Resummation in Cosmological Gravitational Clustering

Roman Scoccimarro (NYU)

Renormalized Perturbation Theory (RPT)

Crocce and Scoccimarro, astro-ph/0509418-9, arXiv:0704.2783

Bernardeau, Crocce and Scoccimarro, in preparation

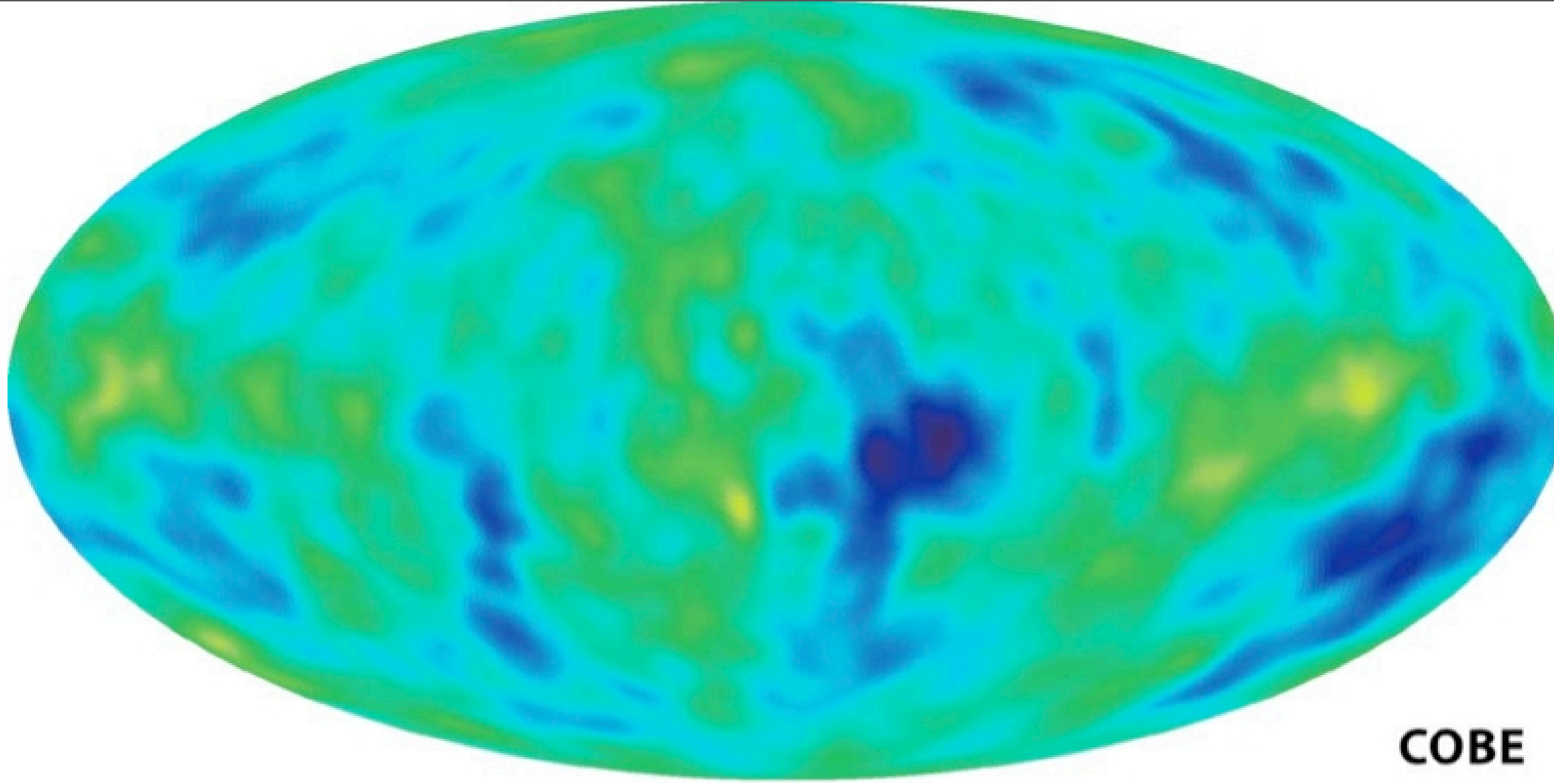
Basic Picture of Cosmological Structure Formation:

- The universe at t=300,000 years (decoupling of photons from baryons) was nearly homogeneous and isotropic with tiny density perturbations.

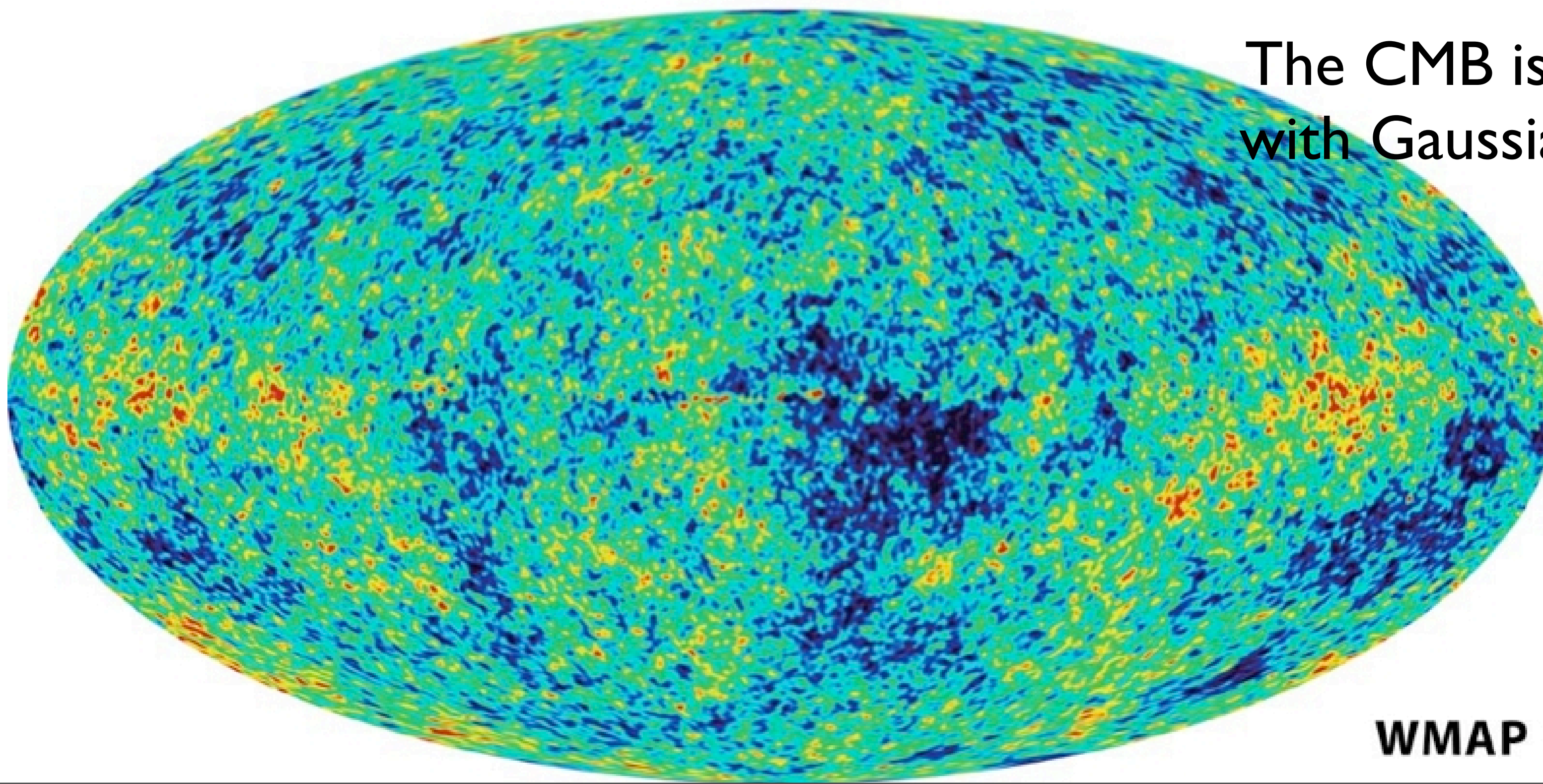
- The best candidate we have for the generation of these fluctuations is inflation in the very early universe, a period of very rapid (nearly exponential) expansion.

