Hybrid Discretization of 4D N=4 Supersymmetric YM Theory

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based on work with

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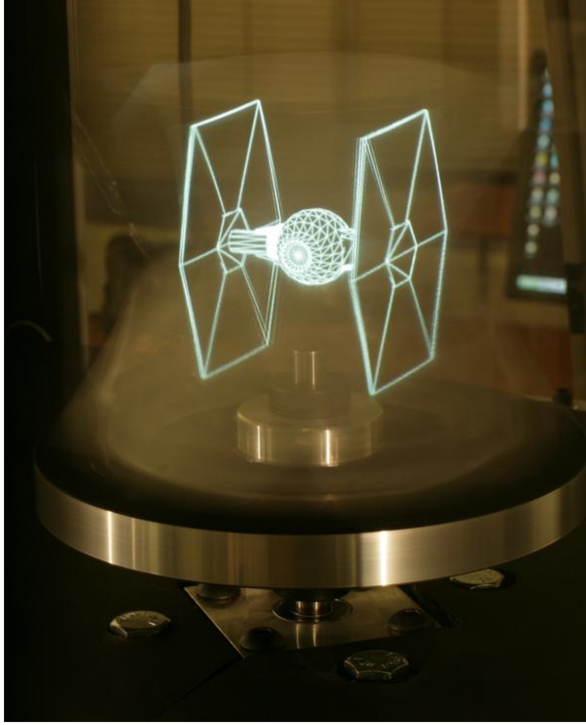
§ 0. Introduction and motivation



Holography

4D N=4 SYM (SU(N), λ) ⟺ supergravity (or superstring) on $AdS_5 \times S^5$

the gauge theory as a candidate of quantum gravity?

So far:
- ✓ BPS operators
- ✓ Wilson loops
- ✓ integrability
- ✓ etc…

What we need next?

- deeper understanding of the theory mathematically.
- **EXPERIMENT!** → **method to numerically simulate the 4D N=4 SYM**

What is geometry?

lattice gauge theory

- lattice: a network of discrete objects
- flat space-time appears in the "continuum limit".
- not merely a technique to compute something
- a definition of the field theory (up to sign-problem)

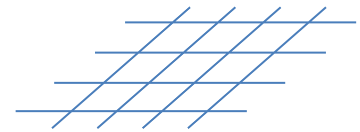
Is lattice only way to create space-time?

NO!

- tachyon condensation from non-BPS D-instantons to Dp-branes
- **Matrix** (appears in my talk soon)
- other smart ways in the future

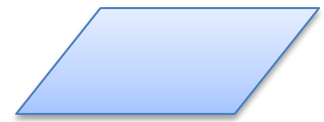
Hybrid discretization of 4d N=4 SYM
(Plan of Today's Talk)

Lattice formulation of
2d N=(8,8) SYM with plane-wave like deformation



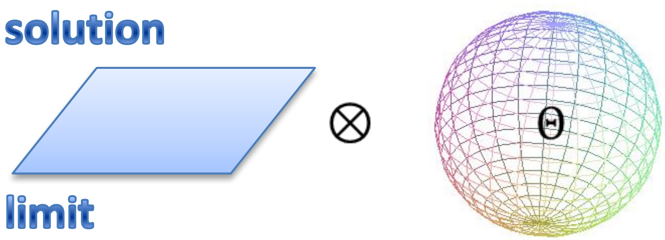
↓ **lattice continuum limit**

Continuum
2d N=(8,8) SYM with plane-wave like deformation



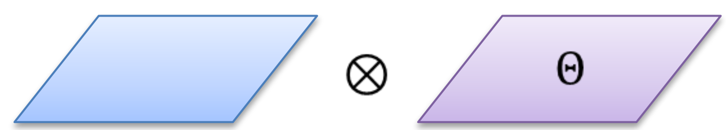
↓ **expand around fuzzy sphere solution**

4d N=4 SYM on $\mathbb{R}^2 \times S^2_\Theta$



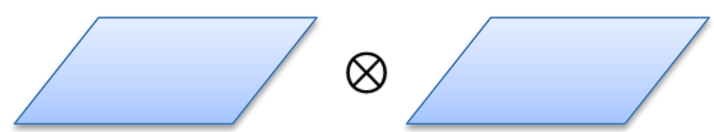
↓ **large N limit (matrix continuum limit)**

4d N=4 SYM on $\mathbb{R}^2 \times \mathbb{R}^2_\Theta$



↓ **$\Theta \to 0$ limit**

4d N=4 SYM on $\mathbb{R}^4$



§ 1. lattice formulation for 2d N=(8,8) SYM with plane-wave like deformation

Euclidean 2d N=(8,8) SYM (dimensional reduction of 10D N=1 SYM)

$$S_0 = \frac{2}{g_{2d}^2}\int d^2x\, Tr\left(\frac{1}{2}F_{12}^2 + \frac{1}{2}\left(D_\mu X^I\right)^2 - \frac{1}{4}[X^I, X^J]^2 + \frac{1}{2}\Psi^T(D_1 + \gamma_2\, D_2)\Psi + \frac{i}{2}\Psi^T\gamma_I[X^I, \Psi]\right)$$

where $\mu = 1,2, \quad I, J = 3,4,\cdots,10.$

fields

- $\boldsymbol{A_\mu}$: gauge field
- $\boldsymbol{X^I}$: 8 scalar fields
- $\boldsymbol{\Psi}$: 16-component spinor

symmetries

- 16 supersymmetries
- SO(8) R-symmetry

REWRITE →

2 SUSY $\boldsymbol{Q_\pm}$
and
$\boldsymbol{SU(2)_R}$
to be manifest

Field redefinition (BTFT form)

$$X^I \Longrightarrow \begin{cases} X_i & (i = 3,4) \\ B_A & (A = 1,2,3) \\ C, \phi_+, \phi_- \end{cases} \qquad \Psi \Longrightarrow \begin{cases} \psi_{+\mu}, \rho_{+i}, \chi_{+A}, \eta_+ \\ \psi_{-\mu}, \rho_{-i}, \chi_{-A}, \eta_- \end{cases}$$

$\begin{pmatrix} \psi_{+\mu} \\ \psi_{-\mu} \end{pmatrix}, \quad \begin{pmatrix} \chi_{+A} \\ \chi_{-A} \end{pmatrix}, \quad \begin{pmatrix} \eta_+ \\ -\eta_- \end{pmatrix}, \quad \begin{pmatrix} Q_+ \\ Q_- \end{pmatrix}$: SU(2) doublets $\qquad \begin{pmatrix} \phi_+ \\ C \\ -\phi_- \end{pmatrix}$: SU(2) triplet

$$\left.\begin{aligned}
&Q_\pm A_\mu = \psi_{\pm\mu}, \quad Q_\pm \psi_{\pm\mu} = \pm i D_\mu \phi_\pm, \quad Q_\mp \psi_{\pm\mu} = \frac{i}{2} D_\mu C \mp \tilde{H}_\mu, \\
&Q_\pm \tilde{H}_\mu = [\phi_\pm, \psi_{\mp\mu}] \mp \frac{1}{2} [C, \psi_{\pm\mu}] \mp \frac{i}{2} D_\mu \eta_\pm, \\
&Q_\pm X_i = \rho_{\pm i}, \quad Q_\pm \rho_{\pm i} = \mp [X_i, \phi_\pm], \quad Q_\mp \rho_{\pm i} = -\frac{1}{2}[X_i, C] \mp \tilde{h}_i, \\
&Q_\pm \tilde{h}_i = [\phi_\pm, \rho_{\mp i}] \mp \frac{1}{2} [C, \rho_{\pm i}] \pm \frac{1}{2} [X_i, \eta_\pm], \\
&Q_\pm B_A = \chi_{\pm A}, \quad Q_\pm \chi_{\pm A} = \pm [\phi_\pm, B_A], \quad Q_\mp \chi_{\pm A} = -\frac{1}{2}[B_A, C] \mp H_A, \\
&Q_\pm H_A = [\phi_\pm, \chi_{\mp A}] \pm \frac{1}{2} [B_A, \eta_\pm] \mp \frac{1}{2} [C, \chi_{\pm A}], \\
&Q_\pm C = \eta_\pm, \quad Q_\pm \eta_\pm = \pm [\phi_\pm, C], \quad Q_\mp \eta_\pm = \mp [\phi_+, \phi_-], \\
&Q_\pm \phi_\pm = 0, \quad Q_\mp \phi_\pm = \mp \eta_\pm.
\end{aligned}\right\} \quad \begin{aligned} &Q_\pm^2 = \{Q_+, Q_-\} = 0 \\ &\text{up to gauge trans.} \end{aligned}$$

Action in Q_+Q_--exact form

$$S_0 = Q_+ Q_- \mathcal{F}^{(0)}$$

$$\mathcal{F}^{(0)} = \frac{1}{g_{2d}^2} \int d^2x \, \mathrm{Tr}\Big\{ -iB_A\Phi_A - \frac{1}{3}\epsilon_{ABC}B_A[B_B, B_C] - \psi_{+\mu}\psi_{-\mu} - \rho_{+i}\rho_{-i} - \chi_{+A}\chi_{-A} - \frac{1}{4}\eta_+\eta_- \Big\},$$

$$\Phi_1 = 2(-D_1X_3 - D_2X_4), \quad \Phi_2 = 2(-D_1X_4 + D_2X_3),$$
$$\Phi_3 = 2(-F_{12} + i[X_3, X_4])$$

manifestly invariant under $Q_\pm$-transformation

deformation of $Q_\pm$ (to obtain plane-wave like deformation)

(A)
$$\begin{cases} Q_\pm A_\mu = \psi_{\pm\mu}, \quad Q_\pm \psi_{\pm\mu} = \pm i D_\mu \phi_\pm, \quad Q_\mp \psi_{\pm\mu} = \frac{i}{2} D_\mu C \mp \tilde{H}_\mu, \\ Q_\pm \tilde{H}_\mu = [\phi_\pm, \psi_{\mp\mu}] \mp \frac{1}{2}[C, \psi_{\pm\mu}] \mp \frac{i}{2} D_\mu \eta_\pm + \frac{M}{3}\psi_{\pm\mu}, \end{cases}$$

