The power of supersymmetry in a lattice model for strongly interacting fermions

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KITP

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Introduction

Strongly interacting electron systems

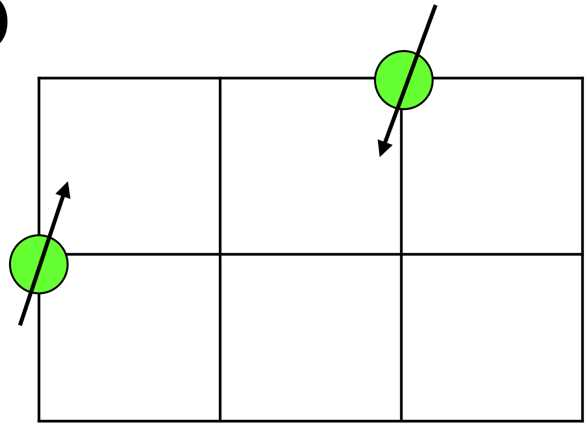
Key examples:

- High T_c superconductors
- Heavy fermion compounds

Challenge conventional theoretical techniques

Introduction

Lattice models (time continuous)



configurations:

electrons located on the sites of an ionic lattice in a solid

Hamiltonian:

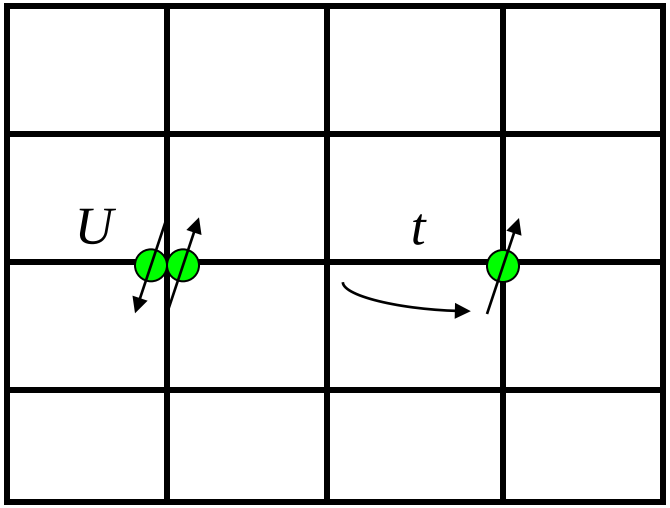
typically a sum of kinetic (hopping) terms and short range repulsive interactions

Introduction

Hubbard model (1963)

Coulomb repulsion → onsite repulsion U

$$H = -t \sum_{\langle ij \rangle} c_{i\sigma}^{\dagger} c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$



$$n_{i\sigma} = c_{i\sigma}^{+} c_{i\sigma}$$

$$\sigma = \uparrow, \downarrow$$

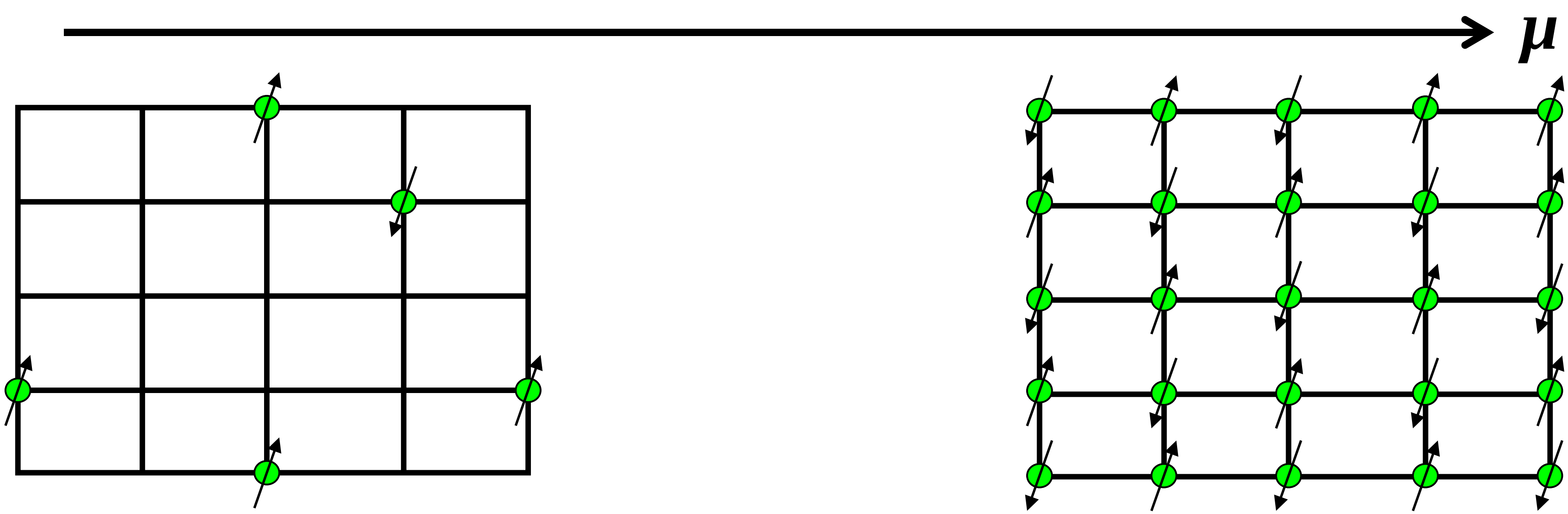
$$\left\{ c_{i\sigma}^{+}, c_{j\sigma} \right\} = \delta_{ij}$$

Introduction

Hubbard model

- Kinetics dominated n<<1, U<<t → Fermi liquid
- Interaction dominated $U>>t$

  → Mott insulator at half filling



Introduction

Strongly interacting electron systems

Challenge: Intermediate densities

Conventional techniques fail

- Mean field results are unreliable
- Bethe Ansatz does not work in D > 1
- Quantum Monte Carlo suffers from sign problem
- …

→ Too difficult

Our work

A model for strongly interacting fermions

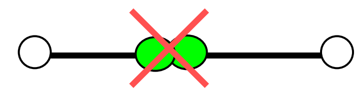
1. Simplifications/adjustments
2. Incorporate supersymmetry

→ Exact result for strongly interacting fermions in $D > 1$ (not accessible via conventional techniques)