- Small fluctuations from decoupling grow to become galaxies by today, t =13.7 billion years. Galaxies and Dark Matter are clustered (large-scale structure).

WMAP Science Team (http://lambda.gsfc.nasa.gov)



COBE



WMAP

The CMB is consistent with Gaussian statistics

Gaussian Random Fields

Simplest form of random fields, just characterized by their second moment,

$$\xi(r) = \langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r}) \rangle \qquad\qquad 1+\delta = \rho/\bar{\rho}$$

The physical interpretation of this two-point function for objects such as galaxies, has to do with probability of finding pairs of objects at some distance from each other

$$\mathrm{d}P_{12} = n^2[1+\xi(x_{12})]\mathrm{d}V_1\mathrm{d}V_2$$

Gaussian Fields are easiest to describe in Fourier space,

$$\delta(\mathbf{x}) = \int \mathrm{d}^3\mathbf{k}\,\delta(\mathbf{k})\,\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})$$

$$\delta(\mathbf{k}) = \delta^*(-\mathbf{k})$$

$$\langle \delta(\mathbf{k})\delta(\mathbf{k}') \rangle = \delta_D(\mathbf{k}+\mathbf{k}')\,P(k)$$

Translation Invariance   Rotational Invariance where the power spectrum P(k) is the Fourier transform of the 2-pt function,

$$\xi(r) = \int \mathrm{d}^3\mathbf{k}\, P(k) \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{r})$$

In a Gaussian field Fourier modes satisfy simple properties,

$$\begin{aligned} \langle \delta(\mathbf{k}_1) \ldots \delta(\mathbf{k}_{2p+1}) \rangle &= 0 \\ \langle \delta(\mathbf{k}_1) \ldots \delta(\mathbf{k}_{2p}) \rangle &= \sum_{\text{all pair associations}} \quad \prod_{p \text{ pairs (i,j)}} \langle \delta(\mathbf{k}_i) \delta(\mathbf{k}_j) \rangle \end{aligned}$$

Thus, "initial" conditions for late structure formation (density/velocities after decoupling) only correlate in pairs due to Gaussianity.

Thus, their power spectrum completely specifies the statistical properties.

"All" we have to do, is evolve forward in time.

What do we know about the power spectrum after decoupling?

For the simplest models of inflation, fluctuations are Gaussian, with a nearly scale-invariant spectrum, i.e. the gravitational potential (~ curvature) spectrum is

$$P_\Phi(k) \propto k^{-3\,+n_s\,-\,1} \qquad\qquad n_s \approx 1$$

So, that after multiplying by number of modes in 3D,

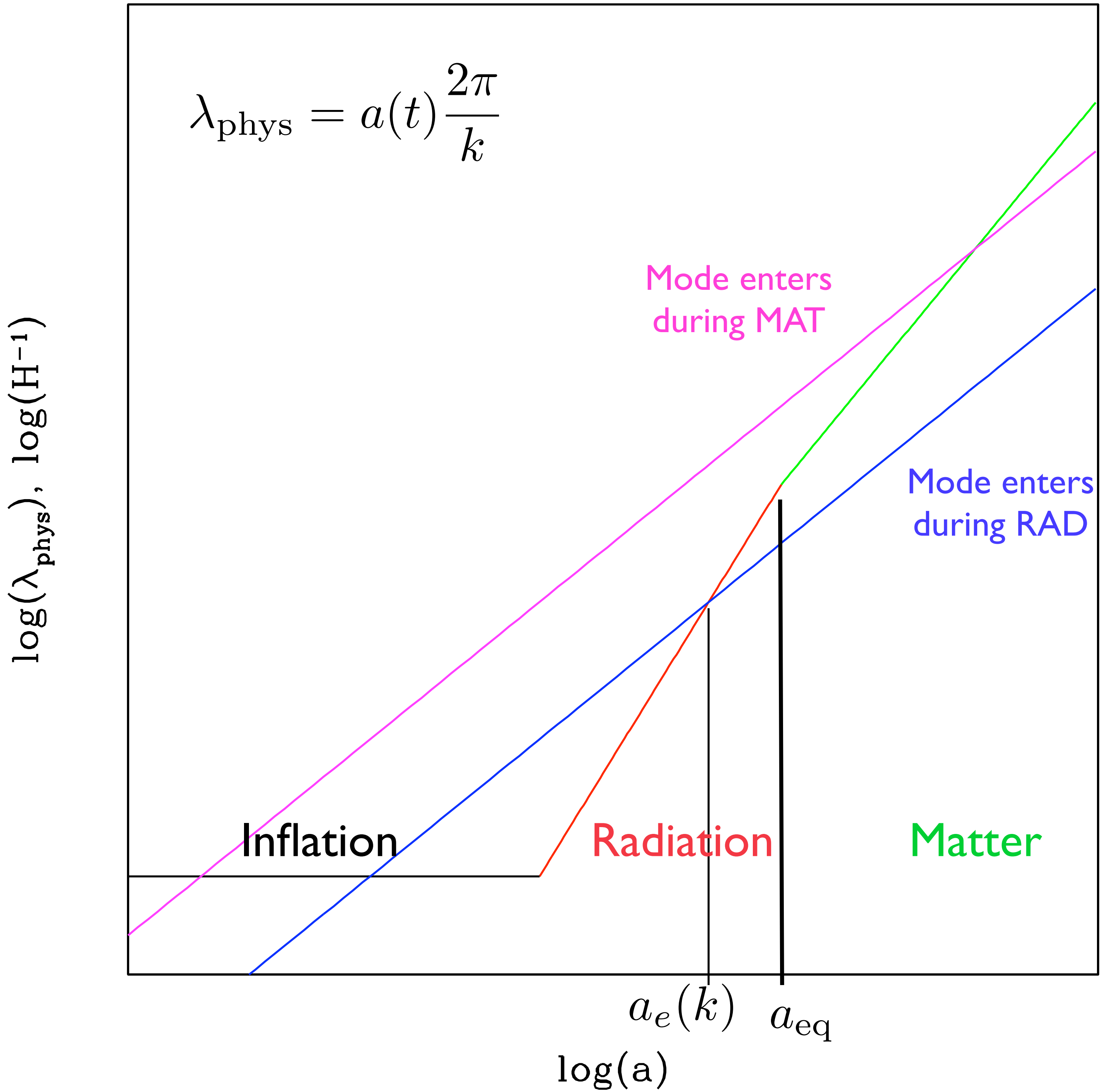
$$k^3 \times P_\Phi(k) = \mathrm{constant} \;\; k^{n_s - 1}$$

