyon—W

lattice	discrete-time LERW looping rate $\rho = \tau + \frac{1}{2}\,\mathrm{Pr}[e \in T]$	sandpile density $\bar{\sigma} = (\delta\rho + \delta - 1)/2$
square 	$5/16$ 0.3125	$17/8$ 2.125
triangular 	$5/18$ $0.277778\ldots$	$10/3$ $3.333333\ldots$
honeycomb 	$13/36$ $0.361111\ldots$	$37/24$ $1.541667\ldots$
kagomé / trihexagonal 	$1/3$ $0.333333\ldots$	$13/6$ $2.166667\ldots$
dice / rhombille 	$5/16$ 0.3125	$17/8$ 2.125
Fisher / truncated hexagonal 	$359/900$ $0.398889\ldots$	$959/600$ $1.598333\ldots$
triakis triangular 	$7/25$ 0.28	$167/50$ 3.34
square-octagon / truncated square 	$\frac{3}{8} - \frac{\mathrm{arcsec}(3)}{12\sqrt{2}\pi} + \frac{\mathrm{arcsec}(3)^2}{8\pi^2}$ $0.371102\ldots$	$\frac{25}{16} - \frac{\mathrm{arcsec}(3)}{8\sqrt{2}\pi} + \frac{3\,\mathrm{arcsec}(3)^2}{16\pi^2}$ $1.556654\ldots$
tetrakis square 	$\frac{7}{24} - \frac{\mathrm{arcsec}(3)}{12\sqrt{2}\pi} + \frac{\mathrm{arcsec}(3)^2}{16\pi^2}$ $0.278174\ldots$	$\frac{27}{8} - \frac{\mathrm{arcsec}(3)}{4\sqrt{2}\pi} + \frac{3\,\mathrm{arcsec}(3)^2}{16\pi^2}$ $3.334521\ldots$

$$\det \underbrace{\begin{bmatrix} w_{1,2}+w_{1,3} & -w_{1,2} & -w_{1,3} \\ -w_{2,1} & w_{2,1}+w_{2,3} & -w_{2,3} \\ -w_{3,1} & -w_{3,2} & w_{3,1}+w_{3,2} \end{bmatrix}}_{\text{graph Laplacian } \Delta} = w_{1,2}w_{2,3} + w_{2,1}w_{1,3} + w_{1,3}w_{2,3}$$



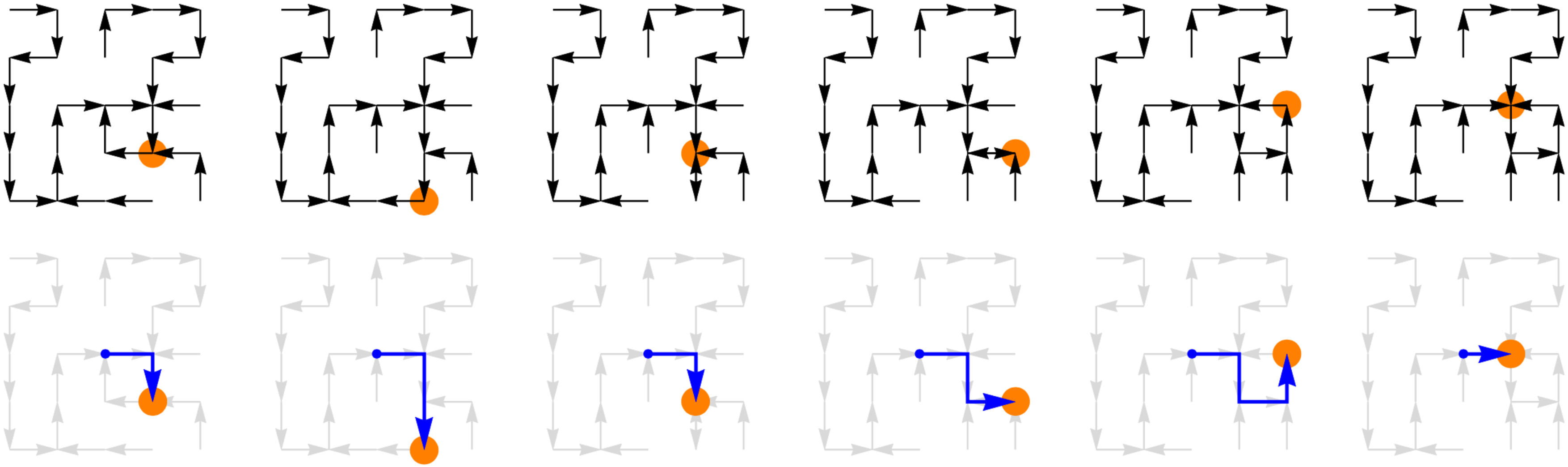
$$F_2(\mathcal{G}) = \sum_{v \neq s} \det \Delta_{\widehat{v,s}}^{\widehat{v,s}} - \sum_{\substack{u \sim v \\ u,v \neq s}} w_{u,v} \det \Delta_{\widehat{u,v,s}}^{\widehat{u,v,s}}$$

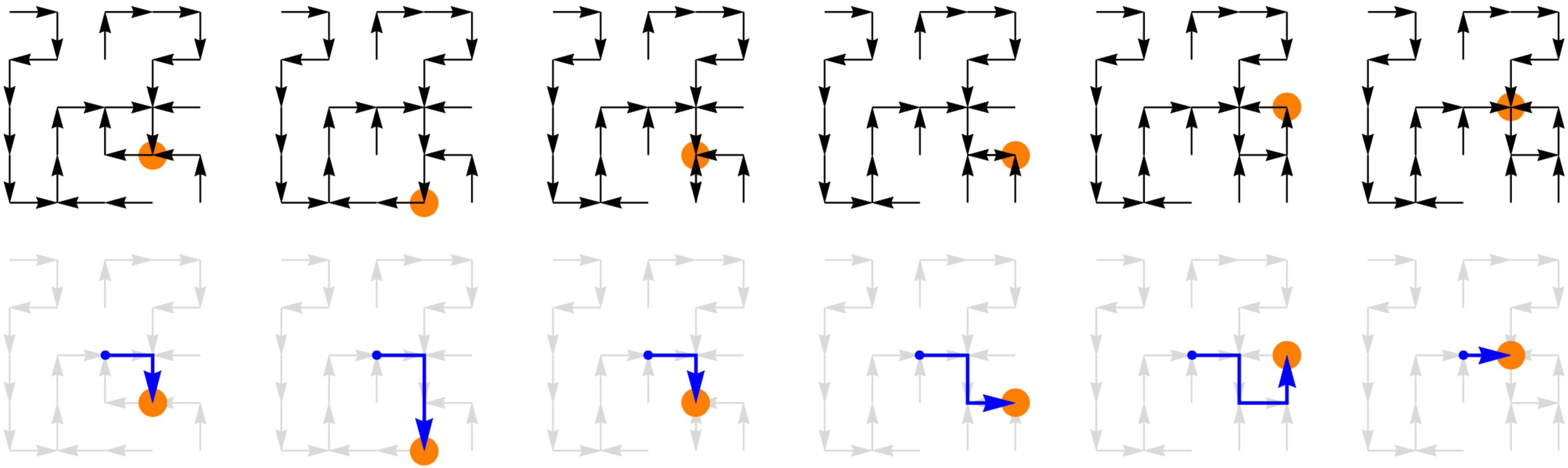
$$\begin{aligned} \frac{F_2(\mathcal{G})}{F_1(\mathcal{G})} &= \sum_{v \neq s} \det G_v^v - \sum_{\substack{u \sim v \\ u,v \neq s}} w_{u,v} \det G_{u,v}^{u,v} \\ &= \sum_{v \neq s} G_{v,v} - \sum_{\substack{u \sim v \\ u,v \neq s}} w_{u,v} \left[G_{u,u} G_{v,v} - G_{u,v}^2 \right] \end{aligned}$$

