Cosmological Perturbations and Structure Formation via

STOCHASTIC GRAVITY

B. L. Hu
University of Maryland

KITP- UCSB Conference on Nonequilibirum Phenomena in Cosmology and Particle Physics (Feb 25-29, 2008)

Reporting on work of Roura and Verdaguer. *slides courtesy E. Verdaguer*

OUTLINE

1. Semiclassical gravity (SCG): SC Einstein Eqn: $<T_{mn}>$

2. Stochastic gravity: Einstein-Langevin Equation

   Noise Kernel: $<T_{mn} T_{rs}>$

3. Influence functional; Stochastic Effective Action.

4. One area of Application:

   Primordial cosmological perturbations

- *Gives result equivalent at* linear order *to usual method of quantizing metric and inflaton perturbations*
- *But can treat quadratic order perturbations needed in* R^2 *e.g., trace anomaly driven (Starobinsky) inflation* .

SEMICLASSICAL GRAVITY

Semiclassical Einstein eq. for classical geometry, metric function g

$$G_{ab}[g] = \kappa \langle \hat{T}_{ab}[g] \rangle_{ren}$$

$$\kappa = 8\pi G = 8\pi / m_P^2$$

Klein-Gordon eq. for quantum matter field ^ϕ

$$(\nabla_g^2 - m^2 - \xi R)\hat{\phi} = 0$$

- Solve for g and ^ϕ self-consistently.
- Backreaction problem is at the heart of semiclassical gravity.

SEMICLASSICAL EINSTEIN EQUATION

Renormalization introduces quadratic tensors

$$G_{ab}[g] + \Lambda g_{ab} - \alpha A_{ab}[g] - \beta B_{ab}[g] = \kappa \langle \hat{T}_{ab}[g] \rangle_{ren}$$

where

$$A^{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \int d^4x \sqrt{-g}\, C_{cdef} C^{cdef}$$

$$B^{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \int d^4x \sqrt{-g}\, R^2$$

$$T^{ab} = \nabla^a \phi \nabla^b \phi - \frac{1}{2} g^{ab} (\nabla^c \phi \nabla_c \phi + m^2 \phi^2) + \xi (g^{ab} \nabla^c \nabla_c - \nabla^a \nabla^b + G^{ab}) \phi^2$$

LIMITS OF SEMICLASSICAL GRAVITY

- Below Planck energy: measurements of time and length intervals

$$\Delta t \gg t_P, \quad \Delta \ell \gg \ell_P$$

- Quantum fluctuations of stress tensor small:

$$\langle \hat{T}^2 \rangle - \langle \hat{T} \rangle^2 \approx 0$$

We want to extend semiclassical Einstein equations to account for fluctuations of $\hat{T}_{ab}$ consistently.

Semiclassical Gravity

Semiclassical Einstein Equation (schematically):

$$\tilde{G}_{\mu\nu}(g_{\alpha\beta}) = \kappa\langle\hat{T}_{\mu\nu}\rangle_q + \kappa\,(T_{\mu\nu})_c$$

$\tilde{G}_{\mu\nu}$ is the Einstein tensor (plus covariant terms associated with the renormalization of the quantum field)

$\kappa = 8\pi G_N$ and G_N is Newton's constant

Free massive scalar field

$$(\Box - m^2 - \xi R)\hat{\phi} = 0.$$

$\hat{T}_{\mu\nu}$ is the stress-energy tensor operator

$\langle\,\rangle_q$ denotes the expectation value

Stochastic Gravity

Einstein- Langevin Equation (schematically):

$$\tilde{G}_{\mu\nu}(g_{\alpha\beta}) = \kappa\left(T^{\rm c}_{\mu\nu} + T^{\rm qs}_{\mu\nu}\right)$$

$T^{c}_{\mu\nu}$ is due to classical matter or fields

$$T^{\rm qs}_{\mu\nu} \equiv \langle \hat{T}_{\mu\nu} \rangle_{\rm q} + T^{\rm s}_{\mu\nu}$$