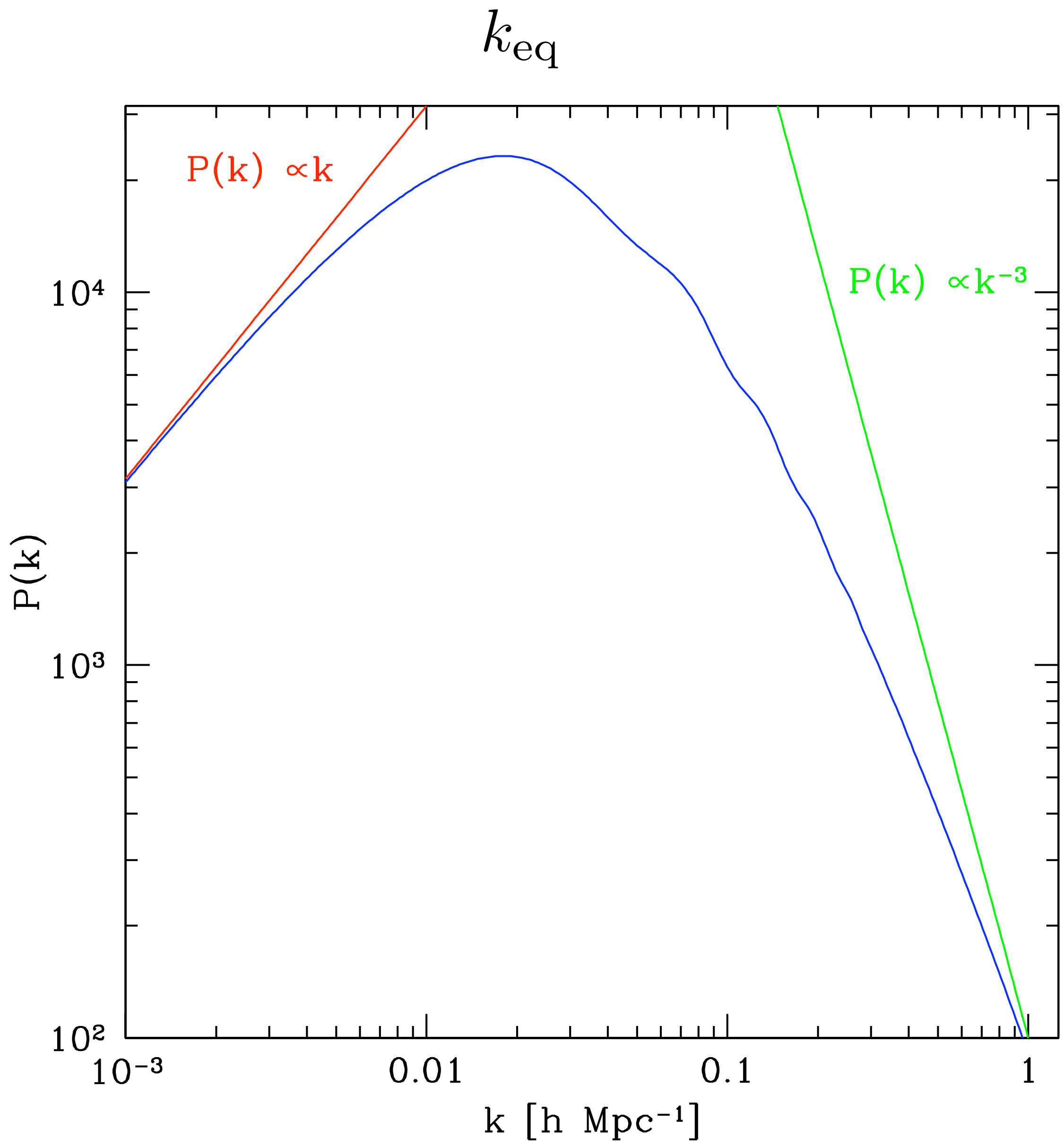
So all scales k have the same amplitude when ns=1, hence "scale-invariant". Now from the Poisson Equation, $\nabla^2\Phi \propto \delta$, hence density spectrum is

$$k^4 P_\Phi \propto P_\delta \sim k^{\,n_s}$$

This is what inflation creates, however, it gets more interesting...

Radiation era suppresses spectrum at high-k

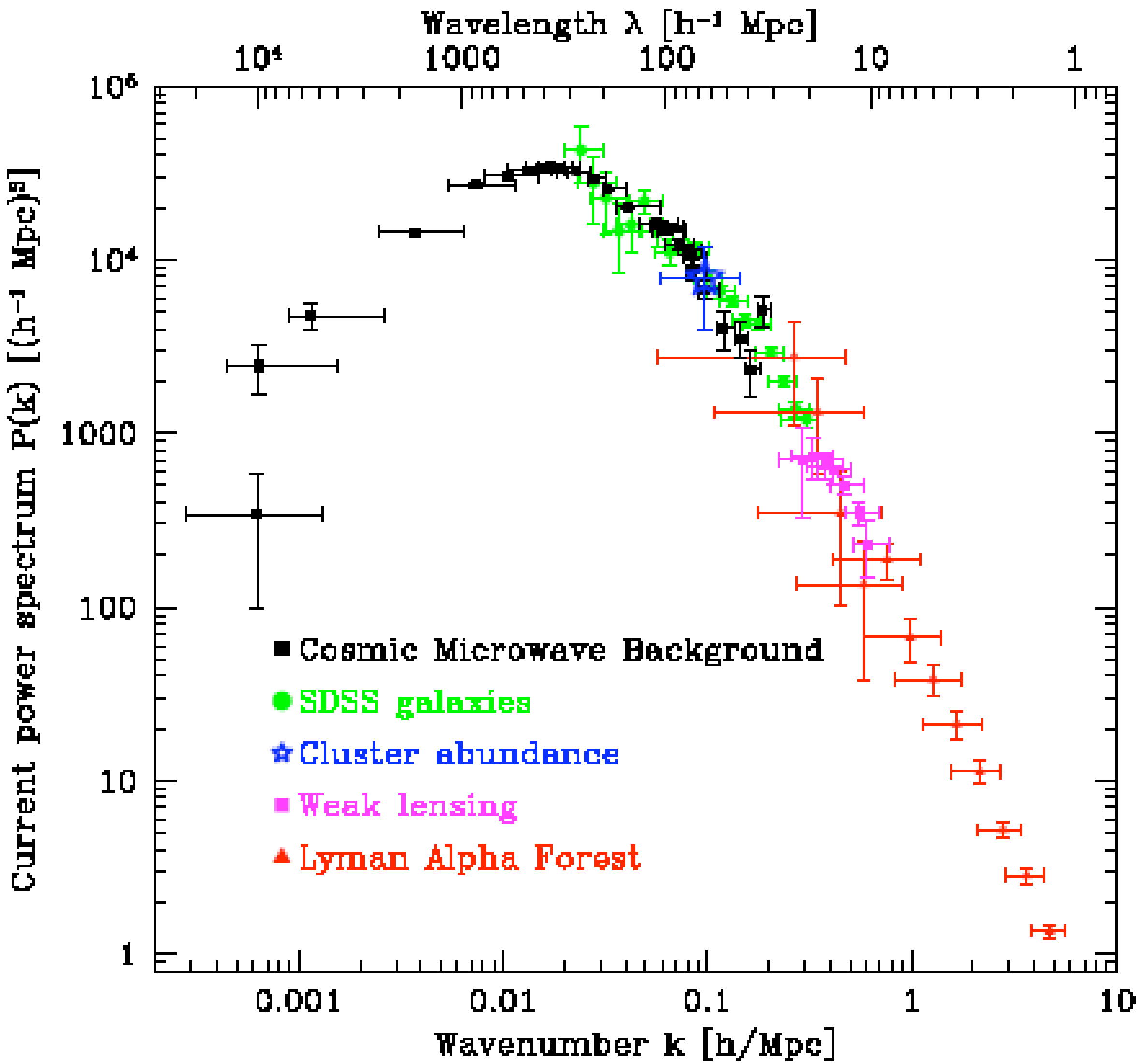




This suppression due to radiation era at high-k is important for RPT

The density/velocity perturbations after decoupling constitute “initial” conditions for the non-linear evolution (structure formation) that follows.