$$G_{u,v}^{(s)} = \begin{cases} \left[(\Delta_{\widehat{s}}^{\widehat{s}})^{-1} \right]_{u,v} & u, v \neq s, \\ 0 & u = s \text{ or } v = s \end{cases}$$

$$\frac{F_2(\mathcal{G})}{F_1(\mathcal{G})} = \sum_{u \sim v} w_{u,v} \left[\left(A^{(s)}_{u,v} - A^{(s)}_{v,u} \right)^2 + A^{(s)}_{u,v} A^{(s)}_{v,u} \right]$$

$$A^{(s)}_{u,v} = G^{(s)}_{u,u} - G^{(s)}_{u,v}$$





$$\rho = \text{discrete-time LERW looping rate} = \frac{\text{weighted sum of oriented CRST's}}{\text{weighted sum of marked oriented CRST's}}$$

$$\tau = \frac{\text{weighted sum of oriented CRST's with cycle length} \geq 3}{\text{weighted sum of marked oriented CRST's}}$$

$$\rho - \tau = \frac{1}{2}\,\mathrm{Pr}[\text{random edge } e \in \text{random tree } T]$$

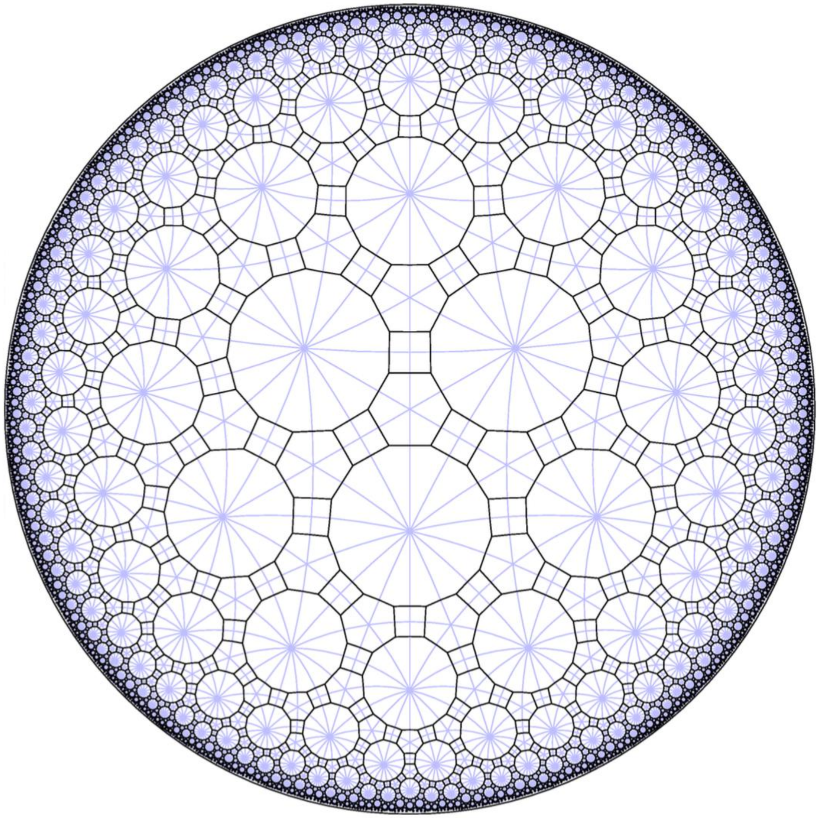
$$\tau = \frac{\displaystyle\sum_{u^* \sim v^*} w_{u^*,v^*} \left(A^{(s^*)}_{u^*,v^*} A^{(s^*)}_{v^*,u^*} + \left(A^{(s^*)}_{u^*,v^*} - A^{(s^*)}_{v^*,u^*} \right)^2 \right)}{\displaystyle\sum_{u^* \sim v^*} 1/w_{u^*,v^*}}$$

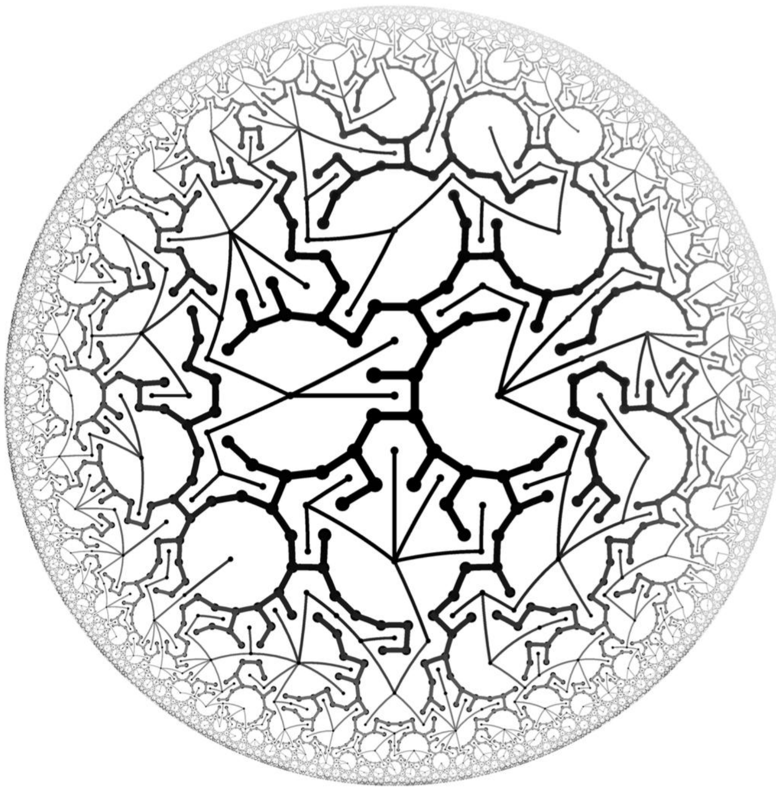
$$T_{\mathcal{G}}(x,y) = \sum_{E' \subseteq E} (x-1)^{k(E')-k(E)} (y-1)^{k(E')+|E'|-|V|}$$

