A new numerical approach to evolution of 5D asymptotically AdS spacetimes

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Outline

- ***Motivation & Background***
  - gauge-gravity dualities, in particular AdS/CFT
- ***A new approach using generalized harmonic (GH) evolution to study AAdS spacetimes (work with Hans Bantilan & Steve Gubser)***
  - overview of the structure of AAdS spacetimes, and why it is not a trivial adaptation of a working asymptotically flat code
  - basics of the GH approach
  - first step: 5D AAdS spacetime with SO(3) symmetry and massless scalar field with "non-deforming" boundary fall-off
  - early results: the non-linear phase of quasi-normal ringdown of black holes, and corresponding boundary dynamics
- ***Conclusion and future work***

Gauge/gravity duality from a non-string theorists perspective

- The main development in the past decade within string theory has been the discovery of gauge/gravity dualities
  - even if string theory is not *the* theory of everything, that such a mapping exists is remarkable, and provides an alternative route to understanding gravity and strongly coupled gauge theories
- The first concrete duality *[Maldacena 1998]* conjectures a 1-1 correspondence between states in type IIB string theory in asymptotically AdS_5 x S^5 spacetimes and $4D$, $\mathcal{N}=4$, $SU(N)$ Yang-Mills theory
  - in the limit of a strongly coupled gauge theory and large AdS radius L relative to the string and Planck scales, the bulk spacetime is well described by Einstein gravity (plus possible form fields)

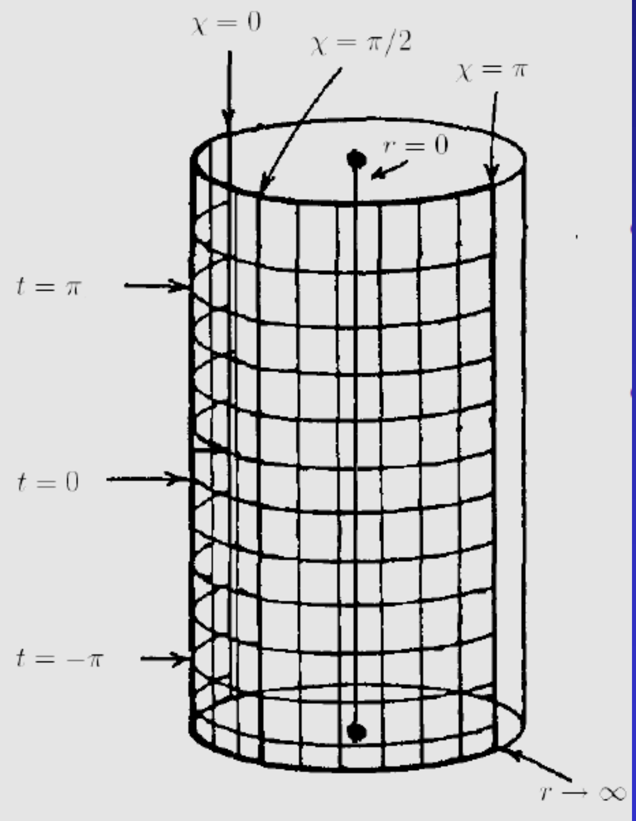
A few demonstrated/conjectured correspondences

- Stationary black holes dual to thermal states
- Perturbations of black holes dual to dynamics of ideal fluids
- Black hole collisions dual to models of the formation and thermalization of the quark-gluon plasma formed in heavy ion collisions
- Various condensed matter dualities: superconductors, superfluids, quantum Hall effect, etc … *[see reviews by S. Hartnol, arXiv:1106.4324, G. Horowitz, arXiv:1010.2784; J. McGreevy, arXiv:0909.0518]*
  - Interesting that black holes are involved in most of these correspondences – are they "merely" supplying a temperature and fluid-like properties, as might have been expected from classical/semiclassical black hole physics, or are there deeper connections?
- Input that numerical relativity can bring to these studies are solutions to the classical gravity side of the correspondence in regimes difficult or impossible to study via analytic methods

5D AdS spacetime

- Global AdS in spherical-polar type coordinates

$$ds^2 = -\left(1 + r^2\big/L^2\right)dt^2 + \left(1 + r^2\big/L^2\right)^{-1} dr^2 + r^2\left(d\chi^2 + \sin^2\chi d\Omega_2^2\right)$$



- spacetime of constant negative curvature ($R=-20/L^2$)
- the boundary metric ($r \to \infty$) is the 4D Einstein static universe ($R \times S^3$)
- Poincare coordinates cover a conformally flat piece of global AdS (the Poincare patch)

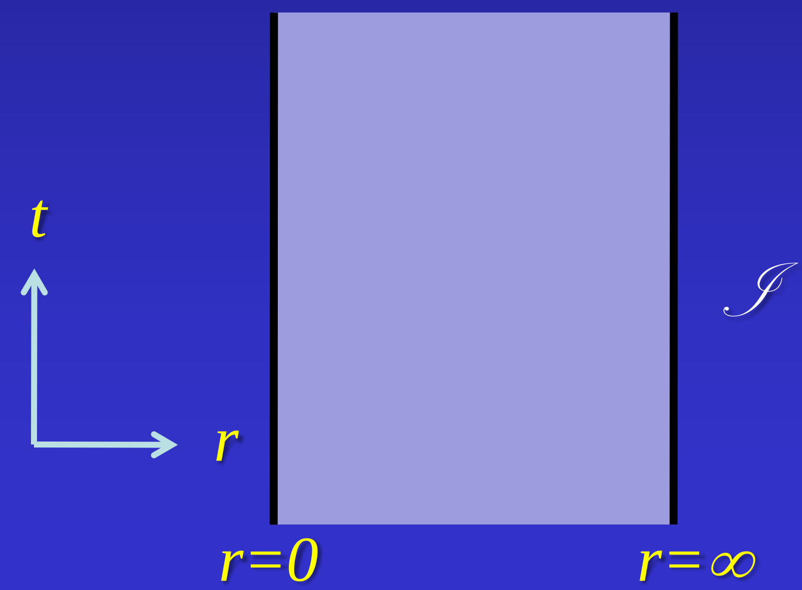
$$ds^2 = -W^2\left(-dt^2 + d\vec{x}_4^2\right)$$
$$W^2 = \sqrt{1 + r^2/L^2}\cos(t/L) + r/L\cos(\chi/L);\ \ W > 0$$

this segment of AdS is usually used for applications with a CFT on $R^{3,1}$; we will solve the equations in global coordinates and transform to a patch as needed

Main source of difficulty evolving AAdS spacetimes

- The boundary ("infinity") is timelike, and correctly solving for the metric behavior approaching it is *crucial* to the problem; however the metric is *singular* in the limit

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right) dt^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2$$



- This is not a "true" geometric singularity, though still has important physical consequences :

  Infinite proper distance from any point in the interior to a point on the boundary on a $t=const.$ slice; null signals will propagate back and forth in *finite proper time*, experiencing *infinite red/blue* shift in the process

Generalized harmonic evolution of AAdS spacetimes

- We want to solve Einstein's equations with a scalar field matter source and cosmological constant $\Lambda = -6/L^2$,

$$R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi\, T_{\alpha\beta}$$

$$\nabla_\gamma \nabla^\gamma \phi = \frac{\partial V(\phi)}{\partial \phi}$$

$$T_{\alpha\beta} = \nabla_\alpha \phi \nabla_\beta \phi - g_{\alpha\beta} \left(\frac{1}{2} \nabla^\gamma \phi \nabla_\gamma \phi + V(\phi) \right)$$

using the GH harmonic scheme *[Garfinkle, PRD 65 (2002), FP CQG 22 (2005)]*, with constraint damping *[Gundlach et al., CQG 22 (2005)]*