(X)
$$\begin{cases} Q_\pm X_i = \rho_{\pm i}, \quad Q_\pm \rho_{\pm i} = \mp [X_i, \phi_\pm], \quad Q_\mp \rho_{\pm i} = -\frac{1}{2}[X_i, C] \mp \tilde{h}_i, \\ Q_\pm \tilde{h}_i = [\phi_\pm, \rho_{\mp i}] \mp \frac{1}{2}[C, \rho_{\pm i}] \pm \frac{1}{2}[X_i, \eta_\pm] + \frac{M}{3}\rho_{\pm i}, \end{cases}$$

(B)
$$\begin{cases} Q_\pm B_A = \chi_{\pm A}, \quad Q_\pm \chi_{\pm A} = \pm [\phi_\pm, B_A], \quad Q_\mp \chi_{\pm A} = -\frac{1}{2}[B_A, C] \mp H_A, \\ Q_\pm H_A = [\phi_\pm, \chi_{\mp A}] \pm \frac{1}{2}[B_A, \eta_\pm] \mp \frac{1}{2}[C, \chi_{\pm A}], + \frac{M}{3}\chi_{\pm A} \end{cases}$$

(C)
$$\begin{cases} Q_\pm C = \eta_\pm, \quad Q_\pm \eta_\pm = \pm [\phi_\pm, C] + \frac{2M}{3}\phi_\pm, \\ Q_\mp \eta_\pm = \mp [\phi_+, \phi_-] \pm \frac{M}{3} C, \quad Q_\pm \phi_\pm = 0, \quad Q_\mp \phi_\pm = \mp \eta_\pm \end{cases}$$

Nilpotency

generators of $SU(2)_R$

$$Q_\pm^2 = (\text{infinitisimal gauge transformation by } \pm\phi_\pm) \ \pm\frac{M}{3} J_{\pm\pm},$$
$$\{Q_+, Q_-\} = (\text{infinitisimal gauge transformation by } C) \ -\frac{M}{3} J_0.$$

corresponding deformation of the action

$$S = \left(Q_+Q_- - \frac{M}{3}\right)(\mathcal{F}_0 + \Delta\mathcal{F}) = S_0 + \Delta S$$

$$\Delta\mathcal{F} = -\frac{1}{g_{2\mathrm{d}}^2}\int d^2x \mathrm{Tr}\left[\sum_{A=1}^{3}\frac{M}{9}B_A^2 + \sum_{i=3}^{4}\frac{2M}{9}X_i^2\right]$$

$$\Delta S = \frac{1}{g_{2d}^2}\int d^2x\ \mathrm{Tr}\Big\{\frac{2M^2}{81}\left(B_A^2 + X_i^2\right) - \frac{M}{2}\epsilon_{abc}X_a[X_b, X_c] + \frac{M^2}{9}\left(X_a^2\right)$$
$$+\frac{2M}{3}\psi_{+\mu}\psi_{-\mu} + \frac{2M}{9}\rho_{+i}\rho_{-i} + \frac{4M}{9}\chi_{+A}\chi_{-A} - \frac{M}{6}\eta_+\eta_-$$
$$-\frac{4iM}{9}B_3\left(F_{12} + i[X_3, X_4]\right)\Big\}.$$

※ This action is not exact w.r.t $Q_\pm$ but invariant by the $Q_\pm$-transformation.

explicit form of the action

$$S_B = \frac{2}{g^2}\int d^2y \mathrm{Tr}\left\{\frac{1}{2}F_{12}^2 + \frac{1}{2}(\mathcal{D}_\mu X_a)^2 + \frac{1}{2}(\mathcal{D}_\mu X_i)^2 - \frac{1}{4}\left[X_i, X_j\right]^2 - \frac{1}{2}[X_a, X_i]^2 + \frac{1}{2}\left(\frac{M}{3}X_a + \frac{i}{2}\epsilon_{abc}[X_b, X_c]\right)^2 + \frac{M^2}{81}X_i^2 - i\frac{2M}{9}X_7\left(F_{12} + i[X_3, X_4]\right)\right\},$$

$$S_F = \frac{2}{g^2}\int d^2y \mathrm{Tr}\left\{\frac{i}{2}\bar\psi_{r\alpha}(\hat\gamma_\mu)_{rs}\mathcal{D}_\mu\psi_{s\alpha} - \frac{1}{2}\bar\psi_{r\alpha}(\sigma_a)_{\alpha\beta}[X_a, \psi_{r\beta}] - \frac{1}{2}\bar\psi_{r\alpha}(\hat\gamma_i)_{rs}[X_i, \psi_{s\alpha}] - \frac{1}{2}m_r\bar\psi_{r\alpha}\psi_{r\alpha},\right\},$$

$$S_{\mathrm{g.f.}} = \frac{2}{g^2}\int d^2y \mathrm{Tr}\,\frac{1}{2}\left(\partial_\mu A_\mu + i[X_a^{(0)}, X_a]\right)^2,$$

$$S_{\mathrm{gh}} = \frac{2}{g^2}\int d^2y \mathrm{Tr}\left\{-\partial_\mu\bar c\mathcal{D}_\mu c + [X_a^{(0)}, \bar c][X_a, c]\right\},$$

fuzzy sphere configuration is a classical solution.

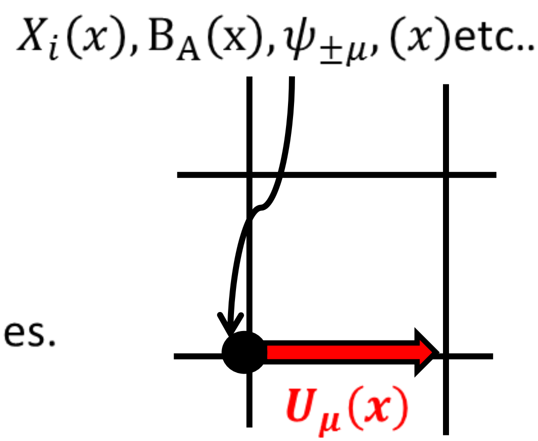
$$\{\hat\gamma_I, \hat\gamma_J\} = -2\delta_{IJ}.$$

$$m_r = \left(\frac{M}{9}, \frac{M}{9}, \frac{M}{3}, \frac{M}{3}, \frac{2M}{9}, \frac{2M}{9}, \frac{2M}{9}, \frac{M}{3}\right),$$

Lattice formulation of this theory (Sugino's formulation)

continuum theory

$$S_{\rm cont} = \left(Q_+ Q_- - \frac{M}{3} \right) \mathcal{F}_{\rm cont}$$



- $A_\mu \to U_\mu$ (link variables)
- others are site variables
- $Q_\pm$ transformation to the lattice variables.

lattice theory

$$S_{\rm lat} = \left(Q_+^{\rm lat} Q_-^{\rm lat} - \frac{M}{3} \right) \mathcal{F}_{\rm lat}$$

Important properties

① The lattice theory preserves 2 supercharges $Q_\pm$.

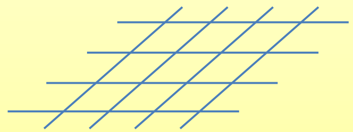
② fuzzy S^2 is still $Q_\pm$-invariant solution

③ All the scalar flat directions are lifted up by the mass deformation, that is, the fuzzy sphere solution is an isolated solution.

④ We do not need any fine tuning in taking the lattice continuum limit.

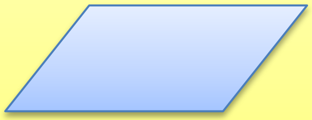
We can take this limit safely!

Lattice formulation of mass deformed 2d N=(8,8) SYM



⇩ lattice continuum limit

Continuum mass deformed 2d N=(8,8) SYM



§ 1.5 From matrix to fuzzy sphere (preparation to the next step)

Let us consider a matrix model with the action,

$$S = \frac{1}{g_0^2} Tr\left[\frac{1}{4}\left(i[X_i, X_j] + \mu\epsilon_{ijk}X_k\right)^2\right]$$

$X_i: nk \times nk$ hermitian matrix

Expanding the matrix X_i around a classical solution as

$$X_i = \mu\hat{L}_i + A_i\,, \quad \left[\hat{L}_i, \hat{L}_j\right] = i\epsilon_{ijk}\hat{L}_k$$

we obtain

$$S = \frac{1}{g_0^2} Tr\left[\frac{1}{4}\left(i\mu\left[\hat{L}_i, A_j\right] - i\mu\left[\hat{L}_j, A_i\right] + \mu\epsilon_{ijk}A_k + i\left[A_i, A_j\right]\right)^2\right]$$

$$X_i = \mu \hat{L}_i + A_i$$

specific solution

$\hat{\boldsymbol{L}}_i = \boldsymbol{1}_k \otimes \boldsymbol{L}_i^{(j)}$, $L_i^{(j)}$: spin j representation of su(2) $(n = 2j + 1)$

$$\left[L_i^{(j)}, L_j^{(j)}\right] = i\epsilon_{ijk} L_k^{(j)}$$

$$L_\pm^{(j)} |j, r> = \sqrt{(j \mp 1)(j \pm 1 + 1)}\, |j, r \pm 1>$$

$$L_3^{(j)} |j, r> = r|j, r>$$

$$\begin{pmatrix} \boldsymbol{L}_i^{(j)} & & \\ & \ddots & \\ & & \boldsymbol{L}_i^{(j)} \end{pmatrix}$$

expand A_i by **fuzzy spherical harmonics**

$$\boldsymbol{A}_i = \sum_{J=0}^{2j} \sum_{m=-J}^{J} \boldsymbol{a}_{Jm,i} \otimes \hat{\boldsymbol{Y}}_{Jm}^{(jj)}$$

$$\hat{Y}_{Jm}^{(jj)} = \sqrt{n} \sum_{r,r'=-j}^{j} (-1)^{-j+r'} C_{jr,j-r}^{Jm} |jr><jr'|$$

satisfying

$$\left[L_\pm^{(j)}, \hat{Y}_{Jm}^{(jj)}\right] = \sqrt{(J \mp m)(J \pm m + 1)}\hat{Y}_{Jm\pm1}^{(jj)}$$

$$\left[L^{(j)2}, \hat{Y}_{Jm}^{(jj)}\right] = J(J+1)\hat{Y}_{Jm}^{(jj)}$$

$$\left(\hat{Y}_{Jm}^{(jj)}\right)^\dagger = (-1)^m \hat{Y}_{J-m}^{(jj)}$$

fuzzy spherical harmonic $\hat{Y}^{(jj)}_{Jm}$ vs spherical harmonics $Y_{Jm}(\Omega)$