This is observed!



Tegmark et al. (2003)

Nonlinear Evolution (Qualitative)

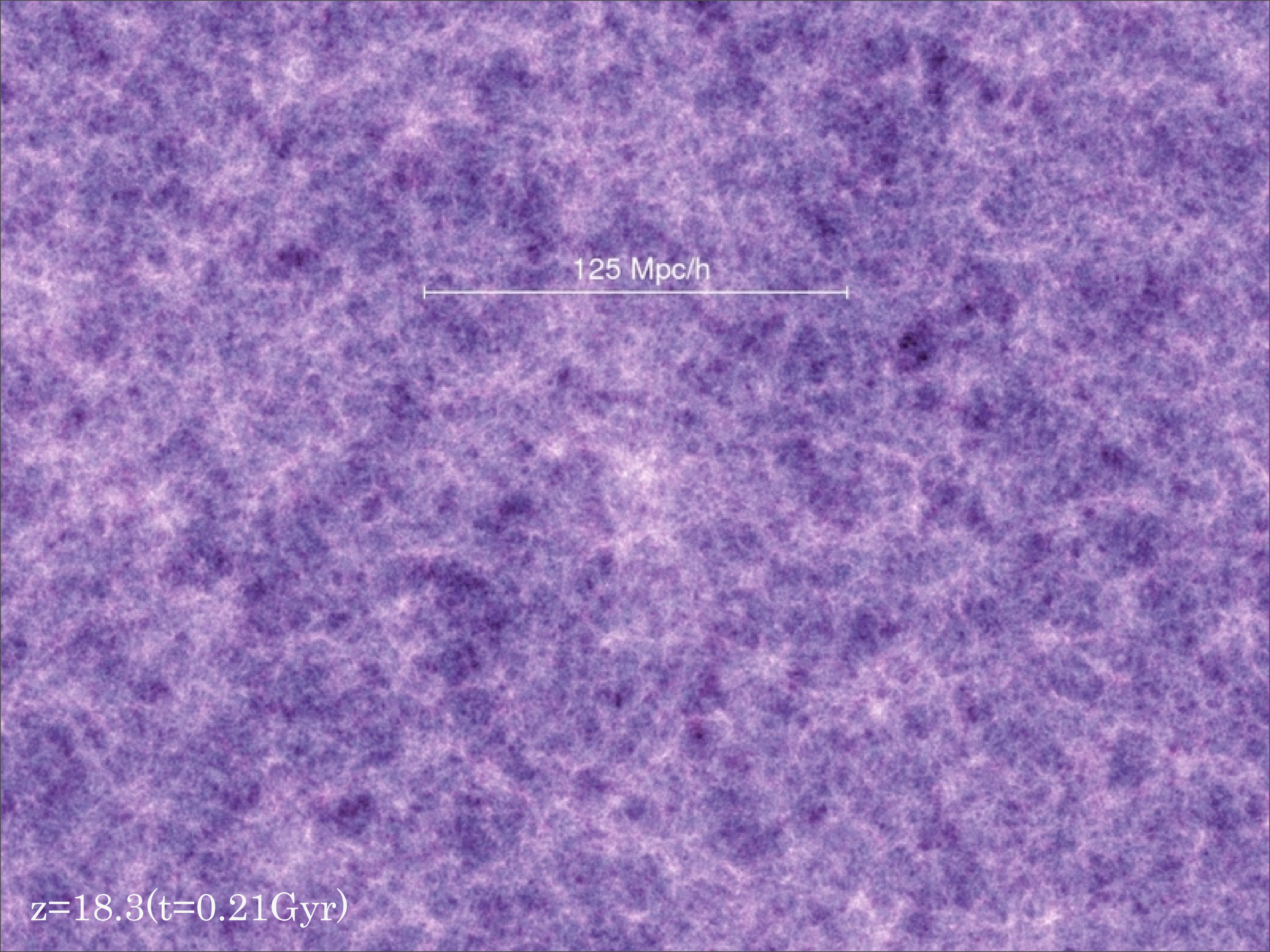
- Nonlinearity modifies the power spectrum

