Instability of Anti-de Sitter Spacetime

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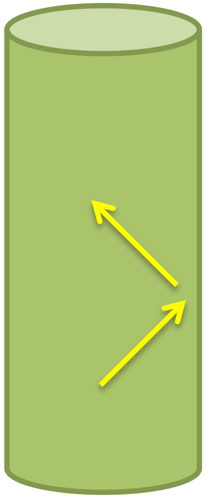
P. Bizon, A. Rostworowski, 1104.3702
O. Dias, J. Santos, G.H., 1109.1825
O. Dias, D. Marolf, J. Santos, G.H., to appear

Outline

1) Statement of the instability and motivation for it
2) Evidence for the instability
3) AdS geons
4) Are all asymptotically AdS solutions unstable?
5) Homework problems for numerical relativists

Consider asymptotically (global) AdS solutions to pure gravity with $\Lambda < 0$ in $D = 4$.



If one requires that the metric (conformally) approach the static cylinder, waves bounce off infinity and return in finite time.

The dual field theory lives on $S^2 \times R$.

At the linearized level, AdS appears just as stable as Minkowski space or de Sitter.

For Minkowski or de Sitter, it has been shown that small but finite perturbations remain small (Christodoulou, Klainerman; Friedrich).

This has never been shown for AdS.

WHY NOT?

It is just not true.

Claim: AdS is nonlinearly unstable

(Anderson, 2006; Dafermos and Holzegel, 2006)

Generic small (but finite) perturbations of AdS become large and eventually form black holes.

The energy cascades from low frequency to high frequency modes in a manor reminiscient of the onset of turbulence.

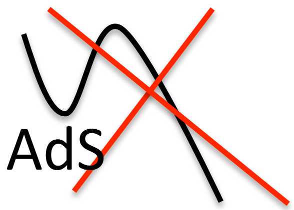
Doesn't this contradict the fact that AdS is supersymmetric?

Doesn't this contradict the fact that there is a positive energy theorem?

No

Positive Energy Theorem: If the matter satisfies a reasonable energy condition, then $E \geq 0$ for all nonsingular, asymptotically AdS initial data, and $E = 0$ if and only if the spacetime is AdS.

This ensures that AdS cannot decay.



It does not ensure that a small amount of energy added to AdS won't generically form a small black hole.

That is usually ruled out by arguing that waves disperse. This doesn't happen in AdS.

Example (Dafermos):

Consider $$S = \int R - (\nabla\phi)^2$$

This has a positive energy theorem and small nonlinear perturbations of Minkowski spacetime remain small.

Now consider $$S = \int R + (\nabla\phi)^2$$

No positive energy theorem, but Minkowski spacetime is still nonlinearly stable.

Why is AdS nonlinearly unstable?

Anderson: AdS boundary conditions act like a confining box. Any finite excitation which is added to this box might be expected to eventually explore all configurations consistent with the conserved quantities – including small black holes.

Dafermos and Holzegel: Since linearized perturbations do not decay, nonlinear corrections are expected to grow in time.

A third motivation (Dias, Santos, G.H.):

Hawking and Penrose proved a singularity theorem showing that closed universes are generically singular. AdS is like a closed universe for the fields inside, so it should be generically singular.

Special solutions need not be singular

For some linearized gravitational modes, there are corresponding nonlinear solutions called geons.

Geons are nonsingular and globally asymptotically AdS. There are an infinite number of them, but they are all special since they are

(1) Exactly periodic in time
(2) Invariant under a continuous symmetry

Perturbative construction of solutions

Expand: $$g = \bar{g} + \sum_i \epsilon^i h^{(i)}$$

At each order, have to solve:

$$\Delta_L h^{(i)}_{ab} = T^{(i)}_{ab}$$

where

$$2\Delta_L h^{(i)}_{ab} \equiv -\bar{\nabla}^2 h^{(i)}_{ab} - 2\bar{R}_a{}^c{}_b{}^d h^{(i)}_{cd} - \bar{\nabla}_a \bar{\nabla}_b h^{(i)} + 2\bar{\nabla}_{(a}\bar{\nabla}^c h^{(i)}_{b)c}.$$

Different types of perturbations

"Scalar type" perturbations: h_{ab} are constructed from spherical harmonics.