$$\left[L^{(j)}_{\pm}, \hat{Y}^{(jj)}_{Jm}\right] = \sqrt{(J \mp m)(J \pm m + 1)}\hat{Y}^{(jj)}_{Jm\pm 1}$$

$$\left[L^{(j)}_{3}, \hat{Y}^{(jj)}_{Jm}\right] = m\hat{Y}^{(jj)}_{Jm}$$

$$\left[L^{(j)2}, \hat{Y}^{(jj)}_{Jm}\right] = J(J+1)\hat{Y}^{(jj)}_{Jm}$$

$$\frac{1}{n}\mathrm{tr}_n\left[\left(\hat{Y}^{(jj)}_{Jm}\right)^{\dagger}\hat{Y}^{(jj)}_{J'm'}\right] = \delta_{JJ'}\delta_{mm'}$$

angular momentum operators

$$\mathcal{L}_{\pm}Y_{Jm}(\Omega) = \sqrt{(J \mp m)(J \pm m + 1)}Y_{Jm\pm 1}(\Omega)$$

$$\mathcal{L}_{3}Y_{Jm}(\Omega) = mY_{Jm}(\Omega)$$

$$\mathcal{L}^{2}Y_{Jm}(\Omega) = J(J+1)Y_{Jm}(\Omega)$$

$$\int d\Omega\left[(Y_{Jm}(\Omega))^{\dagger}Y_{J'm'}(\Omega)\right] = \delta_{JJ'}\delta_{mm'}$$

mapping rule

$$L^{(j)}_{i} \longleftrightarrow \mathcal{L}_i$$

$$\hat{Y}^{(jj)}_{Jm} \longleftrightarrow Y_{Jm}(\Omega)$$

$$\frac{1}{n}tr_n \longleftrightarrow \int d\Omega$$

different point

$$\frac{1}{n}\mathrm{tr}_n\left\{\left(\hat{Y}^{(jj)}_{J_1m_1}\right)^\dagger \hat{Y}^{(jj)}_{J_2m_2}\hat{Y}^{(jj)}_{J_3m_3}\right\}$$
$$=(-1)^{J_1+2j}\sqrt{n(2J_2+1)(2J_3+1)}$$
$$\times C^{J_1m_1}_{J_2m_2,J_3m_3}\begin{Bmatrix} J_1 & J_2 & J_3 \\ j & j & j \end{Bmatrix}$$

$$\int d\Omega\left\{\left(Y_{J_1m_1}(\Omega)\right)^\dagger Y_{J_2m_2}(\Omega)Y_{J_3m_3}(\Omega)\right\}$$
$$=\sqrt{\frac{(2J_2+1)(2J_3+1)}{2J_1+1}}$$
$$\times C^{J_1m_1}_{J_2m_2,J_3m_3}C^{J_10}_{J_20,J_30}$$

$J_1, J_2, J_3 \ll j$

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ j & j & j \end{Bmatrix} \sim (-1)^{-J_1+2j}\sqrt{\frac{1}{n(2J_1+1)}}C^{J_10}_{J_20,J_30}$$

$\hat{Y}^{(jj)}_{Jm}$ gives a matrix regularization of $Y_{Jm}(\Omega)$

claim

$$S=\frac{1}{g_0^2}Tr\left[\frac{1}{4}\left(i\mu[\hat{L}_i,A_j]-i\mu[\hat{L}_j,A_i]+\mu\epsilon_{ijk}A_k+i[A_i,A_j]\right)^2\right]$$

defines a theory on **fuzzy S^2 with radius $1/\mu$**, which is a matrix regularization of a theory on S^2 defined by the action,

$$S=\frac{1}{g^2}\int d\Omega\left[\frac{1}{4}\left(i\mu\mathcal{L}_iA_j(\Omega)-i\mu\mathcal{L}_jA_i(\Omega)+\mu\epsilon_{ijk}A_k(\Omega)+i[A_i(\Omega),A_j(\Omega)]\right)^2\right]$$

Continuum limit of matrix regularization

Let us consider the case $j \gg O(1)$ and look around **the north pole**;

$$|j, j-s> \equiv |s\gg \quad (s \ll j)$$

Around this region,

$$L_+|j,j-s>\sim\sqrt{(2j+1)s}|j,j-s+1>$$
$$L_-|j,j-s>\sim\sqrt{(2j+1)(s+1)}|j,j-s-1>$$

We can regard $L_\pm$ as the **anihilation/creation operator of harmonic oscillator** on $|s\gg$;

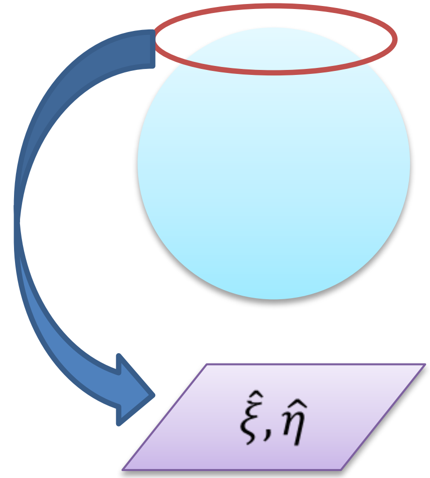
$$a\equiv\frac{L_+}{\sqrt{2j+1}} \qquad a^\dagger\equiv\frac{L_-}{\sqrt{2j+1}} \qquad \begin{array}{l} a|s>>=\sqrt{s}|s-1>> \\ a^\dagger|s>>=\sqrt{s+1}|s+1>> \end{array}$$

In fact,

$$[a,a^\dagger]|s>>=\frac{2L_3}{2j+1}|j,j-s>=\frac{2j-2s}{2j+1}|s>>\sim|s>>$$

$$\Rightarrow \quad [a,a^\dagger]=1$$

Mapping to Moyal plane



a and $a^\dagger$ can be regarded as the coordinate of Moyal plane as

$$\star \begin{cases} \sqrt{2\Theta}a \equiv \hat{\xi} + i\hat{\eta} \equiv \hat{\zeta} \\ \sqrt{2\Theta}a^\dagger \equiv \hat{\xi} - i\hat{\eta} \equiv \hat{\zeta}^\dagger \end{cases} \quad \text{where} \quad [\hat{\xi}, \hat{\eta}] = i\Theta$$

Fuzzy spherical harmonics around the north pole

$$\hat{Y}^{(jj)}_{Jm} = \sqrt{2J+1} \sum_{s,s'=0}^{2j} C^{jj-s}_{jj-s',Jm} |s \gg\ll s'|$$

$$\cong \sqrt{4\pi} \sum_{s=0}^{\infty} Y_{Jm}(\theta, 0) |s \gg\ll s+m|$$

$$\left(\theta \cong 2\sqrt{\frac{s+1/2}{2j+1}}\right) \star$$

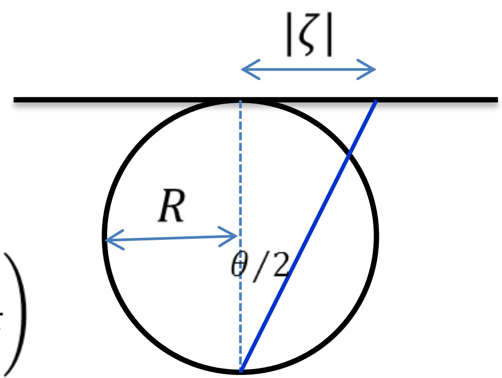
Stereographic transformation

For small θ ,

$$\zeta = R\theta e^{i\varphi}, \ \zeta^\dagger = R\theta e^{-i\varphi}$$

Combining it with ★ and ★, the non-commutativity becomes

$$\Theta = \frac{2}{(2j+1)R^2} = \frac{2}{n}\left(\frac{3}{M}\right)^2 \qquad \left(n \equiv 2j+1, R \equiv \frac{3}{M}\right)$$



Continuum limit

$j \to \infty, M \to 0$, **with fixing** $\Theta = \frac{2}{n}\left(\frac{3}{M}\right)^2$

In this limit, the Hilbert space becomes the Moyal plane with noncommutativity Θ,

and the **fuzzy spherical harmonics** can be expanded by plane wave;

$$\hat{x} = (\hat{\xi}, \hat{\eta})$$

$$\hat{Y}^{(jj)}_{Jm} \cong \begin{cases} \delta_{m,0}\sqrt{2J+1}\int \frac{d^2\tilde{q}}{(2\pi)^2}\delta^2(\tilde{q})e^{i\tilde{q}\cdot\hat{x}} & \text{for } J \leq {}^{\exists}J_\epsilon \\ 2\pi\Theta\sqrt{2J}\int \frac{d^2\tilde{q}}{(2\pi)^2}\frac{(-i)^m}{|\tilde{q}|}\delta(|\tilde{q}| - \frac{MJ}{3})e^{im\phi_{\tilde{q}}}e^{i\tilde{q}\cdot\hat{x}} & \text{for } J \geq J_\epsilon \end{cases}$$

As a result, a matrix ϕ can be regarded as a field on $\mathbb{R}^2_\Theta$:

$$\phi = \sum_{J,m}\phi_{Jm}\otimes Y^{(jj)}_{Jm} \to \int \frac{d^2\tilde{q}}{(2\pi)^2}\tilde{\phi}(\tilde{q})e^{i\tilde{q}\cdot\hat{x}}$$

$$\tilde{\phi}(\tilde{q}) = (2\pi)^2\delta^2(\tilde{q})\sum_{J=0}^{J_\epsilon}\sqrt{2J+1}\phi_{J0} + 2\pi\frac{3}{M}\sqrt{\frac{6}{M}}\sum_{m\in\mathbf{Z}}\frac{(-i)^m}{\sqrt{|\tilde{q}|}}e^{im\phi_{\tilde{q}}}\phi_{J=\frac{3}{M}|\tilde{q}|,m}$$

§ 3 4D N=4 U(k) SYM on R^2 x fuzzy S^2

We repeat the same procedure done in the toy model.

