# Simulating lattice QCD at finite baryon density

Philippe de Forcrand
ETH Zürich & CERN & YITP & KITP

Collaborators: Michael Fromm, Jens Langelage, Kohtaroh Miura, Owe Philipsen, Wolfgang Unger

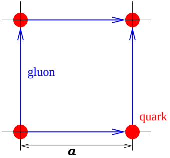
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

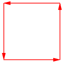
Jan. 30, 2012

# Lattice QCD

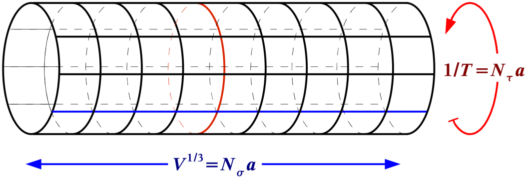
$$Z = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[d^3x d\tau \left(-\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_i(\not{D} + m_i + \mu_i\gamma_0)\psi_i\right)\right]$$

$4d$ hypercubic grid:



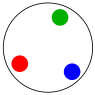

- Discretize action:
  - $\bar{\psi}\not{D}\psi \;\rightarrow\; \bar{\psi}(x)\sum_{\mu}\frac{1}{2a}\gamma_{\mu}\left[U_{+\mu}(x)\psi(x+\mu) - U_{-\mu}(x)\psi(x-\mu)\right]$
  - $-\frac{1}{4}F_{\mu\nu}F_{\mu\nu} \;\rightarrow\; \beta\sum_{x,\mu\nu}\mathrm{ReTr}\,U_P$, $\quad U_P$ plaquette matrix 

- Finite temperature:



# Scope of lattice QCD simulations: Physics of color singlets

* "One-body" physics: confinement
  hadron masses
  form factors, etc..



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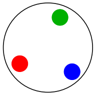
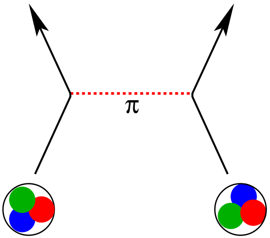


** "Two-body" physics: nuclear interactions
pioneers Hatsuda et al, Savage et al




hard-core
+
pion exchange?

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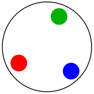

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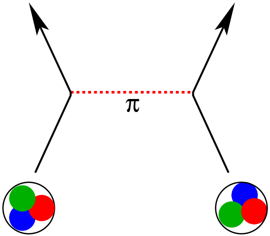


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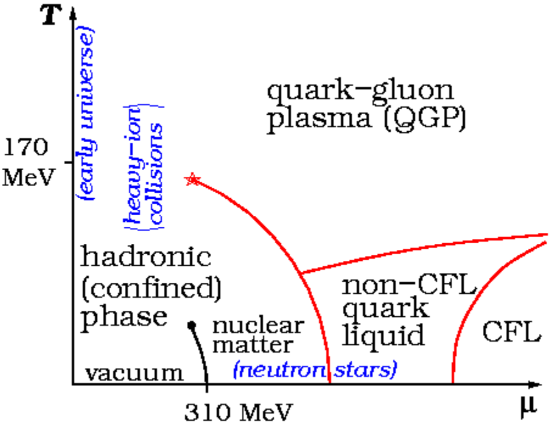
*** Many-body physics: nuclear matter
phase diagram vs (temperature $T$, density $\leftrightarrow \mu_B$)

# Finite $\mu$: why is it important?

The phase diagram of QCD according to Wikipedia



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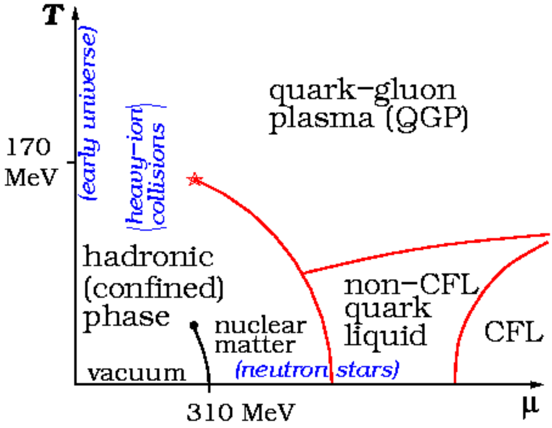



- crystal phase(s)
- quarkyonic phase
- strangelets

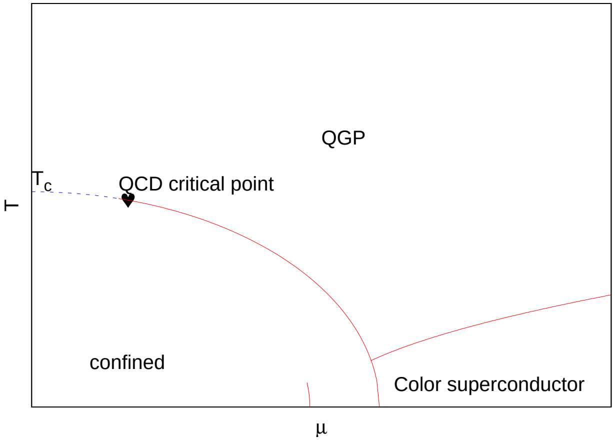
. . .