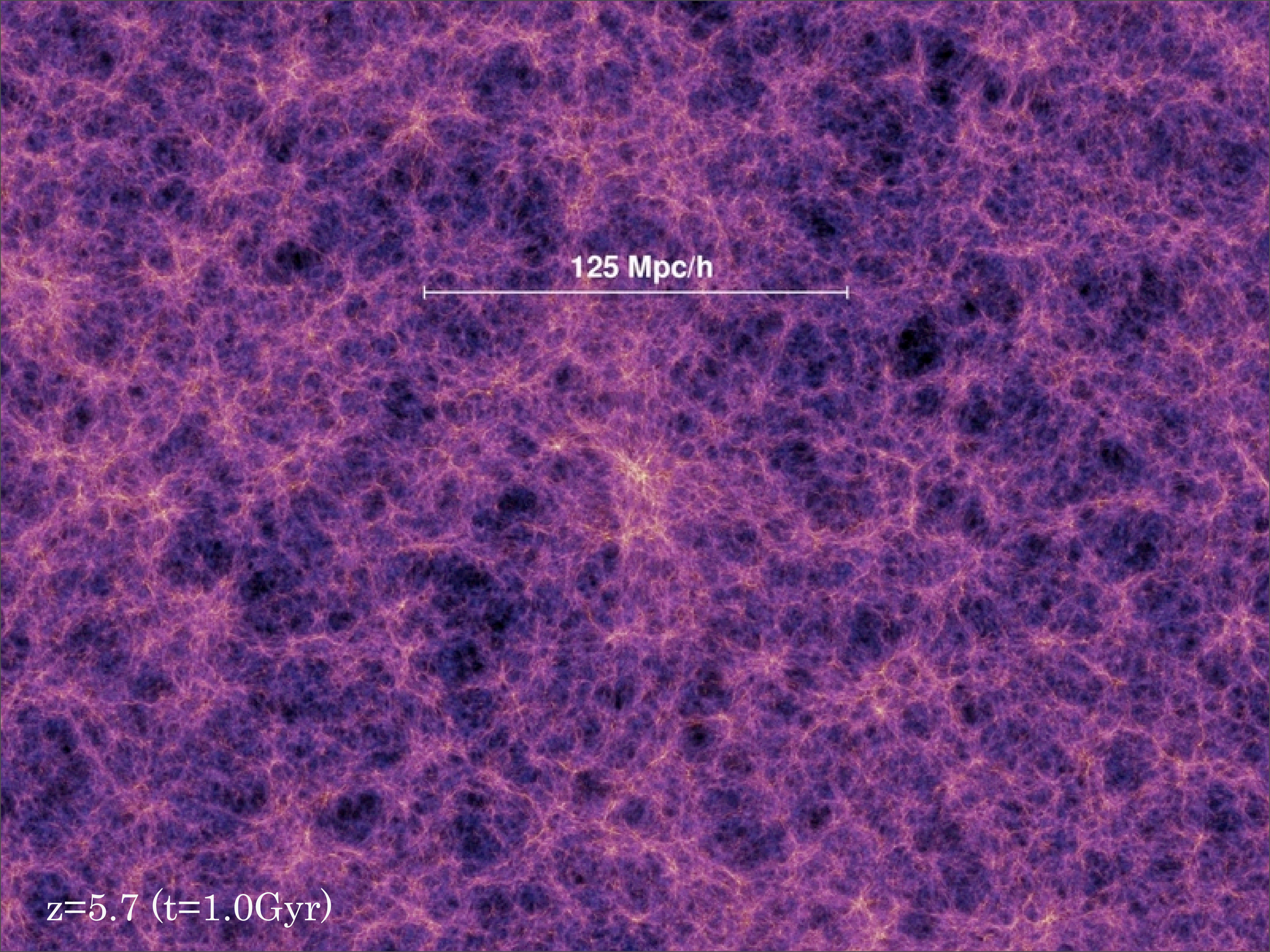
- Creates Non-Gaussianity

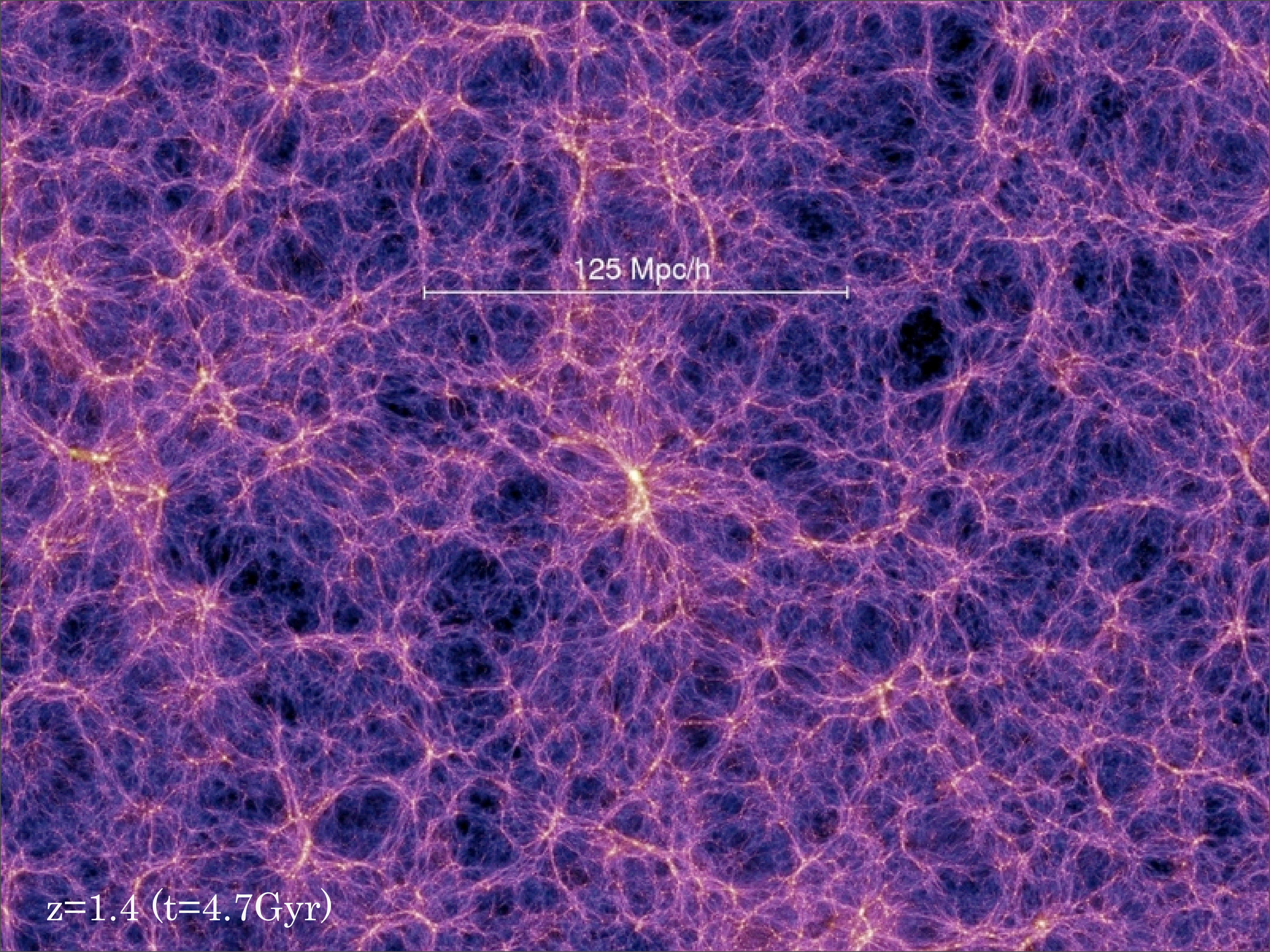
Growth of perturbations: gravity vs. the expansion of the universe

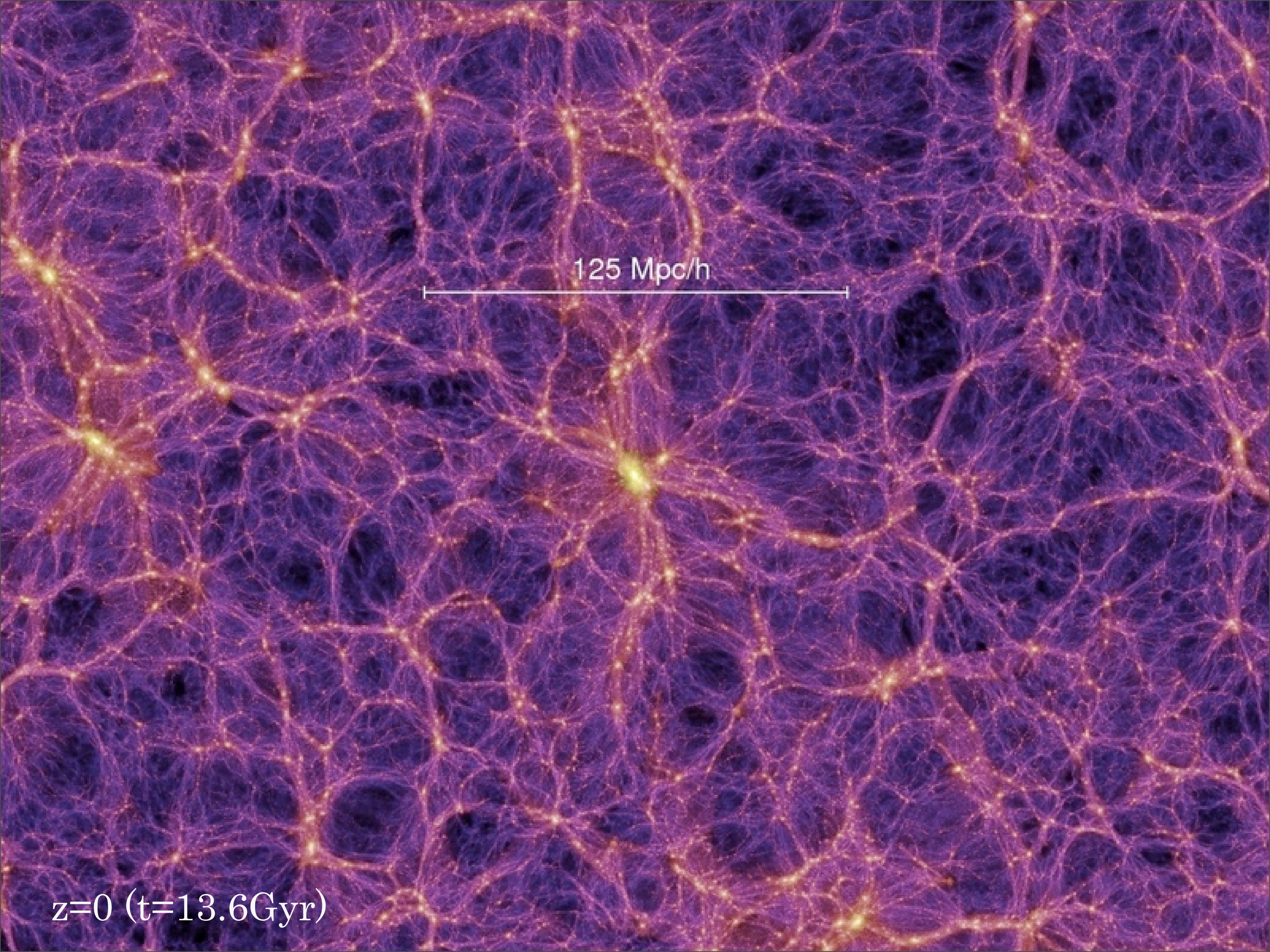
- underdense regions: expansion wins
- overdense regions: gravity wins

