Volker Braun
Dublin Inst. for Adv. Studies

KITP Santa Barbara
March 2, 2012

Geometry of the Compact Directions: A Numerical Approach to Sasaki-Einstein and Calabi-Yau metrics

Near Horizon Limit

Starting point: 10-d supergravity solution

$$ds^2 = H^{-\frac{1}{2}}\left(-dt^2 - d\vec{x}^2\right) + H^{\frac{1}{2}} \underbrace{\left(dr^2 + r^2 d\Omega_5^2\right)}_{\text{flat } \mathbb{C}^3}$$

with

$$H = 1 + \frac{1}{r^4} \quad \sim \quad \frac{1}{r^4}$$

Can replace flat $\mathbb{C}^3$ with any conical

$$dr^2 + r^2 g_{ij} dy^i dy^j$$

Calabi-Yau metric.

Calabi-Yau Cones

On $\mathbb{C}^3 \ni (x_1 + iy_1,\ x_2 + iy_2,\ x_3 + iy_3)$, we thus have:

- The radial vector field

$$r\partial_r = \sum \left(x_i \partial_{x_i} + y_i \partial_{y_i}\right)$$

- The complex structure

$$J(\partial_{x_i}) = \partial_{y_i}, \qquad J(\partial_{y_i}) = -\partial_{x_i}$$

- The Reeb vector field

$$J(r\partial_r) = \sum \left(x_i \partial_{y_i} - y_i \partial_{x_i}\right)$$

in the $r =$ const. plane.

Geometries

Example	Geometry	
$\mathbb{C}^3$	Calabi-Yau	6-dimensional space
S^5	Sasaki-Einstein	at $r = \text{const.}$
$\mathbb{P}^2$	Kähler-Einstein	Quotient by Reeb $U(1)$

- In the following, we'll look at more general Kähler-Einstein surfaces.
- Reeb vector field doesn't always form compact orbits, will come back to this later.

[Martelli-Sparks-Yau, Abreu]

Toric Varieties

What is a toric variety? (in 2-d, say)

- Generalization of projective plane

$$[z_0 : z_1 : z_2] = [\lambda z_0 : \lambda z_1 : \lambda z_3], \quad \lambda \in \mathbb{C}^\times$$

  to n homogeneous coordinates with $(\mathbb{C}^\times)^{n-2}$ rescalings.
- Space with $(\mathbb{C}^\times)^2$ action such that there is a single 2-dimensional orbit.
- Combinatorics of how lower-dimensional orbits ($\simeq \mathbb{C}^\times, \{pt.\}$) compactify $(\mathbb{C}^\times)^2$. This information is equivalent to a *fan* (collection of cones in $\mathbb{R}^2$).

Andrey Novoseltsev and I implemented toric varieties in Sage (see `http://www.sagemath.org`).

Toric Geometry and Sage

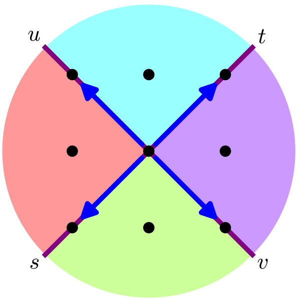
Construct $(\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2$ as a toric variety:

```
sage: square = polytopes.n_cube(2)
sage: fan = FaceFan(square.lattice_polytope())
sage: P1xP1_Z2.<s,u,v,t> = ToricVariety(fan)
sage: P1xP1_Z2
2-d toric variety covered by 4 affine patches
```

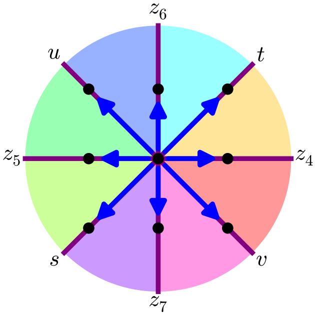
Various quantities that you might want to know:

```
sage: P1xP1_Z2.Mori_cone().rays()
((0, 1, 1, 0, -2), (1, 0, 0, 1, -2))
sage: P1xP1_Z2.cohomology_basis()
(([1],), ([t], [v]), ([v*t],))
sage: P1xP1_Z2.Chow_group().degree()
(Z, C2 x Z^2, Z)
sage: ( -P1xP1_Z2.K() ).sections_monomials()
(s^2*u^2, s^2*v^2, s*u*v*t, u^2*t^2, v^2*t^2)
```

Toric Resolution of Singularities



$(\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2$



After blowing up the 4 singular points.

The Fan knows nothing about the size of the blown-up $\mathbb{P}^1$.

