Twisted Lattice Supersymmetry: A Status Report

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Why lattice SUSY ?

- Non-perturbative definition of supersymmetric (gauge) theories - like lattice QCD for QCD
- New tools e.g. strong coupling, Monte Carlo for uncovering non-perturbative physics .. eg. dynamical SUSY breaking
- Exploring AdS/CFT and quantum gravity ?

Lattice SUSY - the problem

- Lattice contains no infinitessimal translations .... SUSY algebra $\{Q, \bar{Q}\} = \gamma.p$ broken at classical level
  Equivalently: Leibniz rule does not hold for difference operators
- Consequence: generically quantum effects generate (many) SUSY violating terms.
- Couplings to relevant SUSY breaking ops must be fine tuned as $a \to 0$

Potential solution: twisted SUSY

- Twisted theory has subalgebra $Q^2 = 0$.
- Action typically takes form $S = Q\Lambda(\mathcal{A}, \psi)$
- Discretization which preserves nilpotent Q will retain exact SUSY.
- Careful choice of Λ can also preserve gauge invariance and avoid fermion doubling ...
- Great deal of work in this direction: Kaplan, Ünsal, Sugino, Tsuchiya, Hanada, Damgaard, Matsuura, Kawamoto, d'Adda, Giedt, Kanamori, Catterall,...
- Constructions discussed here can also be obtained using orbifolding ...

Example

Example: 2D Yang-Mills with $\mathcal{Q}=4$ SUSY:

- Contains 2 fermions λ^i_α transforming under $SO_{\rm Lorentz}(2)\times SO_{\rm flavor}(2)$

  $$\lambda^i_\alpha \rightarrow R_{\alpha\beta}\lambda^j_\beta (F^T)^{ji}$$

  Under diagonal subgroup $R=F$ λ^i_α transforms like matrix - Ψ!
- Natural to decompose on products of γ matrices

  $$\Psi = \eta I + \psi_i\sigma_i + \chi_{12}\sigma_1\sigma_2$$

  Integer spin p-form fields from spinors Twisting!
  Appearance of scalar fermion ... implies scalar SUSY $\mathcal{Q}$ obeying $\mathcal{Q}^2=0$

What about the bosons ?

- Gauge field A_μ is singlet under flavor: remains vector under twisted rotation group $SO(2)' = \mathrm{diag}(SO(2) \times SO(2))$
- Original theory had 2 scalars ϕ^1, ϕ^2 which transformed in vector rep of flavor – become components of vector B_μ under twisted rotations!!
- In fact all bosons in twisted theory get repackaged as complex gauge field $\mathcal{A}_\mu = A_\mu + iB_\mu$

Twisted action and SUSY

Twisted form of action (adjoint fields with AH generators)

$$S = \frac{1}{g^2}\mathcal{Q}\int \mathrm{Tr}\left(\chi_{\mu\nu}\mathcal{F}_{\mu\nu} + \eta[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] - \frac{1}{2}\eta d\right)$$

$$\begin{aligned}
\mathcal{Q}\ \mathcal{A}_\mu &= \psi_\mu \\
\mathcal{Q}\ \psi_\mu &= 0 \\
\mathcal{Q}\ \overline{\mathcal{A}}_\mu &= 0 \\
\mathcal{Q}\ \chi_{\mu\nu} &= -\overline{\mathcal{F}}_{\mu\nu} \\
\mathcal{Q}\ \eta &= d \\
\mathcal{Q}\ d &= 0
\end{aligned}$$

Note: $\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu$, $\overline{\mathcal{D}}_\mu = \partial_\mu + \overline{\mathcal{A}}_\mu$ $\mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]$

Untwisting

$\mathcal{Q}$-variation, integrate d:

$$S = \frac{1}{g^2}\int \mathrm{Tr}\left(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}_\mu\psi_\mu\right)$$

Rewrite as

$$S = \frac{1}{g^2}\int \mathrm{Tr}\left(-F_{\mu\nu}^2 + 2B_\mu D_\nu D_\nu B_\mu - [B_\mu, B_\nu]^2 + L_F\right)$$

where

$$L_F = \begin{pmatrix} \chi_{12} & \frac{\eta}{2} \end{pmatrix} \begin{pmatrix} -D_2 - iB_2 & D_1 + iB_1 \\ D_1 - iB_1 & D_2 - iB_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