– *talks by Chesler, Garfinkle on alternative approaches*

– The specific spacetimes we will look at here are high density initially time-symmetric, axisymmetric concentrations of scalar field energy, that immediately form distorted black holes that ring down to AdS Schwarzchild black holes

Generalized harmonic evolution of AAdS spacetimes

- Specifically, the Einstein equations in GH form are a set of coupled, qausi-linear hyperbolic PDEs, one for each metric element

$$-\frac{1}{2}g^{\gamma\delta}g_{\alpha\beta,\gamma\delta} - g^{\gamma\delta}{}_{,(\alpha}g_{\beta)\delta,\gamma} - \Gamma^{\gamma}_{\delta\beta}\Gamma^{\delta}_{\gamma\alpha} - H_{(\alpha,\beta)} + H_{\delta}\Gamma^{\delta}_{\alpha\beta}$$
$$-\kappa\left(2n_{(\alpha}C_{\beta)} - (1+P)g_{\alpha\beta}n^{\gamma}C_{\gamma}\right) = \frac{2}{3}\Lambda g_{\alpha\beta} + 8\pi\left(T_{\alpha\beta} - \frac{1}{3}g_{\alpha\beta}T\right)$$

  where

$$C^{\mu} \equiv H^{\mu} - \nabla^{\alpha}\nabla_{\alpha}x^{\mu} = 0;\ \ n_{\mu} = -\alpha\,\partial_{\mu}t$$

  and κ and P are constraint damping parameters

- Because of the singular nature of the AAdS boundary, we cannot directly discretize these equations using the metric $g_{\mu\nu}$ and source functions H_{μ}
  - To describe the regularization, we first need to fix (in part) the gauge

Coordinates

- Choose spherical polar coordinates $(t,\rho,\chi,\theta,\phi)$, with a compactified radial coordinate ρ related to the standard AdS coordinate r via

$$r = \frac{\rho}{1-\rho/\ell}; \;\; r \in [0..\infty] \to \rho \in [0..\ell]$$

  where ℓ is some length scale; for simplicity we choose $\ell=1$ and define a related coordinate $q=1-\rho$ so that the boundary is at $q=0$

- For a first study, consider spacetimes with *SO(3)* symmetry; i.e. non-trivial fields only a function of (t,ρ,χ)
  - corresponds to axisymmetric spacetimes in the bulk, and will be dual to CFT states that have a related symmetry in the stress energy

- AdS vacuum metric in these coordinates:

$$ds^2 = \frac{1}{1-\rho^2}\left[-f(\rho)dt^2 + f(\rho)^{-1}d\rho^2 + \rho^2\left(d\chi^2 + \sin^2\chi d\Omega_2^2\right)\right]$$

$$f(\rho) = (1-\rho)^2 + \rho^2/L^2$$

Regular variables at the AAdS Boundaries

- First, analytically subtract out the divergent parts corresponding to pure AdS

$$g_{\mu\nu} \equiv g_{\mu\nu}^{(AdS)} + \delta g_{\mu\nu}; \;\; H_{\mu} \equiv H_{\mu}^{(AdS)} + \delta H_{\mu}$$

- Second, for simplicity we want to use coordinates where the leading order power-law approach to the AdS boundary takes on the standard (Fefferman-Graham) form used in most of the AdS/CFT literature *[Henneaux and Teitelboim, Comm.Math.Phys 98 (1985)]*

$$\begin{aligned}
\delta g_{rr} &= f_{rr}(t,\chi,\theta,\phi) r^{-6} + O(r^{-7}) \\
\delta g_{rm} &= f_{rm}(t,\chi,\theta,\phi) r^{-5} + O(r^{-6}) \\
\delta g_{mn} &= f_{mn}(t,\chi,\theta,\phi) r^{-2} + O(r^{-3})
\end{aligned}$$

where m,n denote (t,χ,θ,ϕ) components

Regular variables at the AAdS Boundaries

- Third, given the desired fall-off, factor out appropriate powers of q so that we can place a simple Dirichlet boundary condition there on the leading order component *[Garfinkle & Duncan, PRD 63 (2001)]* ; putting all this together (with similar factoring for axis/origin regularity):

$$
\begin{aligned}
g_{tt} &\equiv g_{tt}^{(AdS)} + q(1+\rho)\bar{g}_{tt} \\
g_{t\rho} &\equiv q^2(1+\rho)^2\bar{g}_{t\rho} \\
g_{t\chi} &\equiv q(1+\rho)\bar{g}_{t\chi} \\
g_{\rho\rho} &\equiv g_{\rho\rho}^{(AdS)} + q(1+\rho)\bar{g}_{\rho\rho} \\
g_{\rho\chi} &\equiv q^2(1+\rho)^2\bar{g}_{\rho\chi} \\
g_{\chi\chi} &\equiv g_{\chi\chi}^{(AdS)} + q(1+\rho)\bar{g}_{\chi\chi} \\
g_{\theta\theta} &= \frac{g_{\phi\phi}}{\sin^2\theta} \equiv g_{\theta\theta}^{(AdS)} + q(1+\rho)\left(\rho^2\sin^2\chi\right)\bar{g}_{\psi}
\end{aligned}
$$

$$
\begin{aligned}
H_t &\equiv H_t^{(AdS)} + q^3(1+\rho)^3\bar{H}_t \\
H_\rho &\equiv H_\rho^{(AdS)} + q^2(1+\rho)^2\bar{H}_\rho \\
H_\chi &\equiv H_\chi^{(AdS)} + q^3(1+\rho)^3\bar{H}_\chi
\end{aligned}
$$

$$
\Phi \equiv q^3(1+\rho)^3\bar{\Phi}
$$

- we evolve the barred variables, and *each* one of them variables satisfies a Dirichlet condition at *$\rho=1$*

GH equations approaching the AAdS Boundary

- Unfortunately, just defining variables that are regular and well-behaved in the limit is not good enough
  - the field equations still contain terms that are individually singular, though should conspire to cancel in a well-behaved gauge
  - to see this more clearly, expand the GH equations about $q=0$. define:

$$\bar{g}_{\mu\nu} = \bar{g}_{(1)\mu\nu}(t,x)\,q + \bar{g}_{(2)\mu\nu}(t,x)\,q^2 + O(q^3)$$
$$\bar{H}_{\mu} = \bar{H}_{(1)\mu}(t,x)\,q + \bar{H}_{(2)\mu}(t,x)\,q^2 + O(q^3)$$
$$\bar{\Phi} = \bar{\Phi}_{(1)}(t,x)\,q + \bar{\Phi}_{(2)\mu}(t,x)\,q^2 + O(q^3)$$

  - the field equations in GH form are a set of wave equations, so substitute the above on and solve for a wavelike-operator acting on the leading order term

GH equations approaching the AAdS Boundary

- An illustrative example:

$$\tilde{\nabla}^2_{(tt)}\,\bar{g}_{(1)tt} = \left(-8\bar{g}_{(1)\rho\rho} + 4\bar{H}_{(1)\rho}\right)q^{-2} + O(q^{-1})$$

  where

$$\tilde{\nabla}^2 \sim -c_0\frac{\partial^2}{\partial t^2} + c_1\frac{\partial^2}{\partial \rho^2} + \ldots$$

  and $c_0, c_1, \ldots$ are coefficients that depend on the particular equation, but are finite and regular in the limit $q=0$

- There are a hierarchy (the q^{-2} and q^{-1} terms in all the equations) of "constraints", namely terms that do not contain second time derivatives of the field
  - Note: these are not (entirely) the harmonic constraints
- Implies we are *not* free to choose the asymptotic form of the *regularized* source functions if the evolution is to preserve the desired asymptotic form of the metric
  - e.g. "harmonic" with respect to AdS, i.e. $H=H_{ADS}$ is not allowed (!)