In pictures: Millenium Simulation (Springel et al.)

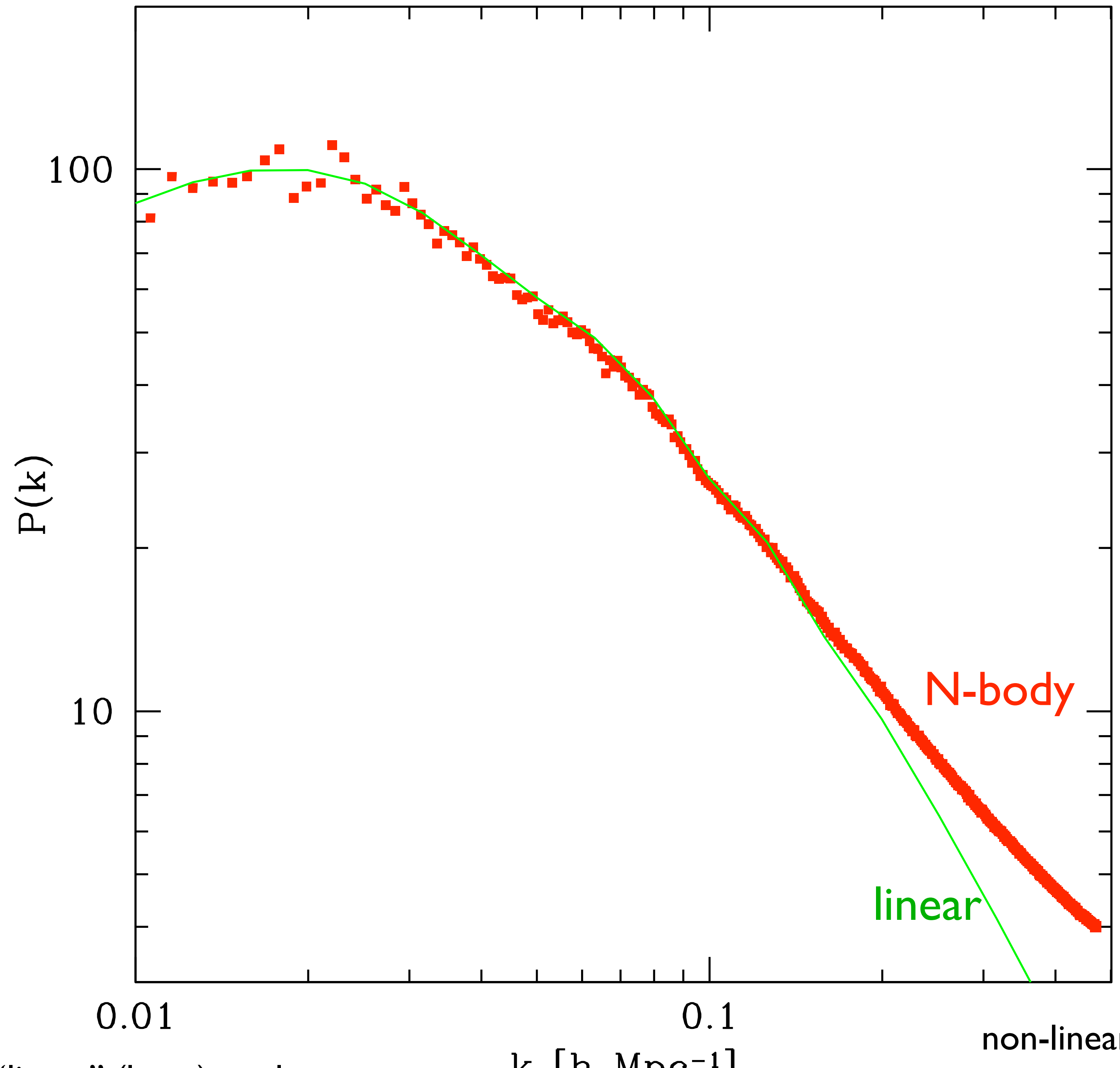

125 Mpc/h
z=18.3(t=0.21Gyr)


125 Mpc/h
z=5.7 (t=1.0Gyr)


125 Mpc/h
z=1.4 (t=4.7Gyr)


125 Mpc/h
z=0 (t=13.6Gyr)

power spectrum: linear PT vs. N-body simulations



"linear" (large) scales

non-linear regime

Nonlinear Evolution (Quantitative)

- matter dominated by collisionless Cold Dark Matter (CDM): pressure-less non-relativistic perfect fluid

- scales smaller than Hubble radius (k/aH >>1): negligible retardation

$$\delta T^{\mu\nu}_{;\nu} = 0$$

$$\frac{\partial \delta}{\partial \tau} + \nabla \cdot [(1+\delta)\mathbf{v}] = 0$$

quadratic nonlinearities

$$\frac{\partial \mathbf{v}}{\partial \tau} + \mathcal{H}\mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \Phi$$

$$\nabla^2 \Phi = \frac{3}{2}\mathcal{H}^2 \Omega_m \delta \qquad \mathcal{H} = Ha$$

Rewrite as a Field theory for a doublet ...

$$\Psi_a(\mathbf{k},\eta) \equiv \left(\delta(\mathbf{k},\eta),\ -\theta(\mathbf{k},\eta)/\mathcal{H}\right), \qquad \eta \equiv \ln a(\tau).$$

(dropped vector modes, i.e. vorticity)

Equations of motion can be written as,

$$\partial_\eta \Psi_a(\mathbf{k},\eta) + \Omega_{ab}\Psi_b(\mathbf{k},\eta) = \gamma_{abc}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\Psi_b(\mathbf{k}_1,\eta)\Psi_c(\mathbf{k}_2,\eta)$$