"Vector type" perturbations: h_{ab} are constructed from vector harmonics. (For S^2, these are ${}^*\nabla Y_{\ell m}$)

"Tensor type" perturbations only exist in higher dimensions.

At each order, can reduce the metric perturbation to two functions satisfying (Kodama, Ishibashi, 2003)

$$\Box_s \Phi^{(i)}_{\ell,m}(t,r) + V^{(i)}_\ell(r)\Phi^{(i)}_{\ell,m}(t,r) = \tilde{T}^{(i)}_{\ell,m}(t,r),$$

where $\Box_s$ is the wave operator associated with

$$ds^2 = -(r^2+1)dt^2 + \frac{dr^2}{r^2+1}$$

Boundary conditions

Regularity at the origin requires: $\Phi_{\ell,m} \sim \mathcal{O}(r^{\ell})$

Asymptotically:

$$\Phi_{\ell,m} \sim R_{\ell,m}(t) + \frac{S_{\ell,m}(t)}{r} + \mathcal{O}(r^{-2})$$

Surprisingly, to keep the metric fixed at infinity, we need to choose

$$S_{\ell,m}(t) = 0$$

First Order

The allowed frequencies are $\omega_\ell = 1 + \ell + 2p$

For p = 0, the solutions are

$$\Phi^{(1)}_{\ell,m}(t,r) = \frac{r^{\ell+1}}{(r^2+1)^{\frac{\ell+1}{2}}} a_{\ell,m} \cos(\omega_\ell\, t)$$

General structure

If the source has harmonic time dependence cos ωt, then the solution will have the same harmonic time dependence, EXCEPT when ω agrees with one of the normal mode frequencies.

Then we get a resonance and the solution grows linearly in time:

$$\Phi(t,r) = \cos(\omega t) R(r) \\ + t \sin(\omega t) L(r).$$

Example 1

Start with a single $\ell = 2, m = 2$ mode.

At second order – no resonances

At third order – one resonant term

but one can set the growing mode to zero by changing the frequency slightly

$$\omega_2 = 3 - \frac{14703}{17920}\epsilon^2$$

Continue in this way to construct geon.

Example 2

Start with a linear combination of $\ell = 2, m = 2$ and $\ell = 4, m = 4$ mode.

At second order – no resonances

At third order – 4 resonant terms

growing mode in two can be removed by adjusting the frequencies of two original modes

growing mode of one is just absent

Last growing mode cannot be removed. This corresponds to $\ell = 6, m = 6$ with ω = 7.

Get a growing mode with higher frequency than we started with.

Energy is transferred to higher frequency modes.

Expect this to continue. When $\ell = 6, m = 6$ mode grows, it will source even higher frequency modes with growing amplitude.

Spherical scalar field collapse in AdS

(Bizon and Rostworowski, 2011)

Recall the situation when $\Lambda = 0$ (Choptuik, Christodoulou):