QCD conserves $u, d, s$ charges separately $\rightarrow$ $Z(T, \mu_u, \mu_d, \mu_s)$

A vast new world to discover!

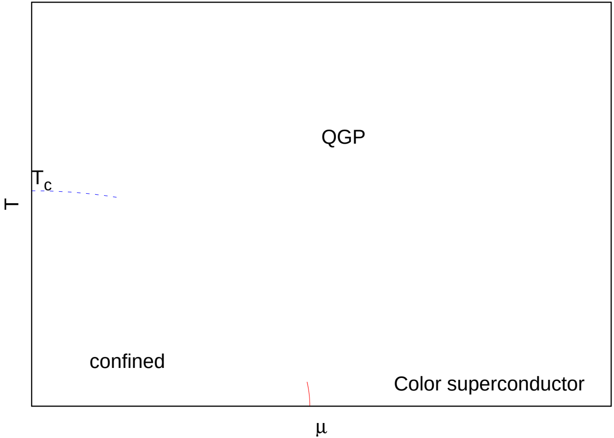
# Finite $\mu$: what is known?

Equilibrium w.r.t. weak interactions ($\beta$-eq.) + electric neutrality $\rightarrow$ single $\mu$




Commonly believed "minimal" phase diagram ("conventional fiction")

# Finite $\mu$: what is known?




Minimal, possible phase diagram

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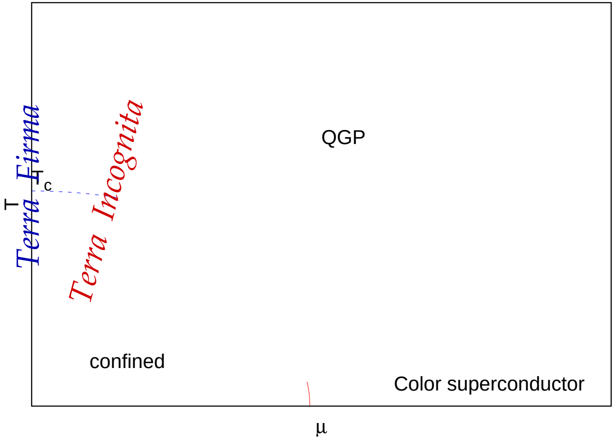
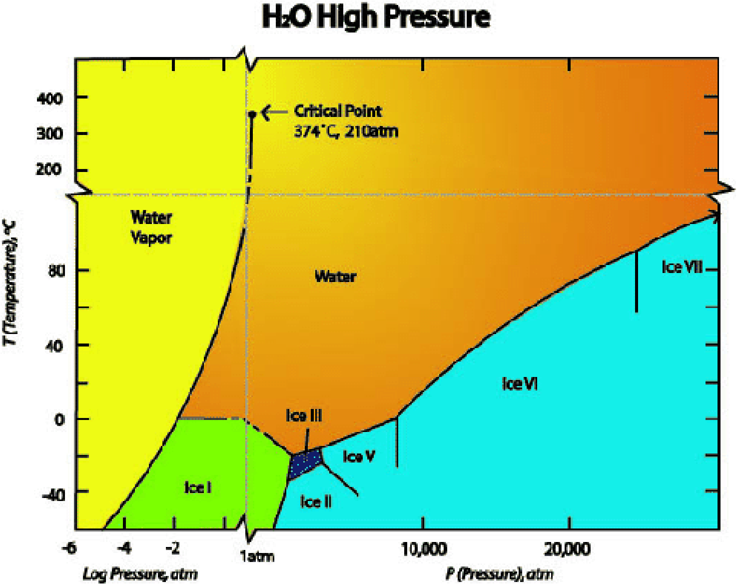



Exploration hampered by sign problem

# Analogy with water



# Why are we stuck at $\mu = 0$? The "sign problem"

- quarks anti-commute → integrate analytically: $\det(\not{D}(U) + m + \mu\gamma_0)$
  $\gamma_5(i\not{p} + m + \mu\gamma_0)\gamma_5 = (-i\not{p} + m - \mu\gamma_0) = (i\not{p} + m - \mu^*\gamma_0)^\dagger$

$$\det \not{D}(\mu) = \det{}^* \not{D}(-\mu^*)$$

det real only if $\mu = 0$ (or $i\mu_i$), otherwise can/will be complex

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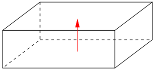
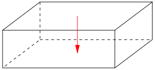


$$\langle \mathrm{Tr\ Polyakov} \rangle = \exp(-\tfrac{1}{T}F_{\mathbf{q}}) = \int \mathrm{Re\ Pol} \times \mathrm{Re}\ d\varpi - \mathrm{Im\ Pol} \times \mathrm{Im}\ d\varpi$$



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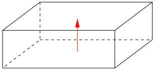

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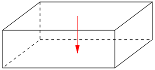

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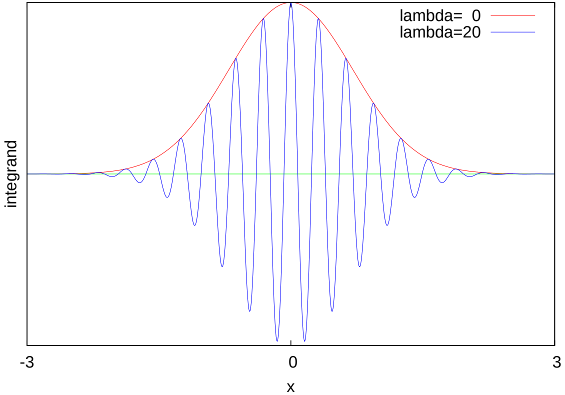
- Origin: $\mu \neq 0$ breaks charge conj. symm., ie. usually complex conj.