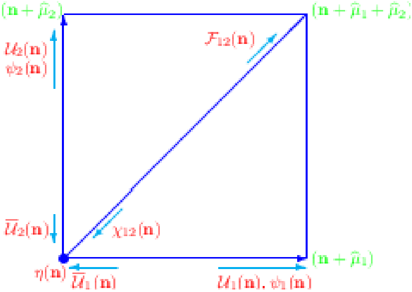
Side comment(s)

- Appearance of $\mathcal{Q}$ reminiscent of BRST gauge fixing. Indeed ...
- vevs of $\mathcal{Q}$ invariant operators independent of coupling and metric. Topological in nature. Useful - sector of lattice theory which can be computed exactly
- However, we do not restrict ourselves to $\mathcal{Q}$-invariant states (vacuum) - just treat twisting as change of variables - more suitable for discretization ...

Discretization

Twisted fields assigned to links of lattice. Under GTs transform like

$$f_p(x) \rightarrow G(x) f_p(x) G^\dagger(x + \mu_p)$$



More lattice construction

- Lattice SUSY transformations same as continuum
- Precise dictionary exists to translate $\mathcal{D}_\mu$ to difference ops. Eg

$$\mathcal{D}_\mu^{(+)}\psi_\nu(x) = \mathcal{U}_\mu(x)\psi_\nu(x+\mu) - \psi_\nu(x)\mathcal{U}_\mu(x+\nu)$$

  Letting $\mathcal{U}_\mu(x) = I + \mathcal{A}_\mu(x)$ yields:

$$\mathcal{D}_\mu^{(+)}\psi_\nu = \psi_\nu(x+\mu) - \psi_\nu(x) + [\mathcal{A}_\mu(x), \psi(x)] + O(a)$$

  Derivative of lattice 1 form yields lattice 2 form !

$$\mathcal{F}_{\mu\nu} = \mathcal{D}_\mu^{+}\mathcal{U}_\nu(x)$$

- Fermion doubling evaded if

$$\mathcal{D}_\mu \overset{curl}{\rightarrow} \mathcal{D}_\mu^{+}$$

$$\mathcal{D}_\mu \overset{div}{\rightarrow} \mathcal{D}_\mu^{-}$$

Twisted $\mathcal{N} = 4$ SYM

- Decompose fields under twisted rotation group $SO(4)' = SO_R(4) \times SO_{rot}(4)$
- Compactly expressed as dimensional reduction of 5D theory
  - 16 fermions: $\Psi = (\eta, \psi_m, \chi_{mn}), m, n = 1 \ldots 5$
  - 10 bosons as 5 complex gauge fields $\mathcal{A}_m, m = 1 \ldots 5$
- Twisted scalar SUSY acts as for 2d YM
- Action $S = \mathcal{Q} \int \left(\chi_{ab} F_{ab} + \eta[\overline{\mathcal{D}}_a, \mathcal{D}_a] - 1/2\eta d\right) + S_{\rm closed}$
- $S_{\rm closed} = \frac{1}{8} \int \epsilon_{abcde} \chi_{ab} \overline{\mathcal{D}}_c \chi_{de}$

(Almost) same as 2D example !

Lattice $\mathcal{N}=4$ theory

- Five complex gauge fields require a lattice with 5 (linearly dep) basis vectors - A_4^* lattice (vectors from center of hypertetrahedron to vertices)
  - $\mathcal{U}_a \quad x \to x+\mu_a$
  - $\eta \quad x \to x$
  - $\psi_a \quad x \to x+\mu_a$
  - $\chi_{ab} \quad x+\mu_a+\mu_b \to x$
- All fields transform as link objects eg: $\psi_a \to G(x)\,\psi_a\,G^\dagger(x+\mu_a)$
- Single exact lattice SUSY $\mathcal{Q}^2=0$
- Prescription for derivatives same as for 2d YM