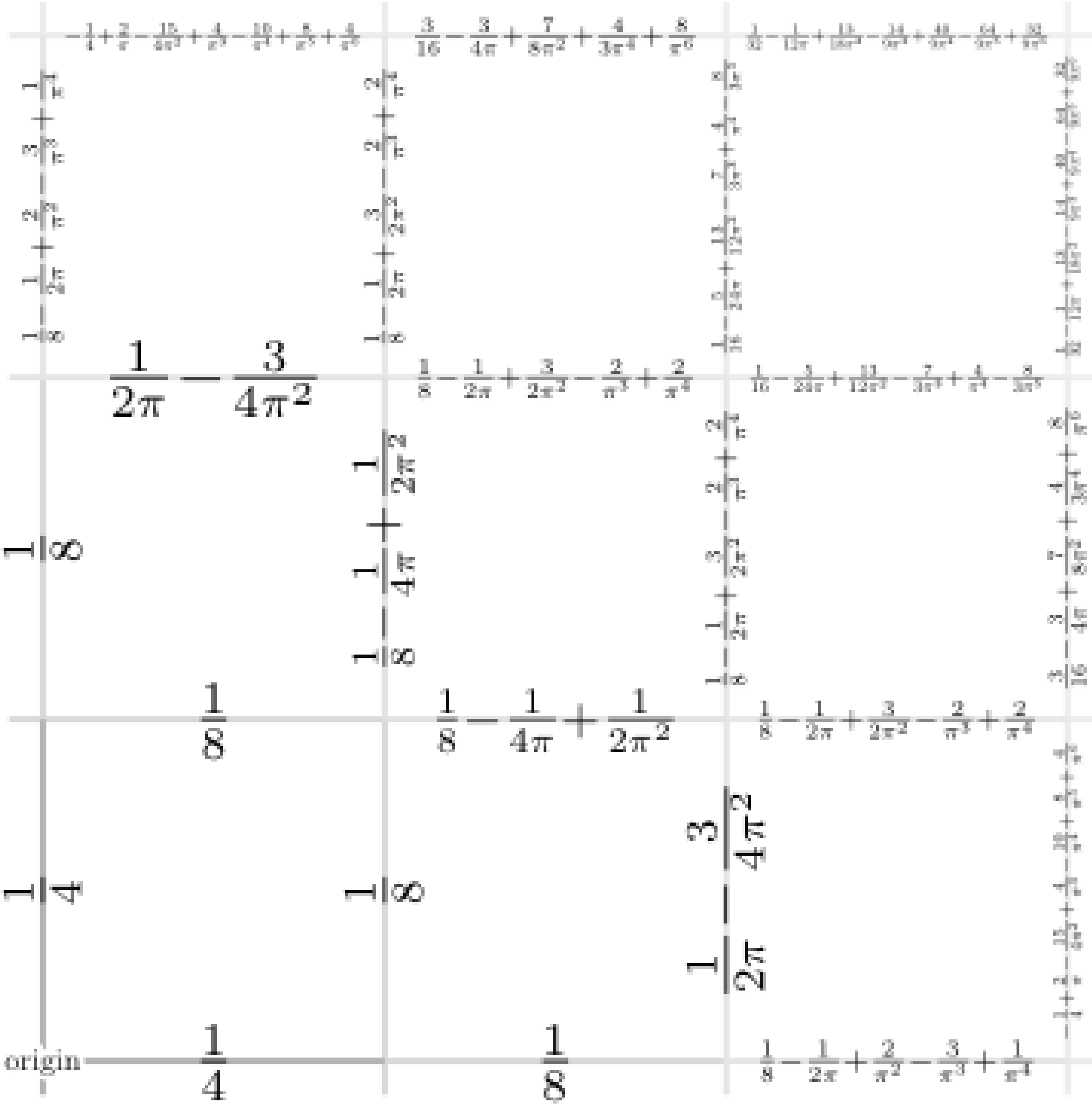
$$\sum_{\substack{\text{recurrent} \\ \text{sandpiles } \sigma}} y^{\text{level}(\sigma)} = T_{\mathcal{G}}(1,y)$$

$$\sum_{\substack{\text{recurrent} \\ \text{sandpiles } \sigma}} \binom{\text{level}(\sigma)}{j} = \frac{1}{j!} \frac{d^j}{dy^j} T_{\mathcal{G}}(1,y)\Big|_{y=1} = \begin{array}{c}\text{\# connected subgraphs of } \mathcal{G} \\ \text{with } |V|+j-1 \text{ edges.}\end{array}$$

$$\mathbb{E}[\text{level}(\sigma)] = \frac{\text{\# unicycles of } \mathcal{G}}{\text{\# spanning trees of } \mathcal{G}} = \tau \times |E|$$