# Sampling oscillatory integrands

- Example: $Z(\lambda) = \int dx \exp(-x^2 + \mathbf{i}\lambda\mathbf{x})$




- $Z(\lambda)/Z(0) = \exp(-\lambda^2/4)$: exponential cancellations

$\rightarrow$ truncating deep in the tail at $x \sim \lambda$ gives $\mathcal{O}(100\%)$ error

"Every $x$ is important" $\leftrightarrow$ How to sample?

# Reweighting and optimal sampling of oscillatory integrand

- To "sample": $Z_f \equiv \int dx\ f(x)$, $f(x) \in \mathbf{R}$, with $f(x)$ sometimes negative

  Sample w.r.t. auxiliary partition function $Z_g \equiv \int dx\ g(x)$, $g(x) \geq 0\ \forall x$

$$\langle W \rangle_f = \frac{\int dx\ W(x) f(x)}{\int dx\ f(x)} = \frac{\int dx\ W(x) \frac{f(x)}{g(x)} g(x)}{\int dx\ \frac{f(x)}{g(x)} g(x)} = \boxed{\frac{\langle W \frac{f}{g} \rangle_g}{\langle \frac{f}{g} \rangle_g}}$$

Reweighting, a.k.a. "put sign in observable"

$\frac{f}{g}$ is the "reweighting factor", $\langle \frac{f}{g} \rangle_g = \frac{Z_f}{Z_g}$ is the "average sign"

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- Statistical error on average sign $\langle \frac{f}{g} \rangle_g$ propagates to any observable $W$

  optimal sampling $\rightarrow$ minimize its relative variance $\frac{\langle (f/g)^2 \rangle_g - \langle f/g \rangle_g^2}{\langle f/g \rangle_g^2}$

Solution when av. sign $\rightarrow 0$: $\boxed{g(x) = |f(x)|}$, ie. $f/g = \mathrm{sign}(f)$ hep-lat/0209126

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- Generically, average sign is exponentially small: $\langle \frac{f}{g} \rangle_g = \frac{Z_f}{Z_g} = \exp(-\frac{V}{T} \underbrace{\Delta f(\mu^2, T)}_{\text{diff. free energy dens.}})$

  Each meas. of $\frac{f}{g}$ gives value $\mathcal{O}(1) \Longrightarrow$ error $\approx \frac{1}{\sqrt{\#\ \text{meas.}}}$

  Constant rel. accuracy $\Longrightarrow$ $\boxed{\text{need statistics} \propto \exp(+2\frac{V}{T}\Delta f)}$

  Large $V$, low $T$ inaccessible

# Sampling for QCD at finite $\mu$

- QCD: sample with $|\mathrm{Re}(\det(\mu)^{N_f})|$ optimal, but not equiv. to Gaussian integral
  Can choose instead: $|\det(\mu)|^{N_f}$, i.e. "phase quenched"
  $|\det(\mu)|^{N_f} = \det(+\mu)^{\frac{N_f}{2}} \det(-\mu)^{\frac{N_f}{2}}$, ie. isospin chemical potential $\mu_u = -\mu_d$
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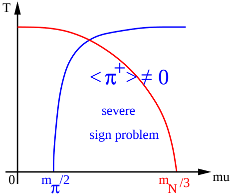


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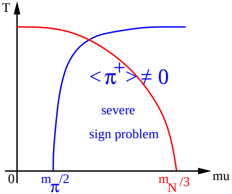


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  $Z_{\mathrm{QCD}} \leftrightarrow Z_{|\mathrm{QCD}|}$ by changing fermion b.c. $\Rightarrow$ ratio UV-finite
  For $T,\ \mu \ll m_\rho$, analytic results via RMT/$\chi$PT Splittorff, Verbaarschot et al.

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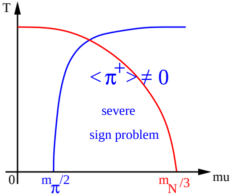


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- Can improve by incorporating baryons via HRG $\rightarrow$ Prediction: 1005.0539

$\langle\mathrm{sign}\rangle \gtrsim 0.1 \Leftrightarrow \mathcal{O}(10)$ baryons max. at $T \lesssim T_c$ (less as $T \searrow$, hardly more as $V \nearrow$)

# Reweighting strategies

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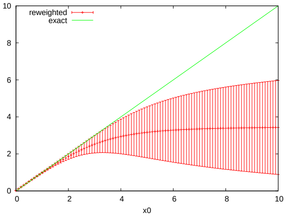
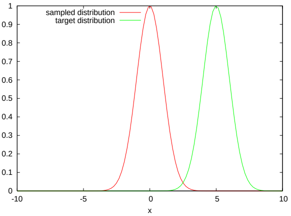
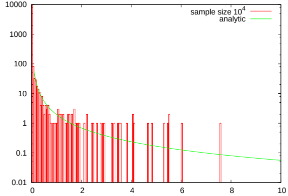
- Sample $\mu = 0$ ensemble? worse, because reweighting factor fluctuates also in magnitude $\rightarrow$ increased statistical errors
- Further danger: "overlap pb." between sampled and reweighted ensembles $\rightarrow$ WRONG estimates in reweighted ensemble for finite statistics
- Example: sample $\exp(-\frac{x^2}{2})$, reweight to $\exp(-\frac{(x-x_0)^2}{2})$ $\rightarrow$ $\langle x \rangle = x_0$ ?