$T^{\rm qs}_{\mu\nu}$ is a new stochastic term
related to the quantum fluctuations of $T_{\mu\nu}$

How could a quantum field give rise to a stochastic source?

via Influence functional (Feynman-Vernon 1963):

- We will assume linear perturbation of semiclassical solution

$g_{ab} + h_{ab}$ But stochastic gravity is NOT restricted to linear perturbations

- Einstein-Langevin equation: $G_{g+h} = \kappa(\langle \hat{T} \rangle_{g+h} + \xi)$

$$G^{(1)}_{ab}[g+h] = \kappa \langle \hat{T}^{(1)}_{ab}[g+h] \rangle_{ren} + \kappa \xi_{ab}[g]$$

$$(\nabla^2_{g+h} - m^2 - \xi R)\hat{\phi} = 0$$

NOISE KERNEL

- Exp Value of 2-point correlations of stress tensor: bitensor
- Noise kernel measures quantum flucts of stress tensor

$$N_{abcd}(x,y) = \frac{1}{2}\langle\{\hat{t}_{ab}(x), \hat{t}_{cd}(y)\}\rangle$$

$$\hat{t}_{ab} \equiv \hat{T}_{ab} - \langle\hat{T}_{ab}\rangle\hat{I}$$

It can be represented by (shown via influence functional to be equivalent to) a classical stochastic tensor source $\xi_{ab}[g]$

$$\langle\xi_{ab}\rangle_s = 0 \qquad \langle\xi_{ab}(x)\xi_{cd}(y)\rangle_s = N_{abcd}(x,y)$$

- **Symmetric, traceless** (for conformal field), **divergenceless**

Noise associated with the fluctuations of a quantum field

- The noise kernel is real and positive semi-definite as a consequence of stress energy tensor being self-adjoint

the ultraviolet behaviour of $\langle \hat{T}_{ab}(x)\hat{T}_{cd}(y)\rangle$ is the same as that of $\langle \hat{T}_{ab}(x)\rangle\langle \hat{T}_{cd}(y)\rangle$.

- Classical Gaussian stochastic tensor field:

$$\langle \xi_{ab}[g;x)\rangle_s = 0, \quad \langle \xi_{ab}[g;x)\xi_{cd}[g;y)\rangle_s = N_{abcd}[g;x,y),$$

$\langle \cdots \rangle_s$

denotes statistical average wrt this noise distribution

Classical Stochastic Field

assoc.with a Quantum Field

- Stochastic tensor is covariantly conserved in the background spacetime (which is a solution of the semiclassical Einstein equation).

$$\nabla^a \xi_{ab}[g;x) = 0.$$

- For a conformal field ξ_{ab} is traceless:

$$g^{ab}\xi_{ab}[g;x) = 0;$$

Thus there is no stochastic correction to the trace anomaly

INFLUENCE FUNCTIONAL

- Open quantum system *(Feynman-Vernon 63)*

system h_{ab} ⟺ ϕ_j **environment**

$$F_{IF} \equiv e^{iS_{IF}} = \int D[\phi_+] D[\phi_-] \exp(S_m[\phi_+, g^+] - S_m[\phi_-, g^-])$$

$$S_{IF}(g+h^{\pm}) = \frac{1}{2}\int \langle \hat{T}_x \rangle [h_x] - \iint [h_x] H_{xy} \{h_y\} + \frac{i}{8}\iint [h_x] N_{xy} [h_y]$$

$$[h] \equiv h^+ - h^- \qquad \{h\} \equiv (h^+ + h^-)/2$$

(x,y denotes ab.cd)

$$H'_{xy} = \frac{1}{4}\mathrm{Im}\langle T^*(\hat{T}_x \hat{T}_y)\rangle - \frac{i}{8}\langle [\hat{T}_x, \hat{T}_y] \rangle$$

$$\langle T^{(1)}_{ab}[g+h]\rangle_{ren} = -2\int d^4y \sqrt{-g}\, H_{abcd}(x,y) h^{cd}(y)$$

INFLUENCE FUNCTIONAL

- Closed Time Path effective action at tree level in metric pert.

$$\Gamma_{CTP}^{(0)}\left[h^+,h^-\right]=S_g\left[h^+\right]-S_g\left[h^-\right]+S_{IF}\left[h^+,h^-\right]+O\left(h^3\right)$$

S_g is EH action plus quadratic terms.