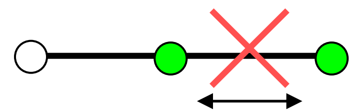
The model

Hardcore spinless fermions

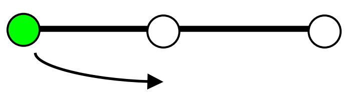
- spinless fermions

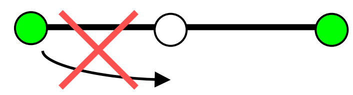


- hardcore $V_1 \to \infty$



- hopping t

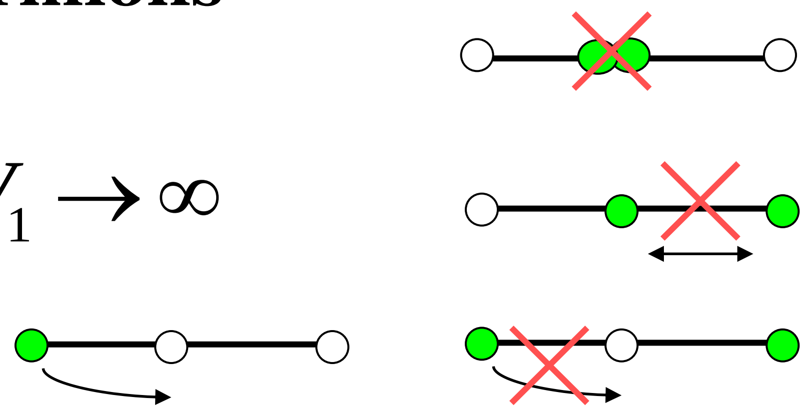


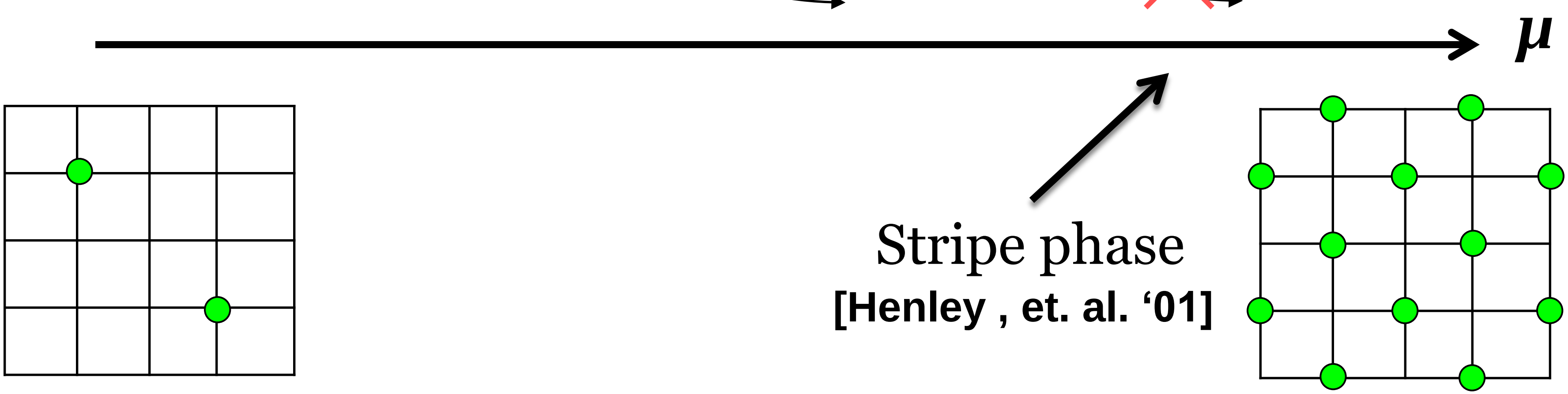


The model

Hardcore spinless fermions

- spinless fermions
- hardcore $V_1 \to \infty$
- hopping t





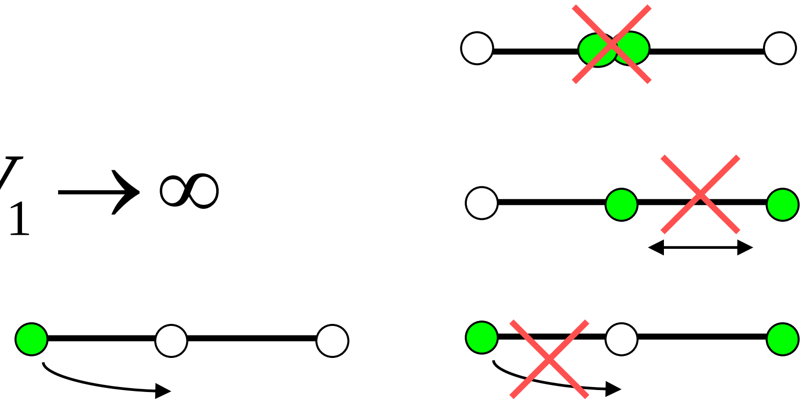
Fermi liquid

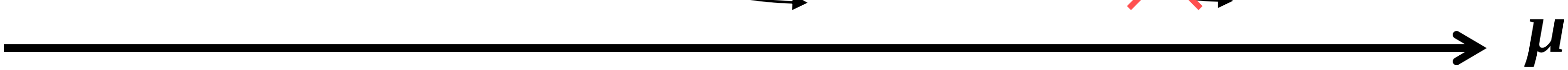
Insulator

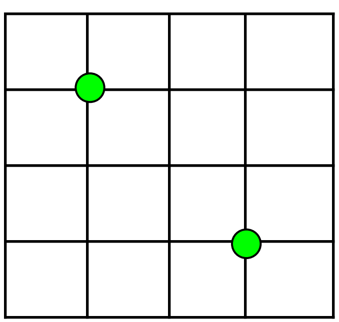
The model

Hardcore spinless fermions

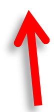
- spinless fermions
- hardcore $V_1 \to \infty$
- hopping t





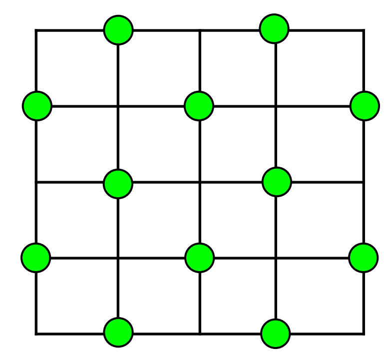


Fermi liquid



Supersymmetry

[Fendley, et. al. '03]



Insulator

Plan of the talk

Benefit of supersymmetry is twofold:

- Powerful tools
- Subtle interplay between kinetic and potential terms leading to quantum criticality and superfrustration

- The model: definition & supersymmetry basics

- Witten index: superfrustration
- Cohomology: quantum ground states as tilings
- Spectral flow: quantum criticality

- Recent developments
- Conclusions

Supersymmetry

Algebraic structure

supercharges Q^+, $Q^-=(Q^+)^\dagger$ and fermion number N_f :

$$(\mathrm{Q}^+)^2 = 0, \quad (\mathrm{Q}^-)^2 = 0, \quad [N_f, \mathrm{Q}^\pm] = \pm \mathrm{Q}^\pm$$

Hamiltonian defined as

$$H = \left\{\mathrm{Q}^+, \mathrm{Q}^-\right\}$$

satisfies

$$[H, \mathrm{Q}^+] = [H, \mathrm{Q}^-] = 0 \,, \quad [H, N_f] = 0$$

Supersymmetry

Spectrum:

- $E \geq 0$ for all states
- $E > 0$ pair into doublets (superpartners)

$$(|\psi\rangle, Q^{+}|\psi\rangle), \qquad Q^{-}|\psi\rangle = 0$$

- $E = 0$ states are singlets $Q^{+}|\psi\rangle = Q^{-}|\psi\rangle = 0$

High energy physics:

symmetry between bosonic and fermionic particles

Here:

- particles are spinless fermions (f)
- symmetry between "bosonic" (f even) and "fermionic" ($f \pm 1$ odd) states

The model

The supersymmetric lattice model

Supercharges for hardcore spinless fermions:

$$Q^+ = \sum_i c_i^\dagger \prod_{j \text{ next to } i} (1 - n_j), \quad Q^- = (Q^+)^\dagger, \quad n_j = c_j^\dagger c_j$$

Hamiltonian for 1D chain $H = \{Q^+, Q^-\}$

$$H = \sum_i [(1 - n_{i-1}) c_i^\dagger c_{i+1} (1 - n_{i+2}) + \text{h.c.}] + \sum_i n_{i-1} n_{i+1} - 2N_f + L$$

Hamiltonian for general lattice

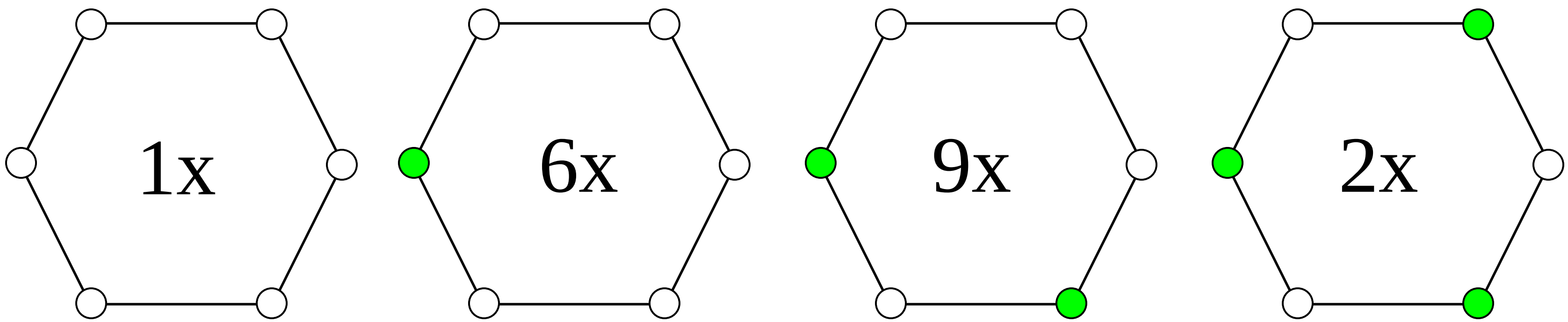
$$H = \sum_{<ij>} P_{<i>} c_i^\dagger c_j P_{<j>} + \sum_i P_{<i>} \qquad P_{<i>} = \prod_{j \text{ next to } i} (1 - n_j)$$

[Fendley - Schoutens - de Boer 2003]

Supersymmetry: example

6-site chain

Possible configurations for hardcore spinless fermions

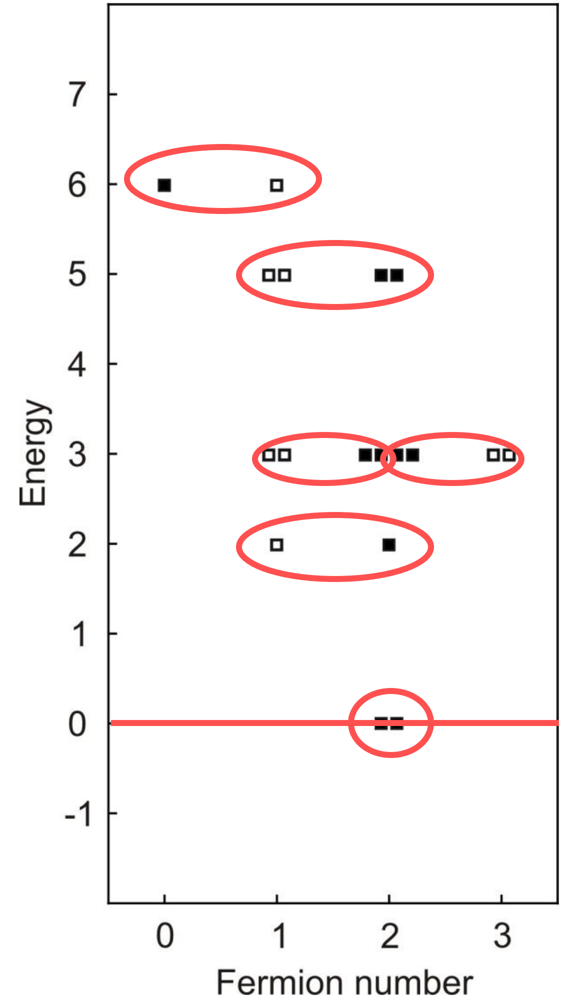


$|0\rangle$ $\quad c_i^+ |0\rangle$ $\quad c_i^+ c_{i+2}^+ |0\rangle$ $\quad c_i^+ c_{i+2}^+ c_{i+4}^+ |0\rangle$

Supersymmetry: example

Manifestly supersymmetric spectrum

- Energy is positive definite
- E>0 states form pairs between "fermionic" and "bosonic" states
- E=0 states are singlets