Laplace transform in time variable, $(\sigma^{-1}_{ab}(\omega) \equiv \omega\delta_{ab} + \Omega_{ab})$

$$\sigma^{-1}_{ab}(\omega)\ \Psi_b(\mathbf{k},\omega) = \underset{\substack{\uparrow \\ \text{Initial Conditions}}}{\phi_a(\mathbf{k})} + \gamma^{(\mathrm{s})}_{abc}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\oint \frac{d\omega_1}{2\pi i}\ \Psi_b(\mathbf{k}_1,\omega_1)\Psi_c(\mathbf{k}_2,\omega-\omega_1),$$

then going back to time,

$$\Psi_a(\mathbf{k},\eta) = g_{ab}(\eta)\phi_b(\mathbf{k}) + \int_0^\eta d\eta' g_{ab}(\eta-\eta')\gamma^{(\mathrm{s})}_{bcd}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\Psi_c(\mathbf{k}_1,\eta')\Psi_d(\mathbf{k}_2,\eta')$$

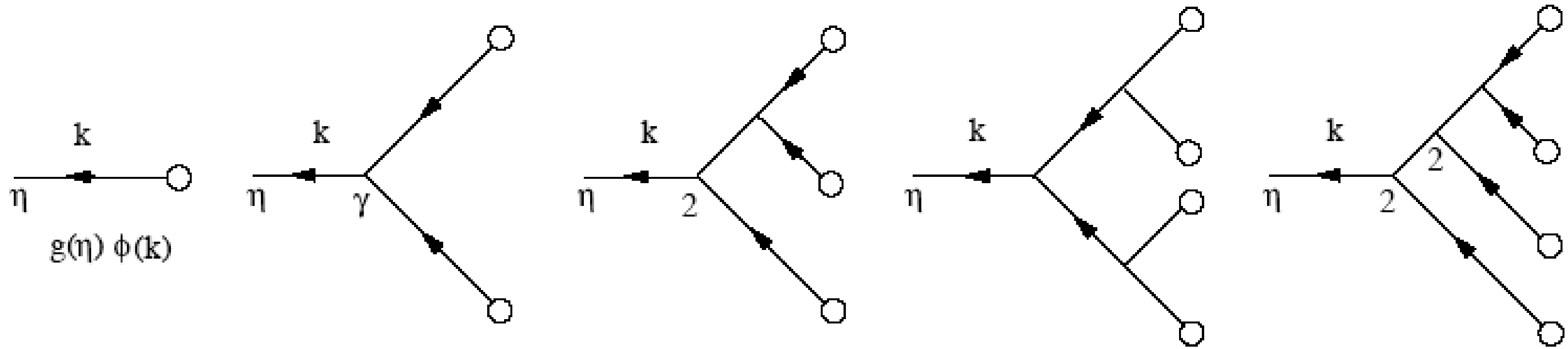
$$\phi_a(\mathbf{k}) = \Psi_a(\mathbf{k},\eta=0)$$

linear propagator $$g_{ab}(\eta) = \frac{\mathrm{e}^\eta}{5}\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} - \frac{\mathrm{e}^{-3\eta/2}}{5}\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix},$$

growing mode
$\phi_a(\mathbf{k}) \propto (1,1)$

decaying mode
$\phi_a(\mathbf{k}) \propto (1,-3/2)$

Diagrammatically:



$\Psi^{(1)}(\mathrm{k},\eta)$ $\Psi^{(2)}(\mathrm{k},\eta)$ $\Psi^{(3)}(\mathrm{k},\eta)$ $\Psi^{(4)}(\mathrm{k},\eta)$

Unusual Field Theory...

- There is no time translation invariance: unstable (non-equilibrium) system, where perturbations grow with time (as ~ power-law).

- "Initial" conditions (density perturbations after decoupling) play a crucial role. They act as a (stochastic) source: observables (expectation values) correspond to averages over the statistics of initial conditions.

- At the fields level, only tree diagrams (no "antiparticles"). Loops appear when we look at correlators such as the power spectrum.

- Vertices have a non-trivial k-dependence.

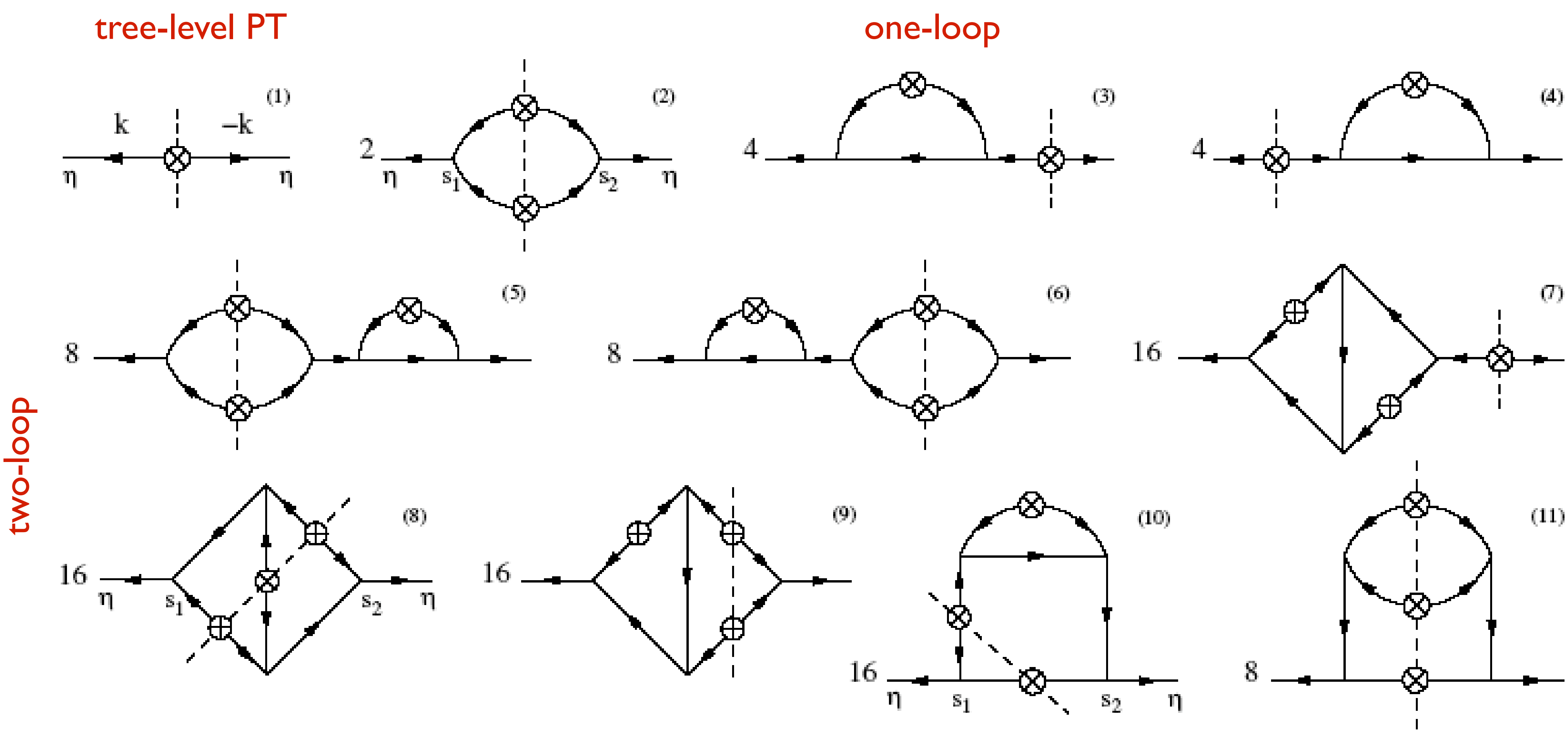
- Propagator has growing and decaying modes, both play important roles in the nonlinear regime.