Asymptotic choice of gauge

- Guided by the $q=0$ expansion, and some trial and error, we found the following asymptotic gauge choice works well (stable, consistent evolution) in the cases we have looked at so far

$$\left.\bar{H}_t\right|_{\rho=1} = \frac{5}{2}\bar{g}_{t\rho}; \quad \left.\bar{H}_\rho\right|_{\rho=1} = 2\bar{g}_{\rho\rho}; \quad \left.\bar{H}_\chi\right|_{\rho=1} = \frac{5}{2}\bar{g}_{\rho\chi}$$

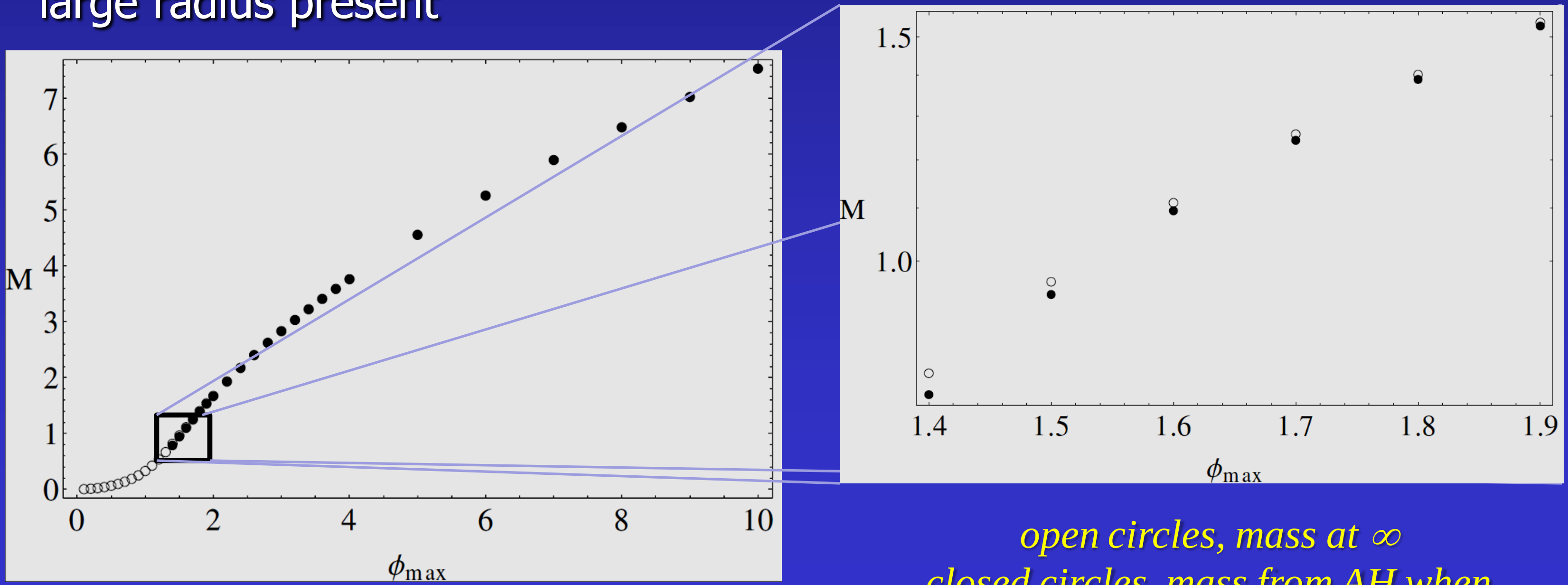
- many open questions regarding boundary conditions:
  - what class of gauges are consistent with a given choice of the asymptotic metric, and constraint preserving?
  - that we are not explicitly setting the leading order metric perturbation is akin to the way axis-regularity conditions are set; i.e., it is not a traditional boundary in the sense where one is free to set the modes coming into the computational domain
    - this corresponds to a boundary theory without "sources"; if we need to add them, in general the leading order metric fall-off would change, and how would this alter the above regularity conditions?

Brief overview of code

- Discretize equations in base second order in space and time from
- Standard, second order accurate finite differences (requires 3 time levels)
- Apparent horizon found via flow method, and used as basis for excision
- Kreiss-Oliger style numerical dissipation
- Berger & Oliger style AMR, multigrid (for initial data) and parallel support through PAMR/AMRD libraries

Initial Data

- Solve the constraints using ADM-based York-Lichnerowitz conformal decomposition
- For this study, restrict to time-symmetric initial data; momentum constraints trivially satisfied, solve the Hamiltonian constraint for a spatial metric that is conformal to pure AdS
- Non trivial initial curvature sourced by the scalar field; interestingly, for a scalar field profile with characteristic width of order the AdS length scale L, can specify arbitrary strong initial data; i.e. trapped surfaces of arbitrarily large radius present



L=1, spherically symmetric ID, width ~ L

open circles, mass at ∞
closed circles, mass from AH when present at t=0

AAdS Black Holes

- The 5D AdS-Schwarzschild black has metric is:

$$ds^2 = -\left(1 + r^2/L^2 - r_0^2/r^2\right)dt^2 + \left(1 + r^2/L^2 - r_0^2/r^2\right)^{-1} dr^2 + r^2 d\Omega_3^2$$

the horizon is at $r = r_H$, where

$$r_0 = r_H \sqrt{1 + r_H^2 / L^2}$$

and it has mass, entropy and temperature

$$M = \frac{3\pi}{8} r_0^2; \; S = \frac{\pi^2 r_H^3}{2}; \; T = \frac{r_H}{\pi L^2}\left(1 + \frac{L^2}{2 r_H^2}\right) \approx \frac{r_H}{\pi L^2}$$

Quasi-normal modes of AAdS Black Holes

- Gravitational and scalar field perturbations of 5D AdS-Schwarzschild black holes exhibit quasi-normal (QN) decay *[Horowitz & Hubeny PRD 62 (2000); Review: Berti, Cardoso & Starinets CQG 26 (2009)]*
  - in general for the metric there are scalar, vector & tensor modes; here due to axisymmetry only scalar modes can be excited
  - decompose scalar perturbation into scalar spherical harmonics on S^3, $S_{klm}(\chi,\theta,\varphi)$; again due to symmetry only $k\neq 0;\ l=m=0$.
  - A given QN mode can then schematically be written as

$$f_{klm}(t,\rho,\chi,\theta,\varphi) = A_{klm}(\rho) S_{klm}(\chi,\theta,\varphi) e^{-i\omega t}$$

$$\omega = \omega_r + i\omega_i$$

  - the decay time (imaginary mode) is of most interest to heavy ion collisions $\leftrightarrow$ thermalization/equilibration time scale of boundary state

Quasi-normal modes of AAdS Black Holes

- For large BHs relative to L ($r_H > L$), there are *fast*

$$\omega \approx (3.0 - 2.7i)\frac{r_H}{L^2} \qquad (k = 2, l = m = 0; \text{fund.mode})$$

  and *slow*

$$\omega \approx 1.6\frac{1}{L} - 0.8i\frac{1}{r_H} \qquad (k = 2, l = m = 0; \text{fund.mode})$$

  gravitational QNMs; the former can be thought of as related to "microscopic" perturbations of the boundary state, the latter "hydrodynamic".