Witten index: superfrustration

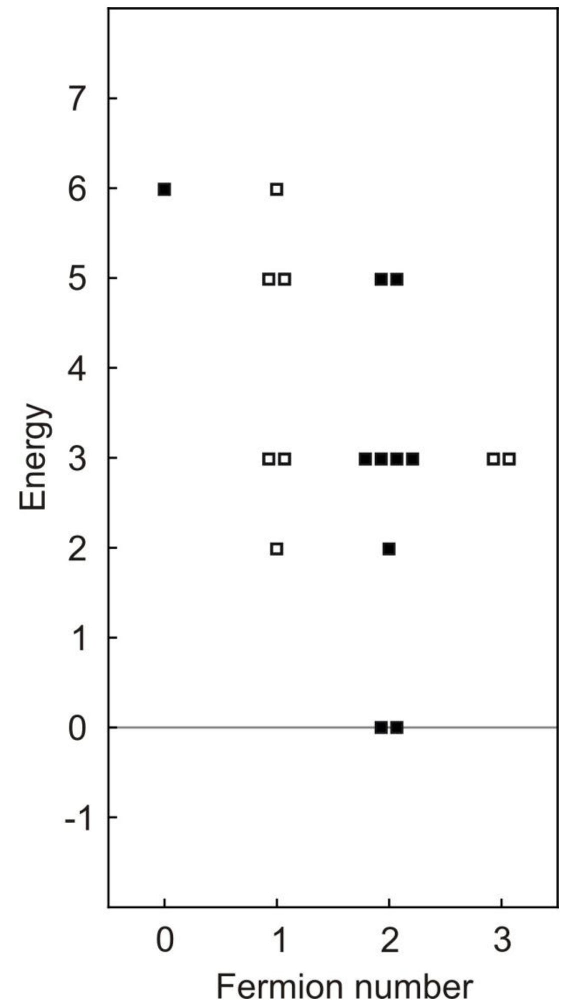
Powerful tool: Witten index

$$W = \mathrm{Tr}(-1)^{N_f}$$

“bosonic” states contribute +1, “fermionic” states contribute -1, so all superpartners cancel

$$\Rightarrow W = \#GS_B - \#GS_F$$

|***W***| is lower bound to number of ground states



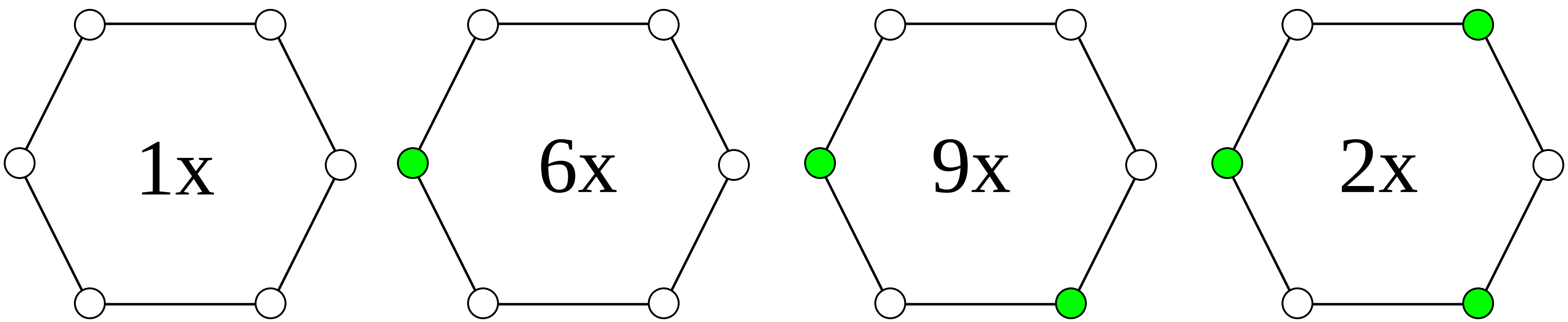
[Witten, ‘82]

Witten index: example

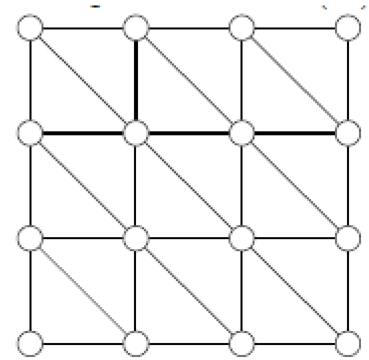
Purely combinatorial problem

$$W = \mathrm{Tr}(-1)^{N_f}$$

$$W = 1 - 6 + 9 - 2 = 2$$



Witten index



Triangular lattice

$N \times M$ sites with periodic BC

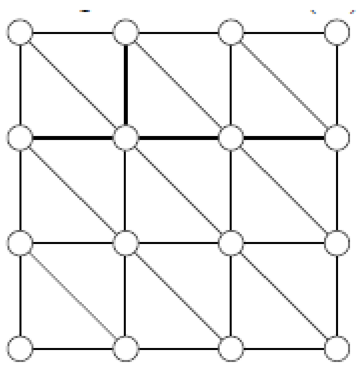
	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	-3	-5	1	11	9	-13	-31	-5	57
3	1	-5	-2	7	1	-14	1	31	-2	-65
4	1	1	7	-23	11	25	-69	193	-29	-279
5	1	11	1	11	36	-49	211	-349	811	-1064
6	1	9	-14	25	[illegible]	[illegible]	[illegible]	-415	1462	-4911
7	1	-13	1	-69	[illegible]	[illegible]	[illegible]	3403	-7055	5237
8	1	-31	31	193	[illegible]	[illegible]	[illegible]	881	-28517	50849
9	1	-5	-2	-29	881	1462	-7055	-28517	31399	313315
10	1	57	-65	-279	-1064	-4911	5237	50849	313315	950592
11	1	67	1	859	1651	12607	32418	159083	499060	2011307
12	1	-47	130	-1295	-589	-26006	-152697	-535895	-2573258	-3973827
13	1	-181	1	-77	-1949	67523	330331	-595373	-10989458	-49705161
14	1	-87	-257	3641	12611	-139935	-235717	5651377	4765189	-232675057
15	1	275	-2	-8053	-32664	272486	-1184714	-1867189	134858383	-702709340

$$|W| \sim 1.14^{NM}$$

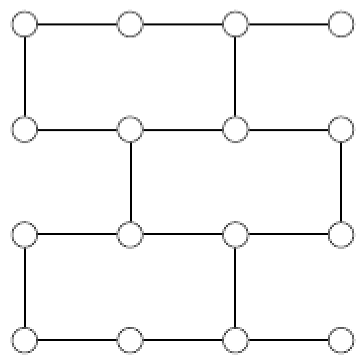
[van Eerten, ‘05]

Superfrustration - examples

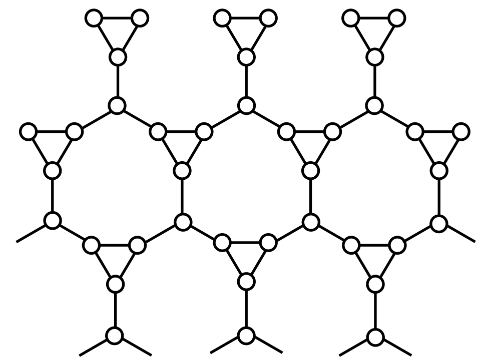
Tri



Hex



Martini



$|W| \sim 1.14^V$

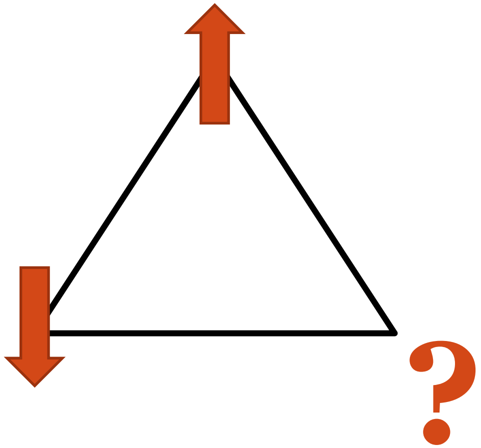
$|W| \sim 1.2^V$

$|W| \sim 1.17^V$

V is number of sites (2D volume)

[van Eerten, '05; Fendley, Schoutens, '05]

Superfrustration



Frustration
Competing terms in hamiltonian
→ multiple ground states

Supersymmetry
Subtle competition between kinetic and potential terms
→ for 2D lattices exponential ground state degeneracy

Violation of 3rd law of thermodynamics
Exponential number of ground states
→ finite zero temperature entropy

Superfrustration

´3-rule´

• repulsive interactions favor 3-site interparticle distance
• chemical potential favors higher densities

Combined with kinetic terms
→ quantum charge frustration at intermediate densities

[Fendley, Schoutens, ‘05]

Cohomology:
quantum ground states as tilings

Powerful tool: Cohomology

- Cohomology of Q
  GS are in 1-1 correspondence with cohomology elements
- More difficult to compute than Witten index
- But gives more information:
  - gives total number of gs
  - gives fermion number of gs
  - often gives relation between gs and geometric object

Cohomology technique

Lemma

Susy ground states are in 1-1 correspondence with the cohomology

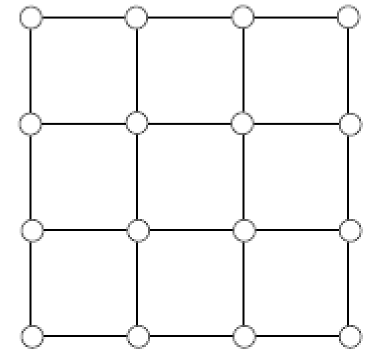
$$H_{Q,N_f} = \mathrm{Ker}[Q^+]_{N_f} / \mathrm{Im}[Q^+]_{N_f-1}$$

of Q^+ in the complex

$$\ldots \xrightarrow{Q^+} H_{N_f} \xrightarrow{Q^+} H_{N_f+1} \xrightarrow{Q^+} \ldots$$