Lattice Bianchi

- Supersymmetry of $\mathcal{Q}$-exact piece trivially true (as for 2d)
- Remarkably discretization of $\mathcal{Q}$-closed piece also invariant since

$$\epsilon_{abcde}\mathcal{D}_a^{(+)}\mathcal{F}_{bc} = 0$$

Bottom line

- Twisting process exposes nilpotent supersymmetry and induces change of variables $(A_\mu, \phi_i, \lambda^i) \to (\mathcal{A}_\mu, \eta, \psi_\mu, \chi_{\mu\nu})$
- Some SUSY can be preserved in lattice theory e.g. $\mathcal{N} = 4$ YM.
- Twisted lattice theories perserve gauge invariance, $\mathcal{Q}$ and are free of doublers (twisted fermions=staggered fermions)
- Completely local. Bosonic action real positive semidefinite. Fermions can be integrated out - $\mathrm{Pf}(M(\mathcal{U}))$. Simulate using standard algs from lattice QCD
- (Hope is) exact SUSY strongly constrains possible fine tunings.

Remaining problems

- Crucial for continuum interpretation that $\mathcal{U}_a = I + \mathcal{A}_a + \ldots$ as lattice spacing $a \to 0$.
    - Interpret I as arising as vev of imaginary part of trace mode of gauge link (need $U(N)$ gauge group).
    - To ensure this add gauge invariant potential (for scalars in untwisted theory) ... Breaks SUSY ...
- Does the model have a sign problem ? If so Monte Carlo ineffective ...
- Fine tuning ? We maintain 1 out 16 supersymmetries; are the others restored automatically as $a \to 0$?

Fixing the vev

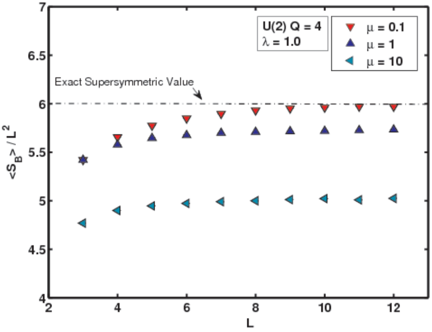
- Add $\Delta S = \mu^2 \sum_x \left[\frac{1}{N}\mathrm{Tr}\left(\mathcal{U}_a^\dagger(x)\mathcal{U}_a(x)\right) - 1\right]^2$
- Writing $\mathcal{U}_a(x) = H_a(x)u_a(x)$ and expanding $H_a = I + h_a$ find

$$\Delta S \sim \mu^2 \sum_x (2h_a^0(x) + h_a^A(x)h_a^A(x))^2$$

- Thus if $\mu^2 = \mu^2 a^2$ held fixed as $a \to 0$ fluctuations in (scalar) trace mode h^0 frozen while traceless modes feel quartic potential
- Send $\mu^2 \to 0$ after continuum limit to restore SUSY

Restoration of $\mathcal{Q}$-symmetry

$\mathcal{Q}=4$ model. $U(2)$ gauge group. Dimensionless 't Hooft coupling $\lambda\beta^2 = 1.0$



Sign problem ...

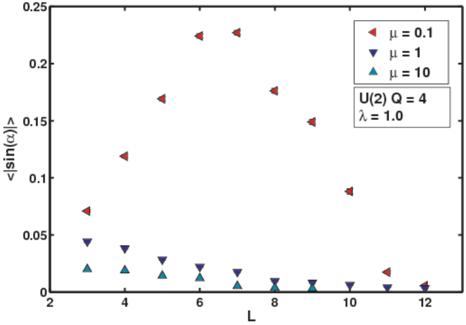
- After integration over twisted fermions - $\mathrm{Pf}(M(\mathcal{U}))$. In general $\mathrm{Pf} = |\mathrm{Pf}(\mathcal{U})|e^{i\alpha(\mathcal{U})}$
- Monte Carlo requires positive definite measure. Can reweight:

$$< O(\mathcal{U}) >= \frac{< O(\mathcal{U})e^{i\alpha(\mathcal{U})} >_{\mathrm{pq}}}{< e^{i\alpha(\mathcal{U})} >_{\mathrm{pq}}}$$

- But if fluctuations in α large statistical error in such measurement grows exponentially with volume - impossible sign problem !!
- Thus, crucial for practical purposes to know if twisted susy lattices suffer from this ...