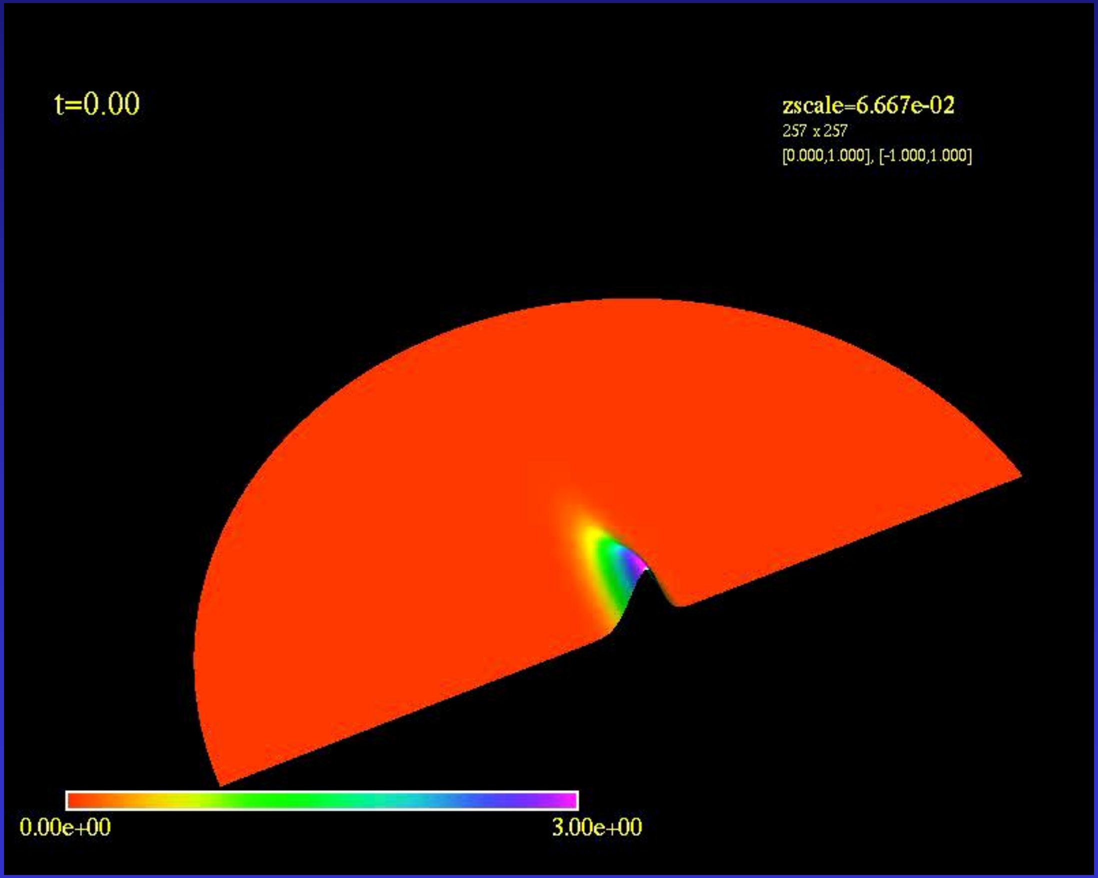
- The scalar field only has fast modes

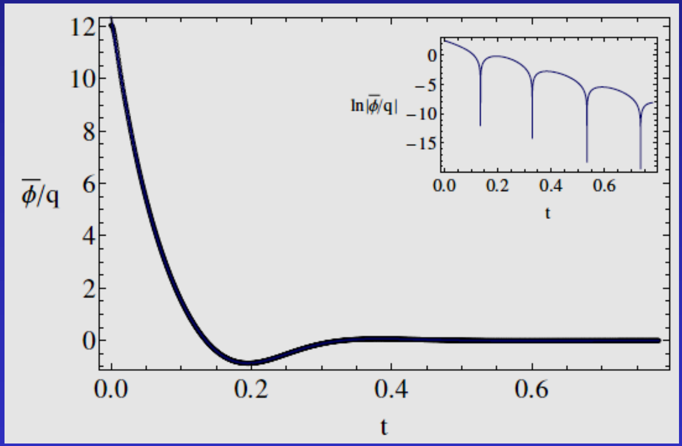
$$\omega \approx (3.0 - 2.7i)\frac{r_H}{L^2} \qquad (k = 0, l = m = 0; \text{fund.mode})$$

Quasi-normal modes of AAdS Black Holes

- form a distorted BH via asymmetric scalar field collapse

$$\overline{\Phi}(\rho,\chi,t=0) = Ae^{-\frac{\rho^2\cos^2\chi}{w_x^2}-\frac{\rho^2\sin^2\chi}{w_y^2}}$$





r_H=5.0; k=0

$$\overline{\Phi}(\rho,\chi,t)$$

r_H=12.2

Quasi-normal modes of AAdS Black Holes

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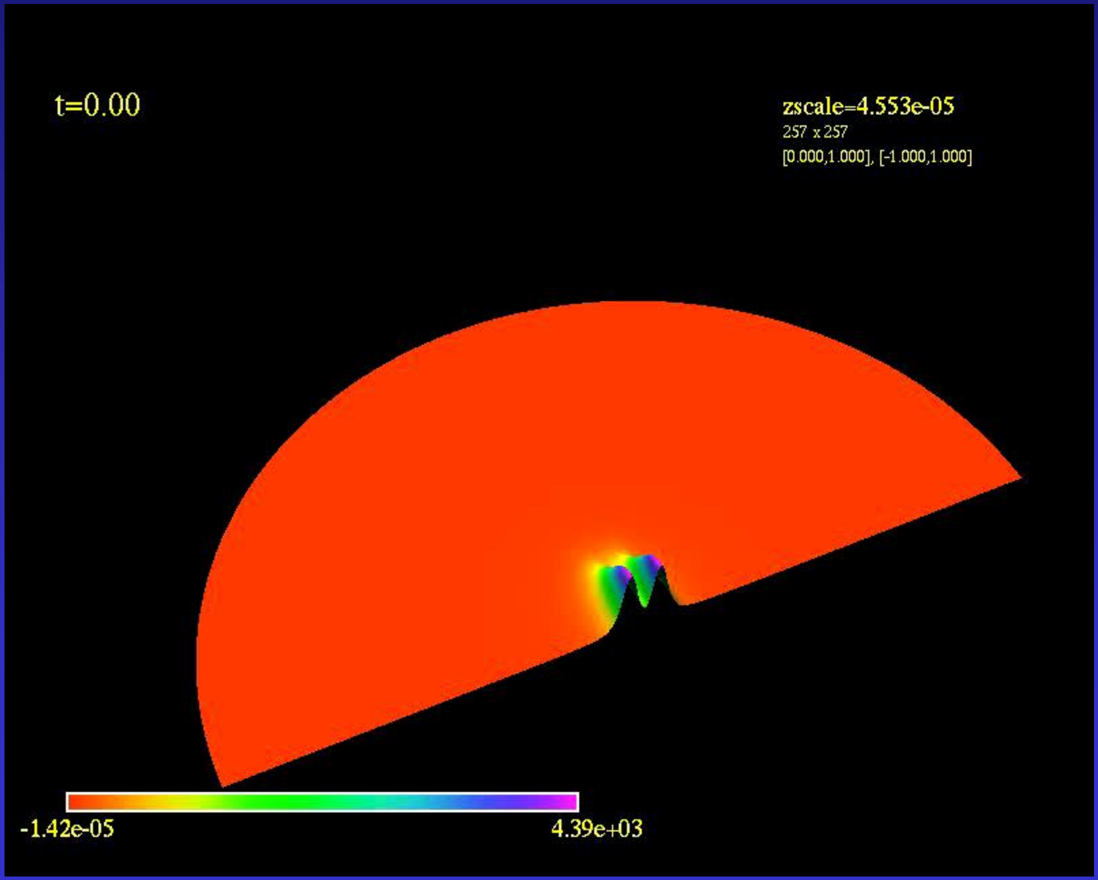
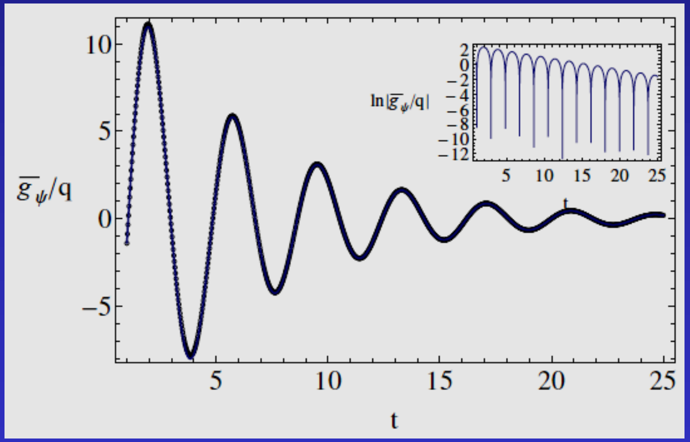


