Quantizing Hořava-Lifshitz Gravity
via
Causal Dynamical Triangulations

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Talk at the Kavli Institute for Theoretical Physics

Work in collaboration with Christian Anderson, Steven Carlip, Petr Hořava, Rajesh Kommu, and Patrick Zulkowski

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Plan

Causal Dynamical Triangulations

Constructing a Quantum Theory of Gravity

Aim Evaluate the path integral for the gravitational action $S[g_{\mu\nu}]$ subject to relevant boundary conditions

$$Z = \int \mathcal{D}[g_{\mu\nu}]\, e^{iS[g_{\mu\nu}]}$$

Means Introduce an appropriate discretization of these geometries and restrict the set of geometries entering the path integral

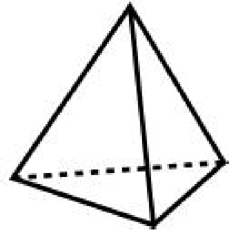
- Approximate continuous spacetimes as piecewise Minkowski simplicial manifolds
- Restriction to spacetimes admitting a global foliation by spacelike hypersurfaces

Hope Well-defined continuum limit emerges at a second order phase transition having correct semiclassical properties on large scales

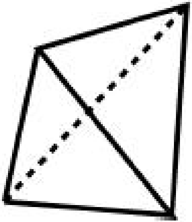
Discretization by Dynamical Triangulations

Construct triangulated spacetimes by appropriately gluing together Minkowskian $(d+1)$-simplices

Simplices in $2+1$ dimensions

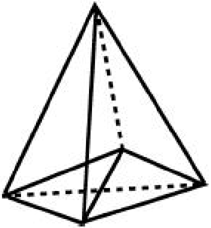


$(3,1)$-simplex

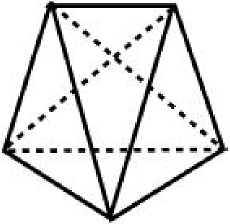


$(2,2)$-simplex

Simplices in $3+1$ dimensions



$(4,1)$-simplex



$(3,2)$-simplex

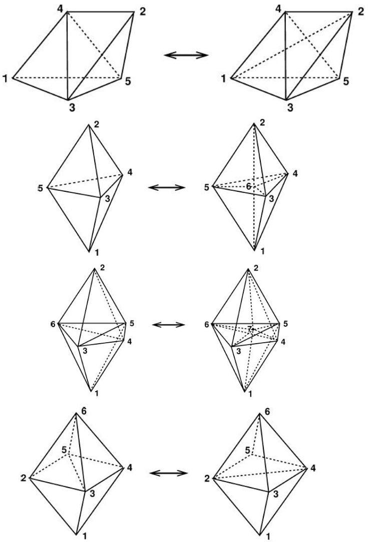
Spacelike edge length $L_{SL}^2 = a^2$
Timelike edge length $L_{TL}^2 = -\eta a^2$

Dynamically move from one triangulated spacetime to another by appropriately removing or inserting Minkowskian $(d+1)$-simplices

[Ambjørn *et al* 2001]

Pachner Moves

$2+1$ dimensions



Pachner Moves

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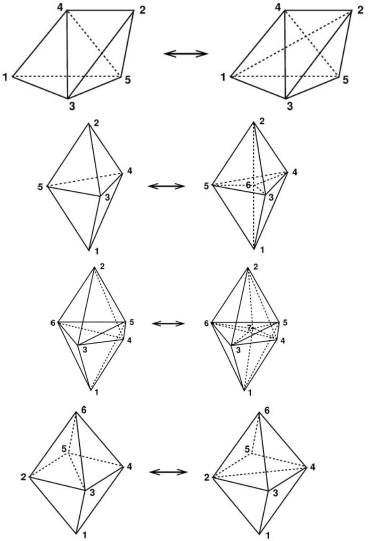

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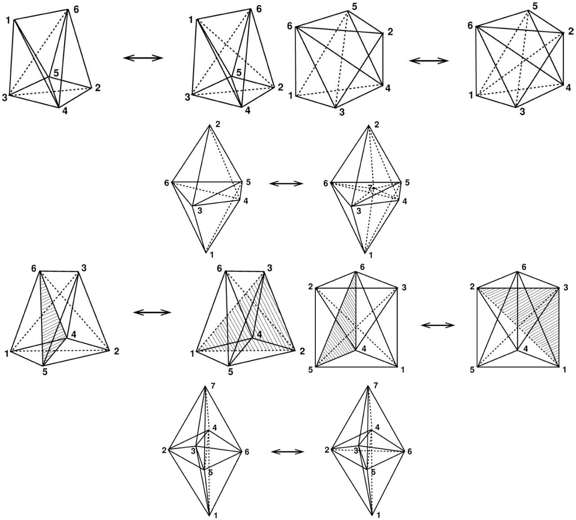

[Ambjørn *et al* 2001]

Restricting to Causal Spacetimes

Only spacetimes admitting a global foliation by spacelike hypersurfaces allowed to enter the path integral

- Interpreted as a sort of causality condition since spatial topology change is prevented
- Allows for Wick rotation to Euclidean signature required for numerical simulations while retaining Lorentzian structure of spacetimes

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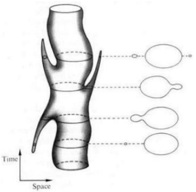
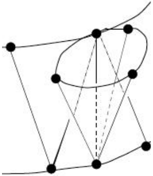
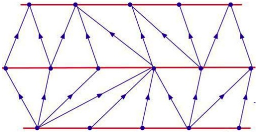