- Integral identity (Feynman Vernon 1963):

$$\mathrm{e}^{-\mathrm{Im}\,S_{IF}}\equiv\exp\left(-\frac{1}{8}\iint\left[h_x\right]N_{xy}\left[h_y\right]\right)\propto\int D\xi\exp\left(-\frac{1}{2}\iint\xi_x N_{xy}^{-1}\xi_y+\frac{i}{2}\int\xi_z\left[h_z\right]\right)$$

- Probability distribution functional of a classical stochastic field $\xi_{ab}(x)$

$$P[\xi]\propto e^{-\frac{1}{2}\iint\xi N^{-1}\xi}$$

$$e^{iS_{IF}[h^+,h^-]}=\int D\xi P[\xi]e^{i\left(\mathrm{Re}\,S_{IF}+\frac{1}{2}\int\xi[h]\right)}\equiv\left\langle e^{i\left(\mathrm{Re}\,S_{IF}+\frac{1}{2}\int\xi[h]\right)}\right\rangle_s$$

STOCHASTIC EFFECTIVE ACTION

- Define a stochastic effective action:

$$\Gamma_{stc}\left[h^{+},h^{-};\xi\right]=S_{g}\left[h^{+}\right]-S_{g}\left[h^{-}\right]+\mathrm{Re}\,S_{IF}+\frac{1}{2}\int\xi_{z}\left[h_{z}\right]$$

- field equation from:

$$\left.\frac{\delta\Gamma_{stc}}{\delta h^{+}}\right|_{h^{\pm}=h}=0$$

➡ the Einstein-Langevin equation

$$G_{ab}^{(1)}[g+h]=\kappa\langle\hat{T}_{ab}^{(1)}[g+h]\rangle_{ren}+\kappa\xi_{ab}[g]$$

SOLUTIONS OF EINSTEIN-LANGEVIN EQUATIONS

- These stochastic equations determine the correlations

$$h_{ab}(x) = h^0_{ab}(x) + \kappa \int d^4x' \sqrt{-g}\, G^{ret}_{abcd}(x,x')\xi^{cd}(x')$$

$$\langle h_{ab}(x) h_{cd}(y)\rangle_s = \langle h^0_{ab}(x) h^0_{cd}(y)\rangle_s + \kappa^2 \iint G^{ret}_{abef}(x,x') N^{efgh}(x',y') G^{ret}_{ghcd}(y',y)$$

Intrinsic fluctuations + **Induced fluctuations**

(flucts in the initial state) (due to matter field flucts)

- Stochastic metric correlations is equivalent to quantum metric correlations in 1/N: (Calzetta, Roura, Verdaguer)

$$\frac{1}{2}\langle \left\{ \hat{h}_{ab}(x), \hat{h}_{cd}(y) \right\} \rangle = \langle h_{ab}(x) h_{cd}(y)\rangle_s$$

Applications of Stochastic Gravity : Fluctuations & Back-reaction Problems

1. Validity of Semiclassical Gravity — Hu, Roura Verdaguer (PRD04)

- Stability of solutions to SC Einstein Eqn with contributions of fluctuations ~ Einstein-Langevin Eqn
- Stochastic Gravity as next-to-leading-order 1/N limit. (Roura Verdaguer 03, Hartle-Horowitz 80)

2. Vacuum Fluctuations of Quantum Fields & Induced effects on Spacetime Dynamics :

- Negative energy density, quantum interest (Ford Roman)
- Re-examine Classical theorems in GR : Energy Dominance Condition with effects of quantum fluctuations

3. Black Hole Horizon Fluctuations & Backreaction (Hu, Raval Sinha 98, 03)

- many speculations on the magnitude of such fluctuations but no quantitative calculations yet.
- Stochastic gravity is the theory for such inquiries

Roura Hu (06, 07)

4. Structure Formation from grav. perturbations (Roura & Verdaguer 03

- particularly useful for trace anomaly-induced inflations.