Kähler Geometry

A Kähler metric is completely determined by the Kähler potential $K(z, \bar{z})$,

$$g_{i\bar{j}}(z, \bar{z}) = \partial_i \bar{\partial}_{\bar{j}} K(z, \bar{z})$$
$$\omega = g_{i\bar{j}}(z, \bar{z})\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}} = \partial\bar{\partial} K(z, \bar{z}).$$

Example (The Fubini-Study metric on $\mathbb{P}^2$)

This is the metric defined by the Kähler potential

$$K(z, \bar{z}) = \ln\left(|z_1|^2 + |z_2|^2 + |z_3|^2\right)$$

Note: $K(\lambda z, \overline{\lambda z}) = K(z, \bar{z}) + \ln(|\lambda|^2)$, so the metric is independent of homogeneous rescalings.

Algebraic Kähler Metrics

For arbitrary toric variety, fix homogeneous (multi)-degree and enumerate all monomials $\{s_\alpha\}$ with this degree.

Theorem (Tian)

The "algebraic Kähler potentials"

$$K(z, \bar{z}) = \ln \sum h^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}}$$

are dense in the space of Kähler potentials.

- $(h^{\alpha\bar{\beta}})$ is a Hermitian matrix.
- $(h^{\alpha\bar{\beta}})$ is diagonal $\Leftrightarrow$ the metric is $U(1)^2$-invariant.
- Actually works for any algebraic variety.

The Moment Map

For $U(1)^2$-invariant metrics, we can define the moment map

$$\mu(z,\bar{z}) = \frac{\sum_\alpha m_\alpha h^{\alpha\bar{\alpha}} |s_\alpha|^2}{\sum_\alpha h^{\alpha\bar{\alpha}} |s_\alpha|^2} \qquad \in \mathbb{R}^2$$

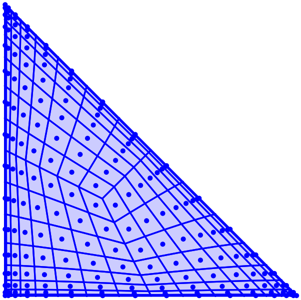
where m_α is the exponent vector (or $U(1)^2$-weight) of the homogeneous monomial s_α.

Theorem (Moment Map)

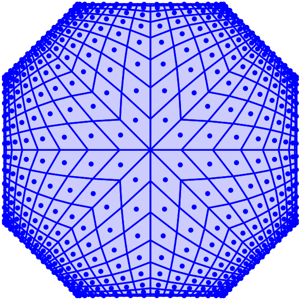
- *The domain of the moment map is a polygon and only depends on the Kähler class* $[\omega]$.
- *Lower-dimensional* $(\mathbb{C}^\times)^2$*-orbits map to the faces of the polygon.*

The Moment Polytope

For example, $h^{\alpha\bar{\beta}} = \mathrm{diag}(1, 1, \ldots, 1)$:



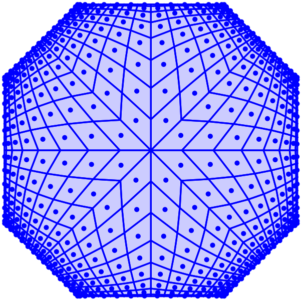
$(\mathbb{P}^2, -K)$



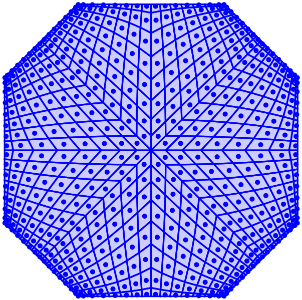
$\left(Bl(\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2^2,\ -3K - \sum E_i\right)$

- One coordinate patch for each vertex
- Quadrangles are images of constant-sized squares in inhomogeneous coordinates.

Different Moment Maps



$h^{\alpha\bar{\alpha}} = (1, 1, \ldots, 1)$



Balanced $h^{\alpha\bar{\alpha}}$

- Different metrics in the same Kähler class $[\omega]$
- Toric variety = $U(1)^2$-fibration over the moment polytope

Volume Form and Integration

- Moment map depends only on absolute value $\xi_i = \operatorname{Re} \ln z_i$ of homogeneous coordinates z_i
- In ξ_i coordinates, the moment map is $\mu(z) = \frac{\partial K}{\partial \xi_i} \stackrel{\text{def}}{=} x_i$
- Legrendre dual is the symplectic potential

$$u(x) = \sum x_i \xi_i - K(\xi)$$

- Metric is $d^2 s = u_{,ij}\, dx^i dx^j + u^{,ij}\, d\varphi_i d\varphi_j$
- Integration is easy:

$$\int f(x)\, \mathrm{dVol}_4 = \int d^2x \int d^2\varphi\, f(x) \sqrt{\det g} = \int f(x)\, d^2x$$

Metrics on Line Bundles

Homogeneous polynomials s_α	$\longleftrightarrow$	Sections of a line bundle $\mathcal{L}$