Implementation in dynamically triangulated spacetimes

- Triangulate each leaf of the foliation with regular spacelike d-simplices
- Connect adjacent leaves of the foliation with timelike edges so that only allowed $(d+1)$-simplices formed
- Pachner moves designed to respect foliation



Motivating the Causality Condition

1. Monte Carlo simulations of the Euclidean path integral for gravity
   - Quantum Regge calculus: Two phases of geometry present—smooth and rough—but uncorroborated evidence for a second order phase transition
   - Dynamical triangulations: Two phases of geometry present—crumpled and polymeric—but convincing evidence for a first order phase transition

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2. Relationship between Euclidean and Lorentzian dynamical triangulations in $1+1$ dimensions
   - Euclidean and Lorentzian theories in different universality classes
     - Continuum limit of Euclidean theory is Liouville gravity
     - Continuum limit of Lorentzian theory is proper time gauge canonically quantized gravity
   - Euclidean theory obtained from Lorentzian theory upon integrating in spatial topology change
   - Lorentzian theory obtained from Euclidean theory upon integrating out spatial topology change

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3. Promising results from causal dynamical triangulations

Discrete Action for Path Integral

Aim to evaluate

$$Z = \int \mathcal{D}[g_{\mu\nu}]\, e^{iS[g_{\mu\nu}]} \quad \text{for} \quad S[g_{\mu\nu}] = \frac{1}{16\pi G} \int_{\mathcal{M}} \mathrm{d}^{d+1}x\, \sqrt{-g}\, (R - 2\Lambda)$$

Use Regge calculus

Prescription for R term

$$\int_{\mathcal{M}} \mathrm{d}^{d+1}x\, \sqrt{-g} R \longrightarrow 2 \sum_{h \in \mathcal{T}} A_h \delta_h$$

- Local curvature carried by $(d+1-2)$-dimensional simplices ("hinges" or "bones" h) of the triangulation $\mathcal{T}$
- Amount of local curvature proportional to deficit angle δ_h about the hinge h and to area A_h of the hinge

Prescription for Λ term

$$\int_{\mathcal{M}} \mathrm{d}^{d+1}x\, \sqrt{-g}\, 2\Lambda \longrightarrow 2\Lambda \sum_{s \in \mathcal{T}} V_s.$$

- Total spacetime volume is simply the sum of the $(d+1)$-volumes V_s of all $(d+1)$-simplices s in the triangulation $\mathcal{T}$

Regge Calculus for Causal Dynamical Triangulations

$2 + 1$ dimensions

$$\int_{\mathcal{M}} \mathrm{d}^3x \, \sqrt{-g} \, (R - 2\Lambda) \longrightarrow 2 \sum_{e \in \mathcal{T}} A_e \delta_e - \Lambda \sum_{s \in \mathcal{T}} V_s$$

Sum $\sum_{e \in \mathcal{T}}$ ranges over all spacelike and timelike edges e in the triangulation $\mathcal{T}$

$3 + 1$ dimensions

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Sum $\sum_{f \in \mathcal{T}}$ ranges over all spacelike and timelike faces e in the triangulation $\mathcal{T}$

Numerical Implementation

1. Wick rotation consists in $\eta \to -\eta$ in the lower half complex plane

$$Z_{CDT} = \sum_{\mathcal{T}} \frac{1}{C(\mathcal{T})} e^{iS_{CDT}[\mathcal{T}]} \quad \longrightarrow \quad Z_{CDT}^{(E)} = \sum_{\mathcal{T}} \frac{1}{C(\mathcal{T})} e^{-S_{CDT}^{(E)}[\mathcal{T}]}$$

2. Simplify discrete Euclidean action with Dehn-Somerville relations
   - In $2+1$ dimensions

$$S_{CDT}^{(E)} = -k_0 N_0 + k_3 N_3$$

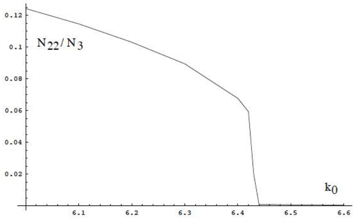
   - In $3+1$ dimensions

$$S_{CDT}^{(E)} = -(\kappa_0 + 6\Delta) N_0 + 2\kappa_4 (N_4^{(3,2)} + N_4^{(4,1)}) + 2\Delta (N_4^{(3,2)} + 2N_4^{(4,1)})$$

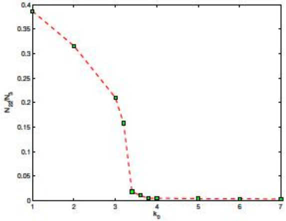
3. Run Monte Carlo simulation of the partition function $Z_{CDT}^{(E)}$
   - Updates with Pachner moves based on weighting by discrete Euclidean action
   - Approximately fix total number N_{d+1} of $(d+1)$-simplices with damping term $\epsilon |N_{d+1} - N_{d+1}^{(target)}|$ for $\epsilon \ll 1$
   - Tune coupling constant k_3 or κ_4 to critical value
   - Collect ensembles of representative spacetimes after simulation has thermalized