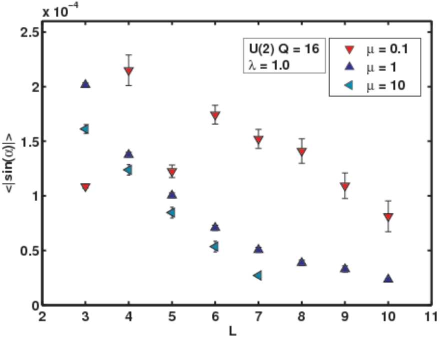
Pfaffian phase $\mathcal{Q} = 4$ model in 2d

$U(2)$, $\lambda\beta^2 = 1.0$



Pfaffian phase $\mathcal{Q} = 16$ model in 2d

$U(2)$, $\lambda\beta^2 = 1.0$



Theoretical understanding ..

- Surprising. Certainly Pf is complex on generic backgrounds ...
- Notice, however that $e^{i\alpha(U)}$ is related to a topological object - Z – Witten index.

$$< e^{i\alpha(\mathcal{U})} >= Z/Z_{\rm phase\ quenched}$$

- Furthermore $\mathcal{Q}$-invariance means that Z can be computed exactly at 1-loop.
- Find $Z = Z_{\rm phase\ quenched}$
- This implies $< e^{i\alpha} >= 1$ and hence $\alpha \sim 0$ as observed. Of course our simulations use a SUSY breaking potential so this argument is only formal ...

Fine tuning ...

To tackle this question must understand renormalization lattice
Lattice symmetries strongly constrain possible counterterms

- Gauge invariance
- $\mathcal{Q}$-symmetry.
- Point group symmetry - eg. S^5 for A_4^* - subgroup of $SO(4)'$
- Exact fermionic shift symmetry $\eta \to \eta + \epsilon I$

Conclusion:

- S^5 PGS guarantees twisted $SO(4)'$ restored as $a \to 0$
- Power counting: only relevant ops correspond to 4 $\mathcal{Q}$-invariant terms already present in classical lattice action!

$$S = \mathcal{Q} \sum \alpha_1 \chi_{ab} F_{ab} + \alpha_2 \eta [\overline{\mathcal{D}}_a, \mathcal{D}] + \frac{\alpha_3}{2} \eta d + \alpha_4 L_{\text{closed}}$$

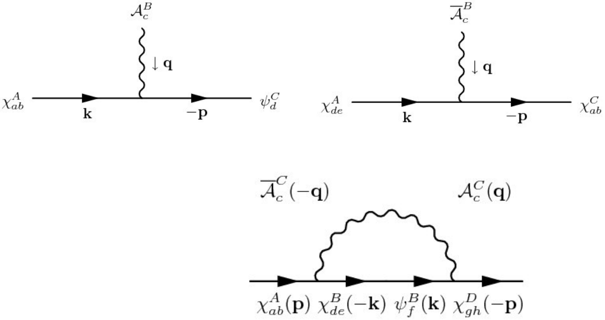
- $\mathcal{Q}$-symmetry ensures that $\Gamma_{\text{eff}}(\mathcal{A}^{\text{classical}}) = 0$ to all orders in p. theory ...

Ingredients for perturbation theory

Lattice rules for A_4^* lattice (Feynman gauge):

- Boson propagator $<\overline{\mathcal{A}}_a^C(k)\mathcal{A}_b^D(-k)>=\frac{1}{\hat{k}^2}\delta_{ab}\delta^{CD}$ with $\hat{k}^2=4\sum_a \sin^2(k_a/2)$
- Fermion propagator $M_{\mathrm{KD}}^{-1}(k)=\frac{1}{\hat{k}^2}M_{\mathrm{KD}}(k)$ with $M(k)$ a 16×16 block matrix acting on (η,ψ_a,χ_{ab})
- Vertices: $\psi\eta$, $\psi\chi$ and $\chi\chi$.
- Four one loop Feymann graphs needed to renormalize three fermion propagators. Yields 3 α's.
- One additional bosonic propagator for remaining α.

Example: chi-chi propagator



- $\Sigma_i(0)=0; \quad \frac{\partial \Sigma_i}{\partial p} = Ag^2 \ln \mu a + \text{finite} + \mathcal{O}(a)$
- Implies $\sqrt{\alpha_i} = Z_i = 1 + \frac{A}{2} g^2 \ln \mu a + \ldots$

Why so simple ?