Roura & Verdaguer (99, 07), Urakawa and Maeda (07)

e → 5. A platform towards Quantum Gravity. —

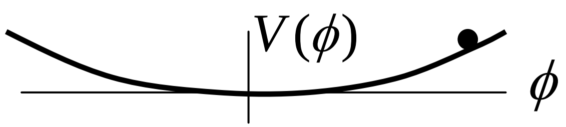
defined as a theory of the micro-scopic structure of ST

NOT quantizing GR

Hu 03

STOCHASTIC GRAVITY AND PRIMORDIAL COSMOLOGICAL PERTURBATIONS

- Quantum fluctuations of inflaton are seeds for structure formation
- Simplest chaotic inflationary model (Linde): Massive minimally coupled inflaton field, initially at average value larger than Planck scale



$$\dot{\phi}^2 \ll V(\phi) \qquad m_P \lesssim \phi_0$$

$$L(\phi) = \frac{1}{2}\partial_a\phi\partial^a\phi + \frac{1}{2}m^2\phi^2$$

- Background inflaton field and FRW metric

$$\phi(\eta) = \langle\hat{\phi}\rangle \qquad ds^2 = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$$

PERTURBATIONS

- Inflaton and scalar metric perturbations

$$\hat{\phi}(x) = \phi(\eta) + \hat{\varphi}(x) \qquad \langle \hat{\varphi} \rangle_g = 0$$

$$ds^2 = a^2(\eta)[-(1+2\Phi)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j]$$

- Einstein-Langevin equations

$$G^{(0)}_{ab}[g] - \kappa \langle \hat{T}^{(0)}_{ab}[g] \rangle + G^{(1)}_{ab}[h] - \kappa \langle \hat{T}^{(1)}_{ab}[h] \rangle = \kappa \xi_{ab}[g]$$

zeroth order metric g is assumed to be quasi- de Sitter.

$$\hat{T}_{ab} = \tilde{\nabla}_a \hat{\phi} \tilde{\nabla}_b \hat{\phi} - \frac{1}{2} \tilde{g}_{ab} (\tilde{\nabla}_c \hat{\phi} \tilde{\nabla}^c \hat{\phi} + m^2 \hat{\phi}^2) \qquad \tilde{g} = g + h$$

$$\langle \xi(x) \xi(y) \rangle_s = \frac{1}{2} \langle \{ \hat{t}(x), \hat{t}(y) \} \rangle [g] \qquad \hat{t} \equiv \hat{T} - \langle \hat{T} \rangle$$

STRESS TENSOR CORRELATIONS

$$\langle \hat{T}[g+h] \rangle = \langle \hat{T}[g+h] \rangle_{\phi\phi} + \langle \hat{T}[g+h] \rangle_{\phi\varphi} + \langle \hat{T}[g+h] \rangle_{\varphi\varphi}$$

$$\langle \{\hat{t},\hat{t}\} \rangle [g] = \langle \{\hat{t},\hat{t}\} \rangle [g]_{\phi^2\varphi^2} + \langle \{\hat{t},\hat{t}\} \rangle [g]_{\varphi^2\varphi^2} \equiv \langle \xi^{(1)} \xi^{(1)} \rangle_s + \langle \xi^{(2)} \xi^{(2)} \rangle_s$$