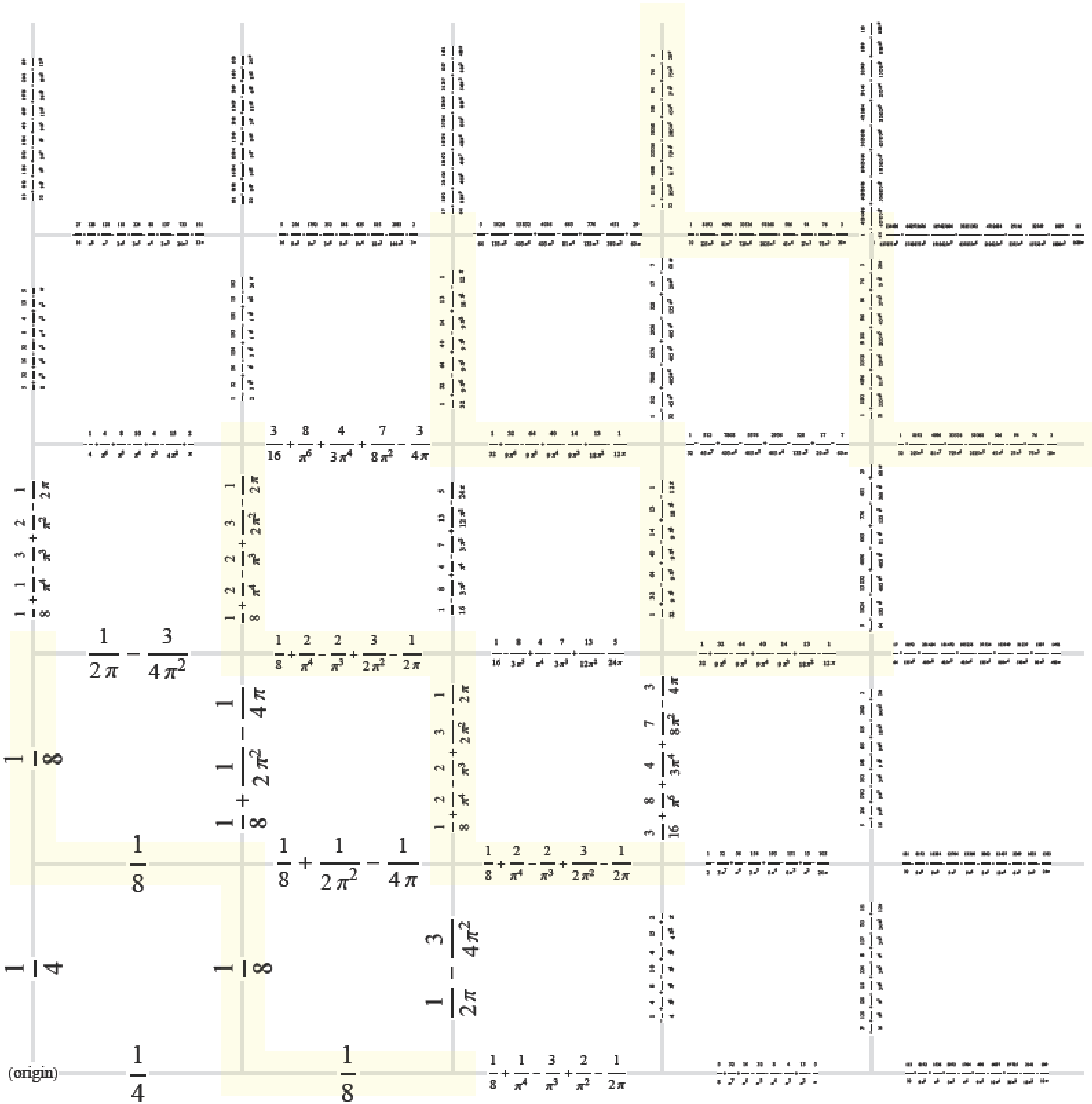




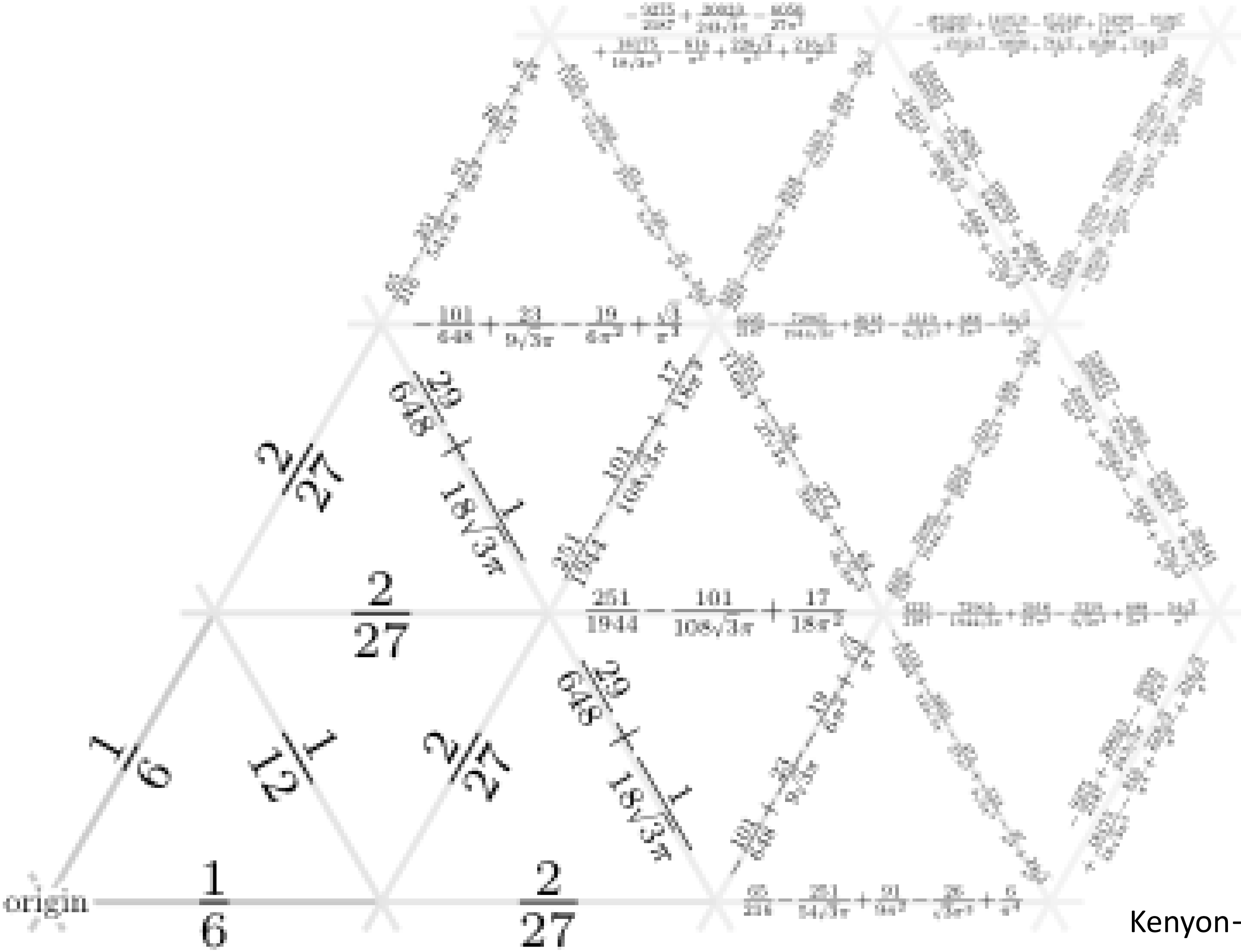


origin
1/4
1/8
1/4
1/8
1/8
1/8
1/8 − 1/(4π) + 1/(2π²)
1/(2π) − 3/(4π²)
1/(2π) − 3/(4π²)
1/(4π) + 1/(2π²)
1/8 − 1/(2π) + 3/(2π²) − 2/π³ + 2/π⁴
3/16 − 3/(4π) + 7/(8π²) + 4/(3π⁴) + 8/π⁶