Phase Structure

$2+1$ dimensions



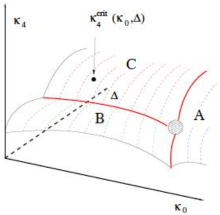
[Ambjørn *et al* 2001]



[Kommu 2011]

First order A-C phase transition

$3+1$ dimensions



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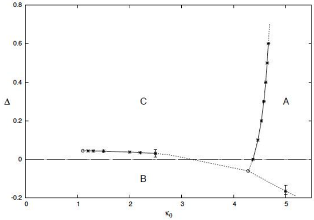


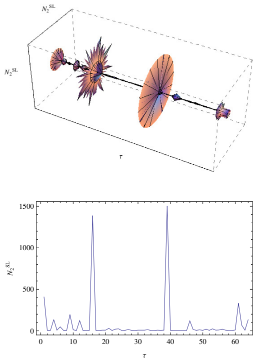
[Ambjørn *et al* 2010]

First order A-C phase transition

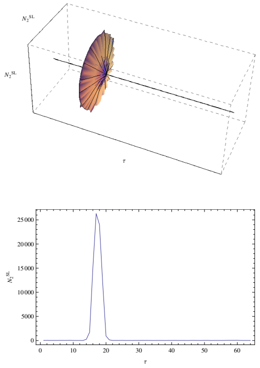
Second order B-C phase transition

Depictions of Representative Spacetimes

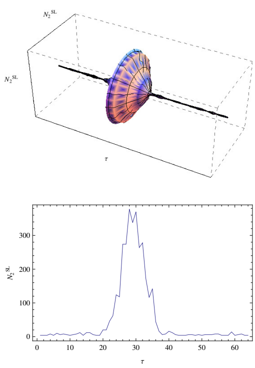
Discrete spatial volume as a function of discrete time



Phase A
Present in $2+1$ and $3+1$ dimensions



Phase B
Present in $3+1$ dimensions only



Phase C
Present in $2+1$ and $3+1$ dimensions

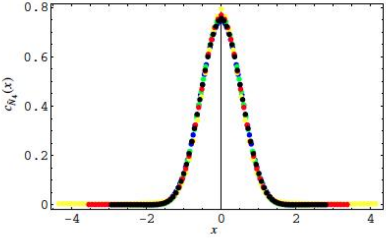
Classical Nature of Phase C: Spatiotemporal Scaling

Behavior of the discrete spatial 3-volume–spatial 3-volume correlator

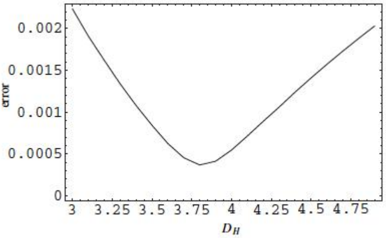
$$\mathcal{C}_{N_4}(x) = \langle N_3(t) N_3(t+x) \rangle$$

under spatiotemporal scalings

$$\tau = \frac{t}{N_4^{1/D_H}} \quad \text{and} \quad n_3(\tau) = \frac{N_3(t)}{N_4^{1-1/D_H}}$$



$\mathcal{C}_{N_4}(x)$ for several values of N_4 scaled naively with $D_H = 4$



Best estimate for D_H from overlap of scaled $\mathcal{C}_{N_4}(x)$ for several values of N_4

[Ambjørn *et al* 2005]

Classical Nature of Phase C: Minisuperspace Model

Effective action for discrete spatial 3-volume $N_3(t)$ determined primarily from measurements of phase C spacetimes

- Kinetic term determined from distributions of differences of adjacent spatial volumes
- Potential term deduced from expected scaling behavior of the spatial volume given this distribution

$$S_D[N_3(t)] = k_1 \sum_{t=1}^{N} \left\{ \frac{\left[N_3(t+1) - N_3(t)\right]^2}{N_3(t)} + k_2 N_3^{1/3}(t) \right\}$$

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Minisuperspace model

- Euclidean metric $\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + a^2(t)\mathrm{d}\Omega_3^2$
- Classical action in terms of the continuous spatial 3-volume $V_3(t) = 2\pi^2 a^3(t)$

$$S_C[V_3(t)] = \frac{1}{24\pi G} \int \mathrm{d}t \sqrt{g_{tt}} \left[\frac{g^{tt}\dot{V}_3^2(t)}{V_3(t)} + 9\left(2\pi^2\right)^{2/3} V_3^{1/3}(t) \right]$$

[Ambjørn *et al* 2005], [Ambjørn *et al* 2008]

Classical Nature of Phase C: Minisuperspace Solution

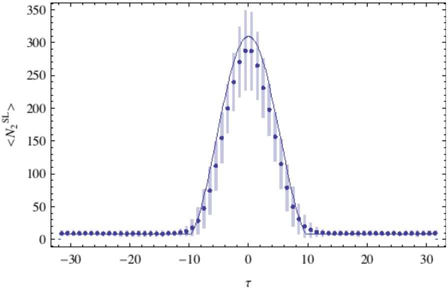
$2+1$ dimensions

Continuous de Sitter solution

$$V_2(t) = \frac{2}{\pi}\frac{V_3}{l_{dS}}\cos^2\left(\frac{t}{l_{dS}}\right)$$

Discretized de Sitter solution

$$N_2(\tau) = \frac{2}{\pi}\frac{N_3^{(1,3)}}{\tilde{s}_0\left(N_3^{(1,3)}\right)^{1/3}}\cos^2\left[\frac{\tau}{\tilde{s}_0\left(N_3^{(1,3)}\right)^{1/3}}\right]$$