For any initial scalar field profile $\phi = \alpha f(r)$,

large α → large black hole

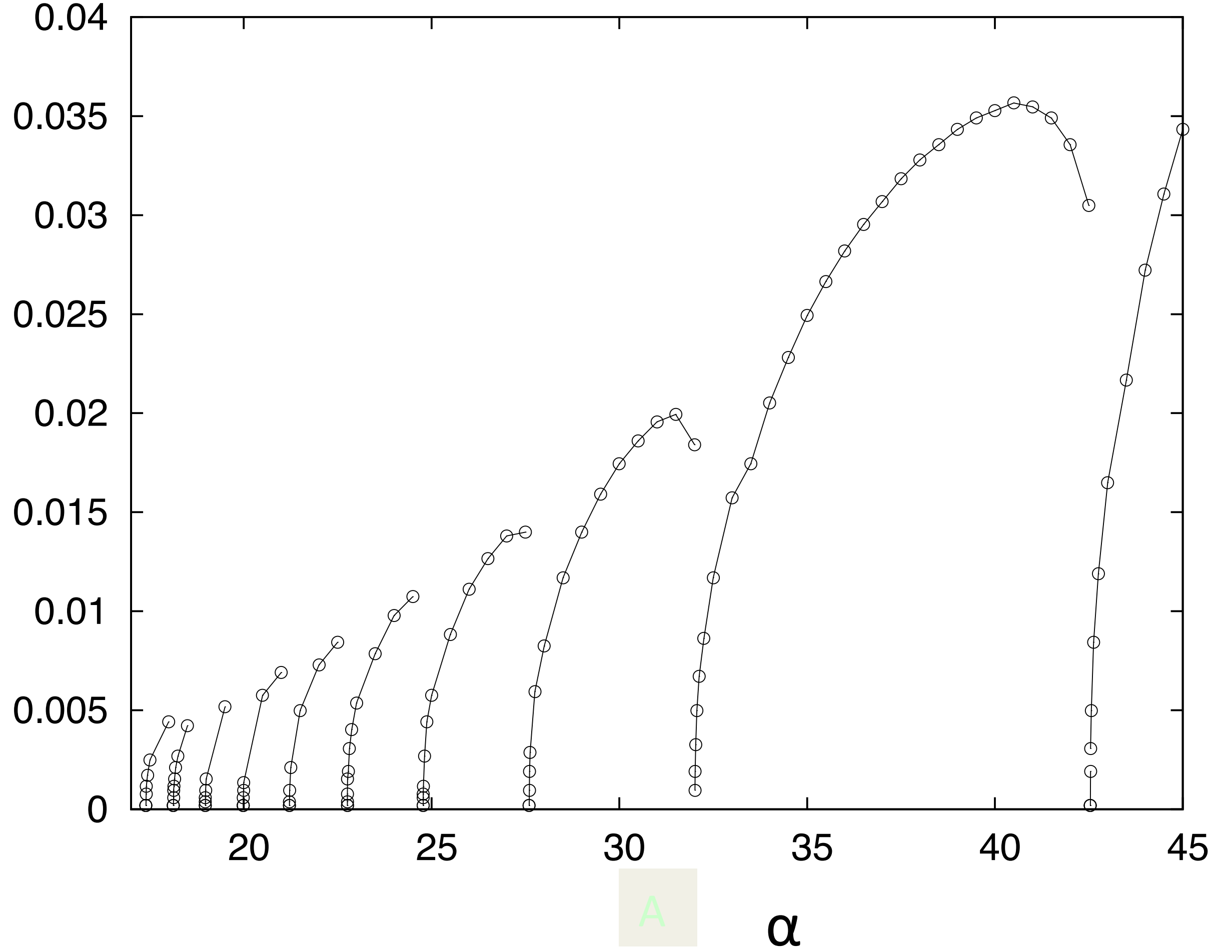
small α → waves scatter and go off to ∞

For a critical value α_*, the collapse forms a "zero mass black hole" i.e. a naked singularity. Near α_*:

$$M_{BH} \sim (\alpha - \alpha_*)^\gamma \text{ with } \gamma = .37$$

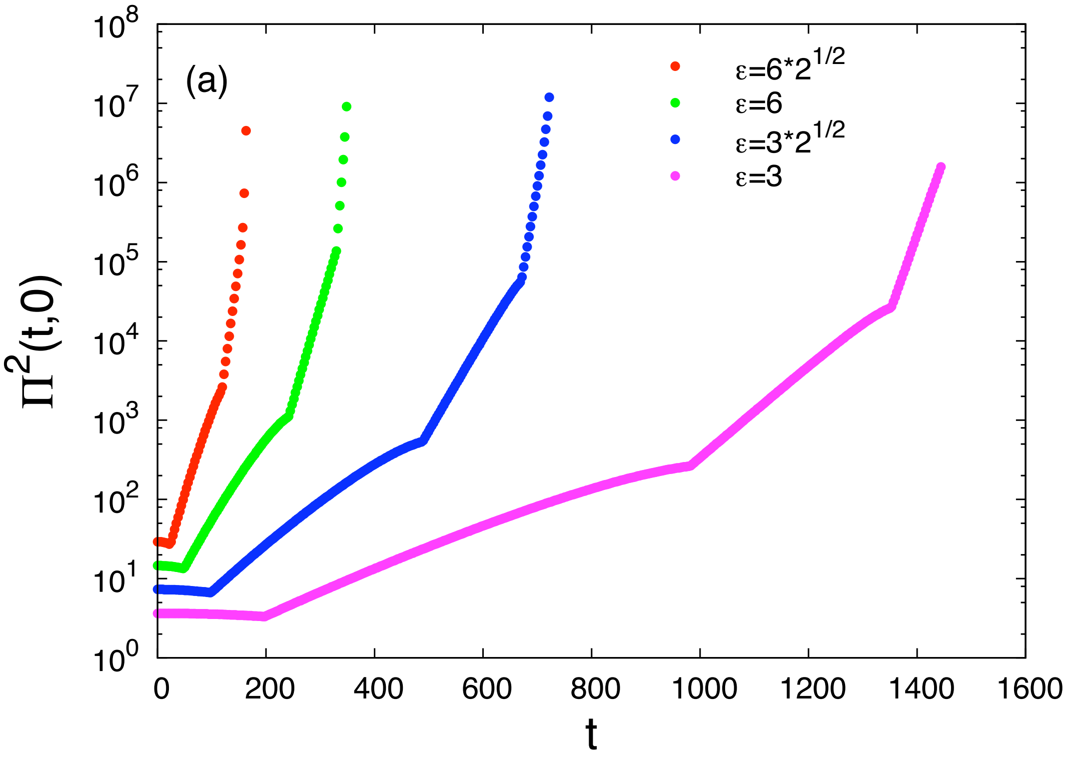
Repeating this in AdS one finds

Radius of initial black hole



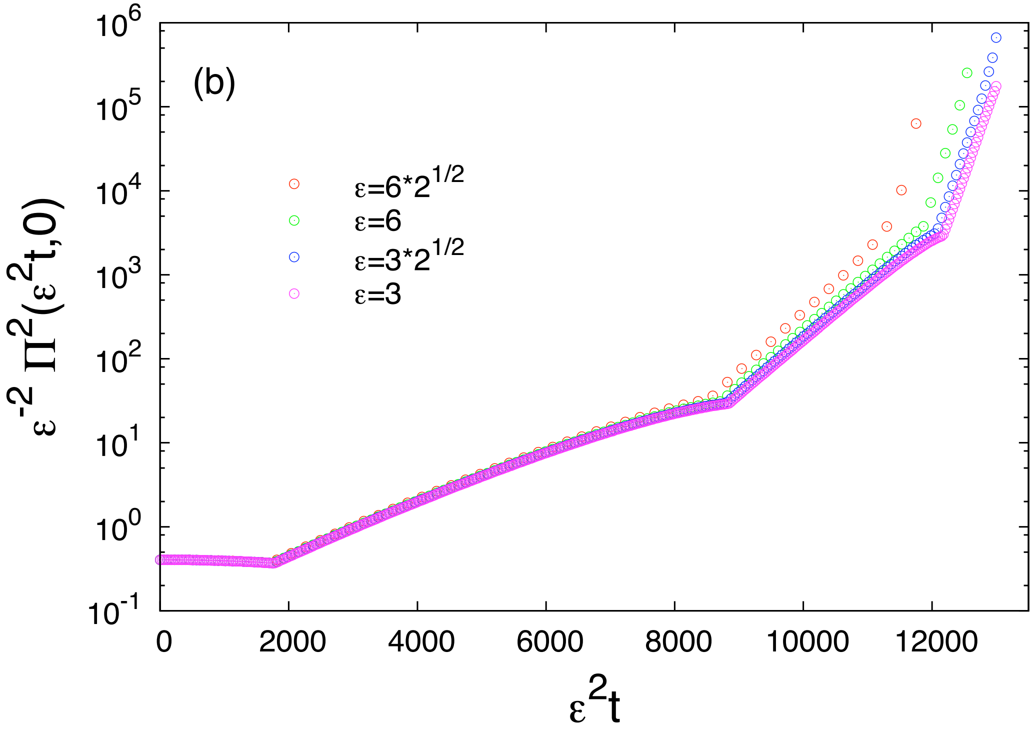
(Bizon and Rostworowski, 2011)

The scalar curvature R at the origin oscillates with period about 2π. Starting with small amplitude initial data, the maximum of R behaves as follows:



(Bizon and Rostworowski, 2011)

If you rescale time and scalar curvature by the amplitude, these curves all agree:



(Bizon and Rostworowski, 2011)

Note: the time to form a black hole scales like (amplitude)$^{-2}$. This is much faster than an ergotic process.

This is because the normal mode frequencies in AdS are all integer multiples of a fundamental frequency. So there are lots of resonances.

AdS is much more unstable than a random box.

Conclusion of spherically symmetric scalar field evolution in AdS:

No matter how small you make the initial amplitude, the curvature at the origin grows and you eventually form a small black hole.