Consider the following specific fuzzy sphere solution:

$$X_a^{(\mathrm{cl})}(y) = \frac{M}{3} 1_k \otimes L_a^{(j)}, \qquad (N = k(2j+1) \equiv kn)$$

$$\begin{aligned} S_B = \frac{2}{g^2}\int d^2y \mathrm{Tr}\Big\{ & \frac{1}{2}F_{12}^2 + \frac{1}{2}(\mathcal{D}_\mu X_a)^2 + \frac{1}{2}(\mathcal{D}_\mu X_i)^2 - \frac{1}{4}\left[X_i, X_j\right]^2 \\ & - \frac{1}{2}[X_a, X_i]^2 + \frac{1}{2}\left(\frac{M}{3}X_a + \frac{i}{2}\epsilon_{abc}[X_b, X_c]\right)^2 \\ & + \frac{M^2}{81}X_i^2 - i\frac{2M}{9}X_7\left(F_{12} + i[X_3, X_4]\right)\Big\}, \\ S_F = \frac{2}{g^2}\int d^2y \mathrm{Tr}\Big\{ & \frac{i}{2}\bar{\psi}_{r\alpha}(\widehat{\gamma}_\mu)_{rs}\mathcal{D}_\mu\psi_{s\alpha} - \frac{1}{2}\bar{\psi}_{r\alpha}(\sigma_a)_{\alpha\beta}[X_a, \psi_{r\beta}] \\ & - \frac{1}{2}\bar{\psi}_{r\alpha}(\widehat{\gamma}_i)_{rs}[X_i, \psi_{s\alpha}] - \frac{1}{2}m_r\bar{\psi}_{r\alpha}\psi_{r\alpha}, \Big\}. \end{aligned}$$

Expand the fields by fuzzy spherical harmonics

vector fuzzy spherical harmonics

$$A_\mu(y) = \int \frac{d^2p}{(2\pi)^2} A_\mu(p) e^{ip\cdot y} = \sum_{Jm} \int \frac{d^2p}{(2\pi)^2} \widetilde{A}_{\mu,Jm}(p) \otimes \hat{Y}^{(jj)}_{Jm} e^{ip\cdot y}$$

$$X_a(y) = \frac{M}{3} \mathbf{1}_k \otimes L^{(j)}_a + \int \frac{d^2p}{(2\pi)^2} V_a(p) e^{ip\cdot y}$$

$$= \frac{M}{3} \mathbf{1}_k \otimes L^{(j)}_a + \sum_{\rho Jm} \int \frac{d^2p}{(2\pi)^2} \tilde{V}_{\rho,Jm}(p) \otimes \hat{Y}^{\rho(jj)}_{Jm,a} e^{ip\cdot y},$$

$$X_i(y) = \int \frac{d^2p}{(2\pi)^2} X_i(p) e^{ip\cdot y} = \sum_{Jm} \int \frac{d^2p}{(2\pi)^2} \widetilde{X}_{i,Jm}(p) \otimes \hat{Y}^{(jj)}_{Jm} e^{ip\cdot y}$$

$$\psi_{r\alpha}(y) = \int \frac{d^2p}{(2\pi)^2} \psi_{r\alpha}(p) e^{ip\cdot y} = \sum_{\kappa Jm} \int \frac{d^2p}{(2\pi)^2} \tilde{\psi}_{\alpha\kappa,Jm}(p) \otimes \hat{Y}^{\kappa(jj)}_{Jm,\alpha} e^{ip\cdot y},$$

$$c(y) = \int \frac{d^2p}{(2\pi)^2} c(p) e^{ip\cdot y} = \sum_{Jm} \int \frac{d^2p}{(2\pi)^2} \tilde{c}_{Jm}(p) \otimes \hat{Y}^{(jj)}_{Jm} e^{ip\cdot y}$$

$$\bar{c}(y) = \int \frac{d^2p}{(2\pi)^2} \bar{c}(p) e^{ip\cdot y} = \sum_{Jm} \int \frac{d^2p}{(2\pi)^2} \tilde{\bar{c}}_{Jm}(p) \otimes \hat{Y}^{(jj)}_{Jm} e^{ip\cdot y}$$

spinor fuzzy spherical harmonics

Repeating the discussion for the toy model, we obtain the action of (matrix regularized) 4d N=4 SYM on $\mathbb{R}^2 \times S^2_\Theta$ by inserting this expansion in the action.

Toward explicit expression

(1) action of $L_i^{(j)}$ to the fuzzy spherical harmonics $\left(\partial_a \equiv i\left[L_i^{(j)},\ \cdot\ \right]\right)$

$$\partial_a \hat{Y}^{(jj)}_{Jm} = i\sqrt{J(J+1)}\hat{Y}^{\rho=0}_{Jm(jj)a},$$
$$\partial_a \hat{Y}^{\rho}_{Jm(jj)a} = i\sqrt{(J(J+1))}\delta_{\rho 0}\hat{Y}^{(jj)}_{Jm},$$
$$\partial_a^2 \hat{Y}^{(jj)}_{Jm} = -J(J+1)\hat{Y}^{(jj)}_{Jm},$$
$$(\vec{\partial}\times\vec{Y}^{\rho}_{Jm} + \vec{Y}^{\rho}_{Jm})_a = \rho(J+1)\hat{Y}^{\rho}_{Jm(jj)a},$$
$$\left(-i\,(\sigma_a)_{\alpha\beta}\,\partial_a + \frac{3}{4}\delta_{\alpha\beta}\right)\hat{Y}^{\kappa}_{Jm(jj)\beta} = \kappa\left(J+\frac{3}{4}\right)\hat{Y}^{\kappa}_{Jm(jj)\alpha}.$$

(2) vertex coefficients

$$\hat{C}^{J_1m_1(jj)}_{J_2m_2(jj);J_3m_3(jj)} \equiv \frac{1}{n}\mathrm{tr}_n\left\{\left(\hat{Y}^{(jj)}_{J_1m_1}\right)^{\dagger}\hat{Y}^{(jj)}_{J_2m_2}\hat{Y}^{(jj)}_{J_3m_3}\right\},$$
$$\hat{\mathcal{D}}^{J_1m_1(jj)}_{J_2m_2(jj)\rho_2;J_3m_3(jj)\rho_3} \equiv \sum_{a=1}^{3}\frac{1}{n}\mathrm{tr}_n\left\{\left(\hat{Y}^{(jj)}_{J_1m_1}\right)^{\dagger}\hat{Y}^{\rho_2}_{J_2m_2(jj)a}\hat{Y}^{\rho_3}_{J_3m_3(jj)a}\right\},$$
$$\hat{\mathcal{E}}_{J_1m_1(jj)\rho_1;J_2m_2(jj)\rho_2;J_3m_3(jj)\rho_3} \equiv \sum_{a,b,c=1}^{3}\epsilon_{abc}\frac{1}{n}\mathrm{tr}_n\left\{\hat{Y}^{\rho_1}_{J_1m_1(jj)a}\hat{Y}^{\rho_2}_{J_2m_2(jj)b}\hat{Y}^{\rho_3}_{J_3m_3(jj)c}\right\},$$
$$\hat{\mathcal{F}}^{J_1m_1(jj)\kappa_1}_{J_2m_2(jj)\kappa_2;J_3m_3(jj)} \equiv \sum_{\alpha=\pm\frac{1}{2}}\frac{1}{n}\mathrm{tr}_n\left\{\left(\hat{Y}^{\kappa_1}_{J_1m_1(jj)\alpha}\right)^{\dagger}\hat{Y}^{\kappa_2}_{J_2m_2(jj)\alpha}\hat{Y}^{(jj)}_{J_3m_3}\right\},$$
$$\hat{\mathcal{G}}^{J_1m_1(jj)\kappa_1}_{J_2m_2(jj)\kappa_2;J_3m_3(jj)\rho_3} \equiv \sum_{\alpha,\beta=\pm\frac{1}{2}}\sum_{a=1}^{3}\sigma^a_{\alpha\beta}\frac{1}{n}\mathrm{tr}_n\left\{\left(\hat{Y}^{\kappa_1}_{J_1m_1(jj)\alpha}\right)^{\dagger}\hat{Y}^{\kappa_2}_{J_2m_2(jj)\beta}\hat{Y}^{\rho_3}_{J_3m_3(jj)a}\right\}.$$

explicit form of the action of the modes (1)

The kinetic part

$$S_B^{\text{kin}} = \frac{2n}{g^2}\text{tr}_k \int \frac{d^2p}{(2\pi)^2} \sum_{J,m}$$

$$\times \Bigg\{ \frac{(-1)^m}{2}\left(p^2 + \left(\frac{M}{3}\right)^2 J(J+1)\right)\tilde{A}_{\mu,J-m}(-p)\tilde{A}_{\mu,Jm}(p)$$

$$+ \frac{(-1)^{m+1}}{2}\sum_\rho \left(p^2 + \left(\frac{M}{3}\right)^2 (J+\rho^2)(J+1)\right)\tilde{V}^\rho_{J-m}(-p)\tilde{V}^\rho_{Jm}(p)$$

$$+ \frac{(-1)^m}{2}\left(p^2 + \left(\frac{M}{3}\right)^2 J(J+1) + \frac{2M^2}{81}\right)\widetilde{X}_{i,J-m}(-p)\widetilde{X}_{i,Jm}(p)$$

$$+ \frac{2M}{9}(-1)^m p_1 X_{7,J-m}(-p)A_{2,Jm}(p) - \frac{2M}{9}(-1)^m p_2 X_{7,J-m}(-p)A_{1,Jm}(p)\Bigg\},$$

$$S_F^{\text{kin}} = \frac{2n}{g^2}\text{tr}_k \int \frac{d^2p}{(2\pi)^2} \sum_{J,m,\kappa}$$

$$\times \left\{ \frac{i\kappa(-1)^{m-1}}{2}\left(p_\mu(\widehat{\gamma}_\mu)_{rs} + \frac{M}{3}\left(\kappa(J+\frac{3}{4}) + \widetilde{m}_r - \frac{3}{4}\right)\delta_{rs}\right)\tilde{\psi}^\kappa_{r,J-m}(-p)\tilde{\psi}^\kappa_{s,Jm}(p)\right\},$$

$$S_{\text{gh}}^{\text{kin}} = \frac{2n}{g^2}\text{tr}_k \int \frac{d^2p}{(2\pi)^2} \sum_{J,m}\left\{(-1)^m\left(-p^2 - \left(\frac{M}{3}\right)^2 J(J+1)\right)\tilde{\bar{c}}_{J-m}(-p)\tilde{c}_{Jm}(p)\right\}$$