- Estimated $\langle x \rangle$ saturates at largest sampled $x$-value
- Error estimate too small



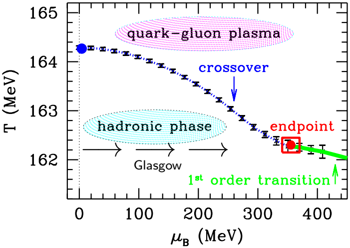

Insufficient overlap ($x_0 = 5$)

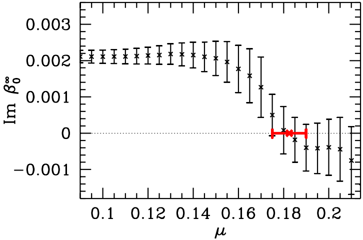



Very non-Gaussian distribution of reweighting factor
Log-normal Kaplan et al.

# Reweighting from $\mu = 0$: Glasgow and multi-parameter

- "Glasgow": $\beta$ fixed, reweight with $\frac{\det(\mu)}{\det(\mu=0)}$ $\rightarrow$ overlap pb.
- Fodor & Katz: sample $(\mu = 0, \beta = \beta_c)$ and reweight with $\frac{\det(\mu)}{\det(\mu=0)} \times e^{-\Delta\beta S_{YM}}$ along pseudo-critical line $T_c(\mu)$
  - less fluctuations in reweighting factor
  - improved (ensured?) overlap: both phases sampled




hep-lat/0402006 (physical quark masses, $N_t = 4$) $\rightarrow$ $(\mu_E^q, T_E) = (120(13), 162(2))$MeV

- Abrupt qualitative change near $\mu_E$:

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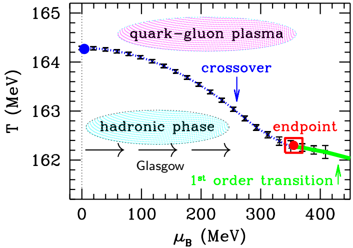
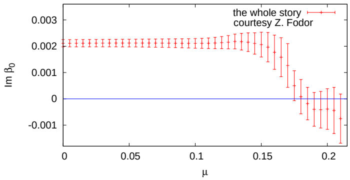
  - less fluctuations in reweighting factor
  - improved (ensured?) overlap: both phases sampled



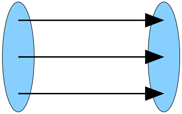

hep-lat/0402006 (physical quark masses, $N_t = 4$) $\rightarrow$ $(\mu_E^q, T_E) = (120(13), 162(2))$MeV

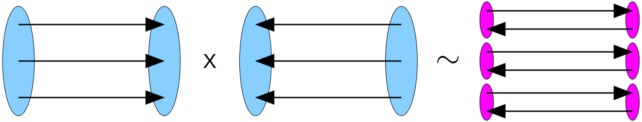
- Abrupt qualitative change near $\mu_E$:
  abrupt change of physics or breakdown of reweighting ?
- Revival (esp. Wilson fermions): Ukawa et al., Nakamura et al., Fodor & Katz

# Alternative at $T \approx 0$: $\mu = 0$ + baryonic sources/sinks

Signal-to-noise ratio of $N$-baryon correlator $\propto \exp(-N(m_B - \frac{3}{2}m_\pi)t)$

Lepage 1989

$C_B(t) =$  $\sim \mathbf{e^{-m_B t}}$

$|C_B(t)|^2 =$ 

$\sim \mathbf{e^{-3m_\pi t}}$

- Mitigated with variational baryon ops. $\to m_{\text{eff}}$ plateau for 3 or 4 baryons ?

  Savage et al., 1004.2935

  At least 2 baryons $\to$ nuclear potential    Aoki, Hatsuda et al., eg. 1007.3559

- Beautiful results with up to 12→72 *pions or kaons*    Detmold et al., eg. 0803.2728

  (cf. isospin-$\mu$: no sign pb.)

# Change of strategy

Reweighting gives exact answer in small volumes (work $\sim \exp(V)$)

Try instead: approximate answer in large volume ?

Improvement: reliability hard to assess $\rightarrow$ full confidence?