[JHC, Anderson *et al* 2011]

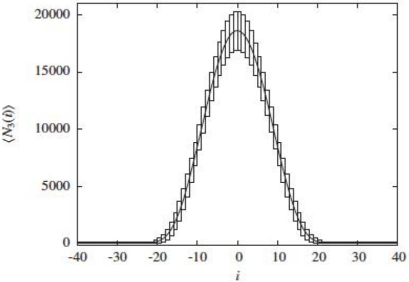
$3+1$ dimensions

Continuous de Sitter solution

$$V_3(t) = \frac{3}{4}\frac{V_4}{l_{dS}}\cos^3\left(\frac{t}{l_{dS}}\right)$$

Discretized de Sitter solution

$$N_3(\tau) = \frac{3}{8}\frac{N_4^{(1,4)}}{\tilde{s}_0\left(N_4^{(1,4)}\right)^{1/4}}\cos^3\left[\frac{\tau}{\tilde{s}_0\left(N_4^{(1,4)}\right)^{1/4}}\right]$$



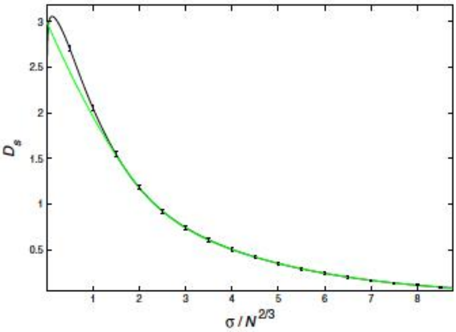
[Ambjørn *et al* 2005]

Classical Nature of Phase C: Spectral Dimension

Spectral dimension is the effective dimensionality of spacetime as measured by a random walker

$$d_s(\sigma) = -2\frac{\mathrm{d}\ln P_r(\sigma)}{\mathrm{d}\ln\sigma}$$

$P_r(\sigma)$ is the return probability as a function of diffusion time σ



- Black points: Measured scaled spectral dimension of phase C in $2+1$ dimensions
- Green curve: Fit of spectral dimension computed from $(2+1)$-dimensional minisuperspace model

[Benedetti and Henson 2009]

Semiclassical Nature of Phase C: Minisuperspace Model

Fluctuations in discrete spatial 3-volume well fit by minisuperspace model at second order

- Covariance of deviations of discrete spatial 3-volume from the mean

$$C(\tau,\zeta) = \langle \delta N_3(\tau)\delta N_3(\zeta)\rangle \quad \text{with} \quad \delta N_3(\tau) = N_3(\tau) - \langle N_3(\tau)\rangle$$

- Expansion of the discrete minisuperspace action to second order

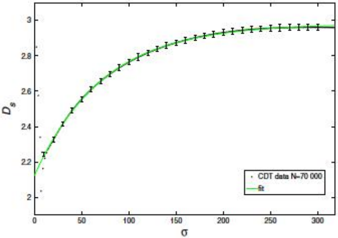
$$S_D\left[\langle N_3(\tau)\rangle + \delta N_3(\tau)\right] = S_D\left[\langle N_3(\tau)\rangle\right] + \frac{1}{2}\sum_{\tau=1}^{T}\sum_{\zeta=1}^{T} P(\tau,\zeta)\delta N_3(\tau)\delta N_3(\zeta)\cdots$$

- To a good degree of accuracy

$$C(\tau,\zeta) = P^{-1}(\tau,\zeta)$$

Quantum Nature of Phase C: Spectral Dimension

$2+1$ dimensions



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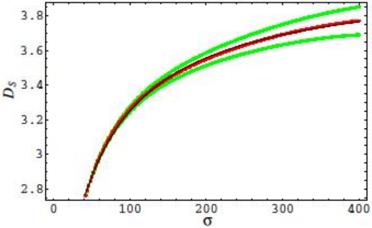

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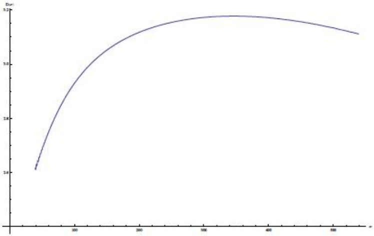

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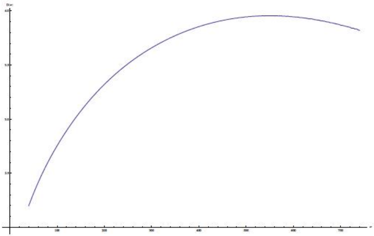

[Kommu 2011]

Hořava-Lifshitz Gravity

Methodology of Hořava-Lifshitz Gravity

Aim Construct a power-counting renormalizable unitary classical theory of gravity

- Power-counting renormalizability requires higher derivative terms
- Unitarity requires that terms contain no more than two time derivatives

Means Abandon manifest spacetime covariance and permit violation of Lorentz invariance

- Metric characterizes geometry of spacetime manifolds carrying a global foliation by spacelike hypersurfaces
- Action only required to be invariant under foliation preserving diffeomorphisms

$$t \longrightarrow \tilde{t} = f(t) \qquad \mathbf{x} \longrightarrow \tilde{\mathbf{x}} = \zeta(t, \mathbf{x})$$

Hope Well-defined quantum theory with renormalization group flow to general relativity in the infrared