- assume Gaussian state: $\langle \hat{\varphi} \rangle = 0 \qquad \langle \hat{\varphi}\hat{\varphi}\hat{\varphi} \rangle = 0$
- two independent stochastic sources: $\xi^{(1)}, \xi^{(2)}$
  independently conserved
- Including only the first (linear) term: $\langle \cdots \rangle_{\phi^2\varphi^2}$

we will show that the stochastic gravity formulation gives equivalent results as the traditional quantized metric and scalar field perturbations

E-L eqn for linear perturbations

$$\frac{\kappa}{2}a^2\left(\langle\delta\hat{\mathcal{T}}^0_0\rangle_\Phi+\xi^0_0\right) = 3\mathcal{H}(\mathcal{H}\Phi+\Psi')-\nabla^2\Psi,$$

$$\frac{\kappa}{2}a^2\left(\langle\delta\hat{\mathcal{T}}^i_0\rangle_\Phi+\xi^i_0\right) = \partial_i(\Psi'+\mathcal{H}\Phi),$$

$$\frac{\kappa}{2}a^2\left(\langle\delta\hat{\mathcal{T}}^j_i\rangle_\Phi+\xi^j_i\right) = \Big[\left(2\mathcal{H}'+\mathcal{H}^2\right)\Phi+\mathcal{H}\Phi'+ \Psi''+2\mathcal{H}\Psi'+\frac{1}{2}\nabla^2 D\Big]\delta^j_i-\frac{1}{2}\delta^{jk}\partial_k\partial_i D.$$

where $\mathcal{H}=a'(\eta)/a(\eta),\ D=\Phi-\Psi,\ \nabla^2=\delta^{ij}\partial_i\partial_i$

- Since $\langle \hat{T}_{ij} \rangle = 0, (i \neq j)$ ⇒ $\xi_{ij} = 0, (i \neq j)$

⇒ metric perturbations $\Phi = \Psi$

- Fourier transf. of 0i-component: (neglecting non-local term):

$$2k_i (H\Phi_k + \Phi'_k) = \kappa \xi_{k(0i)} \qquad H \equiv \frac{a'(\eta)}{a(\eta)}$$

- Retarded propagator for Φ_k

$$G^k_{ret}(\eta, \eta') = \frac{\kappa}{2k_i} \left(\theta(\eta - \eta') \frac{a(\eta)}{a(\eta')} + f(\eta, \eta') \right)$$

With $\Psi = \Phi$ we get for the ii component of E-L eqn:

$$\frac{\kappa}{2} a^2 \left(\langle \delta \hat{\mathcal{T}}_i^{\,i} \rangle_\Phi + \xi_i^i \right) = \left(2\mathcal{H}' + \mathcal{H}^2 \right) \Phi + 3\mathcal{H}\Phi' + \Phi''.$$

Two unknowns 1. scalar metric perturbations $\Phi(x)$

2. $\langle \hat{\varphi} \rangle_\Phi$ the expectation value of the quantum operator for the inflaton perturbations on the spacetime with the perturbed metric, $\langle \hat{\varphi}[g+h] \rangle$

These three equations reduce to two because of the Bianchi Identity, which holds here since the averaged and stochastic sources in the EL eqn are separately conserved.

the one hand, the conservation of $\langle \delta \hat{\mathcal{T}}_{ab} \rangle_\Phi$ is equivalent to the Klein-Gordon equation for the expectation value $\langle \hat{\varphi} \rangle_\Phi$, which is completely analogous to Eq. (36):

$$\langle \hat{\varphi} \rangle_\Phi'' + 2\mathcal{H} \langle \hat{\varphi} \rangle_\Phi' - \nabla^2 \langle \hat{\varphi} \rangle_\Phi + m^2 a^2 \langle \hat{\varphi} \rangle_\Phi - 4\phi' \Phi' + 2m^2 a^2 \phi \Phi = 0. \qquad (41)$$

On the other hand, the conservation of the stochastic source is a consequence of the conservation of the noise kernel, which in turn relies on the fact that the quantum operator for the inflaton perturbations $\hat{\varphi}[g]$ satisfies the Klein-Gordon equation on the background spacetime, $\left(\nabla_a \nabla^a - m^2 \right) \hat{\varphi}(x) = 0$.