Square lattice: Witten index

$N \times M$ sites with periodic BC

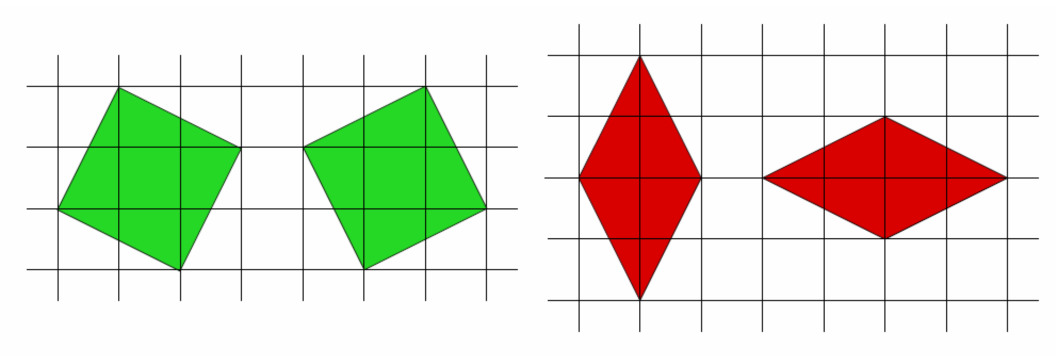


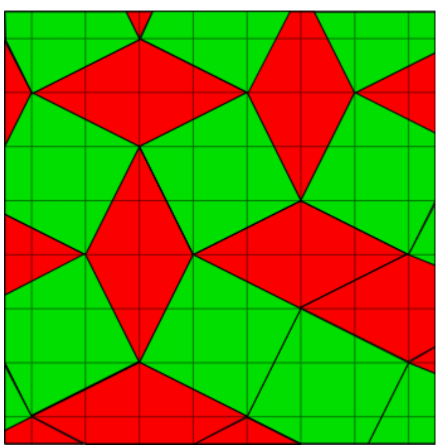
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	−1	1	3	1	−1	1	3	1	−1	1	3	1	−1	1	3	1	−1	1	3
3	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1
4	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1	7
5	1	1	1	1	−9	1	1	1	1	11	1	1	1	1	−9	1	1	1	1	11
6	1	−1	4	3	1	14	1	3	4	−1	1	18	1	−1	4	3	1	14	1	3
7	1	1	1	1	1	1	1	1	1	1	1	1	1	−27	1	1	1	1	1	1
8	1	3	1	7	1	3	1	7	1	43	1	7	1	3	1	7	1	3	1	47
9	1	1	4	1	1	4	1	1	40	1	1	4	1	1	4	1	1	76	1	1
10	1	−1	1	3	11	−1	1	43	1	9	1	3	1	69	11	43	1	−1	1	13
11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	1	3	4	7	1	18	1	7	4	3	1	166	1	3	4	7	1	126	1	7
13	1	1	1	1	1	1	1	1	1	1	1	1	−51	1	1	1	1	1	1	1
14	1	−1	1	3	1	−1	−27	3	1	69	1	3	1	55	1	451	1	−1	1	73
15	1	1	4	1	−9	4	1	1	4	11	1	4	1	1	174	1	1	4	1	11

[Fendley - Schoutens - van Eerten ‘05; Jonsson ‘06]

Square lattice (periodic BC)

- Cohomology of Q gives direct relation between ground states and rhombus tilings
- # of tiles = # of fermions





[LH - Schoutens '10]

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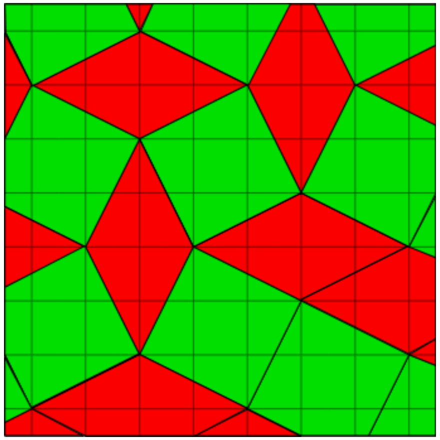


  - GS at intermediate filling

$$\nu = \frac{N_f}{NM} \in [1/5, 1/4] \cap \mathbb{Q}$$

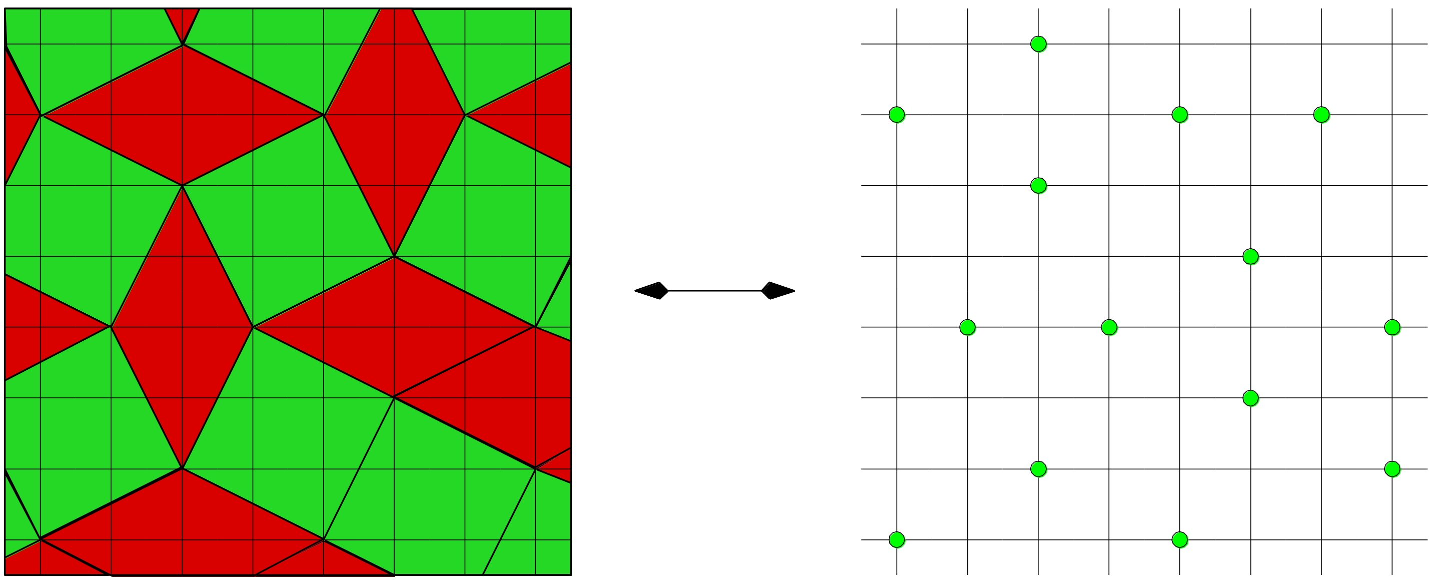
  - Sub-extensive GS entropy

$$S_{gs} \sim 0.46(N+M)$$

[LH - Schoutens '10]

Square lattice: ground states

- # gs grows exponentially with the linear size of the system
- zero energy ground states found at intermediate filling
- compelling evidence for critical edge modes
- what is the nature of these states?



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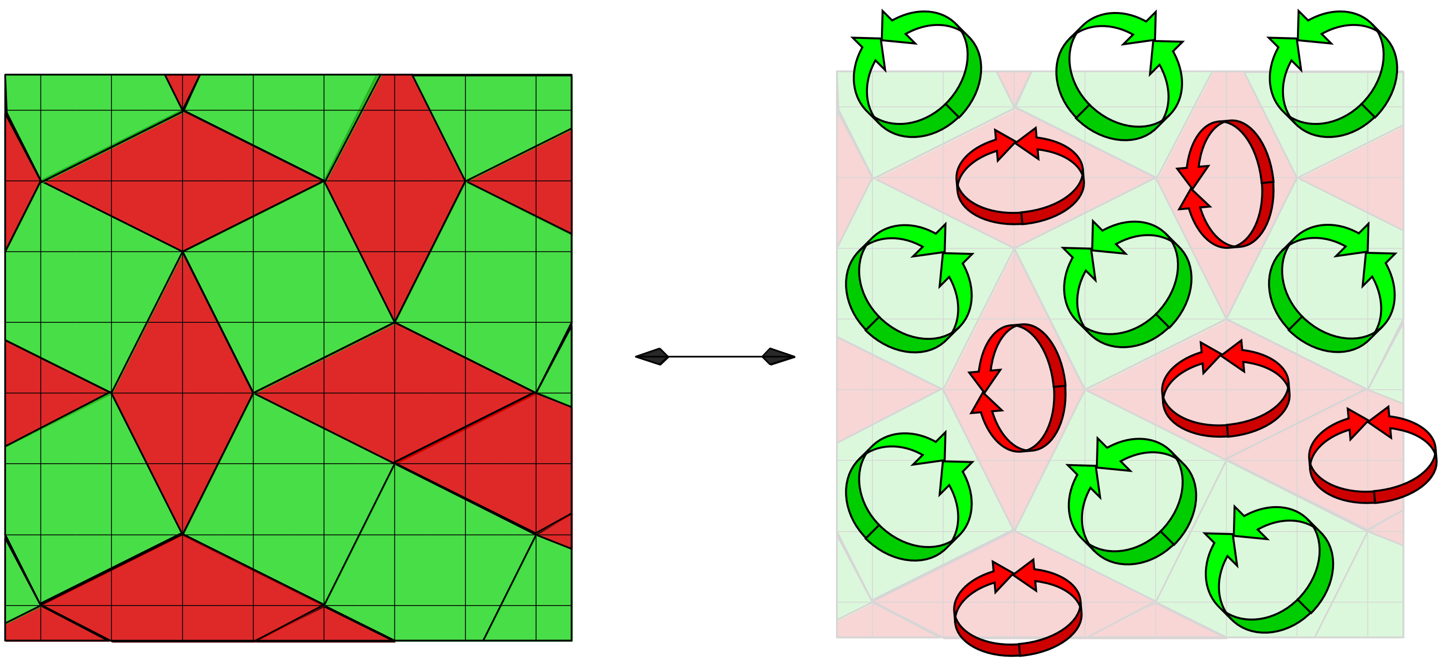
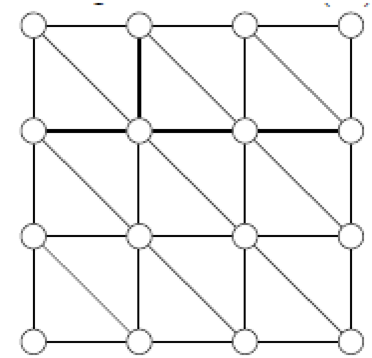