Defining Hořava-Lifshitz Gravity

ADM decomposition of the metric adapted to a foliation of spacetime by spacelike hypersurfaces

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + \gamma_{ij} \left(\mathrm{d}x^i + N^i \mathrm{d}t\right) \left(\mathrm{d}x^j + N^j \mathrm{d}t\right)$$

- N is the lapse function
- N^i is the shift vector
- γ_{ij} is the spatial metric tensor

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Most general action

$$S_{HL} = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}t\, \mathrm{d}^d x \sqrt{\gamma} N \left\{K_{ij}K^{ij} - \lambda K^2 - V\left[\gamma_{ij}, N\right]\right\}$$

- $K_{ij} = \frac{1}{N}\left(\partial_t \gamma_{ij} - D_i N_j - D_j N_i\right)$ is the extrinsic curvature tensor
- $K = \gamma^{ij} K_{ij}$ is the trace of the extrinsic curvature tensor
- $V\left[\gamma_{ij}, N\right]$ is a scalar functional of γ_{ij}, N, and their space derivatives up to order $2z = 2d$
- z is the dynamical critical exponent

Phenomenological Viability of Hořava-Lifshitz Gravity

Projectable version of Hořava-Lifshitz gravity

- Defined by restriction on lapse function $N = N(t)$
- Extra propagating scalar degree of freedom
- Constraints
  - Experimental validity of Newton's law on small scales generically requires λ very close to 1
  - Constraints on Cherenkov radiation from scalar mode essentially render theory unviable
  - Strong dynamics could render theory viable but the scale of such dynamics is uncomfortably low

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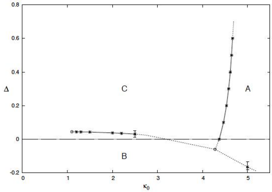
Nonprojectable version of Hořava-Lifshitz gravity

- Defined by lack of restriction on lapse function $N = N(t, \mathbf{x})$
- Extra propagating scalar degree of freedom
- Constraints
  - Reasonable range of coupling values in which theory is viable
  - Strong coupling scale is sufficiently high for perturbative regime to elude limits on violations of Lorentz invariance

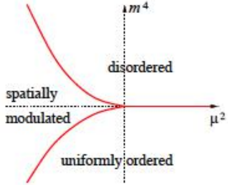
Connections

Connections: Phase Structure

Resemblance of the phase diagram of causal dynamical triangulations to the phase diagram of the Lifshitz scalar field



[Ambjørn *et al* 2010]



[Hořava 2011]

$$S_L = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^d x \left[(\partial_t\phi)^2 - (\nabla^2\phi)^2 - \mu^2\partial_i\phi\,\partial^i\phi - m^4\phi^2 - \lambda\phi^4\right]$$

Identification of respective phases

- Phase A corresponds to the spatially modulated phase
- Phase B corresponds to the disordered phase
- Phase C corresponds to the uniformly ordered

Matching of phase transition structure

- A-C and modulated-ordered phase transitions are first order
- B-C and disordered-ordered phase transitions are second order

Connections: Spectral Dimension

Consistency of the spectral dimension computed in Hořava-Lifshitz gravity with the spectral dimension measured in causal dynamical triangulations

- Generic prediction from Hořava-Lifshitz gravity

$$d_s = 1 + \frac{d}{z}$$

- Expected that z flows from d in the ultraviolet to 1 in the infrared [Hořava 2009]

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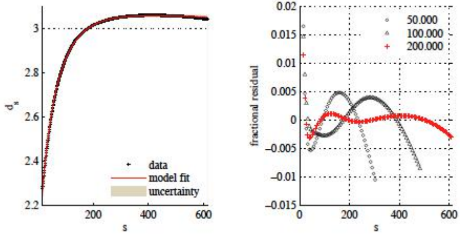
- Expected that z flows from d in the ultraviolet to 1 in the infrared [Hořava 2009]

Fit of the spectral dimension of causal dynamical triangulations on small scales to a dispersion relation for scalar mode of nonprojectable $(2+1)$-dimensional Hořava-Lifshitz gravity



$$\omega^2(k) = \frac{Ak^2(1 + Bk^2 + Ck^4)}{1 + Dk^2}$$

[Sotiriou *et al* 2011]

Connections: Semiclassical Effective Action

Compatibility of solutions of Hořava-Lifshitz gravity with the minisuperspace model fit to the expectation value of geometry in phase C of causal dynamical triangulations

- de Sitter spacetime is a solution of Hořava-Lifshitz gravity [Benedetti and Henson 2009]
- Low energy limit of generic Hořava-Lifshitz minisuperspace model compatible with semiclassical effective action for phase C of causal dynamical triangulations [Ambjørn *et al* 2010]

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Evidence that the semiclassical effective action of causal dynamical triangulations for topology $\mathrm{T}^2 \times \mathcal{S}^1$ has Hořava-Lifshitz-like form [Budd 2011]

Quantizing Hořava-Lifshitz Gravity

Model

$(2+1)$-dimensional projectable Hořava-Lifshitz gravity

Continuum action

$$\begin{aligned} S_{HL}[\mathbf{g}(t,\mathbf{x})] &= \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}t\,\mathrm{d}^2x\sqrt{\gamma(t,\mathbf{x})}N(t)\left[K_{ij}(t,\mathbf{x})K^{ij}(t,\mathbf{x})\right. \\ &\quad \left.-\lambda K^2(t,\mathbf{x}) - \alpha R_2^2(t,\mathbf{x}) + \beta R_2(t,\mathbf{x}) - 2\Lambda\right] \end{aligned}$$