- The matrix $h^{\alpha\bar{\beta}}$ defines an inner product on sections $\{s_\alpha\}$ via

$$(\sigma,\tau)(z) = \frac{\sigma(z)\bar{\tau}(\bar{z})}{\sum h^{\alpha\bar{\beta}} s_\alpha(z)\bar{s}_{\bar{\beta}}(\bar{z})} \qquad \in C^\infty(X,\mathbb{C})$$

- Metric on the space of sections $H^0\big(X,\mathcal{L}\big)$

$$\big\langle \sigma,\tau\big\rangle = \int_X \big(\sigma,\tau\big)(z,\bar{z})\,\mathrm{dVol} \qquad \in \mathbb{C}$$

Balanced Metrics

$h^{\alpha\bar{\beta}}$ is balanced if the matrices representing the metrics coincide, that is:

$$\Big(\big\langle s_\alpha, s_\beta \big\rangle\Big)_{1\leq\alpha,\bar{\beta}\leq N} = h^{-1}$$

Theorem (Donaldson, ...)

Assume that the balanced metric h_k exists for each $\mathcal{L}^k$, $k \geq k_0$. Then the sequence of metrics

$$\omega_k = \partial\bar{\partial} \ln \sum h_k^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}}$$

converges to an extremal metric as $k \to \infty$

T-Operator

Donaldson's T-operator:

$$T(h)_{\alpha\bar{\beta}} = \langle s_\alpha, s_\beta \rangle = \int_X \frac{s_\alpha \bar{s}_{\bar{\beta}}}{\sum h^{\alpha\bar{\beta}} s_\alpha(x) \bar{s}_{\bar{\beta}}(\bar{x})} \, \mathrm{dVol}$$

The balanced condition is then $h_{\mathrm{bal}}^{-1} = T(h_{\mathrm{bal}})$.

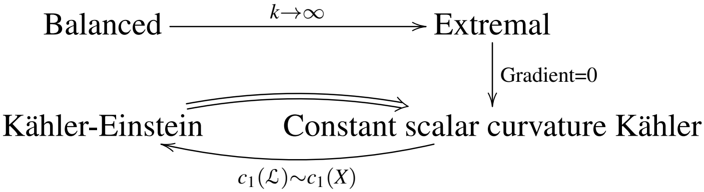
Theorem (Donaldson)

The sequence converges $h_{n+1} = T(h_n)^{-1} \longrightarrow h_{bal}$ if the balanced metric exists.

Metrics Overview

Definition (Extremal Metric)

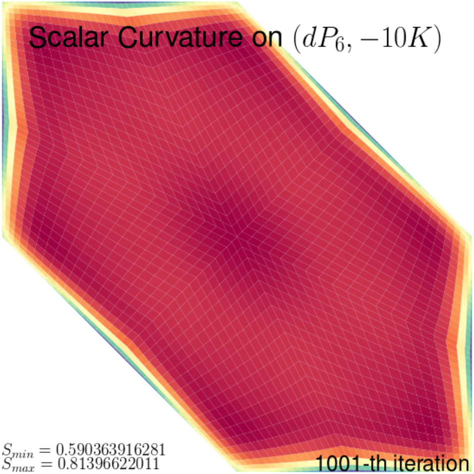
An extremal metric as a Kähler metric whose scalar curvature has a holomorphic gradient vector field. Critical points of $\int R^2$.



- Problems with existence only for $c_1(X) > 0$ (e.g. toric Fano)
- On a Calabi-Yau manifold, the sequence of balanced metrics
  - always exists, and
  - converges towards the Calabi-Yau metric

Kähler Einstein Metric on dP_6

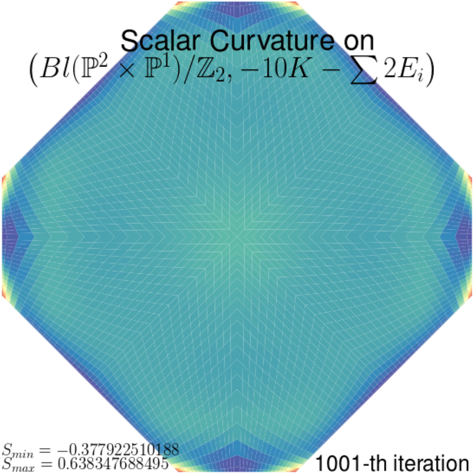
- $dP_6 = \mathbb{P}^2$ blown up at 3 points
- $c_1(\mathcal{L}) \sim c_1(X) \Leftrightarrow$ all 6 rigid $\mathbb{P}^1$ are of same size
- Numerically well-studied:
- Ricci flow Doran-Kantor-Headrick-Herzog-Wiseman
- T-iteration Bunch-Donaldson
- Improved T-iterations Keller