r_H=12.2

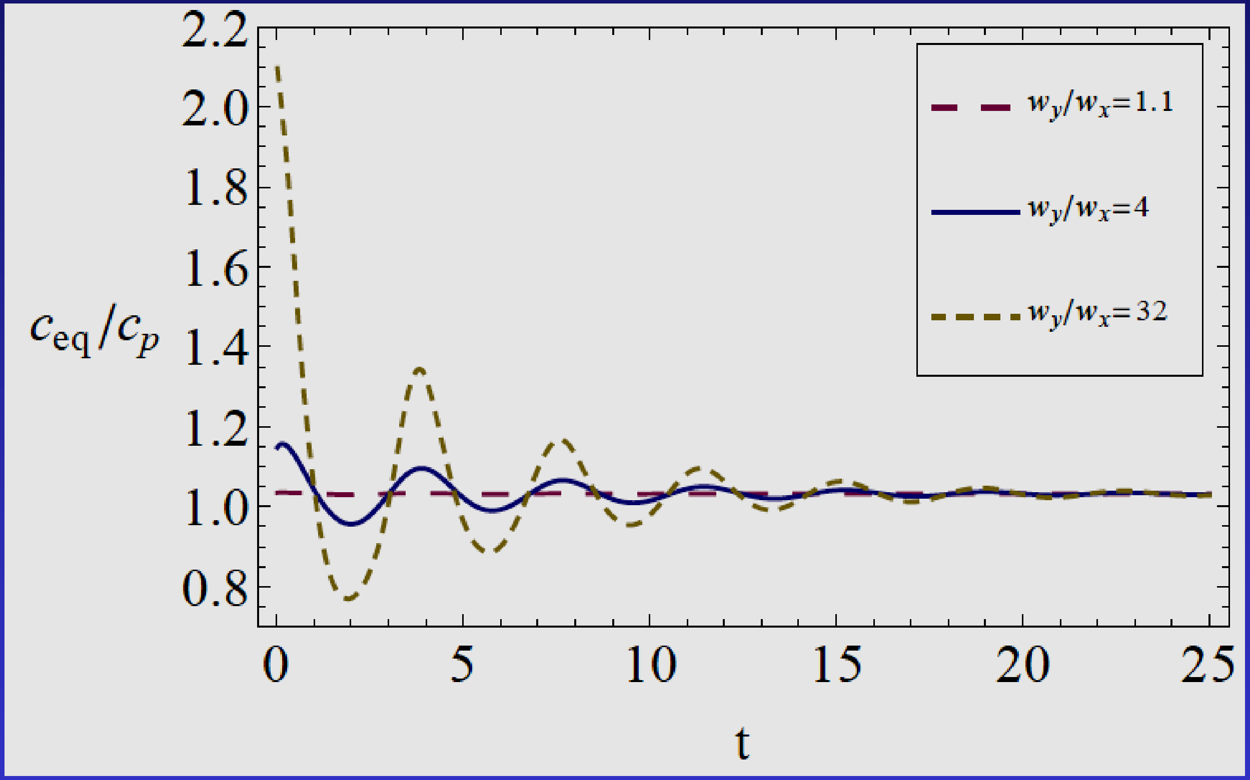


r_H=5.0; k=2

$$\overline{g}_{\chi\chi}(\rho,\chi,t)$$

Quasi-normal modes of AAdS Black Holes

- To give some idea of how "non-linear" we are, below is the ratio of equatorial to polar circumference of AH for increasing asymmetric ID



- Can describe asymptotic behavior of fields as a superposition of linear QN modes, plus what appears to be a gauge mode (a purely decaying exponential); the non-linearity manifests in higher k-number modes through the appearance harmonics of the lower k-modes

Boundary stress energy

- The AdS/CFT dictionary says

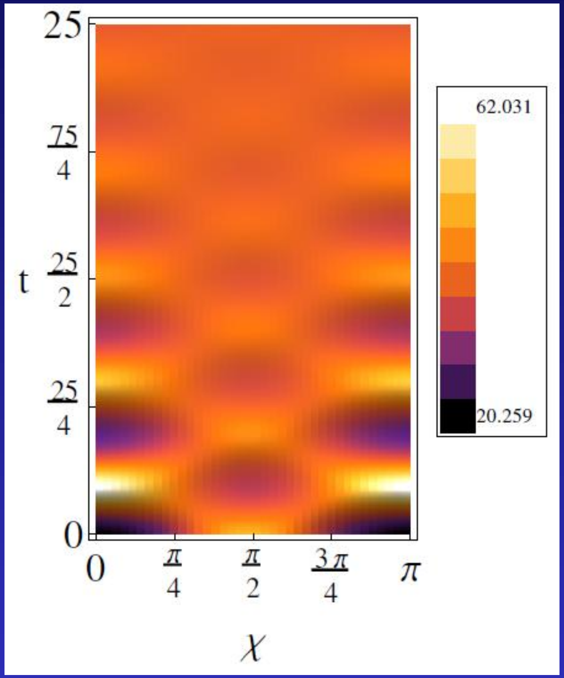
$$\langle T_{\mu\nu} \rangle_{CFT} = \lim_{q\to 0} \frac{1}{q^2}\left({}^{(q)}T_{\mu\nu} - {}^{(q)}T_{\mu\nu}^{ADS} \right)$$

where ${}^{(q)}T_{uv}$ is the Brown-York quasi-local stress energy tensor associated with a q=const. surface (with intrinsic metric Σ_{uv}, extrinsic curvature K_{uv}, and intrinsic Einstein tensor G_{uv})

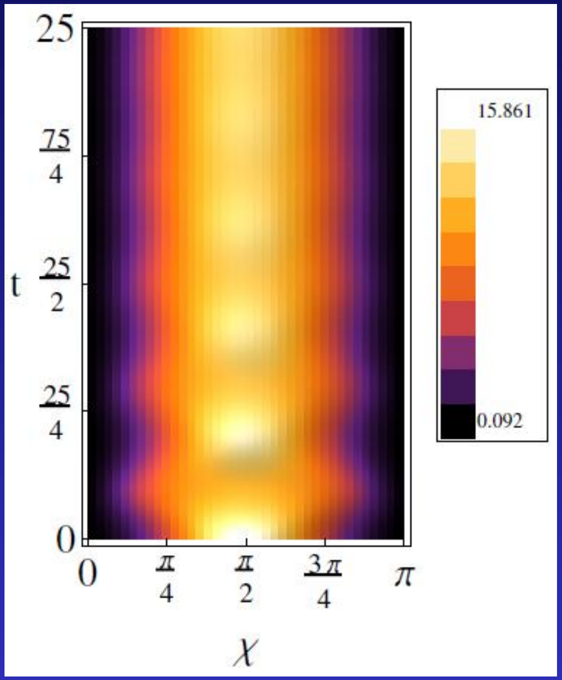
$${}^{(q)}T_{\mu\nu} = \frac{1}{8\pi}\left({}^{(q)}K_{\mu\nu} - \left({}^{(q)}K - \frac{3}{L} \right)\Sigma_{\mu\nu} + {}^{(q)}G_{\mu\nu}\frac{L}{2} \right)$$

and we have subtracted off the AdS Casimir term (arising due to the chosen S^3 topology)