Hamiltonian constraint

$$\begin{aligned} \mathcal{H}_\perp &= \int_\Sigma \mathrm{d}^2x\sqrt{\gamma(t,\mathbf{x})}\left[K_{ij}(t,\mathbf{x})K^{ij}(t,\mathbf{x}) - \lambda K^2(t,\mathbf{x}) + \alpha R_2^2(t,\mathbf{x})\right. \\ &\quad \left.-\beta R_2(t,\mathbf{x}) + 2\Lambda\right] \end{aligned}$$

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Hamiltonian constraint not necessarily enforced

- Lapse function is fixed for any given causal triangulation since all timelike edges have fixed length
- Path integration may or may not dynamically impose the Hamiltonian constraint

Discretization Procedure: Guidelines

1. Use the formalism of causal dynamical triangulations

2. Discrete Hořava-Lifshitz action should reduce to discrete Einstein-Hilbert action when $\lambda = 1$ and $\alpha = 0$
   - Use Gauss-Codazzi equation and discard boundary terms

$$\begin{aligned} S_{HL}[\mathbf{g}(t,\mathbf{x})] \quad = \quad & \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}t\,\mathrm{d}^2x\,\sqrt{-g(t,\mathbf{x})}\,[R(t,\mathbf{x}) - 2\Lambda] \\ & + \frac{1-\lambda}{16\pi G}\int_{\mathcal{M}} \mathrm{d}t\,\mathrm{d}^2x\sqrt{\gamma(t,\mathbf{x})}N(t)K^2(t,\mathbf{x}) \\ & - \frac{\alpha}{16\pi G}\int_{\mathcal{M}} \mathrm{d}t\,\mathrm{d}^2x\sqrt{\gamma(t,\mathbf{x})}N(t)R_2^2(t,\mathbf{x}) \end{aligned}$$

3. Transfer matrix corresponding to the discrete Hořava-Lifshitz action defined on the space of boundary geometries should yield a well-defined Hamiltonian
   - Ensure that the discrete Hořava-Lifshitz action is time-reversal invariant

Discretization Procedure: Volume Sharing

Volume sharing prescription for a squared curvature scalar $\mathscr{R}^2(t,\mathbf{x})$

$$\int_\Sigma \mathrm{d}^d x \sqrt{\gamma(t,\mathbf{x})}\mathscr{R}^2(t,\mathbf{x}) \longrightarrow \sum_{o \in O_\tau(\mathcal{T})} V_o^{(s)} \left(\frac{A_o \delta_o}{V_o^{(s)}}\right)^2$$

- o is the object assigned the curvature
- $O_\tau(\mathcal{T})$ is the set of all objects o on the spacelike hypersurface Σ labelled by discrete time coordinate τ in the triangulation $\mathcal{T}$
- A_o is the appropriate area of the object o
- δ_o is the appropriate deficit angle about the object o
- $V_o^{(s)}$ is the share-volume of the object o, namely the volume of all top-dimensional objects containing o

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Intuition for volume sharing prescription

- Squares of curvature prescriptions are ill-defined as triangulation is refined toward continuum limit
- View curvatures as densities assigned to local volumes

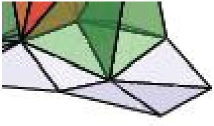
For example $$\int_\mathcal{M} \mathrm{d}^{d+1}x\sqrt{-g}R \longrightarrow 2\sum_{h\in\mathcal{T}} A_h\delta_h = 2\sum_{h\in\mathcal{T}} V_{\mathcal{C}\supset h}\left(\frac{A_h\delta_h}{V_{\mathcal{C}\supset h}}\right)$$

[Hamber and Williams 1984], [Ambjørn *et al* 1993]

Discretization Procedure: R_2^2 Term

Vertices v carry the Ricci curvature R_2 of a 2-dimensional spacelike hypersurface Σ

Volume sharing prescription



[Budd 2011]

$$\int_\Sigma \mathrm{d}^2x \sqrt{\gamma(t,\mathbf{x})} R_2^2(t,\mathbf{x}) \longrightarrow \sum_{v \in V_\tau(\mathcal{T})} V_v^{(s)} \left(\frac{A_v \delta_v}{V_v^{(s)}} \right)^2$$

- Deficit angle about the vertex v with $N_\triangle(v)$ incident spacelike triangles

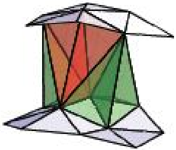
$$\delta_v = 2\pi - \frac{\pi}{3} N_\triangle(v)$$

- Vertex area $A_v = 1$
- Vertex share volume $V_v^{(s)} = \frac{\sqrt{3}}{4} a^2 N_\triangle(v)$

Discretization Procedure: K^2 Term

Spacelike triangles $\triangle$ carry the trace K of the extrinsic curvature of a 2-dimensional spacelike hypersurface Σ

Volume sharing prescription



[Budd 2011]

$$\int_\Sigma \mathrm{d}^2x \sqrt{\gamma(t,\mathbf{x})} K^2(t,\mathbf{x}) \longrightarrow \sum_{\triangle \in T_\tau^{SL}(\mathcal{T})} V_\triangle^{(s)} \left(\frac{A_\triangle \delta_\triangle}{V_\triangle^{(s)}} \right)^2$$