Click for animation

Constant Scalar Curvature Metrics

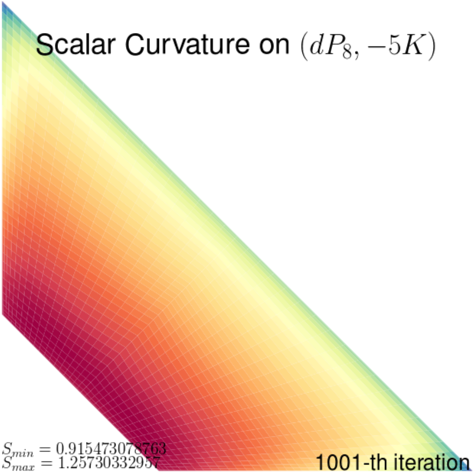
- $\big((\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2$ with 4 orbifold points blown up.
- $\mathcal{L} = -10K - \sum 2E_i$ not proportional to canonical class.



Click for animation

Extremal Metrics that are not cscK

- Scalar curvature is affine function on moment polytope.
- For example: $\mathbb{P}^2$ blown up at a point = dP_8
- Analytic solution Calabi
- Naive T-iteration drives gauge mode Bunch-Donaldson



Click for animation

Monge-Ampère Equation

Pick a reference Kähler metric g_0 with Kähler form ω_0. Kähler forms ω in the same Kähler class can be parametrized as

$$\omega = \omega_0 + i\partial\bar{\partial}\varphi$$

If ω is Kähler-Einstein, $\mathrm{Ric}(g) = \omega$, then

$$[\mathrm{Ric}(g)] = [\omega] = [\omega_0] \quad \Leftrightarrow \quad i\partial\bar{\partial}h_0 = \mathrm{Ric}(g) - \omega_0 \qquad (1)$$

for some function h_0.

Monge-Ampère equation for Kähler-Einstein

The PDE for the Kähler-Einstein metric $g(\varphi)$ is

$$(\omega_0 + i\partial\bar{\partial}\varphi)^d = e^h \omega_0^d$$

Kähler-Ricci Solitons

Definition (KRS)

A Kähler-Ricci soliton is a Kähler metric and holomorphic vector field satisfying

$$\mathrm{Ric}(g) = \omega + L_X \omega$$

- The vector field X defines a function θ_X by $i\bar{\partial}\theta_X = i_X(\omega)$
- The Monge-Ampère equation is modified to

$$(\omega_0 + i\partial\bar{\partial}\varphi)^d = e^{h - \theta_X - X(\varphi)} \omega_0^d$$

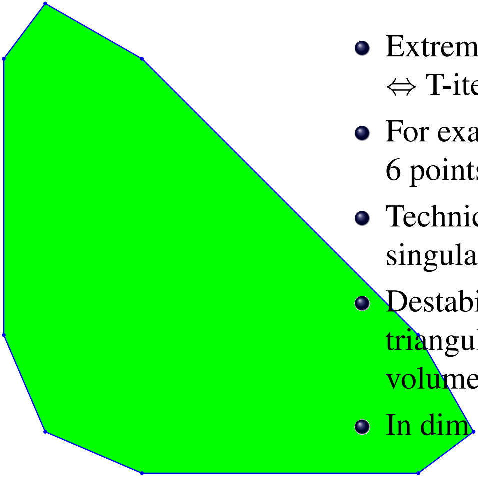
- Expect that the modified T-operator

$$\tilde{T}(h)_{\alpha\bar{\beta}} = \int_X \frac{s_\alpha \bar{s}_{\bar{\beta}}}{\sum h^{\alpha\bar{\beta}} s_\alpha(x) \bar{s}_{\bar{\beta}}(\bar{x})} e^{-\theta_X - X(\varphi)} \,\mathrm{dVol}$$

converges to KRS

Keller

K-Stablity and non-Stability



- Extremal metrics can fail to exist $\Leftrightarrow$ T-iteration does not converge
- For example, this blowup of $\mathbb{P}^2$ at 6 points. Donaldson
- Technical issue: Need to deal with singular toric variety [TODO]
- Destabilizing configuration $\Leftrightarrow$ triangulation with particular volume-to-boundary volume ratio.
- In dim ≥ 3 not proven.

Calabi-Yau Cones and Sasaki-Einstein

- For example, Calabi-Yau cone over dP_8: no KE metric on base.
- Radial direction / Reeb vector not the naive guess.
- In fact: Reeb vector irregular (orbits not compact)
- Analog: $\mathbb{P}^1 \times \mathbb{P}^1$ with irrational ratio of volumes
- Solution: Use rational approximation for Reeb vector
- Toric variety will be singular, need to deal with this.

Outlook

Todo:

- More metrics on toric varieties.
- Solve Laplacian for bundle-valued differential forms
- Develop program that works for “generic” manifolds