Boundary stress energy : $w_y/w_x=4$



$\langle T_{tt}\rangle,\ r_H=5.0$

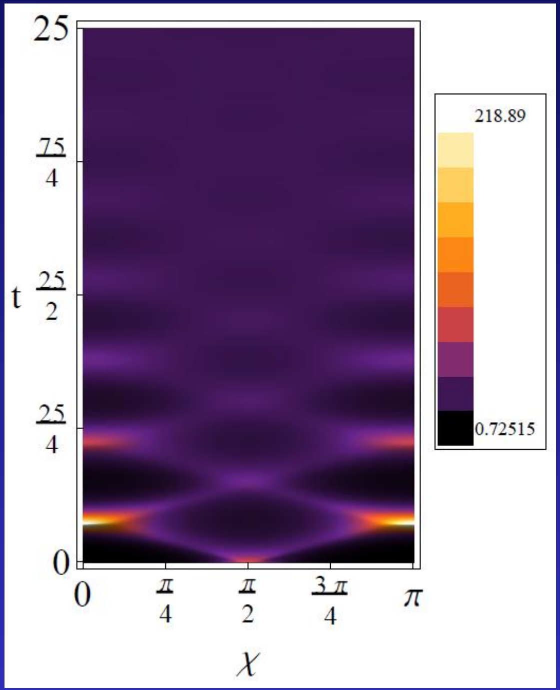


$\langle T_{xx}\rangle,\ r_H=5.0$

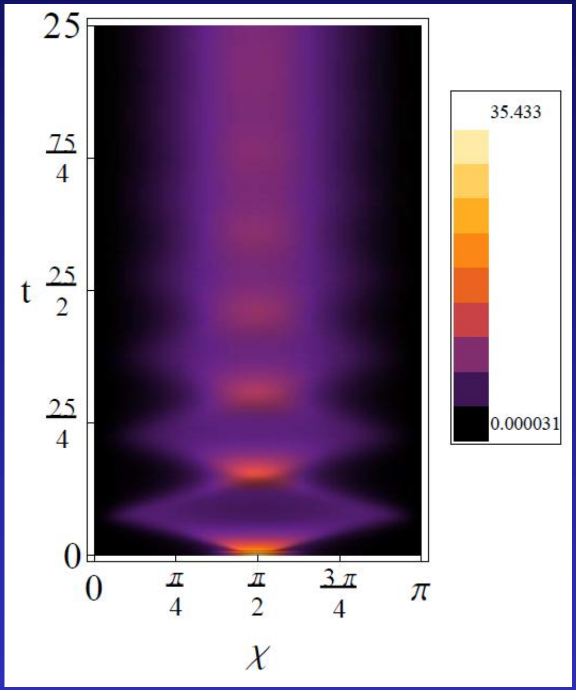
- To compare, The AdS-Schwarzschild solution describes a thermal state on S^3 with ($L=1$):

$$T_{ab} \approx \frac{r_H^4}{16\pi} \cdot \mathrm{diag}\left[3, 1, \sin^2\chi, \sin^2\chi \sin^2\theta\right]$$

Boundary stress energy : $w_y/w_x=32$



$<T_{tt}>,\ r_H=5.0$



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Hydrodynamics of the boundary SET

- Correspondence suggests if the bulk is dual to a thermal state on the boundary, the boundary SET should behave like a *$\mathcal{N}$=4 SYM* conformal fluid

$$T_{\mu\nu} = \sum_{i=0}^{\infty} T^{(i)}_{\mu\nu}$$

where up to 2nd order in a derivative expansion of the fluid velocity

$$\begin{aligned}
T^{(0)}_{\mu\nu} &= \epsilon u_\mu u_\nu + P \perp_{\mu\nu} \\
T^{(1)}_{\mu\nu} &= -2\eta\sigma_{\mu\nu} \\
T^{(2)}_{\mu\nu} &= -2\eta\left[-\tau_\pi u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} + \tau_\omega\left({\omega_\mu}^\lambda \sigma_{\lambda\nu} + {\omega_\nu}^\lambda \sigma_{\lambda\mu}\right)\right] \\
&\quad + \xi_\sigma\left[{\sigma_\mu}^\lambda \sigma_{\lambda\nu} - \frac{\perp_{\mu\nu}}{3}\sigma^{\alpha\beta}\sigma_{\alpha\beta}\right] + \xi_C C_{\mu\alpha\nu\beta} u^\alpha u^\beta
\end{aligned}$$

Hydrodynamics of the boundary SET

- with equation of state and transport coefficients given by

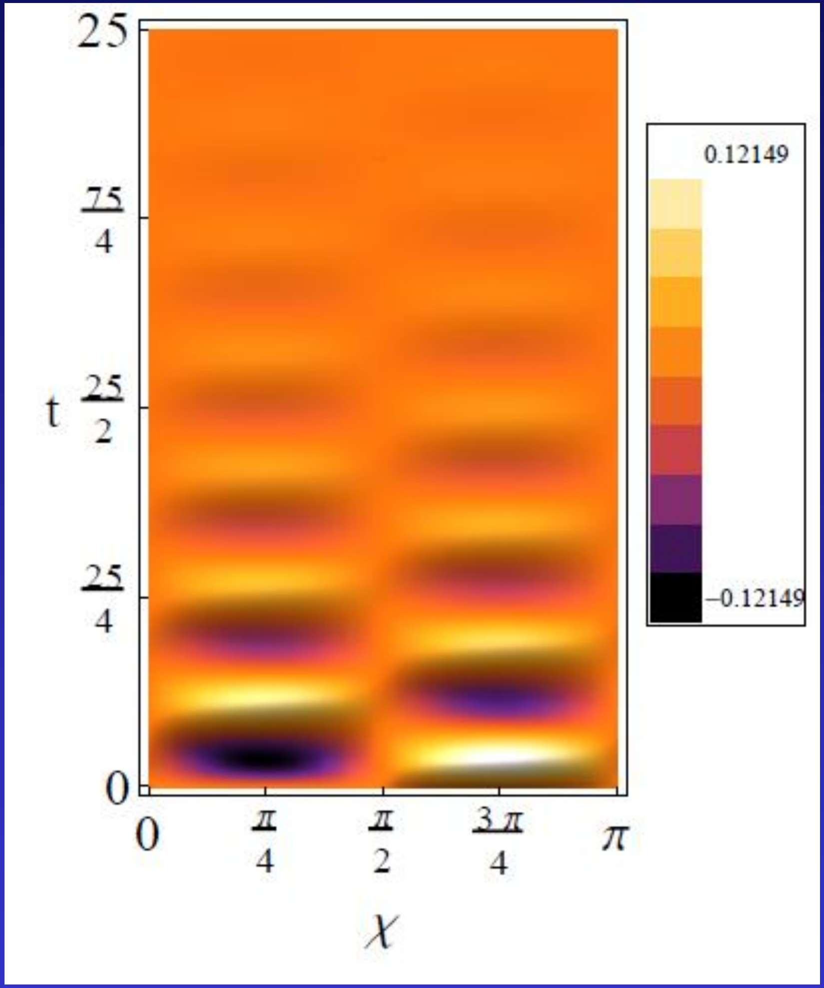
$$
\begin{aligned}
\epsilon &= \frac{3{N_c}^2}{8\pi^2}(\pi T)^4 = 3P \\
\eta &= \frac{{N_c}^2}{8\pi^2}(\pi T)^3 \\
\tau_\pi &= \frac{2-\ln 2}{2\pi T} \\
\tau_\sigma &= \frac{\ln 2}{2\pi T} \\
\xi_\sigma &= \xi_C = \frac{4\eta}{2\pi T}
\end{aligned}
$$

with energy density ε, pressure P, fluid 4-velocity υ^α, shear tensor $\sigma_{\upsilon\nu}$, Weyl curvature tensor $C_{\upsilon\alpha\beta\nu}$, temperature T, number of fields Nc (relate to G), shear viscosity η, stress relaxation time τ_π, shear vorticity coupling τ_σ, shear-shear coupling ξ_σ and Weyl curvature coupling ξ_C.