Consider expansion parameter $\frac{\mu}{T} \lesssim 1$:

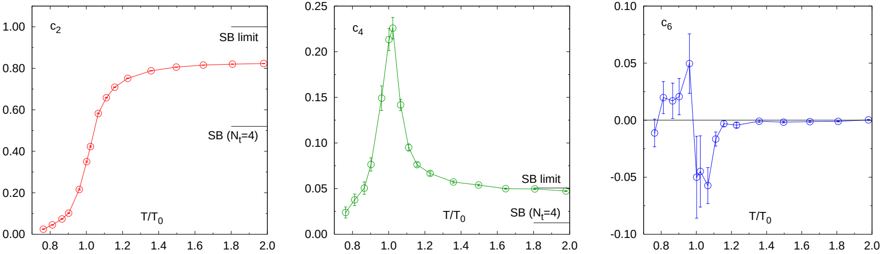
- Taylor expansion about $\mu = 0$
- Imaginary $\mu$ + polynomial fit + analytic continuation

# Taylor expansion

$$P(T,\mu) = \underbrace{P(T,\mu=0)}_{\text{indep. calc.}} + \Delta P(T,\mu), \qquad \boxed{\frac{\Delta P(T,\mu)}{T^4} = \sum_{k=1} c_{2k}(T) \left(\frac{\mu}{T}\right)^{2k}}$$

$$c_{2k} = \langle \mathrm{Tr}(\text{ degree } 2k \text{ polynomial in } \not{D}^{-1}, \frac{\partial \not{D}}{\partial \mu}) \rangle_{\mu=0} \rightarrow \text{vanilla HMC}$$

- From $\{c_{2k}\}$, obtain all thermodynamic info: EOS *and* $T_c(\mu)$ *and* crit. pt. *and* ...
- As $\frac{\mu}{T}$ increases, need higher-order $c_{2k}$'s to control truncation error




C. Schmidt, hep-lat/0610116

# Taylor expansion: nitty-gritty

hep-lat/0501030, Appendix A for generic observable $\mathcal{O}$:

$$\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}U\mathcal{O}(\det M)^{n_f/4}e^{-S_g} \quad ,$$

$$\frac{\partial\langle\mathcal{O}\rangle}{\partial\mu} = \left\langle\frac{\partial\mathcal{O}}{\partial\mu}\right\rangle + \frac{n_f}{4}\left(\left\langle\mathcal{O}\,\frac{\partial(\ln\det M)}{\partial\mu}\right\rangle - \langle\mathcal{O}\rangle\left\langle\frac{\partial(\ln\det M)}{\partial\mu}\right\rangle\right)$$

$$\begin{aligned}
\frac{\partial\ln\det M}{\partial\mu} &= \mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}\right),\\
\frac{\partial^2\ln\det M}{\partial\mu^2} &= \mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) - \mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\right),\\
\frac{\partial^3\ln\det M}{\partial\mu^3} &= \mathrm{tr}\left(M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) - 3\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right)\\
&\quad +2\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\right), \quad \text{etc...}
\end{aligned}$$

# Taylor expansion: nitty-gritty

$$
\begin{aligned}
\frac{\partial^6 \ln\det M}{\partial \mu^6} \; &= \; \mathrm{tr}\left(M^{-1}\frac{\partial^6 M}{\partial\mu^6}\right) - 6\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^5 M}{\partial\mu^5}\right) \\
&-15\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^4 M}{\partial\mu^4}\right) - 10\mathrm{tr}\left(M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) \\
&+30\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^4 M}{\partial\mu^4}\right) + 60\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) \\
&+60\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) + 30\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) \\
&-120\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) \\
&-180\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) \\
&-90\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) \\
&+360\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) \\
&-120\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\right) .
\end{aligned}
$$

Now estimate all Traces by sandwiching between noise vectors... GPUs

# Complexity of Taylor expansion approach?

Effects of increasing Taylor order $k$:

- $c_{2k} = \langle \mathrm{Tr}(\text{ degree } 2k \text{ polynomial in } \not{D}^{-1}, \frac{\partial \not{D}}{\partial \mu})\rangle_{\mu=0}$ $\rightarrow$ nb. terms $\sim 6^{2k}$
- Cancellations: $c_{2k}$ finite as $V \rightarrow \infty$, but sum of terms possibly $\sim V^{2k}$

  ie. the sign problem fights back!
- $c_{2k}$ obtained as average over less and less Gaussian dist. $\rightarrow$ stat. error?
- $c_{2k} \sim 2k$-point function $\rightarrow$ need larger volumes

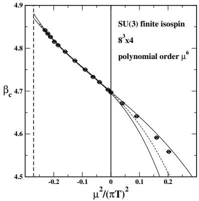
Current best: $N_t = 6$, 8th order Gavai & Gupta, 0806.2233

Need *much* higher order to estimate convergence radius $\rightarrow$ critical point

Karsch, Schaefer et al, 1009.5211

# Imaginary $\mu$: same, but simpler

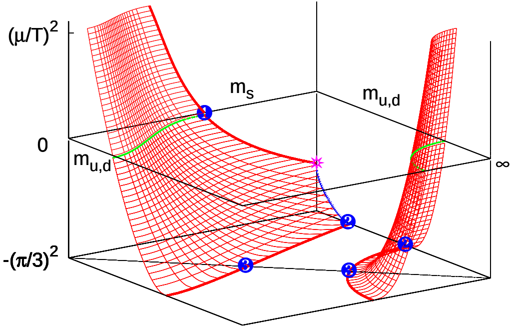
- Simulate at several values of $\mu = i\mu_I$: no sign pb.
  ($|\mu_I| < \frac{\pi T}{3}$, Roberge-Weiss singularity)
- Fit $\langle \mathcal{O} \rangle(\mu_I) = \sum_k \frac{d_k}{k!} \mu_I^k \quad \rightarrow \quad d_k$ is estimator of $\frac{\partial^k \mathcal{O}}{\partial \mu_I^k}$
  Analytic continuation trivial: $i\mu_I \rightarrow \mu$
- For pressure, take eg. $\mathcal{O} = n_B = \frac{\partial P}{\partial \mu_B}$ and integrate fitted polynomial
- Error analysis simple: data at different $\mu_I$'s uncorrelated
- No free lunch: $k^{\text{th}}$ derivative damped by $k!$
- Data fitted by *truncated* Taylor series or Pade $\rightarrow$ systematic error?
  Conformal mapping to unit disk    Morita et al., 1008.4549