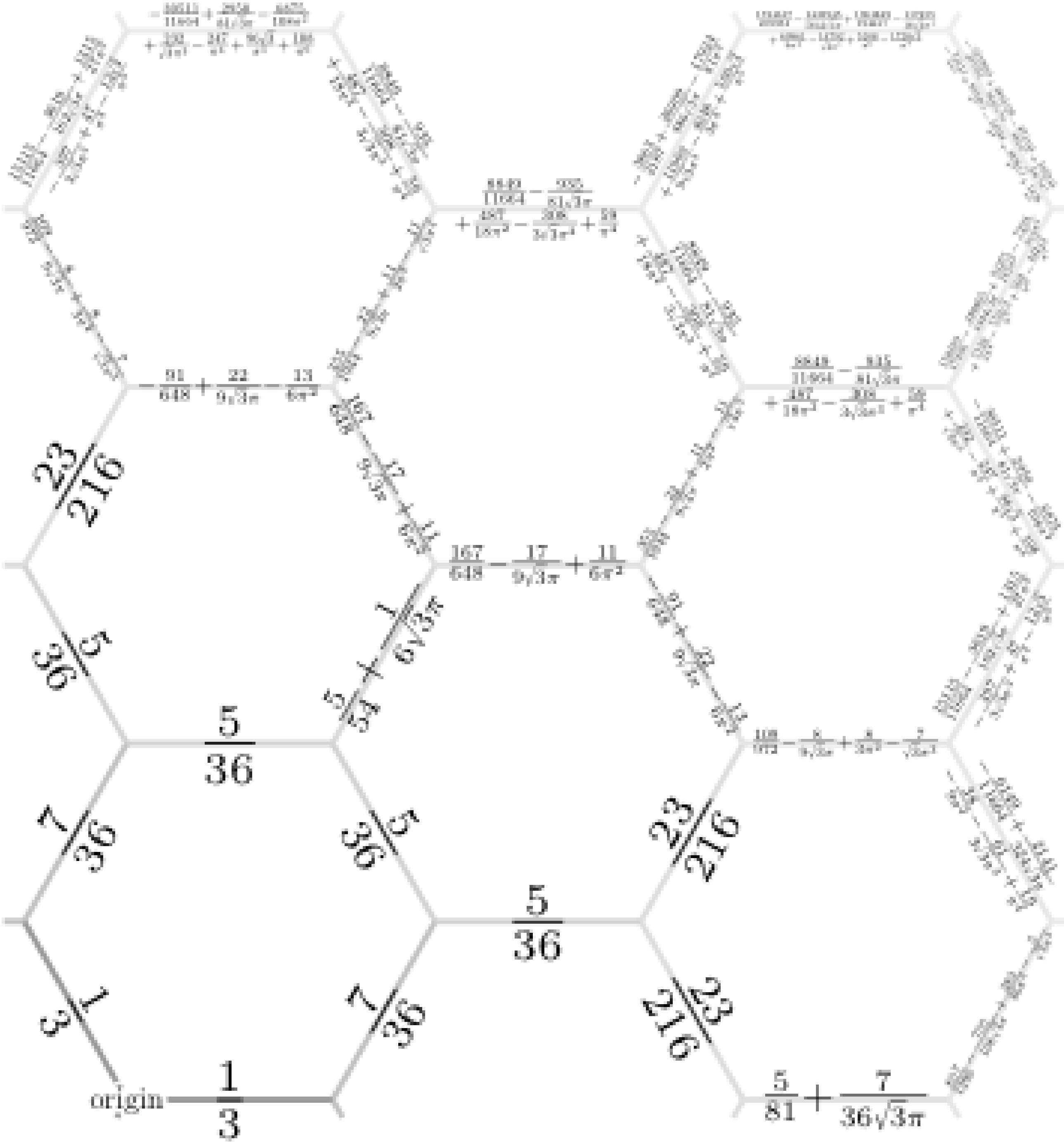
Kenyon—W


(origin)
1/4
1/8
1/8 + 1/(2π²) − 1/(4π)
1/(2π) − 3/(4π²)
1/8 + 1/π⁴ − 3/π³ + 2/π² − 1/(2π)
1/8 + 2/π⁴ − 2/π³ + 3/(2π²) − 1/(2π)
3/16 + 8/π⁶ + 4/(3π⁴) + 7/(8π²) − 3/(4π)

Kenyon—W


origin
1/6
1/6
1/12
2/27
2/27
2/27
2/27
20/648 + 1/(18√3π)
20/648 + 1/(18√3π)
251/1944 − 101/(108√3π) + 17/(18π²)
−101/648 + 23/(9√3π) − 19/(6π²) + √3/π³

Kenyon—W


origin
1/3
7/36
5/36
23/216
5/54 + 1/(6√3π)
167/648 − 17/(9√3π) + 11/(6π²)
−91/648 + 22/(9√3π) − 13/(6π²)
5/81 + 7/(36√3π)

Kenyon—W

Joint distribution of heights at two neighboring vertices

0	$\frac{9}{32}-\frac{9}{2\pi}+\frac{47}{2\pi^2}-\frac{48}{\pi^3}+\frac{32}{\pi^4}$	$-\frac{33}{64}+\frac{191}{32\pi}-\frac{383}{16\pi^2}+\frac{81}{2\pi^3}-\frac{47}{2\pi^4}$	$\frac{15}{64}-\frac{47}{32\pi}+\frac{39}{16\pi^2}+\frac{7}{2\pi^3}-\frac{17}{2\pi^4}$
$\frac{9}{32}-\frac{9}{2\pi}+\frac{47}{2\pi^2}-\frac{48}{\pi^3}+\frac{32}{\pi^4}$	$-\frac{69}{32}+\frac{453}{16\pi}-\frac{1059}{8\pi^2}+\frac{521}{2\pi^3}-\frac{167}{\pi^4}-\frac{32}{\pi^5}$	$\frac{83}{32}-\frac{243}{8\pi}+\frac{537}{4\pi^2}-\frac{256}{\pi^3}+\frac{619}{4\pi^4}+\frac{59}{\pi^5}$	$-\frac{15}{32}+\frac{97}{16\pi}-\frac{227}{8\pi^2}+\frac{111}{2\pi^3}-\frac{79}{4\pi^4}-\frac{27}{\pi^5}$
$-\frac{33}{64}+\frac{191}{32\pi}-\frac{383}{16\pi^2}+\frac{81}{2\pi^3}-\frac{47}{2\pi^4}$	$\frac{83}{32}-\frac{243}{8\pi}+\frac{537}{4\pi^2}-\frac{256}{\pi^3}+\frac{619}{4\pi^4}+\frac{59}{\pi^5}$	$-\frac{107}{32}+\frac{617}{16\pi}-\frac{1259}{8\pi^2}+\frac{291}{\pi^3}-\frac{375}{2\pi^4}-\frac{108}{\pi^5}$	$\frac{105}{64}-\frac{421}{32\pi}+\frac{753}{16\pi^2}-\frac{175}{2\pi^3}+\frac{225}{4\pi^4}+\frac{49}{\pi^5}$
$\frac{15}{64}-\frac{47}{32\pi}+\frac{39}{16\pi^2}+\frac{7}{2\pi^3}-\frac{17}{2\pi^4}$	$-\frac{15}{32}+\frac{97}{16\pi}-\frac{227}{8\pi^2}+\frac{111}{2\pi^3}-\frac{79}{4\pi^4}-\frac{27}{\pi^5}$	$\frac{105}{64}-\frac{421}{32\pi}+\frac{753}{16\pi^2}-\frac{175}{2\pi^3}+\frac{225}{4\pi^4}+\frac{49}{\pi^5}$	$-\frac{33}{32}+\frac{129}{16\pi}-\frac{161}{8\pi^2}+\frac{65}{2\pi^3}-\frac{28}{\pi^4}-\frac{22}{\pi^5}$

0.	0.0103411	0.0238479	0.0394473
0.0103411	0.0260442	0.0525221	0.0849925
0.0238479	0.0525221	0.0930601	0.136861
0.0394473	0.0849925	0.136861	0.184871