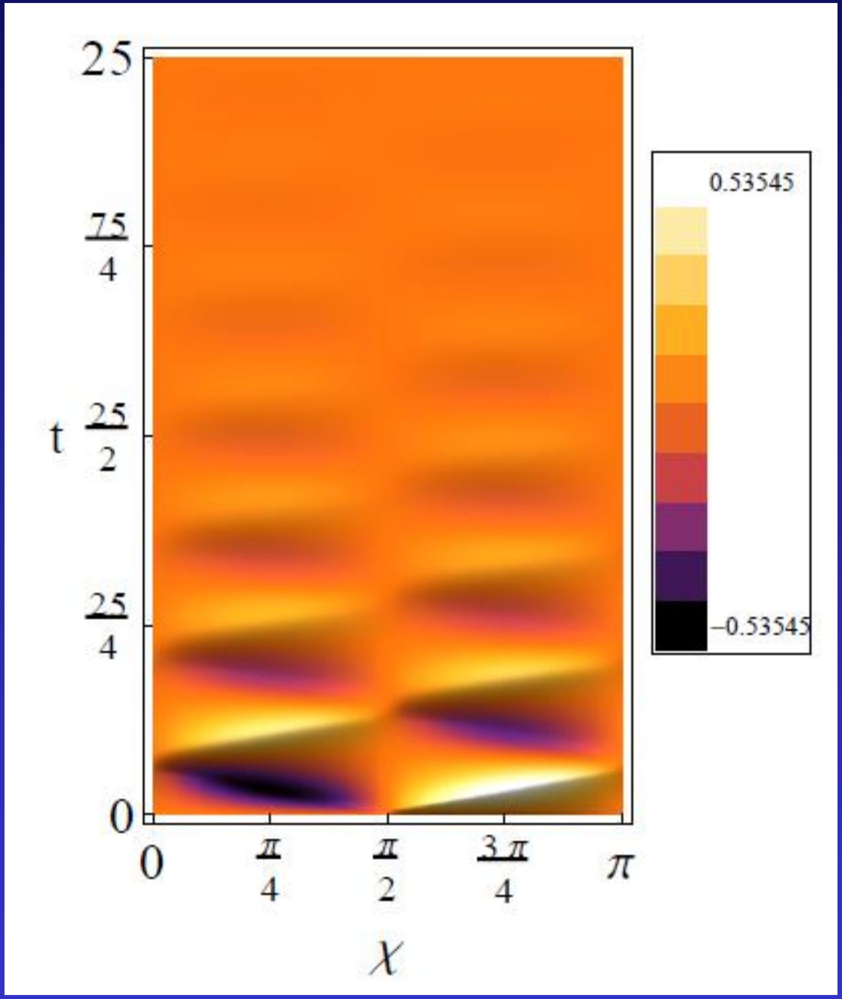
Hydrodynamics of the boundary SET

- Strategy to test for consistency
  - evaluate divergence of the extracted boundary SET … is it converging to zero? Yes.
  - from the extracted boundary SET, compute an energy density and 4 velocity … using this and the constitutive relations, reconstruct the SET order by order in the derivative expansion and compare to the remaining components of the extracted SET
    - i.e. we have 4 independent components of the extracted SET ($\varepsilon, v, P_\chi, P_{\theta/\phi}$), but if the dynamics is of that of a thermal fluid, only 2 are independent