Equivalence with Quantum approach:

Can show that EL eqn reduces to (Roura and Verdaguer 2007)

$$\Phi'' + 2\left(\mathcal{H} - \frac{\phi''}{\phi'}\right)\Phi' - \nabla^2\Phi + 2\left(\mathcal{H}' - \mathcal{H}\frac{\phi''}{\phi'}\right)\Phi = 0,$$

Same as the conventional approach via quantized linear perturbations, e.g., Eq. (6.48) of

V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).

Comments:

1. In Fourier space nonlocal terms in the integro-differential equation in the spatial sector simplify to products. Non-locality in time in this equation disappears due to an exact cancellation of the different contributions from $\langle \delta \hat{\mathcal{T}}_a^b \rangle_\Phi$
2. Seems like there is no dependence on the stochastic source.
   But the solutions to ELeqn should also satisfy the constraint eqn at the initial time in addition to the dynamical eqn. The initial conditions for $\Phi_k(\eta_0)$ and $\Phi'_k(\eta_0)$ have dependence on the stochastic source.

CORRELATIONS FOR METRIC PERTURBATIONS

- Solutions of E-L equation:

$$\langle \Phi_k(\eta)\Phi_{k'}(\eta')\rangle_s = (2\pi)^2 \delta(\vec{k}+\vec{k}')\iint G_{ret}^{k} \langle \xi_k \xi_{k'}\rangle_s G_{ret}^{k'}$$

$$\langle \xi_k \xi_{k'}\rangle_s \equiv \frac{1}{2}\langle \left\{ \hat{t}_k, \hat{t}_{k'} \right\} \rangle$$

$$\langle \left\{ \hat{t}_{0i}^{k}(\eta_1), \hat{t}_{0i}^{-k}(\eta_2) \right\} \rangle = k_i k_i \phi'(\eta_1)\phi'(\eta_2)\langle \left\{ \hat{\varphi}_k(\eta_1), \hat{\varphi}_{-k}(\eta_2) \right\} \rangle$$

$$\langle \left\{ \hat{\varphi}_k(\eta_1), \hat{\varphi}_{-k}(\eta_2) \right\} \rangle = G_k^{(1)}(\eta_1, \eta_2)$$

is the Hadamard function for free scalar field on de Sitter, in Euclidean vacuum

$$a(\eta) = -\frac{1}{H\eta}; -\infty < \eta < 0$$

METRIC PERTURBATION CORRELATIONS

Computing $G_k^{(1)}$ perturbatively in m/m_P

assuming slow roll $\dot{\phi}(t) \simeq -m_P^2 (m/m_P)$

taking (rather insensitive to initial conds.) $\eta_0 \to -\infty$

$$\langle \Phi_k(\eta)\Phi_{k'}(\eta')\rangle_s \simeq 8\pi^2 \left(\frac{m}{m_P}\right)^2 k^{-3}(2\pi)^3 \delta(\vec{k}+\vec{k}')\cos[k(\eta-\eta')]$$

- Harrison-Zel'dovich scale inv. spectrum large scales $k\eta \leq 1$
- Amplitude of CMB anisotropies $\Rightarrow$ $\frac{m}{m_P} \simeq 10^{-6}$
- Agreement with linear perturbations approach (Mukhanov 92)
- Stochastic gravity can go beyond linear app. in inflaton flucts. (Weinberg 05) and deal with Starobinsky (tr anomaly) inflation

Summary: Main Features

1. Semiclassical gravity depends on e.v. of q. stress tensor

   S.G fails when flucts. of quantum stress tensor are large

2. Stochastic gravity incorporates these fluctuations (at Gaussian level) through the noise kernel acting as source for the Einstein-Langevin equation