Witten index



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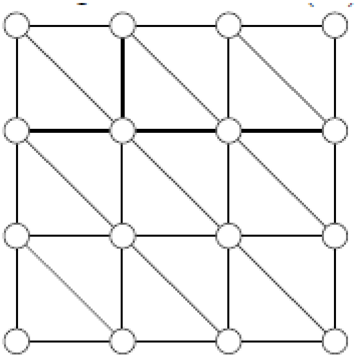
[van Eerten, '05]

Triangular lattice: ground states

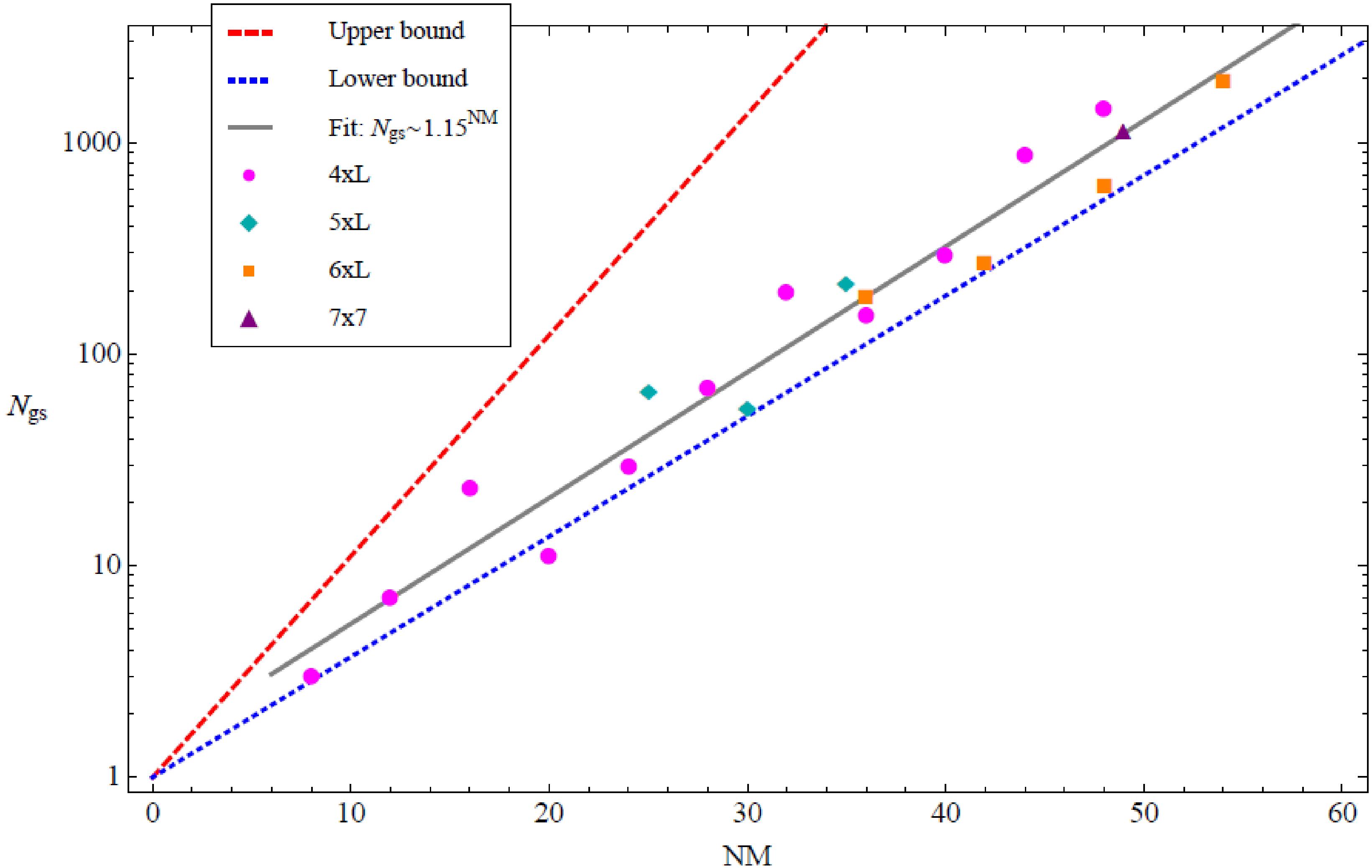


Full cohomology problem is very hard, but intermediate results:

- Upper bound on #gs $\#\text{gs} \lesssim (\sqrt{\phi})^{MN} \sim 1.27^{MN}$
- Lower bound on filling $\nu \in [1/7, 1/5] \cap \mathbb{Q}$
- Upper bound on filling $\nu \in [1/8, 1/4] \cap \mathbb{Q}$

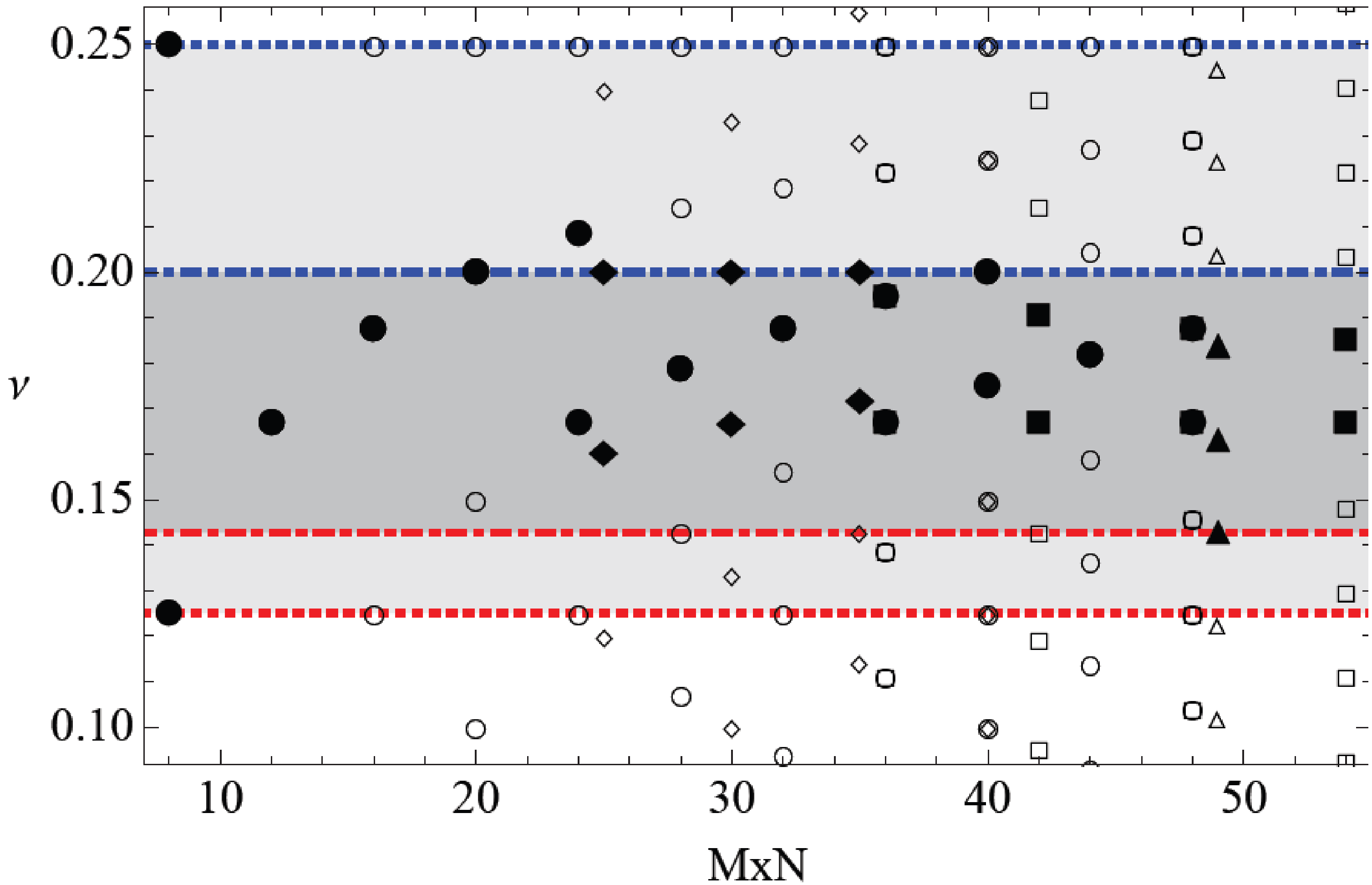
[Jonsson, '10; Engström, '09; LH, Mehta, Moran, Schoutens, Vala, '11]

Numerical results: Ground state degeneracy



[LH, Mehta, Moran, Schoutens, Vala, '11]

Numerical results: Filling fraction



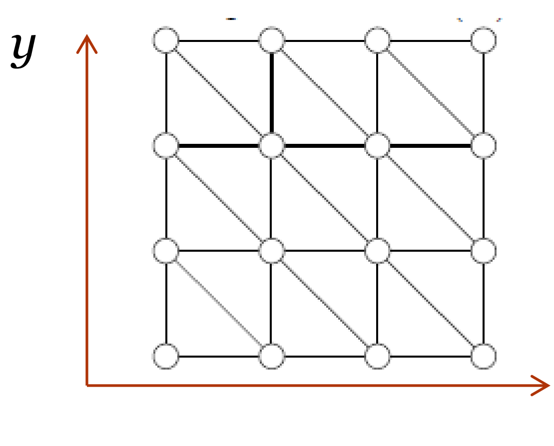
[LH, Mehta, Moran, Schoutens, Vala, '11]