- Past- (Future-) directed deficit angle about the spacelike triangle $\triangle$ with $N_{(2,2)}^{\downarrow(\uparrow)}(\triangle)$ $(2,2)$-simplices in its immediate past (future)

$$\delta_\triangle = \delta_{e_1} + \delta_{e_2} + \delta_{e_3} = 3\pi - 6\theta_L^{(3,1)} - \theta_L^{(2,2)} N_{(2,2)}^{\downarrow(\uparrow)}(\triangle)$$

- Spacelike triangle area $A_\triangle = \frac{\sqrt{3}}{4} a^2$
- Spacelike triangle share volume $V_\triangle^{(s)} = 4V_L^{(3,1)} + V_L^{(2,2)} N_{(2,2)}^{\downarrow(\uparrow)}(\triangle)$

[Hartle and Sorkin 1981]

Causal Dynamical Triangulated Hořava-Lifshitz Gravity

Wick rotated path integral for spacetime topology $\mathcal{S}^2 \times \mathcal{S}^1$

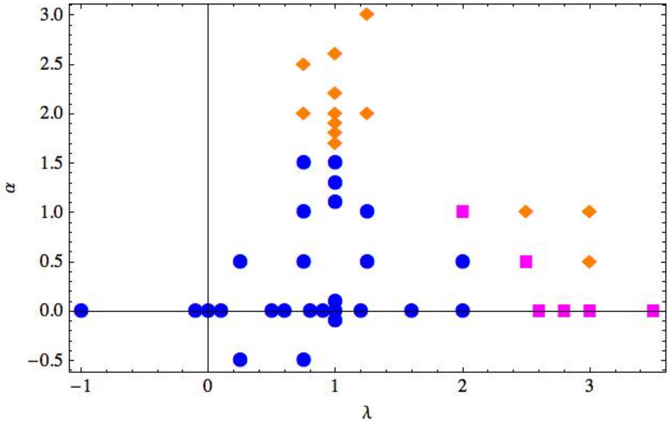
$$Z_{HL}^{(E)} = \sum_{\mathcal{T}} \frac{1}{C(\mathcal{T})} e^{-S_{HL}^{(E)}}$$

$$\begin{aligned}
S_{HL}^{(E)} &= -k_0 N_0 + k_3 N_3 \\
&\quad + \frac{1-\lambda}{16\pi G} \sum_{\tau} \sum_{\triangle \in T_{\tau}^{SL}(\mathcal{T})} a^4 \left[\frac{\left(3\pi - 6\theta_E^{(3,1)} - \theta_E^{(2,2)} N_{(2,2)}^{\uparrow}(\triangle)\right)^2}{4V_E^{(3,1)} + V_E^{(2,2)} N_{(2,2)}^{\uparrow}(\triangle)} \right. \\
&\qquad \left. + \frac{\left(3\pi - 6\theta_E^{(3,1)} - \theta_E^{(2,2)} N_{(2,2)}^{\downarrow}(\triangle)\right)^2}{4V_E^{(3,1)} + V_E^{(2,2)} N_{(2,2)}^{\downarrow}(\triangle)} \right] \\
&\quad + \frac{\alpha}{16\pi G} \sum_{\tau} \sum_{v \in V_{\tau}(\mathcal{T})} \frac{\sqrt{\eta}}{a} \frac{\left(6 - N_{\triangle}(v)\right)^2}{N_{\triangle}(v)}
\end{aligned}$$

Run Monte Carlo simulations of the partition function $Z_{HL}^{(E)}$

Phase Structure

Critical surface for $k_0 = 1$ projected onto the $\lambda - \alpha$ plane

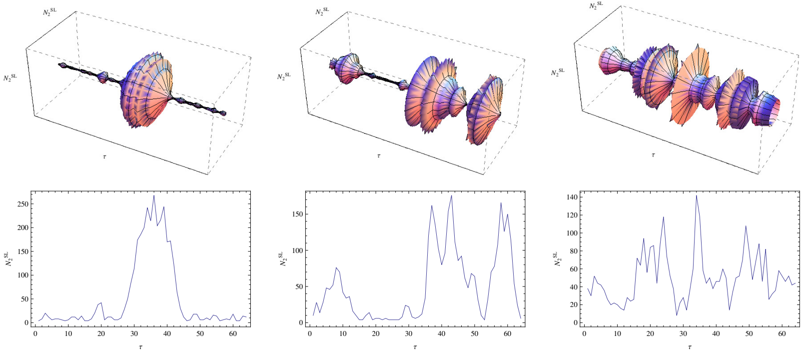


Phase C: blue circles
Phase D: magenta squares
Phase E: orange diamonds

[JHC, Anderson *et al* 2011]

Depictions of Representative Spacetimes

Discrete 2-volume as a function of discrete time

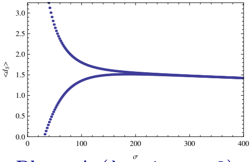


Phase C Phase D Phase E

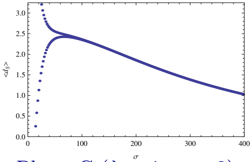
[JHC, Anderson *et al* 2011]