3. Stochastic two-point metric correlations agree with quantum two-point metric correlations to order 1/N in large N expansion

4. Cosmological Perturbation and Structure Formation:

Agreement with linear perturbations approach (e.g.,Mukhanov 92)

- But can go beyond linear order in inflaton fluctuations

necessary for trace-anomaly driven inflations (e.g., Starobinsky 1980)

Stochastic Gravity program

- *(since 1994)* *E. Calzetta (Buenos Aires),*
  B. L. Hu, A. Matacz, N.G. Phillips, S. Sinha (Maryland)
  A. Campos, R. Martin, A. Roura, Enric Verdaguer (Barcelona);

- *Current work:*
  with *A. Roura (Los Alamos), Enric Verdaguer (Barcelona)*

- cosmological perturbations work // by Urakawa and Maeda (Waseda)

- *Review:*

B. L. Hu and E. Verdaguer, "Stochastic gravity: Theory and Applications", in Living Reviews in Relativity 7 (2004) 3.

[update in arXiv:0802.0658]

SEMICLASSICAL GRAVITY

- Gravitational field classical g_{ab}, Matter fields quantum $\hat{\phi}_j$ j=1,..,N
- Quantum field theory in curved spacetime: $\hat{\phi}_j$ **test field**

  particle creation: early universe, Hawking radiation
- Semiclassical gravity: **backreaction** of $\hat{\phi}_j$ on g_{ab}

  cosmology, inflation, black hole evaporation

  - Axiomatic approach (Wald 77)
  - Large N expansion (Hartle-Horowitz 81)

Einstein-Langevin Equation

- Consider a weak gravitational perturbation h off a background $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, The ELE is given by

$$G_{ab}[g+h] + \Lambda(g_{ab} + h_{ab}) - 2(\alpha A_{ab} + \beta B_{ab})[g+h]$$

$$= 8\pi G\left(\langle \hat{T}^{R}_{ab}[g+h]\rangle + \xi_{ab}[g]\right).$$

- The ELE is Gauge invariance

Noise Kernel

A physical observable that describes these fluctuations to the lowest order is the noise kernel which is the vacuum expectation value of the two-point correlation function of the stress-energy operator

$$N_{abcd}[g; x, y) = \frac{1}{2}\langle\{\hat{t}_{ab}[g; x), \hat{t}_{cd}[g; y)\}\rangle,$$

$$\hat{t}_{ab}[g; x) \equiv \hat{T}_{ab}[g; x) - \langle\hat{T}_{ab}[g; x)\rangle.$$

Stochastic Gravity in relation to *Quantum* and *Semiclassical*

(w. Enric Verdaguer, Peyresq 98)

As an example, let's consider

gravitational perturbations $h_{\mu\nu}$ in a FLRW universe with background metric $g_{\mu\nu}$

The Semiclassical Einstein Equation is $\Box h = \langle \hat{T} \rangle$

where < > denotes the quantum vacuum expectation value

With solutions $h = \int G \langle \hat{T} \rangle, \qquad h_1 h_2 = \int \int G_1 G_2 \langle \hat{T} \rangle \langle \hat{T} \rangle$

The Quantum (Heisenberg) Equation is $\Box \hat{h} = \hat{T}$

With solutions $\hat{h} = \int G \hat{T}, \qquad \langle \hat{h}_1 \hat{h}_2 \rangle = \int \int G_1 G_2 \langle \hat{T} \hat{T} \rangle_{\hat{h}, \hat{\phi}}$

Where the average is taken with respect to the quantum fluctuations of both the gravitational and matter fields

For **stochastic gravity**, the Einstein Langevin equation is of he form

$$\Box h = \langle \hat{T} \rangle + \tau$$

With solutions

$$h = \int G \langle \hat{T} \rangle + \int G\tau, \qquad h_1 h_2 = \int \int G_1 G_2 [\langle \hat{T} \rangle \langle \hat{T} \rangle + (\langle \hat{T} \rangle \tau + \tau \langle \hat{T} \rangle) + \tau\tau]$$