Boundary Hydrodynamics : extracted velocities

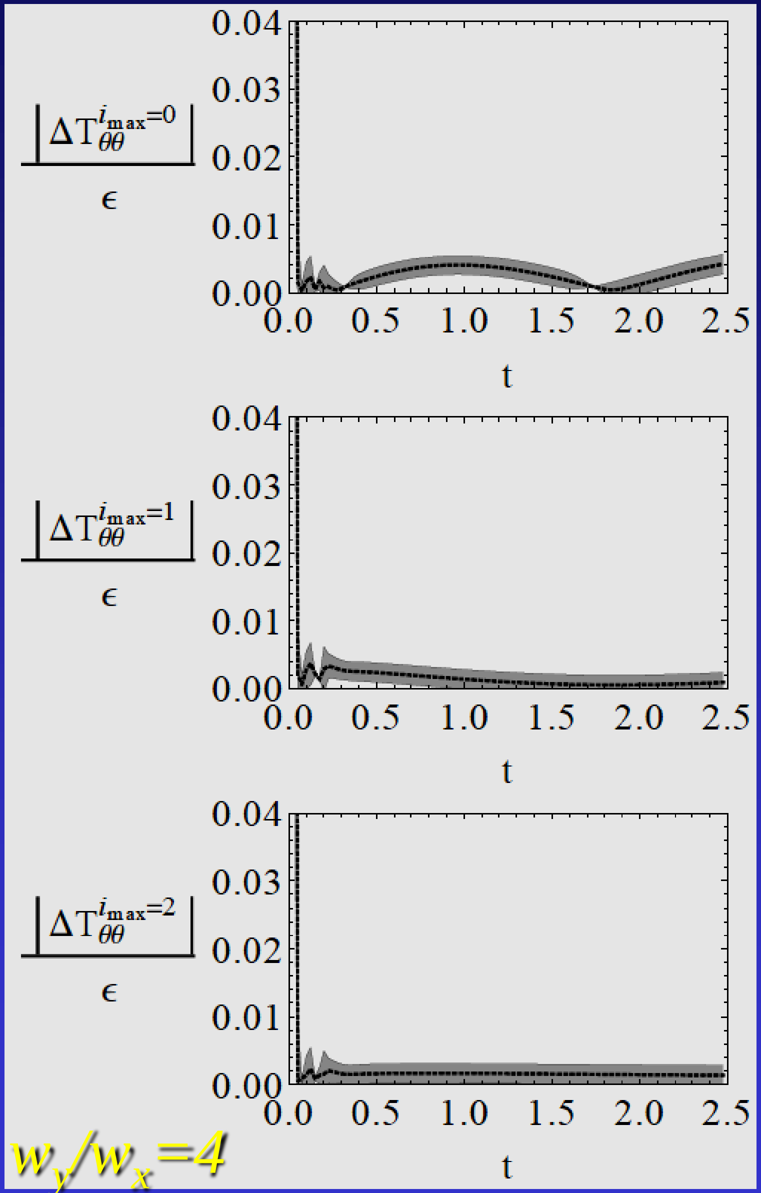


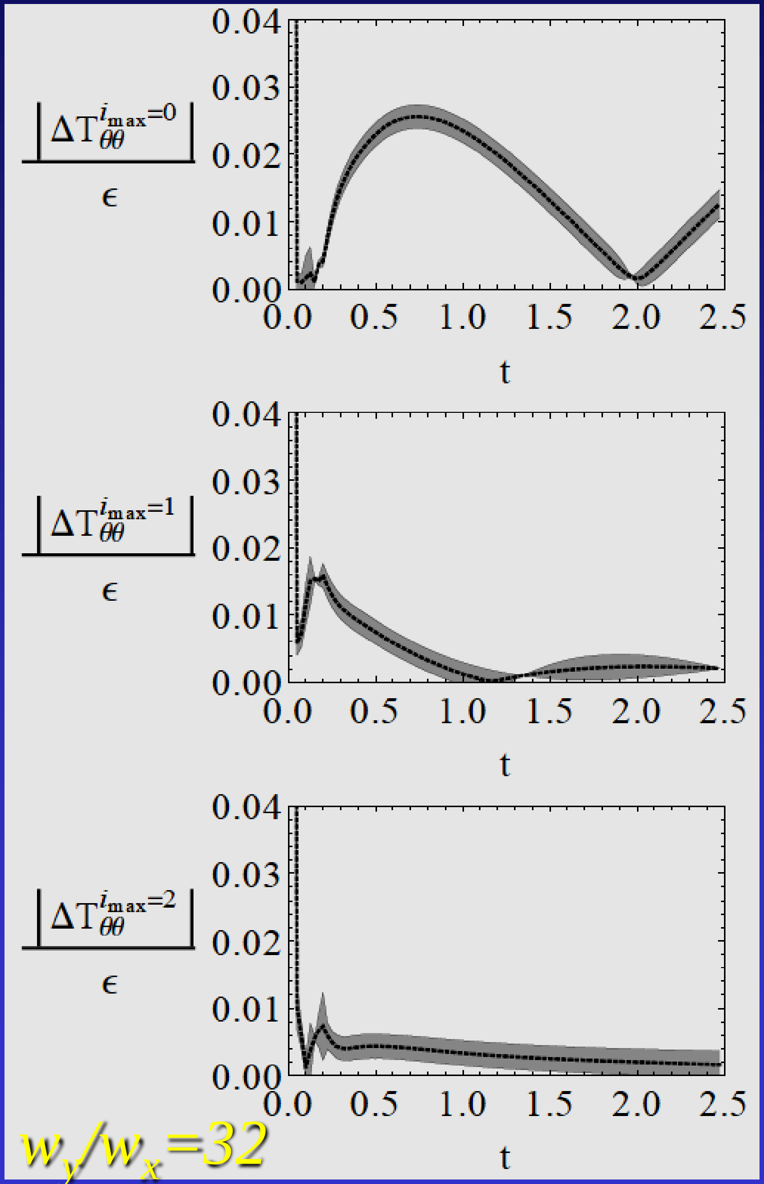
$w_y/w_x=4$



$w_y/w_x=32$

Boundary Hydrodynamics : consistency with a SYM fluid

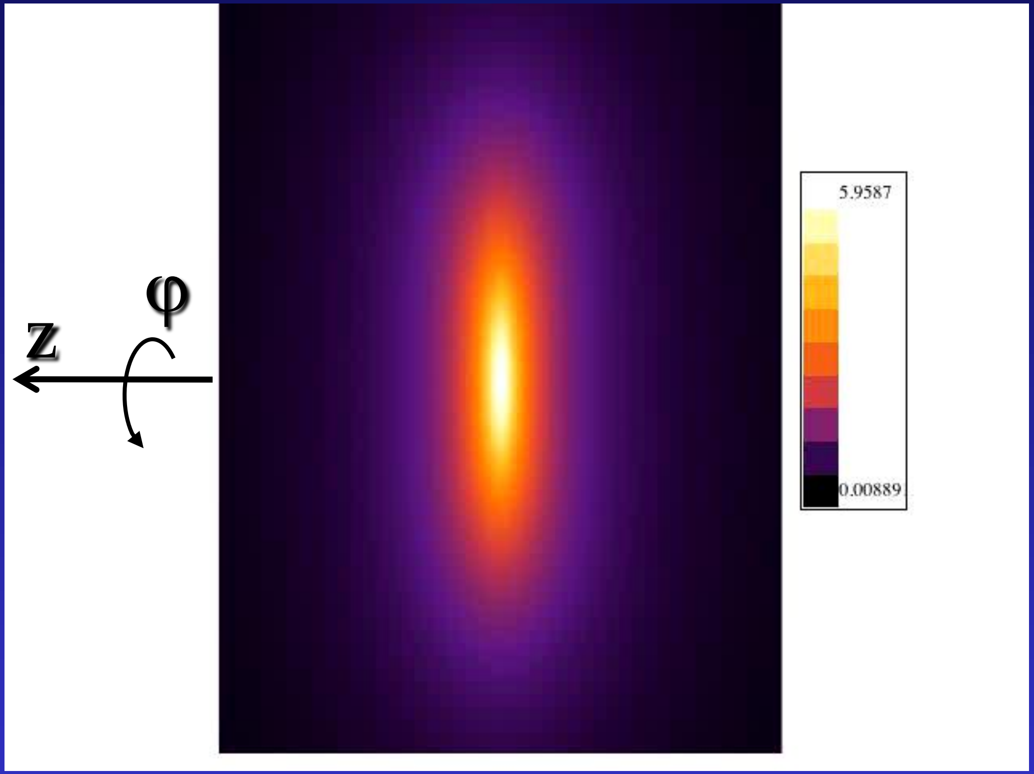


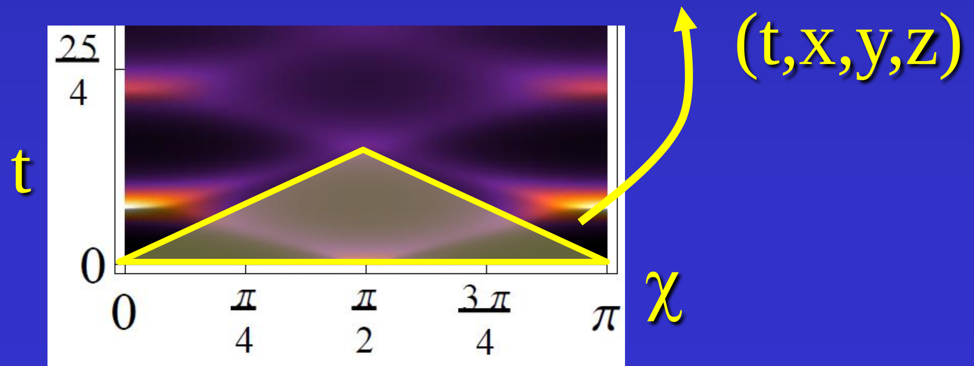


Connecting to QGP flows

- The simulations were performed in global coordinates; to relate to hydrodynamics in Minkowski spacetime we need to extract a Poincare patch of the boundary
  - some freedom in terms of which patch to use and the conformal transformation from $S^3 x R$ to $R^{3,1}$: use a transformation by Gubser *[PRD82, 2010]* designed to capture deviations from translational invariance orthogonal to the collision axis in the Bjorken flow picture
  - time-symmetric conditions suggest t=0 is a decent approximation to the "moment of collision" (though we're starting with a thermal state)

Temperature, w_y/w_x=32





Conclusions

- Future extensions/applications
  - connect simulation results to QGP experiments by some post-process description of particle production (e.g. Cooper-Frye)
    - based on this tune gravity initial conditions to best model experiments
  - relax symmetries and initial data to model non-central collisions, and possibly a pre-thermalization stage of the collision (soliton collisions?)
  - theoretical questions : how far can the gravity/fluid duality be pushed, "instability" of AdS in the sense of Bizon et al., etc.
  - adding various matter fields, including those corresponding to operator insertions in the CFT and hence "deformed" AdS asymptotics

GH equations approaching the AAdS Boundary

- An illustrative example:

$$\tilde{\nabla}^2_{(tt)}\,\bar{g}_{(1)tt} = \left(-8\bar{g}_{(1)\rho\rho} + 4\bar{H}_{(1)\rho}\right)q^{-2} + O(q^{-1})$$

$$\tilde{\nabla}^2_{(t\rho)}\,\bar{g}_{(1)t\rho} = \begin{pmatrix} -60\,\bar{g}_{(1)t\rho} - 8\cot\chi\,\bar{g}_{(1)t\chi} + 24\,\bar{H}_{(1)t} - \bar{g}_{(1)tt,t} \\ +2\bar{g}_{(1)t\chi,\chi} + 2\bar{g}_{(1)\rho\rho,t} - \bar{g}_{(1)\chi\chi,t} - 2\bar{g}_{(1)\psi,t} - 2\bar{H}_{(1)\rho,t} \end{pmatrix} q^{-2} + O(q^{-1})$$

where

$$\tilde{\nabla}^2 \sim -c_0\frac{\partial^2}{\partial t^2} + c_1\frac{\partial^2}{\partial \rho^2} + \ldots$$

and c_0, c_1, … are coefficients that depend on the particular equation, but are finite and regular in the limit $q=0$.