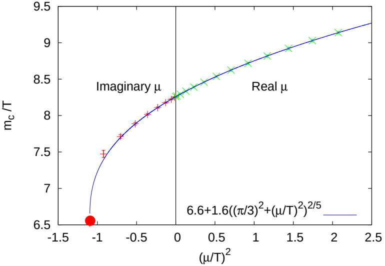

Frequent problem (here for $T_c(\mu)$):
the series in $(i\mu_I)^2$ is *alternating*
D'Elia et al., 0905.1292

# New hope with imaginary $\mu$: tricritical scaling

- Rich phase diagram as a function of $(\mu = i\mu_I, m_{u,d}, m_s)$:
  - Roberge-Weiss transition at $\frac{\mu_I}{T} = \frac{2\pi}{3}(2k+1)$
  - Two tricritical lines in Columbia plot $(m_{u,d}, m_s)$ for $\frac{\mu_I}{T} = \frac{2\pi}{3}$
  - Associated tricritical scaling window may be broad 1004.3144




Phase diagram $(m_{u,d}, m_s, \left(\frac{\mu}{T}\right)^2)$
1201.2769

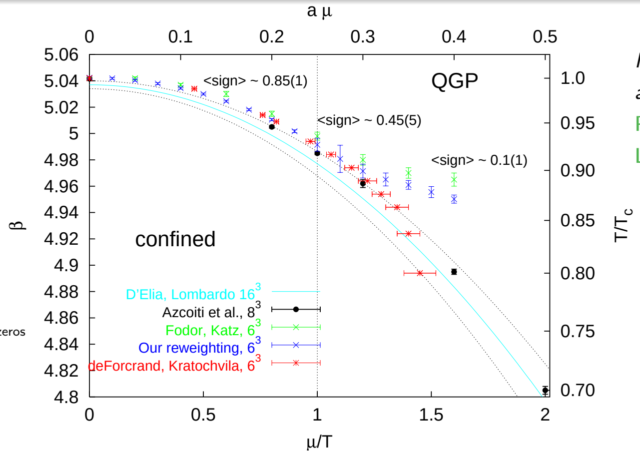



Tricritical scaling for heavy quarks (Potts)
1004.3144

# Crosschecks

All methods agree for $\mu/T \lesssim \mathcal{O}(1)$

imaginary $\mu$
2 param. imag. $\mu$
dble reweighting, LY zeros
Same, susceptibilities
canonical



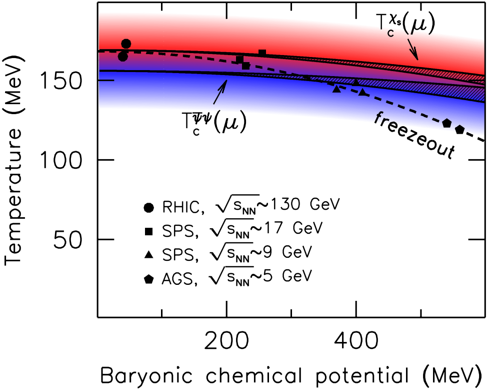

$N_f = 4$ staggered,
$am_q = 0.05$, $N_t = 4$
PdF & Kratochvila
LAT05

More recent crosschecks (Wilson fermions):

- Reweighting $\leftrightarrow$ Taylor expansion — Nagata & Nakamura
- Reweighting $\leftrightarrow$ canonical — Takeda, Kuramashi & Ukawa

# State of the art I

- Curvature of $T_c(\mu)$ in continuum limit (5 deriv. of $P$) Fodor, Katz et al.

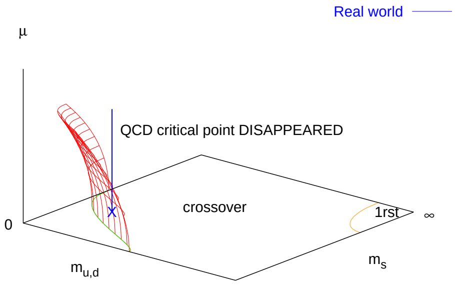



- $T_c(\mu)$ very flat $\rightarrow$ critical point far from freeze-out curve
- continuum curvature $\approx$ same as $N_t = 4 \rightarrow$ small discretization error ?
- No evidence of critical point for $\mu_q/T \lesssim \mathcal{O}(1)$

# State of the art II

- Curvature of critical surface on coarse lattices ($N_t = 4$) to $\mathcal{O}(\mu/T)^4$ (8 deriv. of $P$) PdF & Philipsen

Region ($m_{u,d}, m_s$) of first-order transition shrinks as $\mu$ is turned on




Results I and II use same numerical method (small imag. $\mu$) 0711.0262

# Tame the sign problem at strong coupling

Avoid complex determinant by reversing order of integration: links, then fermions