Higher dimensional marginals of sandpile heights

Pr[3,2,1,0 in 4x1 rectangle] =

$$-\frac{61815}{128}+\frac{1856395}{128\pi}-\frac{99783277}{576\pi^2}+\frac{964096235}{864\pi^3}-\frac{5588534021}{1296\pi^4}+\frac{5014047485}{486\pi^5}$$
$$-\frac{10884136816}{729\pi^6}+\frac{25765891840}{2187\pi^7}-\frac{23058546688}{6561\pi^8}-\frac{319225856}{729\pi^9}\doteq 0.00169649.$$

Sandpiles on hexagonal lattice

$$\Pr[h = 0] = \frac{1}{12}$$

$$\Pr[h = 1] = \frac{7}{24}$$

$$\Pr[h = 2] = \frac{5}{8}$$

(One-site probabilities also computed by Ruelle)

$$0 \qquad -\frac{5}{54} + \frac{7}{6\sqrt{3}\pi} - \frac{1}{\pi^2} \qquad \frac{19}{108} - \frac{7}{6\sqrt{3}\pi} + \frac{1}{\pi^2}$$

$$-\frac{5}{54} + \frac{7}{6\sqrt{3}\pi} - \frac{1}{\pi^2} \qquad \frac{11}{54} - \frac{7}{3\sqrt{3}\pi} + \frac{3}{\pi^2} \qquad \frac{13}{72} + \frac{7}{6\sqrt{3}\pi} - \frac{2}{\pi^2}$$

$$\frac{19}{108} - \frac{7}{6\sqrt{3}\pi} + \frac{1}{\pi^2} \qquad \frac{13}{72} + \frac{7}{6\sqrt{3}\pi} - \frac{2}{\pi^2} \qquad \frac{29}{108} + \frac{1}{\pi^2}.$$

Sandpiles on triangular lattice

$$\Pr[h=0] = -\frac{25}{648} - \frac{55}{72\sqrt{3}\pi} + \frac{7}{3\pi^2} + \frac{11\sqrt{3}}{\pi^3} - \frac{90}{\pi^4} + \frac{54\sqrt{3}}{\pi^5} \doteq 0.053623$$

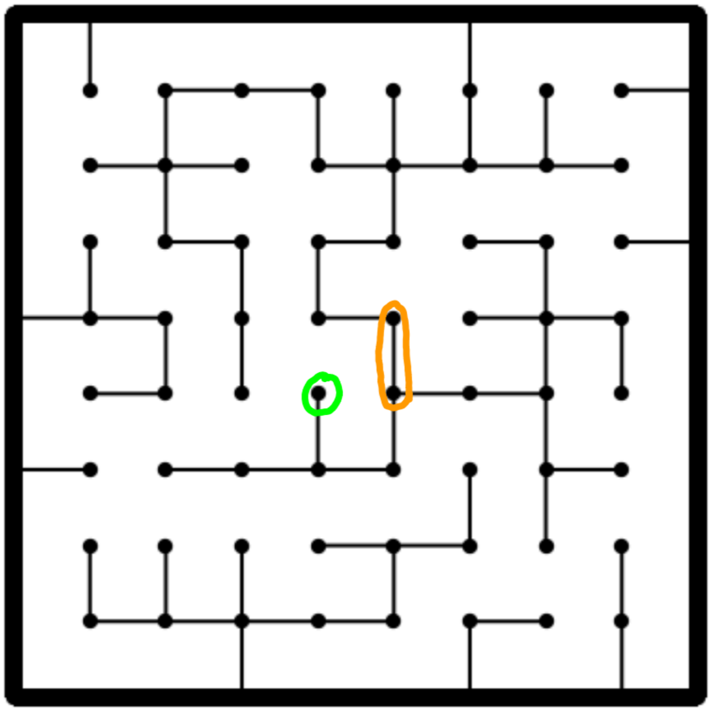
$$\Pr[h=1] = \frac{47}{1296} + \frac{301}{24\sqrt{3}\pi} - \frac{193}{6\pi^2} - \frac{29\sqrt{3}}{\pi^3} + \frac{405}{\pi^4} - \frac{270\sqrt{3}}{\pi^5} \doteq 0.091525$$

$$\Pr[h=2] = \frac{3}{8} - \frac{5929}{144\sqrt{3}\pi} + \frac{1441}{12\pi^2} - \frac{9\sqrt{3}}{\pi^3} - \frac{720}{\pi^4} + \frac{540\sqrt{3}}{\pi^5} \doteq 0.137356$$

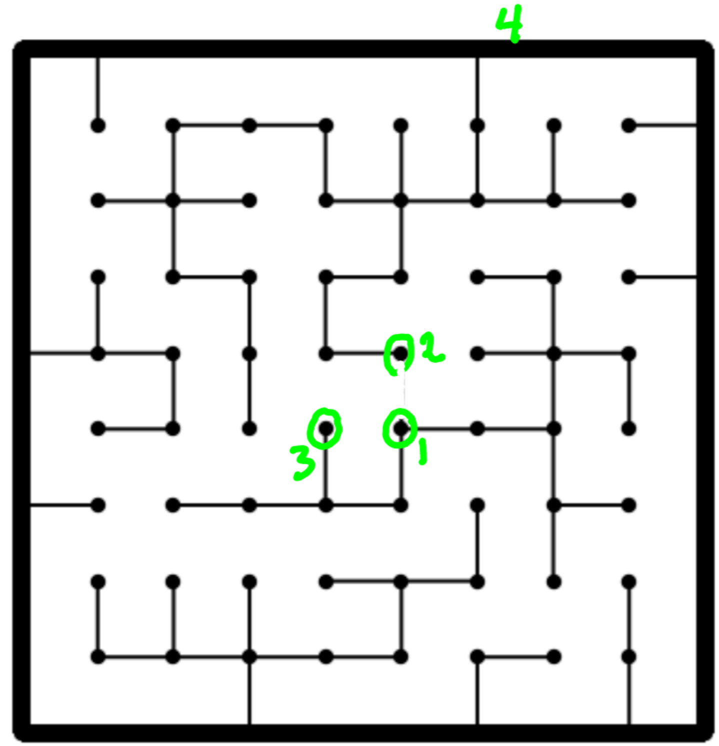
$$\Pr[h=3] = \frac{3427}{2592} + \frac{6515}{144\sqrt{3}\pi} - \frac{2125}{12\pi^2} + \frac{91\sqrt{3}}{\pi^3} + \frac{630}{\pi^4} - \frac{540\sqrt{3}}{\pi^5} \doteq 0.189037$$

$$\Pr[h=4] = -\frac{2663}{1296} - \frac{71\sqrt{3}}{16\pi} + \frac{1331}{12\pi^2} - \frac{94\sqrt{3}}{\pi^3} - \frac{270}{\pi^4} + \frac{270\sqrt{3}}{\pi^5} \doteq 0.242307$$

$$\Pr[h=5] = \frac{1175}{864} - \frac{365}{144\sqrt{3}\pi} - \frac{289}{12\pi^2} + \frac{30\sqrt{3}}{\pi^3} + \frac{45}{\pi^4} - \frac{54\sqrt{3}}{\pi^5} \doteq 0.286152$$