We now take the noise average

$$\langle \rangle \xi$$

Recall

$$\hat{t}_{\mu\nu}(x) \equiv \hat{T}_{\mu\nu}(x) - \langle \hat{T}_{\mu\nu}(x) \rangle \hat{I}$$

Hence

$$\langle \tau \rangle_\xi = 0, \qquad \langle \tau_1 \tau_2 \rangle_\xi \equiv \langle \hat{T}_1 \hat{T}_2 \rangle - \langle \hat{T}_1 \rangle \langle \hat{T}_2 \rangle$$

We get,

$$\langle h_1 h_2 \rangle_\xi = \int \int G_1 G_2 \langle \hat{T} \hat{T} \rangle_{\hat{\phi}}$$

Note this **has the same form as in quantum** gravity except that the Average is taken with respect to matter field fluctuations only.

Semiclassical Gravity includes only the **mean value** of the Stress-Energy Tensor of the matter field

Stochastic Gravity includes the two point function of **T**mn in the **Einstein-Langevin equation**

It is **the lowest order in the hierarchy of correlation functions**. The full hierarchy gives full information about the matter field.

At each level of the hierarchy there is a linkage with the gravity sector. The lowest level is the **Einstein equation** relating the **T**mn itself to the Einstein tensor **G**mn

Quantum Fluctuations :: Quantum Correlation :: Quantum Coherence

Thus **stochastic gravity recovers partial quantum coherence in the gravity sector via the metric fluctuations** induced by matter fields

A simple illustrative model

- Classical theory $\Box \phi = 0$ $\Box h = \kappa T = \kappa \partial_a \phi \partial^a \phi$
- Quantum theory (Heisenberg) $\Box \hat{h} = \kappa \hat{T}$

solution $$\hat{h}_x = \hat{h}_x^0 + \kappa \int G_{xy} \hat{T}_y$$

$$\langle \{ \hat{h}_x , \hat{h}_y \} \rangle = \langle \{ \hat{h}_x^0 , \hat{h}_y^0 \} \rangle + \kappa^2 \iint G_{xx'} G_{yy'} \langle \{ \hat{T}_{x'} , \hat{T}_{y'} \} \rangle$$

- Noise kernel:

$$N_{xy} = \frac{1}{2}\langle\{\hat{t}_x, \hat{t}_y\}\rangle,$$

$$\hat{t} \equiv \hat{T} - \langle\hat{T}\rangle$$

Define stochastic Gaussian field $\langle\xi\rangle_s = 0,$ $\langle\xi_x\xi_y\rangle_s = N_{xy}$

- Langevin equation

$$\Box h_x = \kappa(\langle\hat{T}_x\rangle + \xi_x)$$

solution

$$h_x = h_x^0 + \kappa\int G_{xx'}(\langle\hat{T}_{x'}\rangle + \xi_{x'})$$

$$2\langle h_x h_y\rangle_s = 2\langle h_x^0 h_y^0\rangle_s + \kappa^2 \iint G_{xx'}G_{yy'}\langle\{\hat{T}_{x'}, \hat{T}_{y'}\}\rangle$$

A SIMPLE MODEL

- Second terms on r.h.s. are equal
- First terms on r.h.s. are equal provided initial distribution h_x^0

$$\langle h_x^0 h_y^0 \rangle_s = \frac{1}{2} \langle \{ \hat{h}_x^0 , \hat{h}_y^0 \} \rangle$$

- Obtain quantum correlations from stochastic approach.

$$\frac{1}{2} \langle \{ \hat{h}_x , \hat{h}_y \} \rangle = \langle h_x h_y \rangle_s$$

$$\frac{1}{2} \langle [\hat{h}_x , \hat{h}_y] \rangle = i\kappa (G_{yx} - G_{xy})$$

in agreement with large N expansion