No conservation law for sign pb.! Chandrasekharan, Wenger, PdF, ...

$$Z = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[d^3x d\tau \left(-\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_i(\not{D} + m_i + \mu_i\gamma_0)\psi_i\right)\right]$$

- Problem: $-\frac{1}{4}F_{\mu\nu}F_{\mu\nu} \rightarrow \frac{1}{g_0^2}\mathrm{Tr}\,U_{\mathrm{Plaquette}}$, ie. 4-link interaction 
- Solution: set $g_0 = \infty$, strong coupling limit ($\leftrightarrow$ continuum limit)
- Then integral over gauge links *factorizes*: $\sim \int \prod dU \exp(\bar{\psi}_x U_{x,\hat{\mu}}\psi_{x+\hat{\mu}})$

- analytic 1-link integral $\rightarrow$ only color singlets survive
- perform Grassmann integration last $\rightarrow$ hopping of color singlets

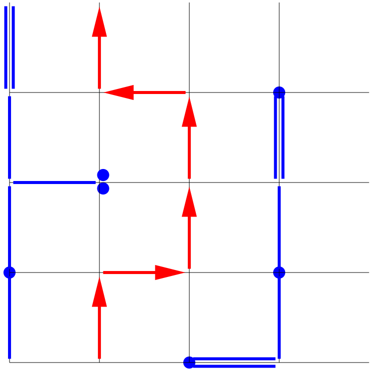
$\rightarrow$ hadron (baryon, meson) worldlines (staggered quarks so far)

- sample gas of worldlines by Monte Carlo
- baryons make *self-avoiding* loops:

Point-like, hard-core baryons in pion bath

No $\pi NN$ vertex: just hard-core repulsion?

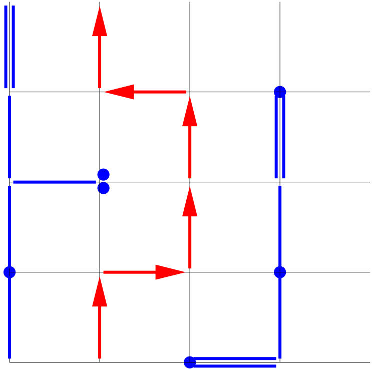
# Worldline configurations in $(1+1)d$



Constraint at every site:
3 blue symbols ($\bullet$ $\bar{\psi}\psi$, meson hop)
or a baryon loop

Sign problem mild at all densities $\rightarrow$ complete numerical solution

# Worldline configurations in $(1+1)d$



Constraint at every site:
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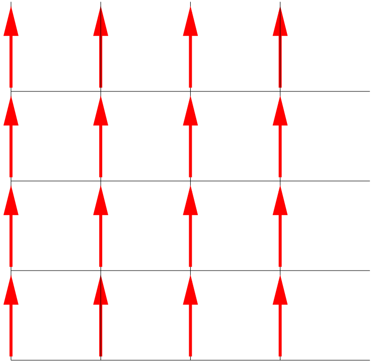
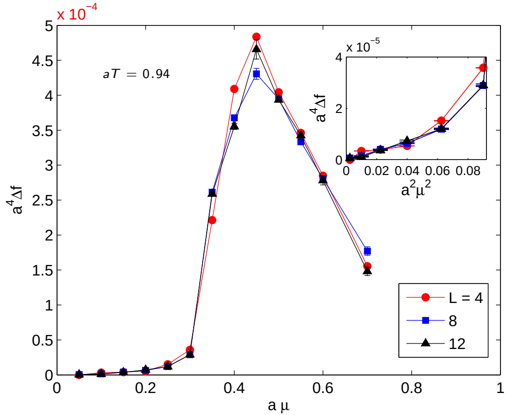
or a baryon loop



The dense (crystalline) phase:
1 baryon per site; no space left
$\rightarrow \quad \langle\bar{\psi}\psi\rangle = 0$

Sign problem mild at all densities $\rightarrow$ complete numerical solution

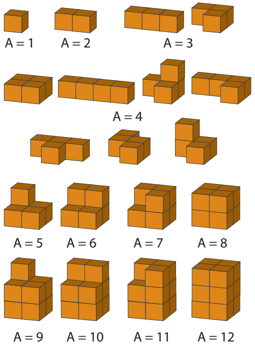
# Sign problem? Monitor $-\frac{1}{V}\log\langle\text{sign}\rangle$

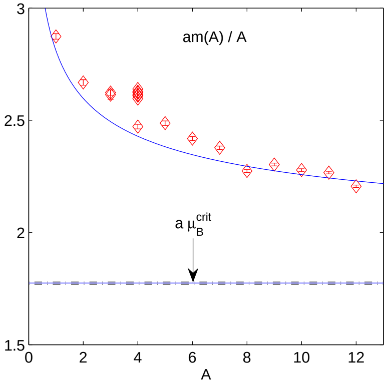



- $\langle\text{sign}\rangle = \frac{Z}{Z_{||}} \sim \exp(-\frac{V}{T}\Delta f(\mu^2))$ as expected; $\Delta f \sim \mu^2 + \mathcal{O}(\mu^4)$
- Determinant method $\rightarrow \Delta f \sim \mathcal{O}(1)$. Why is worldline so much better??
  - no conservation law for sign pb. (eg. use eigenbasis of $H$)
  - negative sign caused by spatial baryon hopping: no baryon $\rightarrow$ no sign pb

    no silver blaze pb.