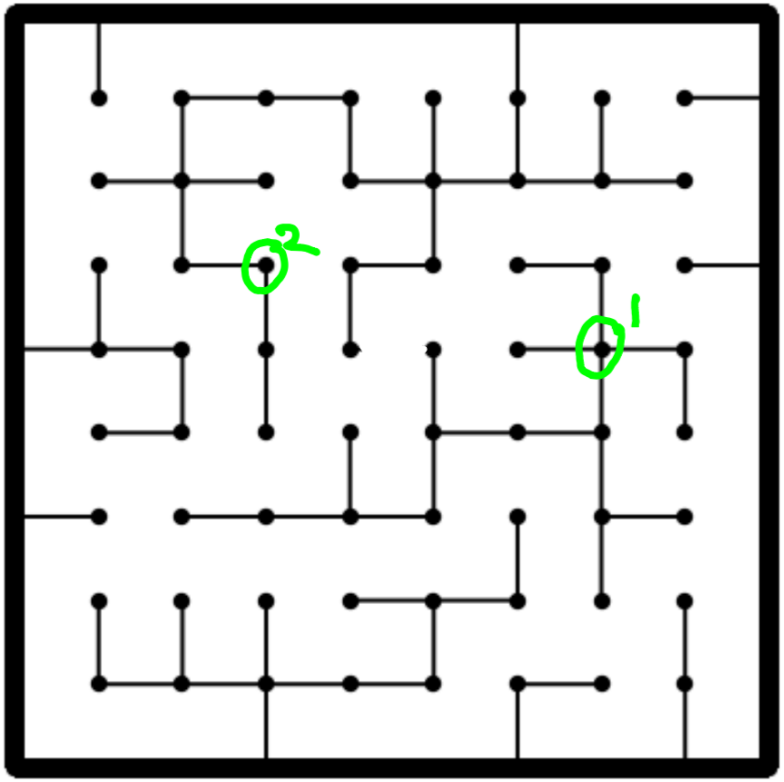


Spanning tree in which path from start to boundary uses given directed edge



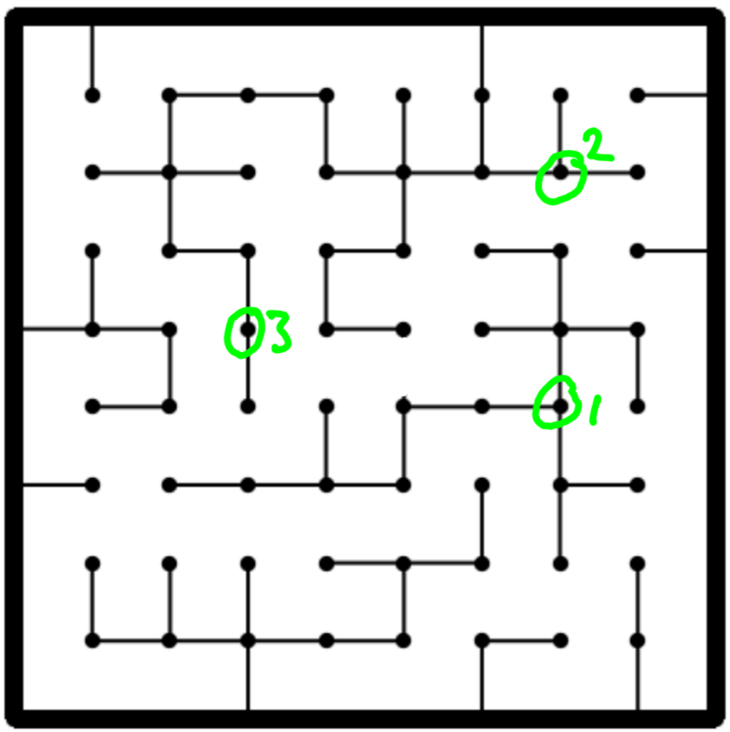
grove of type $1,3 \mid 2,4$

$$\Pr[\text{LERW uses directed edge}] = \frac{Z(1,3 \mid 2,4)}{Z}$$



grove of type 1|2

Kirchhoff: $R_{1,2} = \frac{Z(1|2)}{Z}$



grove of type 1|2,3

$$\frac{Z(1|23)}{Z} = \frac{1}{2}R_{1,2} + \frac{1}{2}R_{1,3} - \frac{1}{2}R_{2,3}$$

$$\frac{Z(1|2|3)}{Z} = \frac{R_{1,2}R_{1,3} + R_{1,2}R_{2,3} + R_{1,3}R_{2,3}}{2} - \frac{R_{1,2}^2 + R_{1,3}^2 + R_{2,3}^2}{4}$$

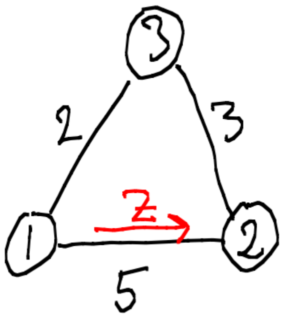
4 (or more) nodes: No formula of the above type that holds for general graphs.

Pairwise resistances do <u>not</u> determine $\frac{Z(1,2|3,4)}{Z}$ for general graphs.

If graph is <u>planar</u> and all nodes on the same face, then $\frac{Z(\sigma)}{Z}$ is a polynomial in the $R_{i,j}$'s.

(KW)

Line Bundle Laplacian



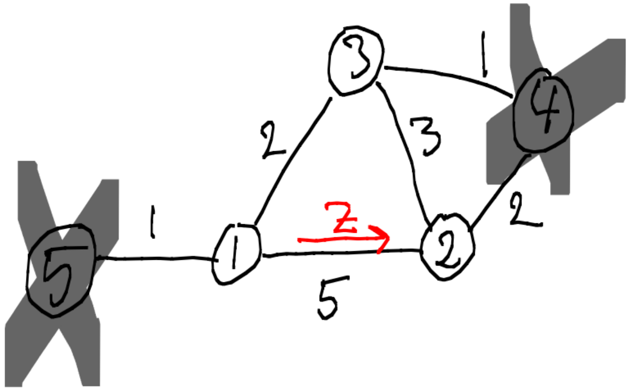
$$\Delta = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 7 & -5/z & -2 \\ -5z & 8 & -3 \\ -2 & -3 & 5 \end{bmatrix} \end{array}$$

Forman:

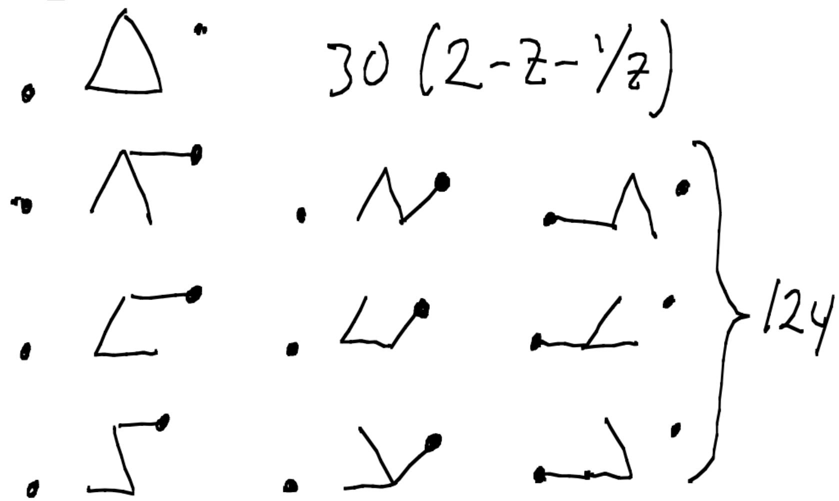
$$\det\Delta = \underset{\text{product of edge weights}}{30}\,(2 - z - 1/z)$$

(monodromy of cycle: z, $1/z$)

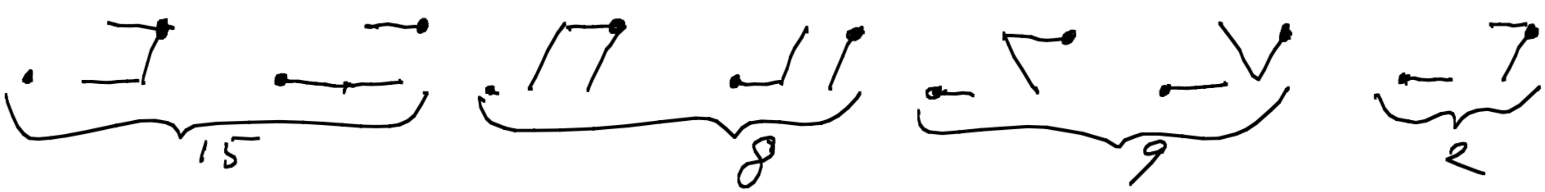
= weighted sum of cycle-rooted spanning forests



$$\Delta = \begin{bmatrix} & 1 & 2 & 3 & 4 & 5 \\ 1 & 8 & -5/z & -2 & 0 & -1 \\ 2 & -5z & 10 & -3 & -2 & 0 \\ 3 & -2 & -3 & 6 & -1 & 0 \\ 4 & 0 & -2 & -1 & 3 & 0 \\ 5 & -1 & & & & 1 \end{bmatrix}$$

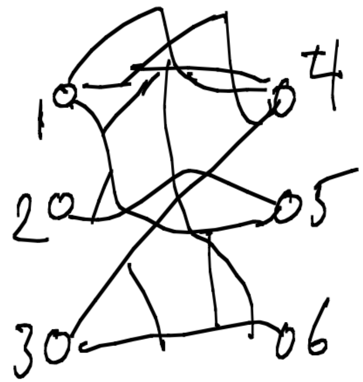


$$\det\Delta = 480 - 30z - 30/z$$
$$- 72 - 150 - 40$$
$$= 30(2 - z - 1/z) + 158$$



Curtis-Ingerman-Morrow

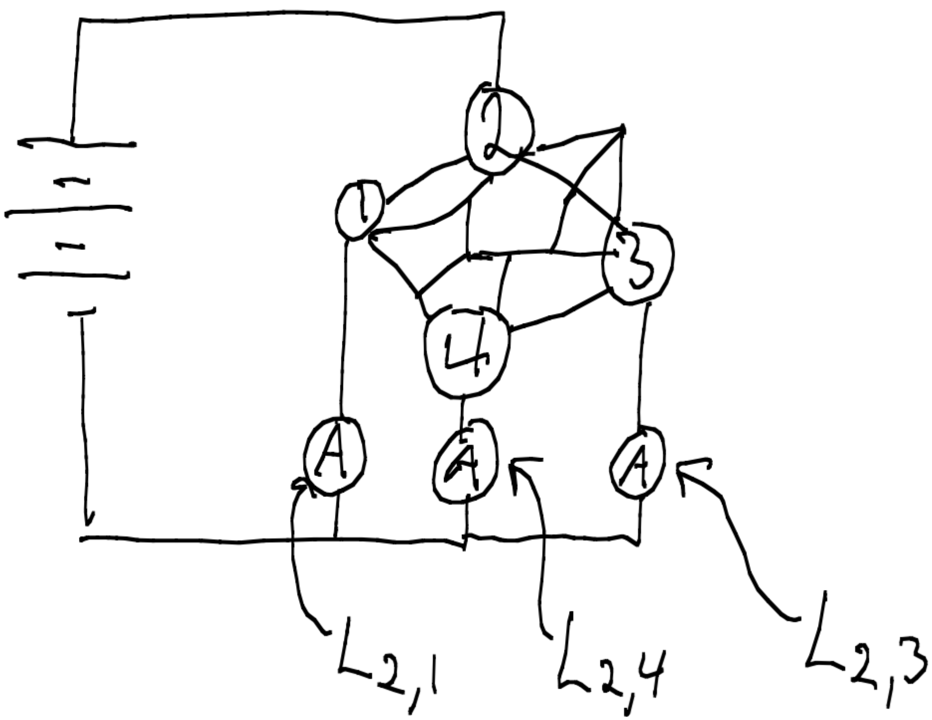
general graph with 2n vertices



$$\det L^{4,5,6}_{1,2,3} = \frac{Z(14|25|36) - Z(14|26|35) + Z(15|26|34) - Z(15|24|36) + Z(16|24|35) - Z(16|25|34)}{Z(1|2|3|4|5|6)}$$

Response matrix

$L_{i,j}$: apply voltage at i; measure current at j.

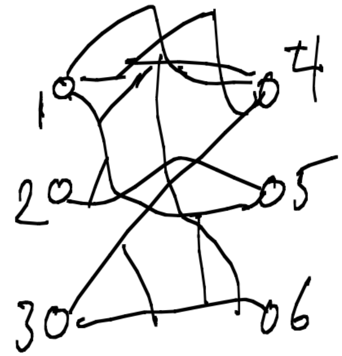


Fact: $L_{i,j} = L_{j,i}$

$\sum_j L_{i,j} = 0$

$\binom{n}{2}$ $L_{i,j}$ variables

$\binom{n}{2}$ $R_{i,j}$ variables



general graph with 2n vertices

$$\det \mathcal{L}^{4,5,6}_{1,2,3} = \frac{Z\left(\begin{array}{c|c|c}4&5&6\\1&2&3\end{array}\right) - Z\left(\begin{array}{c|c|c}4&6&5\\1&2&3\end{array}\right) + Z\left(\begin{array}{c|c|c}5&6&4\\1&2&3\end{array}\right) - Z\left(\begin{array}{c|c|c}5&4&6\\1&2&3\end{array}\right) + Z\left(\begin{array}{c|c|c}6&4&5\\1&2&3\end{array}\right) - Z\left(\begin{array}{c|c|c}6&5&4\\1&2&3\end{array}\right)}{Z(1|2|3|4|5|6)}$$

includes parallel transports

cycle rooted groves

(proof similar to C-I-M proof, but starts with Forman's MTT)