What is the general endpoint of this instability?

It is might not a Kerr AdS black hole, since that may also be unstable!

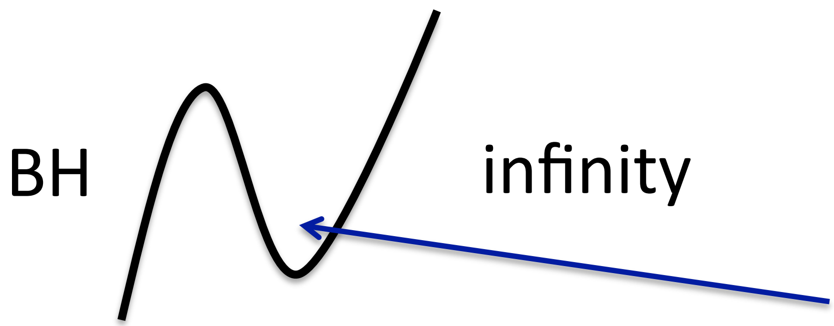
Holzegel and Smulevici (2011) argued that a linear scalar field outside a Kerr-AdS black hole probably decays very slowly, like 1/log t. This was confirmed numerically by Santos.

All asymptotically AdS solutions are probably unstable (Dafermos).

A proposal for the evolution

(Dias, Marolf, Santos, G.H., in progress)

The instability appears similar to the instability of AdS. The effective potential for a large angular momentum mode looks like



The result is a “black moon”.

One approach to showing this:

Quasi-normal mode (QNM) frequencies of Kerr AdS approach those of AdS when ℓ becomes large.

If the time to form a black moon is short enough so that the difference in QNM frequencies doesn't matter, the evolution will be just like in AdS.

The orbit of the black moon will typically decay, and the moon will merge with the original black hole. But it radiates during this process.

End up with smaller amount of energy in radiation around the black hole. This may form a smaller black moon which spirals in, etc.

If you continue to form smaller and smaller black holes, you violate the spirit of cosmic censorship.

Slight complication:

There are non-coalescing binaries in AdS.

The gravitational waves produced become standing waves that support the orbit.

There are special "black moon" solutions that don't decay:

Consider a Kerr AdS black hole with horizon generator

$$\xi = \partial/\partial t + \Omega \; \partial/\partial\phi$$

If $\Omega > 1$, there is a radius for which ξ is tangent to a geodesic.

These are circular orbits which are invariant under the Killing field. They are stable for certain black holes.

Now replace the geodesic by a small black hole. This will create a metric perturbation which will also be invariant under ξ.

Since ξ is null on the horizon, the perturbation will not cause the original horizon to grow. Adding higher order corrections leads to an exact "black moon" solution.

Since the orbit is stable, black moons farther out might be expected to approach this configuration.

Superradiance

If a wave $e^{-i\omega t+im\phi}$ scatters off a rotating black hole with $\Omega > 1$ and $\omega < m\Omega$, it can return with larger amplitude.

In AdS, the outgoing wave is reflected off infinity and the process repeats. Get a superradiant instability.

This is different from the earlier instability, but interacts with it.

Implications for the dual field theory

The fact that perturbations evolve to black holes can be viewed as thermalization (in a microcanonical ensemble).

The instability of AdS black holes is probably not present at finite N.

- What is the dual description of the instability at large N?

Geons are dual to high energy states that do not thermalize.

They are different from the states found by Freivogel, McGreevy, and Suh, 1109.6013.

- Why don't these states thermalize?
- How many states don't thermalize?

Boundary stress tensor for geon

Contains alternating positive and negative energy regions around the equator.

Invariant under $\partial/\partial t + (\omega/m)\partial/\partial\phi$ which is timelike near the poles but spacelike near the equator.

Conclusions

(1) Anti-de Sitter spacetime is nonlinearly unstable: generic small perturbations become large and (probably) form black holes.

(2) The resulting AdS black holes may also be unstable. Their evolution might produce arbitrarily large curvature outside the horizon.

(3) There are exact nonsingular geons, and noncoalescing binaries in AdS.

Homework Problems for Numerical Relativists

(1) Evolve small perturbations of anti-de Sitter and show that they form black holes.

(2) Construct the geons explicitly.

(3) Evolve small perturbations of Kerr AdS and see if a black moon forms.

(4) Construct the non-coalescing binary explicitly.