Numerical results: Momentum

Flatband dispersion: Zero energy states at all momenta

Eigenvalues of translations:

$$t_x = e^{2\pi \imath k_x / N}, \; t_y = e^{2\pi \imath k_y / M}$$



6×9

$N_{gs} = 1926$, $W = 1462$

$f = 9$							$f = 10$						
(k_x, k_y)	0	1	2	3	4	5	(k_x, k_y)	0	1	2	3	4	5
0	9	3	3	9	3	3	0	33	31	31	33	31	31
1	4	5	3	4	5	3	1	32	31	31	32	31	31
2	4	3	5	4	3	5	2	32	31	31	32	31	31
3	7	4	4	7	3	4	3	32	32	31	32	30	31
4	4	5	3	4	5	3	4	32	31	31	32	31	31
5	4	3	5	4	3	5	5	32	31	31	32	31	31
6	7	4	3	7	4	4	6	32	31	30	32	31	32
7	4	5	3	4	5	3	7	32	31	31	32	31	31
8	4	3	5	4	3	5	8	32	31	31	32	31	31

[LH, Mehta, Moran, Schoutens, Vala, '11]

Spectral flow: quantum criticality

1D chain

$$H = \sum_i [(1 - n_{i-1}) c_i^\dagger c_{i+1} (1 - n_{i+2}) + \text{h.c.}] + \sum_i n_{i-1} n_{i+1} - 2N_f + L$$

- Periodic chain of length *L*: 2 gs for *L* mod 3 = 0; 1 gs otherwise

- Fermion number in ground state: $f = [L/3]$
- Bethe Ansatz solution (integrable)
- Continuum limit: $\mathcal{N}$=(2,2) SCFT with central charge c=1 → quantum critical, emergent spacetime supersymmetry

[Fendley-Schoutens-deBoer '03, Fendley-Nienhuis-Schoutens '03, Beccaria-DeAngelis '05, Fendley-Hagendorf '10 & '11, LH '11]

Relations/extensions

- XXZ spin chain – exact mapping
- SUSY Matrix models of Veneziano-Wosiek (via mapping to XXZ spin chain)
- Generalize hard-core constraint
  - Allow k particles to be nearest neighbors, but not k+1: M_k susy model ↔ k-th SCFT minimal model

**[Fendley , Nienhuis , Schoutens '03
Veneziano, Wosiek, '06]**

1D Quantum criticality

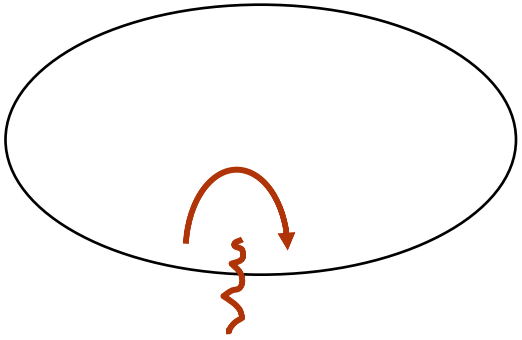
Numerical techniques to identify continuum CFT

- Finite size scaling of energy gap: E~1/L
- Entanglement entropy: S ~ c log L
- Superconformal field theories: Spectral flow:
  Energy depends parabolically on boundary twist
    + Accurate for small systems
    + No scaling required

Boundary twist: spectral flow

wave function picks up a phase exp($2\pi\imath\alpha$)
as a particle hops over a "boundary":

$$e^{2\pi\imath\alpha} c_L^\dagger c_1 + \text{ h.c.}$$

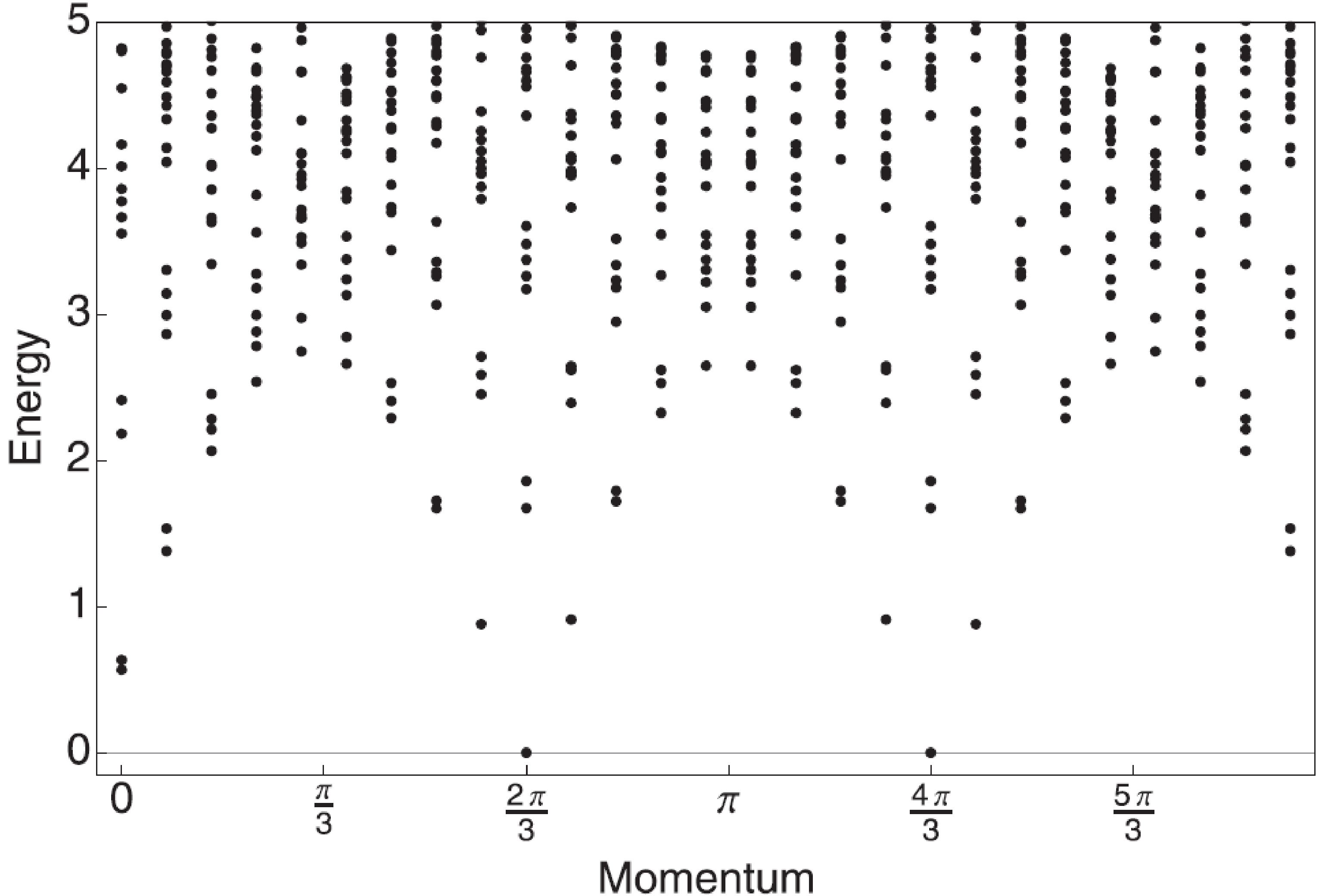


twist: α: 0 ↔ 1/2
"pbc ↔ apbc" = "R ↔ NS sector"

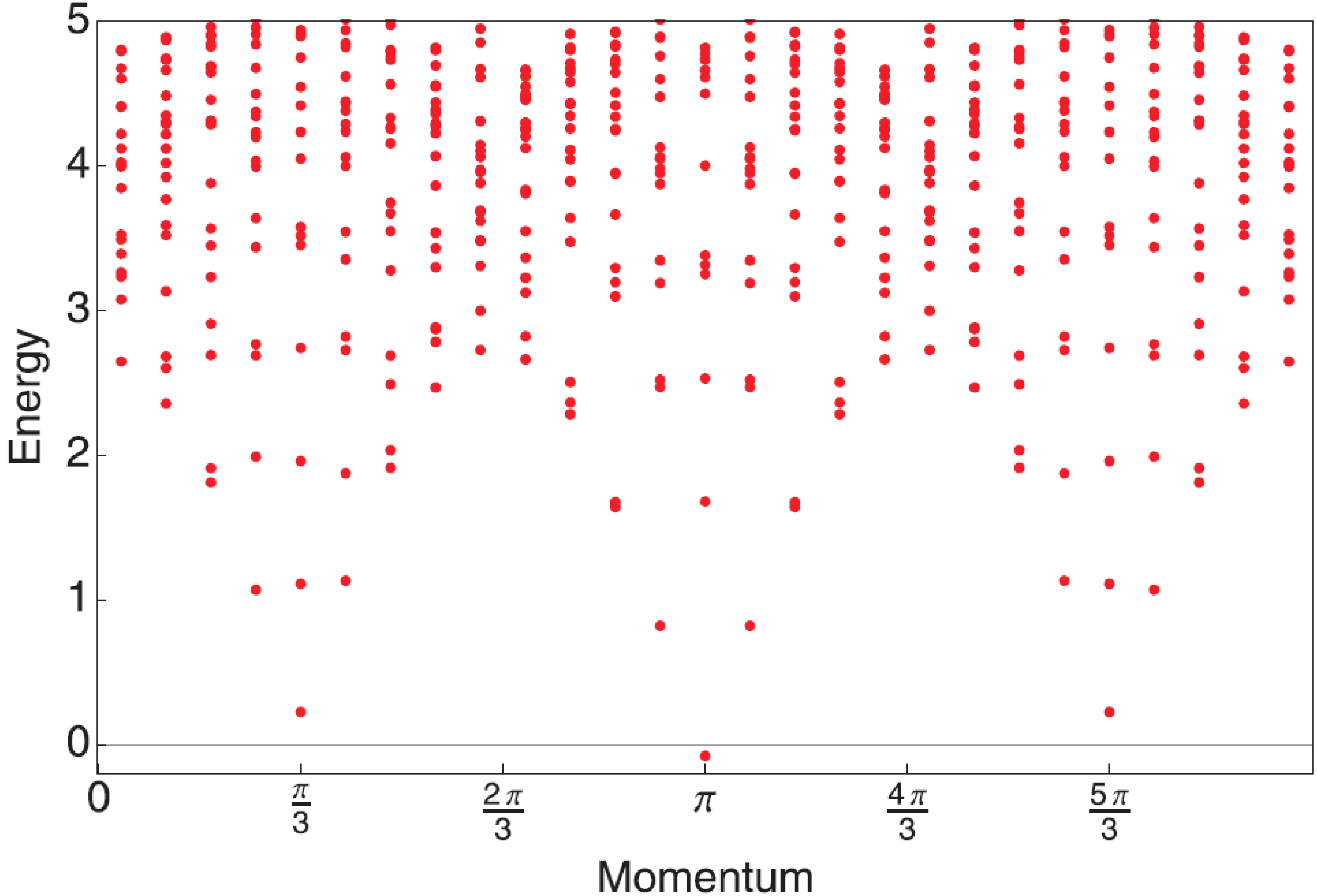
SCFT: energy is parabolic function of twist parameter