Preliminary Evidence for Semiclassicality

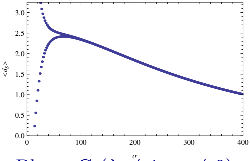
Spectral dimension



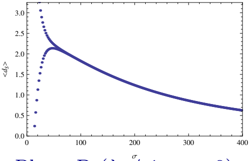
Phase A ($\lambda = 1$, $\alpha = 0$)



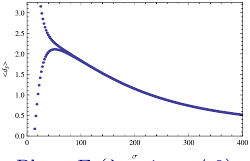
Phase C ($\lambda = 1$, $\alpha = 0$)



Phase C ($\lambda \neq 1$, $\alpha \neq 0$)



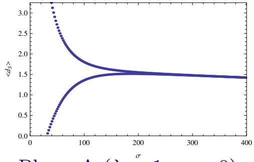
Phase D ($\lambda \neq 1$, $\alpha = 0$)



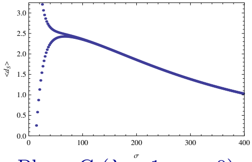
Phase E ($\lambda = 1$, $\alpha \neq 0$)

Preliminary Evidence for Semiclassicality

Spectral dimension



Phase A ($\lambda = 1$, $\alpha = 0$)



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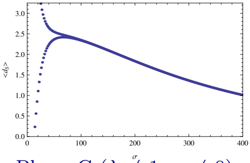
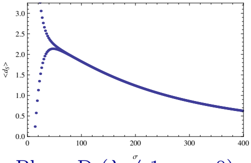
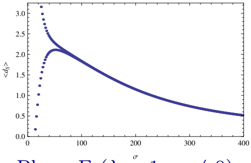


Phase C ($\lambda \neq 1$, $\alpha \neq 0$)

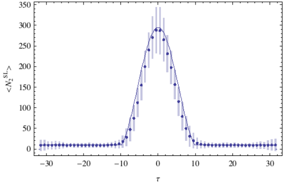


Phase D ($\lambda \neq 1$, $\alpha = 0$)



Phase E ($\lambda = 1$, $\alpha \neq 0$)

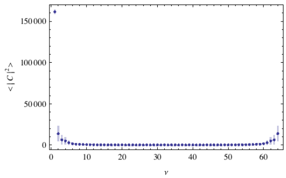
Minisuperspace model fit to phase C ensemble for $\lambda \neq 1$, $\alpha \neq 0$



[JHC, Anderson *et al* 2011]

Preliminary Evidence for Static Nature of Phase E

Fourier transform of discrete 2-volume as a function of discrete time



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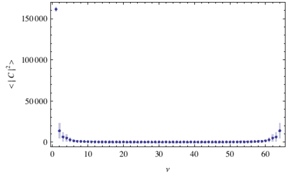

Normalized variance of the discrete 2-volume

$$\left.\frac{\langle \Delta_{N_2^{SL}} \rangle_{max}}{\langle N_2^{SL} \rangle_{max}}\right|_C = 0.20 \quad < \quad \left.\frac{\langle \Delta_{N_2^{SL}} \rangle}{\langle N_2^{SL} \rangle}\right|_E = 0.69 \quad < \quad \left.\frac{\langle \Delta_{N_2^{SL}} \rangle_{min}}{\langle N_2^{SL} \rangle_{min}}\right|_C = 0.78$$

$$\left.\frac{\sqrt{\langle \Delta_{N_2^{SL}} \rangle_{max}}}{\sqrt[3]{\langle N_3 \rangle_{max}}}\right|_C = 0.35 \quad > \quad \left.\frac{\sqrt{\langle \Delta_{N_2^{SL}} \rangle}}{\sqrt[3]{\langle N_3 \rangle}}\right|_E = 0.27 \quad > \quad \left.\frac{\sqrt{\langle \Delta_{N_2^{SL}} \rangle_{min}}}{\sqrt[3]{\langle N_3 \rangle_{min}}}\right|_C = 0.22$$

[JHC, Anderson *et al* 2011]

C-E Phase Transition

Conjecture The C-E phase transition is a confinement-deconfinement transition of the global gravitational charge $\mathcal{H}_\perp$

Hamiltonian constraint

$$\mathcal{H}_\perp = \int_\Sigma \mathrm{d}^2 x \sqrt{\gamma(t,\mathbf{x})} \left[K_{ij}(t,\mathbf{x}) K^{ij}(t,\mathbf{x}) - \lambda K^2(t,\mathbf{x}) + \alpha R_2^2(t,\mathbf{x}) - \beta R_2(t,\mathbf{x}) + 2\Lambda\right]$$

Interpretation of transition in terms of FLRW spacetimes

- Confined phase C: Hamiltonian constraint equation becomes the Friedmann equation for the scale factor, which precludes the ground state geometry from being time-independent
- Deconfined phase E: Hamiltonian constraint measures the energy levels with the ground state identified as the lowest energy, typically static configuration

Continuing Research

Current and Future Research

Regarding causal dynamically triangulated Einstein gravity

- Quantum scalar field theory on curved spacetime
  - Better determination of spectral dimension
  - Dynamical determination of light cone structure
- Renormalization group flow of the cosmological constant
- Fixed metric boundary conditions
- Testing Newton's law of gravitation
  - Introduce bundles of quasilocal mass

Regarding causal dynamically triangulated Hořava-Lifshitz gravity

- Testing the confinement-deconfinement conjecture
- Continuing exploration of the phase diagram
  - Better distinguish phases D and E
  - Ascertain relationships to phase A