explicit form of the action of the modes (2)

bosonic 3-point interactions

$$S_B^3 = \frac{2n}{g^2}\mathrm{tr}_k \int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{d^2r}{(2\pi)^2}(2\pi)^2\delta^2(p+q+r) \sum_{J_1 m_2 J_2 m_2 J_3 m_3}$$

$$\times\Bigg\{(-1)^{m_1}\hat{C}^{J_1 -m_1(jj)}_{J_2 m_2(jj);J_3 m_3(jj)} \sum_{\mu\neq\nu}(q_\mu - r_\mu)\tilde{A}_{\mu,J_1 m_1}(p)\tilde{A}_{\nu,J_2 m_2}(q)\tilde{A}_{\nu,J_3 m_3}(r)$$

$$+\sum_{\rho_2,\rho_3}(-1)^{m_1}\hat{\mathcal{D}}^{J_1 -m_1(jj)}_{J_2 m_2(jj)\rho_2;J_3 m_3(jj)\rho_3}(q_\mu - r_\mu)\tilde{A}_{\mu,J_1 m_1}(p)\tilde{V}^{\rho_2}_{J_2 m_2}(q)\tilde{V}^{\rho_3}_{J_3 m_3}(r)$$

$$+(-1)^{m_1}\hat{C}^{J_1 -m_1(jj)}_{J_2 m_2(jj);J_3 m_3(jj)}(q_\mu - r_\mu)\tilde{A}_{\mu,J_1 m_1}(p)\widetilde{X}_{i,J_2 m_2}(q)\widetilde{X}_{i,J_3 m_3}(r)$$

$$+\frac{M}{3}\sum_{\rho_1}(-1)^{m_3}\sqrt{J_2(J_2+1)}\hat{\mathcal{D}}^{J_3 -m_3(jj)}_{J_1 m_1(jj)\rho_1;J_2 m_2(jj)\rho_2=0}\tilde{V}^{\rho_1}_{J_1 m_1}(p)\tilde{A}_{\mu,J_2 m_2}(q)\tilde{A}_{\mu,J_3 m_3}(r)$$

$$-\frac{M}{3}\sum_{\rho_1}(-1)^{m_2}\sqrt{J_3(J_3+1)}\hat{\mathcal{D}}^{J_2 -m_2(jj)}_{J_3 m_3(jj)\rho_3=0;J_1 m_1(jj)\rho_1}\tilde{V}^{\rho_1}_{J_1 m_1}(p)\tilde{A}_{\mu,J_2 m_2}(q)\tilde{A}_{\mu,J_3 m_3}(r)$$

$$+\frac{M}{3}\sum_{\rho_1}(-1)^{m_3}\sqrt{J_2(J_2+1)}\hat{\mathcal{D}}^{J_3 -m_3(jj)}_{J_1 m_1(jj)\rho_1;J_2 m_2(jj)\rho_2=0}\tilde{V}^{\rho_1}_{J_1 m_1}(p)\widetilde{X}_{i,J_2 m_2}(q)\widetilde{X}_{i,J_3 m_3}(r)$$

$$-\frac{M}{3}\sum_{\rho_1}(-1)^{m_2}\sqrt{J_3(J_3+1)}\hat{\mathcal{D}}^{J_2 -m_2(jj)}_{J_3 m_3(jj)\rho_3=0;J_1 m_1(jj)\rho_1}\tilde{V}^{\rho_1}_{J_1 m_1}(p)\widetilde{X}_{i,J_2 m_2}(q)\widetilde{X}_{i,J_3 m_3}(r)$$

$$+i\frac{M}{3}\sum_{\rho_1,\rho_2,\rho_3}\rho_1(J_1+1)\hat{\mathcal{E}}_{J_1 m_1(jj)\rho_1;J_2 m_2(jj)\rho_2;J_3 m_3(jj)\rho_3}\tilde{V}^{\rho_1}_{J_1 m_1}(p)\tilde{V}^{\rho_2}_{J_2 m_2}(q)\tilde{V}^{\rho_3}_{J_3 m_3}(r)$$

$$+\frac{2M}{9}(-1)^{m_1}\hat{C}^{J_1 -m_1(jj)}_{J_2 m_2(jj);J_3 m_3(jj)}\widetilde{X}_{7,J_1 m_1}(p)\tilde{A}_{1,J_2 m_2}(q)\tilde{A}_{2,J_3 m_3}(r)$$

$$-\frac{2M}{9}(-1)^{m_1}\hat{C}^{J_1 -m_1(jj)}_{J_2 m_2(jj);J_3 m_3(jj)}\widetilde{X}_{7,J_1 m_1}(p)\tilde{A}_{2,J_2 m_2}(q)\tilde{A}_{1,J_3 m_3}(r)$$

$$+\frac{2M}{9}(-1)^{m_1}\hat{C}^{J_1 -m_1(jj)}_{J_2 m_2(jj);J_3 m_3(jj)}\widetilde{X}_{7,J_1 m_1}(p)\widetilde{X}_{3,J_2 m_2}(q)\widetilde{X}_{4,J_3 m_3}(r)$$

$$-\frac{2M}{9}(-1)^{m_1}\hat{C}^{J_1 -m_1(jj)}_{J_2 m_2(jj);J_3 m_3(jj)}\widetilde{X}_{7,J_1 m_1}(p)\widetilde{X}_{4,J_2 m_2}(q)\widetilde{X}_{3,J_3 m_3}(r)\Bigg\},$$

explicit form of the action of the modes (3)

fremionic 3-point interactions

$$S_F^3 = \frac{2n}{g^2}\mathrm{tr}_k \int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{d^2r}{(2\pi)^2}(2\pi)^2\delta^2(p+q+r) \sum_{J_1 m_1 \kappa_1 J_2 m_2 \kappa_2 J_3 m_3}$$

$$\times\Big\{ i\kappa_1(-1)^{m_1-1}\sum_{\mu,r,s}\hat{\mathcal{F}}^{J_1-m_1(jj)\kappa_1}_{J_2m_2(jj)\kappa_2;J_3m_3(jj)}(\hat{\gamma}_\mu)_{rs}\,\tilde{\psi}^{\kappa_1}_{rJ_1m_1}(p)\tilde{\psi}^{\kappa_2}_{sJ_2m_2}(q)\tilde{A}_{\mu,J_3m_3}(r)$$

$$+ i\kappa_1(-1)^{m_1-1}\sum_{i,r,s}\hat{\mathcal{F}}^{J_1-m_1(jj)\kappa_1}_{J_2m_2(jj)\kappa_2;J_3m_3(jj)}(\hat{\gamma}_i)_{rs}\,\tilde{\psi}^{\kappa_1}_{rJ_1m_1}(p)\tilde{\psi}^{\kappa_2}_{sJ_2m_2}(q)\widetilde{X}_{i,J_3m_3}(r)$$

$$+ i\kappa_1(-1)^{m_1-1}\sum_{\rho_3,r}\hat{\mathcal{G}}^{J_1-m_1(jj)\kappa_1}_{J_2m_2(jj)\kappa_2;J_3m_3(jj)\rho_3}\tilde{\psi}^{\kappa_1}_{rJ_1m_1}(p)\tilde{\psi}^{\kappa_2}_{rJ_2m_2}(q)\tilde{V}^{\rho_3}_{J_3m_3}(r)$$

ghost 3-point interactions

$$S_{\mathrm{gh}}^3 = \frac{2n}{g^2}\mathrm{tr}_k \int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{d^2r}{(2\pi)^2}(2\pi)^2\delta^2(p+q+r) \sum_{J_1 m_1 J_2 m_2 J_3 m_3}$$

$$\times\Big\{(-1)^{m_1}p_\mu\hat{C}^{J_1-m_1(jj)}_{J_2m_2(jj);J_3m_3(jj)}\tilde{\tilde{c}}_{J_1m_1}(p)\tilde{A}_{\mu,J_2m_2}(q)\tilde{c}_{J_3m_3}(r)$$

$$-(-1)^{m_1}p_\mu\hat{C}^{J_1-m_1(jj)}_{J_2m_2(jj);J_3m_3(jj)}\tilde{\tilde{c}}_{J_1m_1}(p)\tilde{c}_{J_2m_2}(q)\tilde{A}_{\mu,J_3m_3}(r)$$

$$+\frac{M}{3}\sqrt{J_1(J_1+1)}(-1)^{m_3}\hat{\mathcal{D}}^{J_3-m_3(jj)}_{J_1m_1(jj)\rho_1=0;J_2m_2(jj)\rho_2}\tilde{\tilde{c}}_{J_1m_1}(p)\tilde{V}^{\rho_2}_{J_2m_2}(q)\tilde{c}_{J_3m_3}(r)$$

$$-\frac{M}{3}\sqrt{J_s(J_s+1)}(-1)^{m_3}\hat{\mathcal{D}}^{J_3-m_3(jj)}_{J_1m_1(jj)\rho_1;J_2m_2(jj)\rho_2=0}\tilde{V}^{\rho_1}_{J_1m_1}(r)\tilde{\tilde{c}}_{J_2m_2}(p)\tilde{c}_{J_3m_3}(q)\Big\},$$