# Results I – Crude nuclear matter: spectroscopy

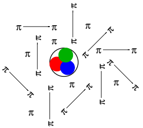


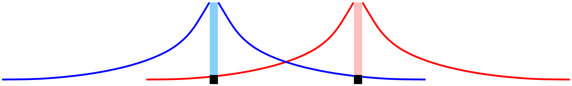

- Can compare masses of differently shaped "isotopes"
- $am(A) \sim a\mu_B^{\rm crit} A + (36\pi)^{1/3} \sigma a^2 A^{2/3}$, ie. (bulk + surface tension)
  Bethe-Weizsäcker   parameter-free   ($\mu_B^{\rm crit}$ and $\sigma$ measured separately)
- "Magic numbers" with increased stability: $A = 4,\ 8,\ 12$ (reduced area)

# Results II – Nuclear interactions and Phase diagram

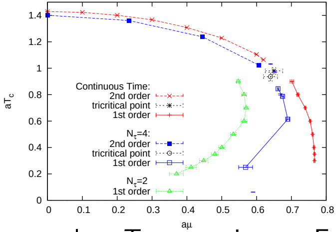
- Baryon: point-like core (self-avoiding loop) disturbs pion bath
  $\Rightarrow$ macroscopic pion cloud $\Delta E_\pi(R) \propto \frac{\exp(-\mathbf{m}_{\rho/\omega} R)}{R} \times (-1)^{x+y+z}$



- Nuclear interaction from nucleon's core disturbing other nucleon's pion cloud
  Linear response $\Rightarrow$ $V_{NN}(R) \approx -2 \times \Delta E_\pi(R)$, ie. Yukawa!



- Phase diagram for $m_q = 0$: chiral transition line, with tricritical point



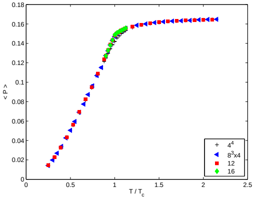

Discretization error, esp. at low $T \rightarrow$ continuous Euclidean time Unger

# Going beyond strong coupling limit

- At $\beta = 0$: measure gauge observables (plaquette, Polyakov loop)

w/Fromm, Langelage, Miura,..

Polyakov loop vs $T$
across chiral transition
$m_q = 0$, $\mu = 0$



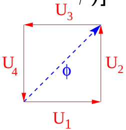

- Simulate $\mathcal{O}(\beta)$ action: in progress

- Beyond $\mathcal{O}(\beta)$: decouple the 4 links in each plaquette by *auxiliary fields*?

Hubbard-Stratonovich: $\int d\phi^* d\phi \exp(-|\phi - \phi_0|^2) = \text{const.}$ indep. of $\phi_0$

Variant: $\exp(\alpha AB) \propto \int d\phi^* d\phi \exp[-\alpha(|\phi|^2 - \phi^* A - B\phi)] \quad \forall \alpha \in R^+$

Take $A = U_1 U_2, B = U_3 U_4$,
$\phi$ along diagonal




Further decoupling to "1-link" action $\rightarrow$ link integration possible $\forall \beta$

# Conclusions (from LAT09 Beijing plenary)

- Finite density QCD is important enough to keep trying
- Analytic understanding of severity of sign problem
- Crosschecks among LQCD methods and with effective models
- Slow but steady progress for small $\mu$: $T_c(\mu)$ OK, crit. pt. ??
  Try to control $a \to 0$ extrapolation
- Confucius: Real knowledge is to know the extent of one's ignorance
- Future: -Start with link integration still vague beyond $\beta = 0$
  -Complex Langevin do miracles really happen?

  - Not covered: - canonical ensemble
    - density of states method

# Backup: complex Langevin 80's revival Aarts, Seiler, Stamatescu, Berges,..

- Real action $S$: Langevin evolution in Monte-Carlo time $\tau$ Parisi-Wu

$\frac{\partial\phi}{\partial\tau} = -\frac{\delta S[\phi]}{\delta\phi} + \eta$, ie. drift force + noise

Can prove: $\langle W[\phi]\rangle_\tau = \frac{1}{Z}\int \mathcal{D}\phi \exp(-S[\phi]) W[\phi]$

- Complex action $S$ ? Parisi, Klauder, Karsch, Ambjorn,..

Drift force complex $\rightarrow$ complexify field $(\phi^R + i\phi^I)$ and simulate as before

With luck: $\langle W[\phi^R + i\phi^I]\rangle_\tau = \frac{1}{Z}\int \mathcal{D}\phi \exp(-S[\phi]) W[\phi]$

- Only change since 1980's: adaptive stepsize $\rightarrow$ runaway sols disappear

- Gaussian example:

$Z(\lambda) = \int dx \exp(-x^2 + \mathbf{i}\lambda\mathbf{x})$

Complexify:

$\frac{d}{d\tau}(x + iy) = -2(x + iy) + i\lambda + \eta$

For any observable $W$,

$\langle W(x + iy)\rangle_\tau = \langle W(x)\rangle_Z$