$$E_\alpha = E_0 - \alpha Q_0 + \alpha^2 c/3$$
$$\text{R}: \quad \alpha = 0$$
$$\text{NS}: \quad \alpha = 1/2$$

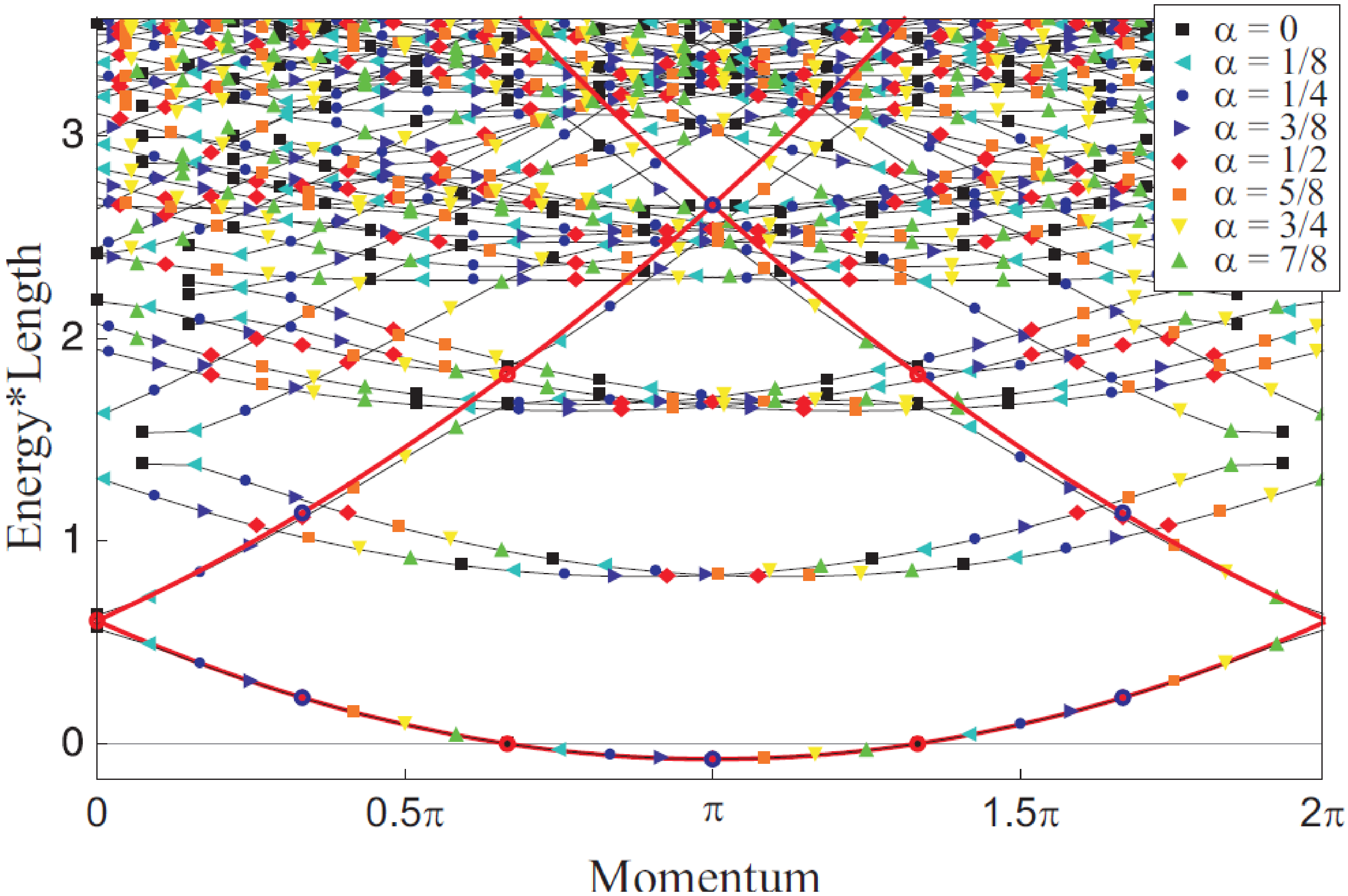
Chain Spectrum, L=27, N_f=9, PBC (α=0)



Chain Spectrum, L=27, N_f=9, APBC

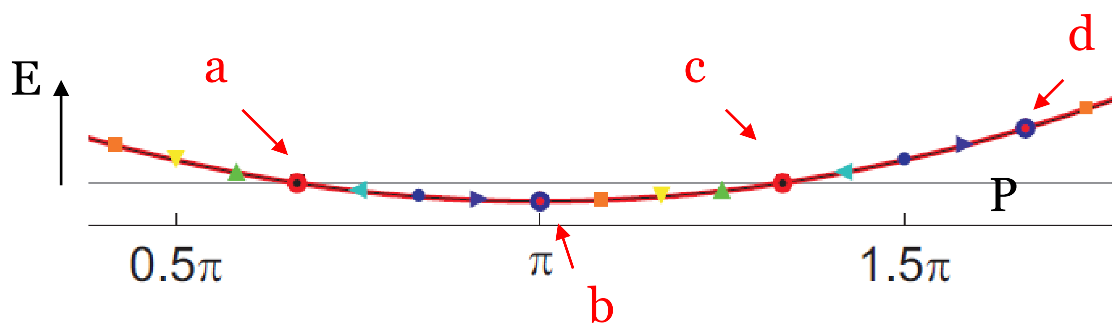


Spectral flow chain, L=27, N_f=9



What can we learn from spectral flow?

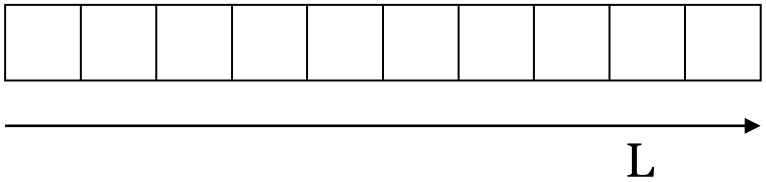
- 3 fit parameters
- 4 unknowns:
  E, Q_o, c and v_F
- → ratios
- for 1D chain we extract:

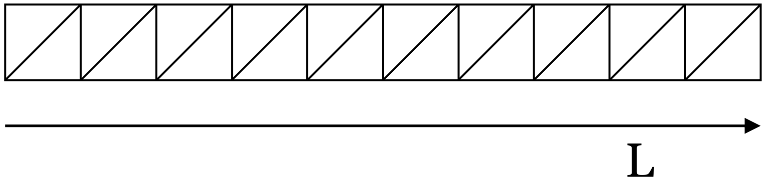


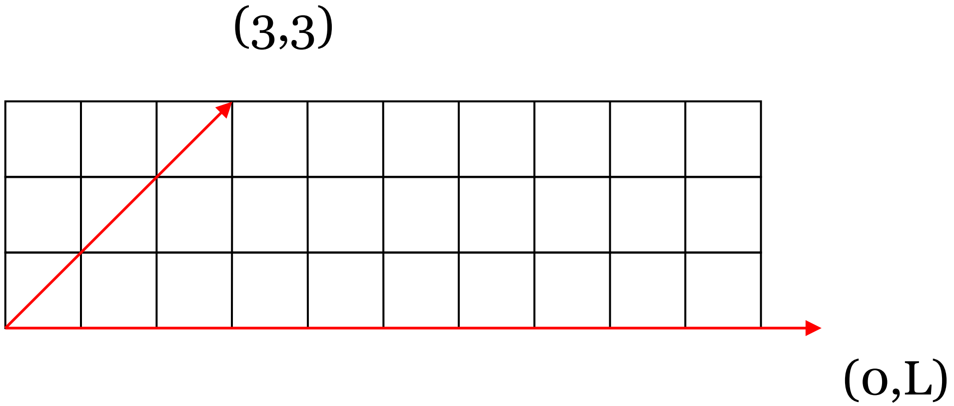
	numerics		SCFT	
state	E/c	Q_o/c	E/c	$Q_{o/c}$
a	0	-0.334	0	-1/3
b	-0.083	0	-1/12	0
c	0	0.342	0	1/3
d	0.254	0.675	1/4	2/3

Spectral flow for the square lattice

- square ladder
  (2,0)x(0,L)
- zigzag ladder
  (2,1)x(0,L)
  GS for $\nu \in [1/5, 1/4]$
- (3,3)x(0,L)
  fermions can hop past each other