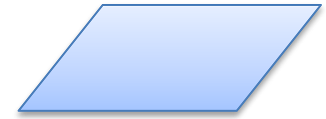
explicit form of the action of the modes (4)

bosonic 4-point interactions

$$
\begin{aligned}
S_B^4 = \frac{2n}{g^2}\mathrm{tr}_k \int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{d^2r}{(2\pi)^2}\frac{d^2s}{(2\pi)^2}(2\pi)^2\delta^2(p+q+r+s) \sum_{J_1m_2J_2m_2J_3m_3J_4m_4} \\
\times\Bigg\{ \sum_{Jm}(-1)^m \hat{C}^{Jm(jj)}_{J_1m_1(jj);J_2m_2(jj)}\hat{C}^{J-m(jj)}_{J_3m_3(jj);J_4m_4(jj)} \\
\times\Big(-\tilde{A}_{1,J_1m_1}(p)\tilde{A}_{2,J_2m_2}(q)\tilde{A}_{1,J_3m_3}(r)\tilde{A}_{2,J_4m_4}(s) \\
+\tilde{A}_{1,J_1m_1}(p)\tilde{A}_{2,J_2m_2}(q)\tilde{A}_{2,J_3m_3}(r)\tilde{A}_{1,J_4m_4}(s) \\
-\tilde{A}_{\mu,J_1m_1}(p)\widetilde{X}_{i,J_2m_2}(q)\tilde{A}_{\mu,J_3m_3}(r)\widetilde{X}_{i,J_4m_4}(s) \\
+\tilde{A}_{\mu,J_1m_1}(p)\widetilde{X}_{i,J_2m_2}(q)\widetilde{X}_{i,J_3m_3}(r)\tilde{A}_{\mu,J_4m_4}(s) \\
-\frac{1}{2}\widetilde{X}_{i,J_1m_1}(p)\widetilde{X}_{j,J_2m_2}(q)\widetilde{X}_{i,J_3m_3}(r)\widetilde{X}_{j,J_4m_4}(s) \\
+\frac{1}{2}\widetilde{X}_{i,J_1m_1}(p)\widetilde{X}_{j,J_2m_2}(q)\widetilde{X}_{j,J_3m_3}(r)\widetilde{X}_{i,J_4m_4}(s)\Big) \\
+\sum_{\rho_2\rho_4}\sum_{Jm\rho}(-1)^{m_1+m_3+m+1}\hat{\mathcal{D}}^{J_1-m_1(jj)}_{J_2m_2(jj)\rho_2;J-m(jj)\rho}\hat{\mathcal{D}}^{J_3-m_3(jj)}_{J_4m_4(jj)\rho_4;Jm(jj)\rho} \\
\times\Big(-\tilde{A}_{\mu,J_1m_1}(p)\tilde{V}^{\rho_2}_{J_2m_2}(q)\tilde{A}_{\mu,J_3m_3}(r)\tilde{V}^{\rho_4}_{J_4m_4}(s)\Big) \\
+\sum_{\rho_2\rho_4}\sum_{Jm\rho}(-1)^{m_1+m_3+m+1}\hat{\mathcal{D}}^{J_1-m_1(jj)}_{J_2m_2(jj)\rho_2;J-m(jj)\rho}\hat{\mathcal{D}}^{J_3-m_3(jj)}_{Jm(jj)\rho;J_4m_4(jj)\rho_4} \\
\times\Big(+\tilde{A}_{\mu,J_1m_1}(p)\tilde{V}^{\rho_2}_{J_2m_2}(q)\tilde{V}^{\rho_4}_{J_4m_4}(r)\tilde{A}_{\mu,J_3m_3}(s)\Big)\Bigg\} \\
+\sum_{\rho_2\rho_4}\sum_{Jm\rho}(-1)^{m_1+m_3+m+1}\hat{\mathcal{D}}^{J_1-m_1(jj)}_{J_2m_2(jj)\rho_2;J-m(jj)\rho}\hat{\mathcal{D}}^{J_3-m_3(jj)}_{J_4m_4(jj)\rho_4;Jm(jj)\rho} \\
\times\Big(-\widetilde{X}_{i,J_1m_1}(p)\tilde{V}^{\rho_2}_{J_2m_2}(q)\widetilde{X}_{i,J_3m_3}(r)\tilde{V}^{\rho_4}_{J_4m_4}(s)\Big) \\
+\sum_{\rho_2\rho_4}\sum_{Jm\rho}(-1)^{m_1+m_3+m+1}\hat{\mathcal{D}}^{J_1-m_1(jj)}_{J_2m_2(jj)\rho_2;J-m(jj)\rho}\hat{\mathcal{D}}^{J_3-m_3(jj)}_{Jm(jj)\rho;J_4m_4(jj)\rho_4} \\
\times\Big(+\widetilde{X}_{i,J_1m_1}(p)\tilde{V}^{\rho_2}_{J_2m_2}(q)\tilde{V}^{\rho_4}_{J_4m_4}(r)\widetilde{X}_{i,J_3m_3}(s)\Big) \\
+\sum_{\rho_1\rho_2\rho_3\rho_4}\sum_{Jm\rho}(-1)^{-m+1}\hat{\mathcal{E}}_{J_1m_1(jj)\rho_1;J_2m_2(jj)\rho_2;Jm(jj)\rho}\hat{\mathcal{E}}_{J_3m_3(jj)\rho_3;J_4m_4(jj)\rho_4;J-m(jj)\rho} \\
\times\Big(-\frac{1}{2}\tilde{V}^{\rho_1}_{J_1m_1}(p)\tilde{V}^{\rho_2}_{J_2m_2}(q)\tilde{V}^{\rho_3}_{J_3m_3}(r)\tilde{V}^{\rho_4}_{J_4m_4}(s)\Big)\Bigg\}
\end{aligned}
$$

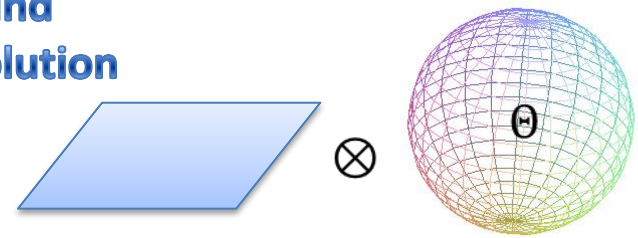
§ 4. Matrix continuum limit

Continuum
mass deformed 2d N=(8,8) SYM



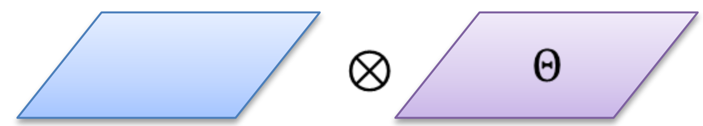
expand around
fuzzy sphere solution

We are now HERE ⇒ **4d N=4 SYM on $\mathbb{R}^2 \times S^2_\Theta$**



$$j \to \infty, M \to 0, \text{ with fixing } \Theta = \frac{2}{n}\left(\frac{3}{M}\right)^2$$

4d N=4 SYM on $\mathbb{R}^2 \times \mathbb{R}^2_\Theta$



QUESTION

Can we take this limit safely?

① Tree level: OK.

② Quantum mechanically: NON-TRIVIAL

If the deformation by the mass parameter M causes soft breaking of 16 supersymmetry, there is no problem:

Superficial degrees of divergence of a graph

$$D = 4 - E_B - \frac{3}{2} E_F$$

$E_B \cdots$ # of bosonic external lines
$E_F \cdots$ # of fermionic external lines

The most severe UV divergences come from $E_B = 2$ (Λ^2)

<u>possible structure of the divergent terms:</u>

$$A \cdot \Lambda^2 + O\left(M^p \left(\log \frac{\Lambda}{M}\right)^q\right) \qquad (p, q = 1, 2, \cdots)$$

- The leading term is canceled because of the original 16 SUSY.
- The next leading terms vanish in the continuum limit:

$$M^p \left(\log \frac{\Lambda}{M}\right)^q \sim M^p (\log n)^q \to 0$$

since $M \propto n^{-\frac{1}{2}} \to 0$.

Unfortunately, the situation is not so simple:

- The parameter M is indeed a soft mass in 2d theory but is it really soft in 4d theory?
- The 4d theory is **non-commutative** gauge theory. UV/IR mixing?
- The remaining SUSY is only two.
- Is the continuous theory really a theory on $R^2 \times R^2_\theta$?

We should check if there is no additional divergence at least perturbatively.

Result of perturbative comuutation

SO(4) momentum!

Effective action of X_i^2 at the 1-loop level

$$g_{4d}^2 \sum_{J,m} \int \frac{d^2p}{(2\pi)^2}(-1)^m \sum_{i=5,6} \left[k\mathrm{tr}_k\left(X_{i,Jm}(p)X_{i,J-m}(-p)\right) - \mathrm{tr}_k\left(X_{i,Jm}(p)\right)\mathrm{tr}_k\left(X_{i,J-m}(-p)\right)\right]$$
$$\times \frac{1}{8\pi^2}\left[\ln\left(\tilde{\Lambda}+0.0510\right)(p^2+u^2) - \frac{1}{2}(p^2+u^2)\ln(p^2+u^2) + (p^2+u^2) + 0.854\right]$$

1-loop correction to the effective action of scalar^2 in 4d N=4 SYM

$$g_{4d}^2 \int \frac{d^4p}{(2\pi)^2}\left[k\mathrm{tr}_k(\phi(-p)^\dagger\phi(p)) - \mathrm{tr}_k(\phi(-p)^\dagger)\mathrm{tr}_k(\phi(p))\right]$$
$$\times \frac{1}{8\pi^2}\left[\ln(\Lambda)p^2 - \frac{1}{2}p^2\ln(p^2) + p^2\right]$$

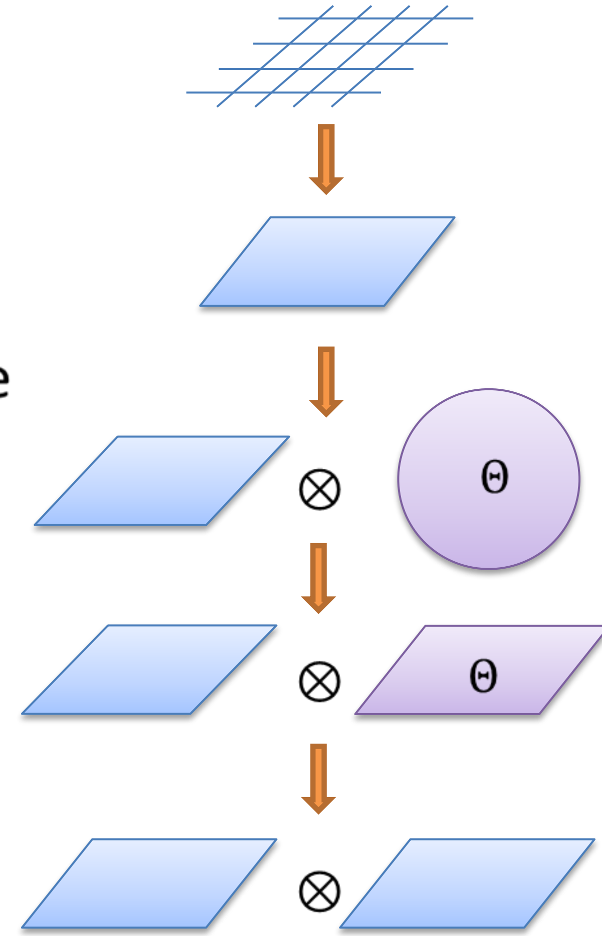
There is no additional divergence to the two-point function at least in the 1-loop level.