- One loop lattice diagrams in 1-1 correspondence with continuum diagrams and have only log divergences.
- Divergences come from region near $pa \sim 0$ where lattice propagators and vertices approach continuum expressions
- Thus (divergent part of) 1-loop lattice diagram - same as continuum !
- In continuum twisted theory equivalent to usual - has full supersymmetry. Requires common wavefunction renormalization all fermions/bosons – log divergences must be same for all α_i ..
- Similar args indicate that $\boxed{\beta_{\rm lattice}(g) = 0}$ at 1-loop

Sketch of $\Gamma_{\rm eff} = 0$

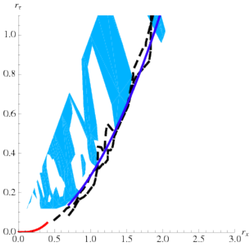
- Classical (boson) vacua correspond to constant commuting complex matrices
- Expand to quadratic order about generic vacuum $\mathcal{U}_b(x) = I + \mathcal{A}_b^{\rm classical} + a_b(x)$. Integrate
- Bosons: $\det^{-5}\left(\overline{\mathcal{D}}_a^{(-)}\mathcal{D}_a^{(+)}\right)$
- Ghosts+Fermions: $\det\left(\overline{\mathcal{D}}_a^{(-)}\mathcal{D}_a^{(+)}\right) + \left(Pf(M_F) \stackrel{Maple}{=} \det^4\left(\overline{\mathcal{D}}_a^{(-)}\mathcal{D}_a^{(+)}\right)\right)$
- Thus $Z_{\rm pbc} = 1$ at 1-loop. $\mathcal{Q}$-exact structure – result good to all orders! Exact quantum moduli space
- Witten index: all states cancel except vacua. Counting indep of g.

Applications: holography

Eg black string in type II SUGRA – dual to large N $\mathcal{N} = 4$ YM on 2D torus (sizes r_x and r_τ)

- Depending on r_x, r_τ black string solution may become less stable than black hole. Supergravity analysis predicts $r_\tau < c r_x^2, \; r_x, r_\tau \to \infty$ (c unknown)
  Gregory-LaFlamme transition
- In dual gauge theory see thermal phase transition associated with breaking of center symmetry - order parameter spatial Polyakov line.

Black hole-black string phase transition



Boundary between confined/deconfined phases corresponds to $\frac{1}{N}|P_s| = 0.5$

Good agreement with supergravity - blue curve - $r_\tau = cr_x^2$ with fitted $\boxed{c \sim 3.5}$.

Good agreement with high T dim reduction - red curve

Conclusions

- Lattice theories with exact SUSY possible. In particular $\mathcal{N} = 4$ SYM.
- Key is to discretize topologically twisted form of continuum theory.
- Exact SUSY enough to ensure moduli space survives to all orders and no fine tuning at 1-loop
- Subtleties remain: necessary to add gauge invariant potential to freeze trace mode and regularize flat directions.
- Possible sign problem in $\mathcal{N} = 4$. Numerically seems safe ...
- Exploration of phase diagram underway...
- Explore theory at strong coupling, AdS/CFT etc

Lattice action

$$S = \beta(S_{\rm exact} + S_{\rm closed})$$

$$\begin{aligned} S_{\rm exact} &= \sum_{\mathbf{x}} {\rm Tr} \left(\mathcal{F}^{\dagger}_{ab}\mathcal{F}_{ab} + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_{a}\mathcal{U}_a\right)^2 \right. \\ &- \left. \chi_{ab}\mathcal{D}^{(+)}_{[a}\psi_{b]} - \eta\overline{\mathcal{D}}^{(-)}_{a}\psi_a \right) \\ S_{\rm closed} &= -\frac{1}{2}\sum_{\mathbf{x}} {\rm Tr}\epsilon_{abcde}\chi_{de}(\mathbf{x}+\mu_{\mathbf{a}}+\mu_{\mathbf{b}}+\mu_{\mathbf{c}})\overline{\mathcal{D}}^{(-)}_{\mathbf{c}}\chi(\mathbf{x}+\mu_{\mathbf{c}}) \end{aligned}$$