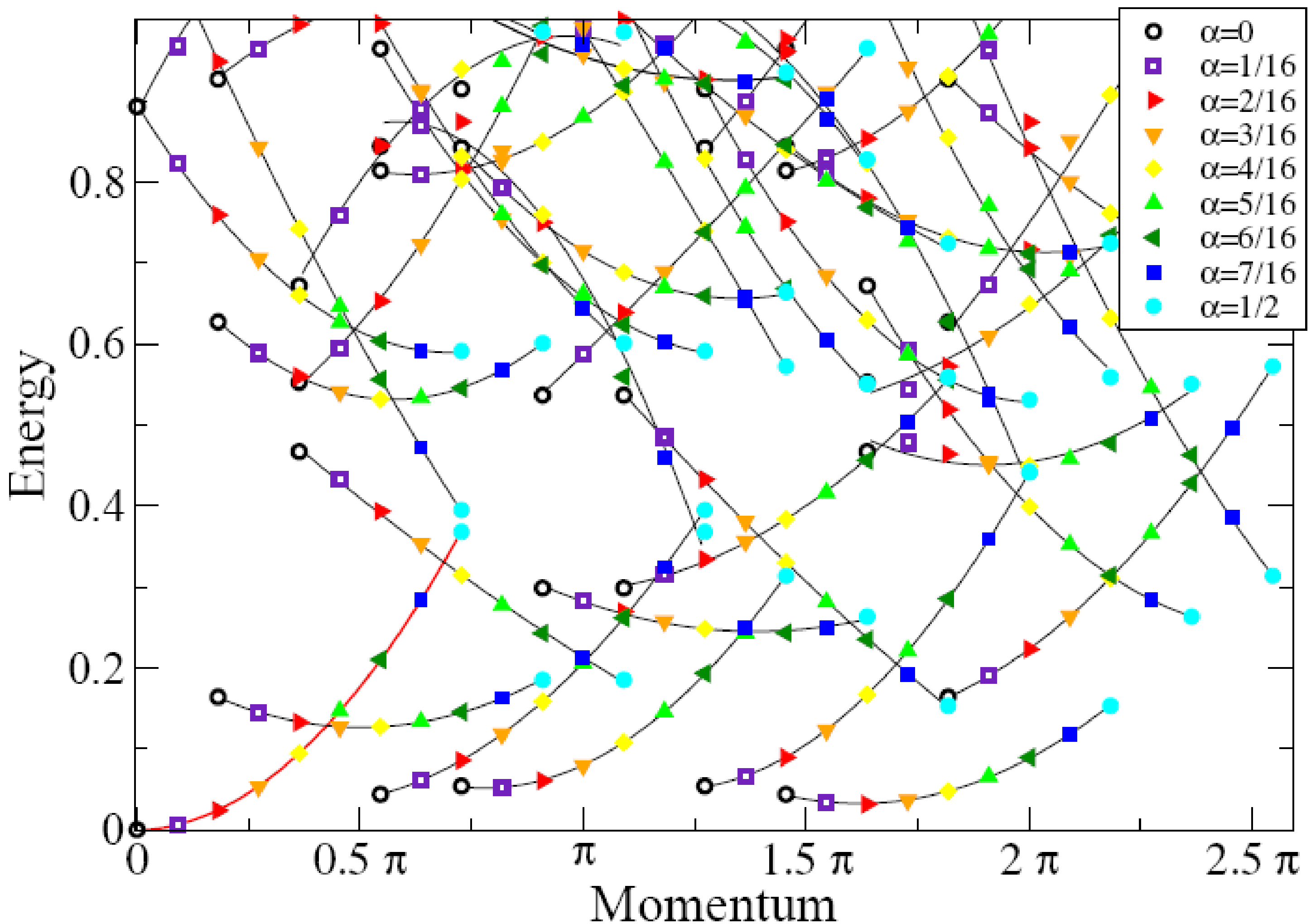






[LH - Halverson - Fendley - Schoutens '08]

Spectral flow results (3,3)x(0,11), N_f=8



[LH - Halverson - Fendley - Schoutens '08]

Spectral flow results

$(L,0) \times (3,3)$

N	f	E/c	Q/c
18	4	-0.0851	0.004
36	8	-0.0841	-0.002
15	4	0.0898	0.349
21	4	0.0850	0.337
24	5	0.0850	0.337
30	7	0.0853	0.338
33	**8**	**0.0855**	**0.338**

$(L,0) \times (1,2)$

N	f	E/c	Q/c
9	2	-0.0858	-0.005
18	4	-0.0842	-0.002
27	6	-0.0839	-0.001
17	4	0.0844	0.336
26	6	0.0840	0.335
35	8	0.0839	0.335
14	3	0.2666	0.701
23	5	0.2458	0.657
32	7	0.2432	0.652

$(L,0) \times (0,2)$

N	f	E/c	Q/c
16	4	-0.0897	-0.014
24	6	-0.0889	-0.012
32	8	-0.0885	-0.011
12	3	0.0911	0.350
20	5	0.0900	0.348
28	7	0.0894	0.347
14	4	0.0855	0.338
22	6	0.0849	0.337
30	8	0.0847	0.336

[LH - Halverson - Fendley - Schoutens '08]

Spectral flow results

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minimal models in SCFT: $c=\dfrac{3k}{k+2}$

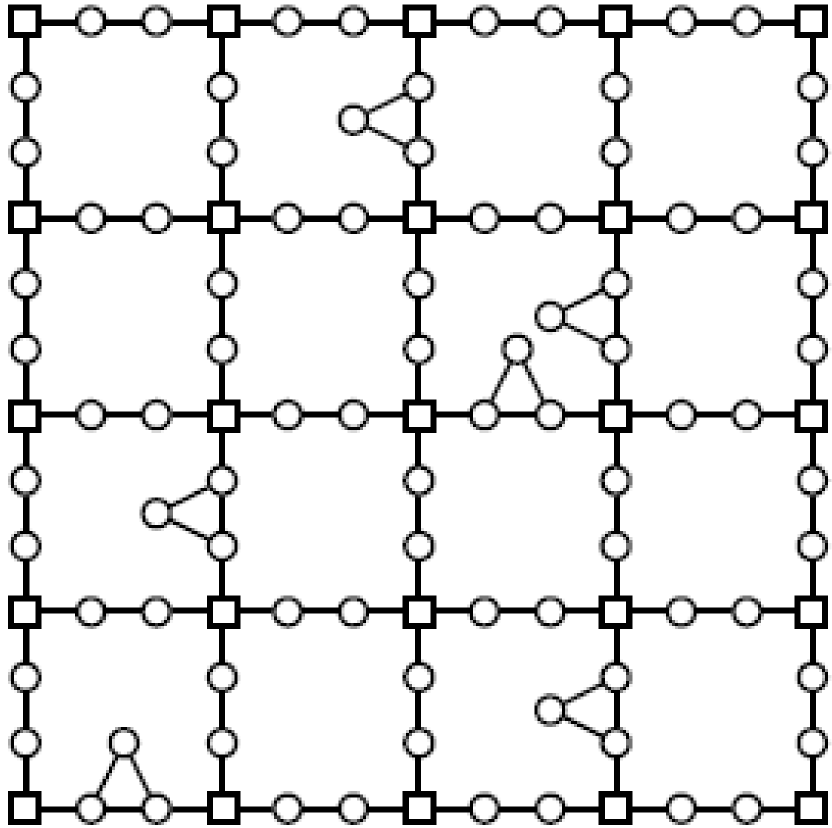
$$E/c=\frac{4l-k}{12k} \text{ and } Q_0/c=\frac{2l}{3k}$$

$$l=0:\ (-1/12,0),\ l=k/2:\ (1/12,1/3),\ l=k:\ (1/4,2/3)$$

Open problems/recent developments

Controllable gs degeneracy

- Triangles (t) as impurities
- $\#gs=2^t$
- Anomalous scaling of entanglement entropy



[LH-Swingle (to appear)]

Staggering: beyond gs counting in 2D

$$Q^+ = \sum \lambda_i^* c_i^\dagger P_{<i>}$$
$$Q^- = \sum \lambda_i c_i P_{<i>}$$

$$\lambda_i = \begin{cases} 1 & \text{for } i \in S_1 \\ y & \text{for } i \in S_2 \end{cases}$$

- Exact ground states from cohomology for lattices with reduced frustration
- Towards understanding the phases in the square lattice: system of decoupled chains in infinite staggering limit

[LH - Moran - Schoutens - Vala '11 ; Fendley – Hagendorf '10; '11; LH-Berg (work in progress)]

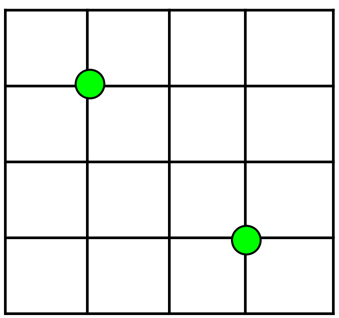
Open questions/future directions

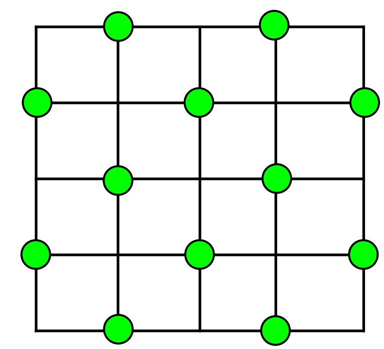
- AdS/CMT: apply gauge/gravity duality to condensed matter systems
- Toy model for AdS/CMT??
- Typical features: extensive ground state entropy, supersymmetry and large N
- What is continuum theory in 2D?
- Is there an emergent gauge symmetry?
- Can we include gauge symmetry explicitly on the lattice?
- …

Conclusions

Exact results for strongly interacting fermions

μ





Supersymmetry

Techniques

Witten index
Cohomology
Spectral flow
Staggering
…

Features

Superfrustration
Quantum criticality
Competing orders
…

Thank you