We can strongly expect to obtain 4d N=4 SYM!!

Comments

- We also calculated the other 2-point functions and confirmed that the situation is the same.
- Honestly speaking, we should confirm that we need only wave function renormalization in the continuum limit, but it is quite hard task even at the 1-loop level.
- It is would be better to check it numerically.
- **It is believed that we can take the limit of $\Theta \to 0$ smoothly for 4d N=4 SYM.**

➡ **We can take the commutative limit safely.**
(the final step is OK!)



This sequence is complete!

Lattice formulation of
mass deformed 2d N=(8,8) SYM

⇩ lattice continuum limit

Continuum
mass deformed 2d N=(8,8) SYM

⇩ expand around fuzzy sphere solution

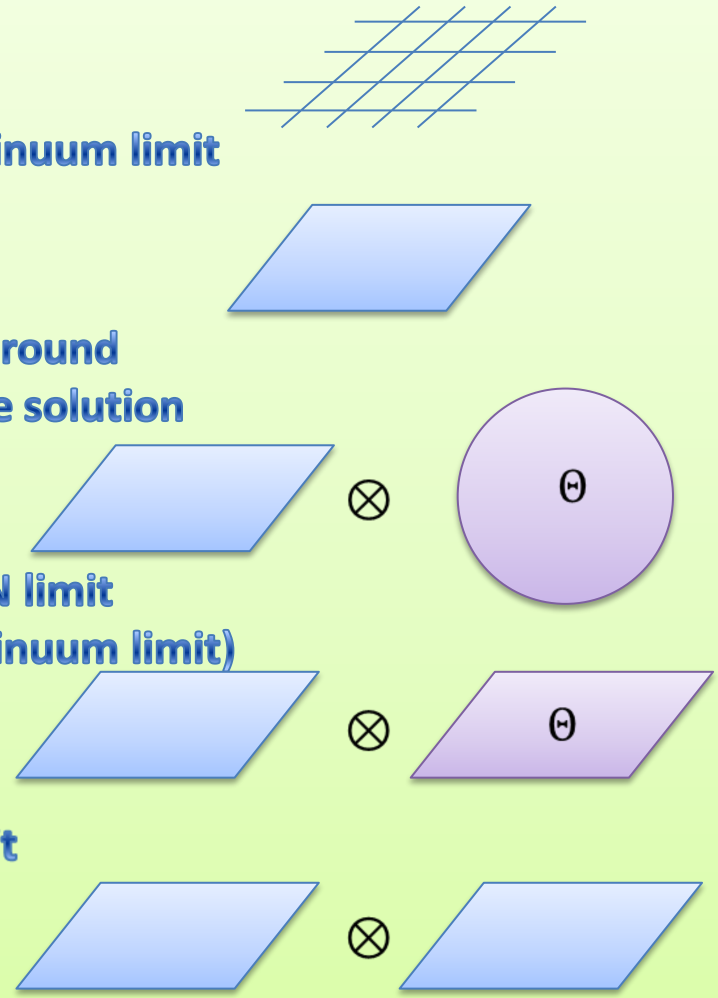
4d N=4 SYM on $\mathbb{R}^2 \times S^2_\Theta$

⇩ large N limit (matrix continuum limit)

4d N=4 SYM on $\mathbb{R}^2 \times \mathbb{R}^2_\Theta$

⇩ $\Theta \to 0$ limit

4d N=4 SYM on $\mathbb{R}^4$



It's time to perform numerical experiment for 4d N=4 SYM !!

Future works

1. Check if this sequence really work.
2. We can (hopefully) carry out numerical simulation of 4d N=4 SYM with finite rank gauge group.
   1. ¾-problem
   2. AdS/CFT correspondence
   3. 4d N=4 SYM as a quantum gravity

3. We can formulate other theories using the same method.

4. Connection to Ω-background? (The deformation is quite similar to that introduced by Nekrasov to discretize the instanton moduli space of 4d N=2 SYM.)

1-loop correction to the effective action of $X_{i,Jm}(p)$:

planar graphs **non-planar graphs**

$$\frac{g_{2d}^2}{2n}\left(\frac{3}{M}\right)^2 \sum_{J,m}\int \frac{d^2p}{(2\pi)^2}\frac{(-1)^m}{4\pi}\sum_{i=5,6}\Big[k\,\mathrm{tr}_k\left(X_{i,Jm}(p)X_{i,J-m}(-p)\right)-\mathrm{tr}_k\left(X_{i,Jm}(p)\right)\mathrm{tr}_k\left(X_{i,J-m}(-p)\right)\Big]$$

$$\times\left(\mathcal{A}_{J,\tilde{p}}+\sum_{J_1=0}^{2j}\sum_{J_2=1}^{2j}\mathcal{B}_{J,\tilde{p}}(J_1,J_2)\right)\Big]$$

$$=\frac{\Theta g_{2d}}{4}=\frac{g_{4d}^2}{8\pi}$$

<u>Notation</u>

$$A(J)\equiv\left(\frac{M}{3}\right)^2\tilde{A}(J)\equiv\left(\frac{M}{3}\right)^2(J+\frac{1}{3})(J+\frac{2}{3}),\quad B(J)\equiv\left(\frac{M}{3}\right)^2\tilde{B}(J)\equiv\left(\frac{M}{3}\right)^2 J(J+\frac{1}{3}),$$

$$C(J)\equiv\left(\frac{M}{3}\right)^2\tilde{C}(J)\equiv\left(\frac{M}{3}\right)^2(J+1)(J+\frac{2}{3}),\quad D(J)\equiv\left(\frac{M}{3}\right)^2\widetilde{D}(J)\equiv\left(\frac{M}{3}\right)^2 J(J+1),$$

$$E(J)\equiv\left(\frac{M}{3}\right)^2\tilde{E}(J)\equiv\left(\frac{M}{3}\right)^2(J+1)^2,\quad F(J)\equiv\left(\frac{M}{3}\right)^2\tilde{F}(J)\equiv\left(\frac{M}{3}\right)^2 J^2,\quad p^2\equiv\left(\frac{M}{3}\right)^2\tilde{p}^2,$$

$$L(A,B;p)\equiv\frac{1}{\sqrt{(p^2)^2+2(A+B)p^2+(A-B)^2}}\times\log\left(\frac{p^2+A+B-\sqrt{(p^2)^2+2(A+B)p^2+(A-B)^2}}{p^2+A+B+\sqrt{(p^2)^2+2(A+B)p^2+(A-B)^2}}\right).$$

IR part

$$\mathcal{A}_{J,\tilde{p}} = \left(\frac{M}{3}\right)^2 \Bigg\{ -\frac{4}{3}\frac{\tilde{p}^2 - \tilde{A}(J)}{\tilde{p}^2 + \tilde{A}(J)} \ln\left(\frac{\tilde{A}(J)\delta}{(\tilde{p}^2 + \tilde{A}(J))}\right) + \frac{1}{3}\frac{\tilde{p}^2 - \tilde{A}(J)}{\tilde{p}^2 + \tilde{A}(J)} \ln(3) + (\tilde{p}^2 + \tilde{A}(J) - 1)\ln\frac{2}{3}$$

$$+ (\tilde{p}^2 + \tilde{A}(J))\ln\frac{\tilde{A}(J)}{(\tilde{p}^2 + A(J))^2} - \frac{4}{3}\frac{\tilde{p}^2}{\tilde{p}^2 + \tilde{A}(J)}$$

$$- \frac{(\tilde{p}^2 + \tilde{A}(J))^2 - \frac{2}{3}\tilde{p}^2 - \frac{10}{3}J(J+1) - \frac{2}{3}}{\sqrt{(\tilde{p}^2 + \tilde{A}(J))^2 + \frac{4}{3}\tilde{p}^2 - \frac{4}{3}\tilde{A}(J) + \frac{4}{9}}} \ln\left(\frac{\tilde{p}^2 + \tilde{A}(J) + \frac{2}{3} - \sqrt{(\tilde{p}^2 + \tilde{A}(J))^2 + \frac{4}{3}\tilde{p}^2 - \frac{4}{3}\tilde{A}(J) + \frac{4}{9}}}{\tilde{p}^2 + \tilde{A}(J) + \frac{2}{3} + \sqrt{(\tilde{p}^2 + \tilde{A}(J))^2 + \frac{4}{3}\tilde{p}^2 - \frac{4}{3}\tilde{A}(J) + \frac{4}{9}}}\right)$$

$$+ \frac{2\tilde{p}^2 - 2J(J+1) + \frac{4}{9}}{\sqrt{(\tilde{p}^2 + \tilde{A}(J))^2 + 2\tilde{p}^2 - 2\tilde{A}(J) + 1}} \ln\left(\frac{\tilde{p}^2 + \tilde{A}(J) + 1 - \sqrt{(\tilde{p}^2 + \tilde{A}(J))^2 + 2\tilde{p}^2 - 2\tilde{A}(J) + 1}}{\tilde{p}^2 + \tilde{A}(J) + 1 + \sqrt{(\tilde{p}^2 + \tilde{A}(J))^2 + 2\tilde{p}^2 - 2\tilde{A}(J) + 1}}\right) \Bigg\}$$

$$\frac{M}{3} \cdot \tilde{p}_\mu = p_\mu, \qquad \frac{\mathrm{M}}{3} \cdot J = u$$

$$M \to 0$$

$$(p^2 + u^2)\left\{ \ln\left(\frac{2}{3}\right) + \ln\frac{u^2\left(\frac{M}{3}\right)^2}{(p^2 + u^2)^2} - \ln\left(\frac{2}{3}\right) - \ln\frac{u^2\left(\frac{M}{3}\right)^2}{(p^2 + u^2)^2} \right\} = 0$$

Main part

We like to evaluate the following expression in the limit of $j \to \infty, M \to 0$ with fixing $u = \frac{MJ}{3}, p_\mu = \frac{M\tilde{p}_\mu}{3}$

$$\sum_{J_1=0}^{2j}\sum_{J_2=1}^{2j} \mathcal{B}_{J,\tilde{p}}(J_1,J_2) = \left(\frac{M}{3}\right)^2 \sum_{J_1=0}^{2j}\sum_{J_2=1}^{2j} (2j+1)(2J_1+1)(2J_2+1)\begin{Bmatrix} J_1 & J_2 & J \\ j & j & j \end{Bmatrix}^2$$

$$\times\Bigg\{\left[\frac{\tilde{p}^2+\tilde{A}(J_1)}{J_2(2J_2+1)} - \frac{J_2}{2J_2+1}\right]\ln\tilde{B}(J_2) + \left[\frac{\tilde{p}^2+\tilde{A}(J_1)}{(J_2+1)(2J_2+1)} - \frac{J_2+1}{2J_2+1}\right]\ln\tilde{C}(J_2)$$
$$+\left[-\frac{\tilde{p}^2+\tilde{A}(J_1)}{J_2(J_2+1)} + \frac{J_2+(J_2+1)}{2J_2+1}\right]\ln\tilde{D}(J_2) + \frac{1}{2J_2+1}\ln\tilde{E}(J_2) - \frac{1}{2J_2+1}\ln\tilde{F}(J_2)$$

log part

$$+\left(\frac{M}{3}\right)^2 L(A(J_1),B(J_2);p)$$
$$\times\left[-\frac{(\tilde{p}^2+\tilde{A}(J_1))^2}{J_2(2J_2+1)} + \frac{1}{2J_2+1}\left(-\frac{2}{3}\tilde{p}^2 + 2(J_1+\frac{1}{3})(J_1+\frac{2}{3})(J_2+\frac{1}{3}) - J_2(J_2+\frac{1}{3})^2 \right.\right.$$
$$\left.\left. + 2J(J+1)(J_2+\frac{1}{6}) - \frac{1}{3}J_1(J_1+1) + \frac{1}{3}J_2(J_2+\frac{5}{3})\right)\right]$$
$$+\left(\frac{M}{3}\right)^2 L(A(J_1),C(J_2);p)$$
$$\times\left[-\frac{(\tilde{p}^2+\tilde{A}(J_1))^2}{(J_2+1)(2J_2+1)} + \frac{1}{2J_2+1}\left(\frac{2}{3}\tilde{p}^2 + 2(J_1+\frac{1}{3})(J_1+\frac{2}{3})(J_2+\frac{2}{3}) - (J_2+1)(J_2+\frac{2}{3})^2 \right.\right.$$
$$\left.\left. + 2J(J+1)(J_2+\frac{5}{6}) + \frac{1}{3}J_1(J_1+1) - \frac{1}{3}(J_2+1)(J_2-\frac{2}{3})\right)\right]$$
$$+\left(\frac{M}{3}\right)^2 L(A(J_1),D(J_2);p)$$
$$\times\left[\frac{(\tilde{p}^2+\tilde{A}(J_1))^2}{J_2(J_2+1)} - \frac{1}{J_2(J_2+1)}\Big(J(J+1)-J_1(J_1+1)\Big)^2\right]$$
$$+\left(\frac{M}{3}\right)^2 L(A(J_1),E(J_2);p)$$
$$\times\left[\frac{2(J_2+1)}{2J_2+1}\left(\tilde{p}^2+\tilde{A}(J)\right) - \frac{(J_1+J_2+J+2)(J_1-J_2+J)(-J_1+J_2+J+1)(J_1+J_2-J+1)}{(J_2+1)(2J_2+1)}\right]$$
$$+\left(\frac{M}{3}\right)^2 L(A(J_1),F(J_2);p)$$
$$\times\left[\frac{2J_2}{2J_2+1}\left(\tilde{p}^2+\tilde{A}(J)\right) - \frac{(J_1+J_2+J+1)(J_1-J_2+J+1)(-J_1+J_2+J)(J_1+J_2-J)}{J_2(2J_2+1)}\right]\Bigg\}.$$

L part

[1] log part

$$\left(\frac{M}{3}\right)^2 \sum_{J_1=0}^{2j} \sum_{J_2=1}^{2j} (2j+1)(2J_1+1) \begin{Bmatrix} J_1 & J_2 & J \\ j & j & j \end{Bmatrix}^2$$

$$\times \left\{ (\tilde{p}^2 + \tilde{A}(J_1)) \underbrace{\left[\frac{1}{J_2} \ln \frac{J_2+1}{J_2+1/3} - \frac{1}{J_2+1} \ln \frac{J_2+2/3}{J_2} \right]}_{g(J_2)} \underbrace{- J_2 \ln \frac{J_2+1}{J_2+1/3} + (J_2+1) \ln \frac{J_2+2/3}{J_2} - 2 \ln \frac{J_2+1}{J_2}}_{h(J_2)} \right\}$$

formulae

★ $$\sum_{J_1=0}^{2j} (2j+1)(2J_1+1) \begin{Bmatrix} J_1 & J_2 & J \\ j & j & j \end{Bmatrix}^2 = 1$$

★ $$f_j(J) \equiv \sum_{J_1=0}^{2j} \sum_{J_2=1}^{2j} (2j+1)(2J_1+1) \begin{Bmatrix} J_1 & J_2 & J \\ j & j & j \end{Bmatrix}^2 \tilde{A}(J_1) g(J_2)$$

$$\Rightarrow f_j(J) = f_j(0) + \left[\sum_{J_2=1}^{2j} \left(2 - \frac{J_2(J_2+1)}{j(j+1)}\right) g(J_2) \right] \frac{J(J+1)}{2}$$

$$= p^2 \left(\sum_{J_2=1}^{2j} g(J_2) \right) + u^2 \sum_{J_2=1}^{2j} \left(1 - \frac{J_2(J_2+1)}{2j(j+1)}\right) g(J_2) + \left(\frac{M}{3}\right)^2 \sum_{J_2=1}^{2j} g(J_2)\tilde{A}(J_2) + \left(\frac{M}{3}\right)^2 \sum_{J_2=1}^{2j} h(J_2)$$

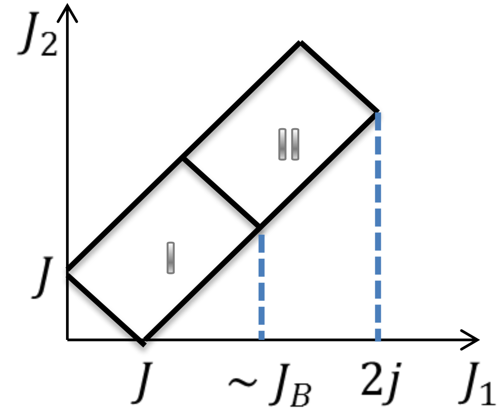
$$\to (p^2+u^2) \underbrace{\left(\sum_{J_2=1}^{\infty} g(J_2) \right)}_{\cong 0.2042} + \underbrace{\sum_{J_2=1}^{\infty} g(J_2)\tilde{A}(J_2)}_{\cong 3.413}$$

[2] **L-part** We separate the region of (J_1, J_2) into

Region I: $J \le J_1 + J_2 \le J_B,\ -J \le J_1 - J_2 \le J$

Region II: $J_B \le J_1 + J_2 \le 4j, -J \le J_1 - J_2 \le J$

$(J_B = O(j^\epsilon)\ \ (0 < \epsilon < 1/2))$



Region I

In this region, Wigner 6j-symbol can be approximated as

$$\begin{Bmatrix} J_1 & J_2 & J \\ j & j & j \end{Bmatrix}^2 \cong \frac{2}{\pi}\frac{1}{2j+1}\frac{1}{(J_1+J_2+J)(-J_1+J_2+J)(J_1-J_2+J)(J_1+J_2-J)}$$

We can estimate the summation by the integral,

$$[\mathrm{L-part}]_{\mathrm{region\ I}} \sim \frac{2}{\pi}(p^2+u^2)\int_{u\le u_1+u_2\le u_B,\ -u\le u_1-u_2\le u} du_1 du_2$$

$$\times \frac{u_1 u_2}{\sqrt{(p^2+(u_1+u_2)^2)(p^2+(u_1-u_2)^2)((u_1+u_2)^2-u^2)(u^2-(u_1-u_2)^2)}}$$

$$\times \ln\left(\frac{p^2+u_1^2+u_2^2-\sqrt{(p^2+(u_1+u_2)^2)(p^2+(u_1-u_2)^2)}}{p^2+u_1^2+u_2^2+\sqrt{(p^2+(u_1+u_2)^2)(p^2+(u_1-u_2)^2)}}\right)$$

$$\left(u_B = \left(\frac{M}{3}\right)J_B\right)$$

$$= 4(p^2+u^2)\left(\ln u_B - \frac{1}{2}\ln(p^2+u^2) + 1 - \ln(2)\right)$$

Region II

In this region, Wigner 6j-symbol can be approximated as

$$\begin{Bmatrix} J_1 & J_2 & J \\ j & j & j \end{Bmatrix}^2 \cong \frac{1}{(2j+1)(2J_2+1)} \frac{1}{2^{2J}} \frac{J!}{(J+\Delta)!(J-\Delta)!} \left[(1+X)^{\frac{\Delta}{2}} (1-X)^{\frac{\Delta}{2}} \sum_r (-1)^2 \binom{J-\Delta}{r} \binom{J+\Delta}{J-r} \left(\frac{1+X}{1-X} \right)^r \right]^2$$

$$\left(\Delta = J_1 - J_2, X = \frac{1}{2}\sqrt{\frac{J_2(J_2+1)}{j(j+1)}} \cong \frac{J_1+J_2}{4j} \right)$$

By using

1. $L\left(\tilde{A}(J_1), \tilde{B}(J_2)\right) \cong -\frac{4}{(J_1+J_2)^2}, \cdots$
2. $X < 1$
3. $\sum_\Delta \frac{1}{2^{2J}} \frac{(J!)^2}{(J+\Delta)!(J-\Delta)!} \sum_r (-1)^r \binom{J-\Delta}{r} \binom{J+\Delta}{J-r} = 1$

We see the most singular part of the summation is

$$[\mathrm{L-part}]^{\text{most singular}}_{\text{Region II}} = 4(p^2+u^2) \sum_{n=J_B/2}^{2j} \frac{1}{n}$$

$$\cong 4(p^2+u^2) \left(\ln\left(\frac{2jM}{3}\right) - \ln(u_B) + \ln(2) \right)$$

(We can show the other contributions vanish in the continuum limit.)