- Due to the shape of CDM spectrum, there are no UV divergences (nor IR). "Renormalizations" are all finite.

Nonlinear Evolution of the Power Spectrum

$$\langle\, \Psi_a(k)\Psi_b(k')\,\rangle = \delta_{\rm D}(k+k')\, P_{ab}(k)$$

We just "glue" two Ψ trees together according to Gaussian initial conditions:



Tree-level PT gives the asymptotic behavior at large-scales for all statistics.

- How about nonlinear corrections (“loop” diagrams) to tree-level results?

- Once these become important, one basically needs to sum up all orders in PT (i.e. number of loops) to obtain meaningful answers, since the expansion parameter becomes of order unity or larger.

$$P(k,z) = \underbrace{D_+^2(z)\,P_0(k)}_{P_{\rm lin}} + P_{\rm 1loop}(k,z) + P_{\rm 2loop}(k,z) + \ldots$$

$$P_{\rm 1loop} \sim \mathcal{O}(P_{\rm lin}\,\Delta_{\rm lin}), \qquad P_{\rm 2loop} \sim \mathcal{O}(P_{\rm lin}\,\Delta_{\rm lin}^2), \qquad \Delta_{\rm lin} \equiv 4\pi k^3 P_{\rm lin}$$

Renormalized Perturbation Theory (RPT)

- partial resummation of PT contributions

- resulting expansion is *not* an expansion in amplitude of fluctuations

- large-scales are effectively ``shielded'' from small scales

- truncation of RPT expansion accounts for all nonlinearities down to a given scale (the impact of smaller scales is highly suppressed).

- In RPT, the linear propagator gets ``renormalized'' due to nonlinearities,

Final density / velocity div.

$$G_{ab}(k,\eta)\ \delta_{\rm D}(\mathbf{k}-\mathbf{k}') \equiv \left\langle \frac{\delta\Psi_a(\mathbf{k},\eta)}{\delta\phi_b(\mathbf{k}')} \right\rangle$$

Initial Conditions

For Gaussian initial conditions, the nonlinear propagator can be related to the cross-correlation between initial and final conditions,

$$G_{ab}(k,\eta)\,\langle\phi_b(\mathbf{k})\phi_c(\mathbf{k}')\rangle = \langle\Psi_a(\mathbf{k},\eta)\,\phi_c(\mathbf{k}')\rangle.$$

In this sense the propagator measures the memory of perturbations to their initial conditions. The asymptotics are,

$$G_{ab}(k\to 0,\eta) = g_{ab}(\eta), \qquad G_{ab}(k\to\infty,\eta) = 0$$

impossible to recover at fixed order in PT!

$$g_{ab}(\eta) = \frac{\mathrm{e}^{\eta}}{5}\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} - \frac{\mathrm{e}^{-3\eta/2}}{5}\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix},$$

growing mode $\phi_a(\mathbf{k}) \propto (1,1)$

decaying mode $\phi_a(\mathbf{k}) \propto (1,-3/2)$

The resummation of the propagator can be carried out exactly in the high-k limit, yielding correct asymptotics. Let me just sketch how it is done.

Propagator: high-k limit resummation

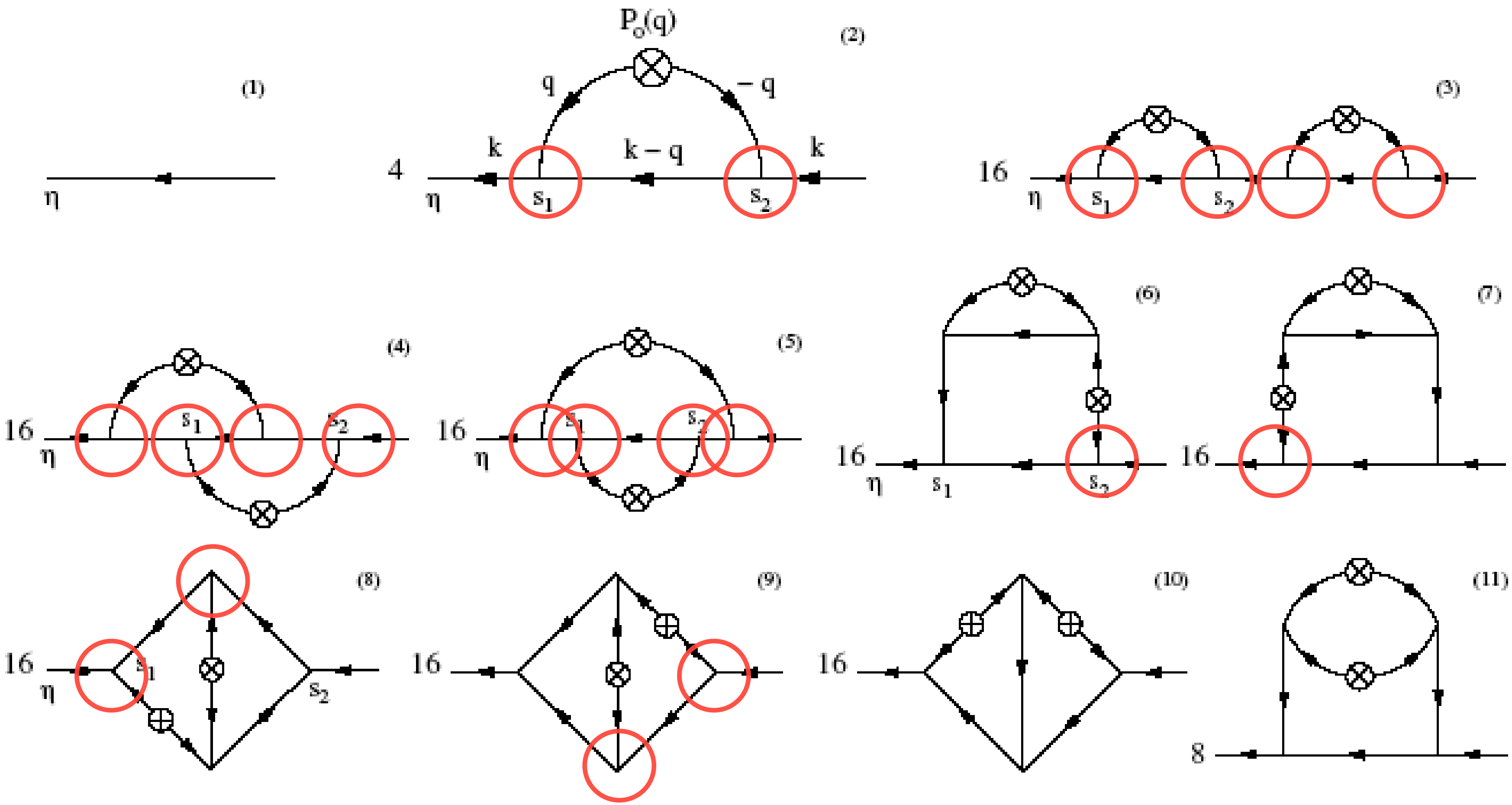


FIG. 2: Diagrams for the non linear propagator $G(k, \eta)$ up to two loops.

The dominant contributions arise from max number of interactions in growing mode at each order...

The dominant contributions can be resummed *exactly* in high-k limit!

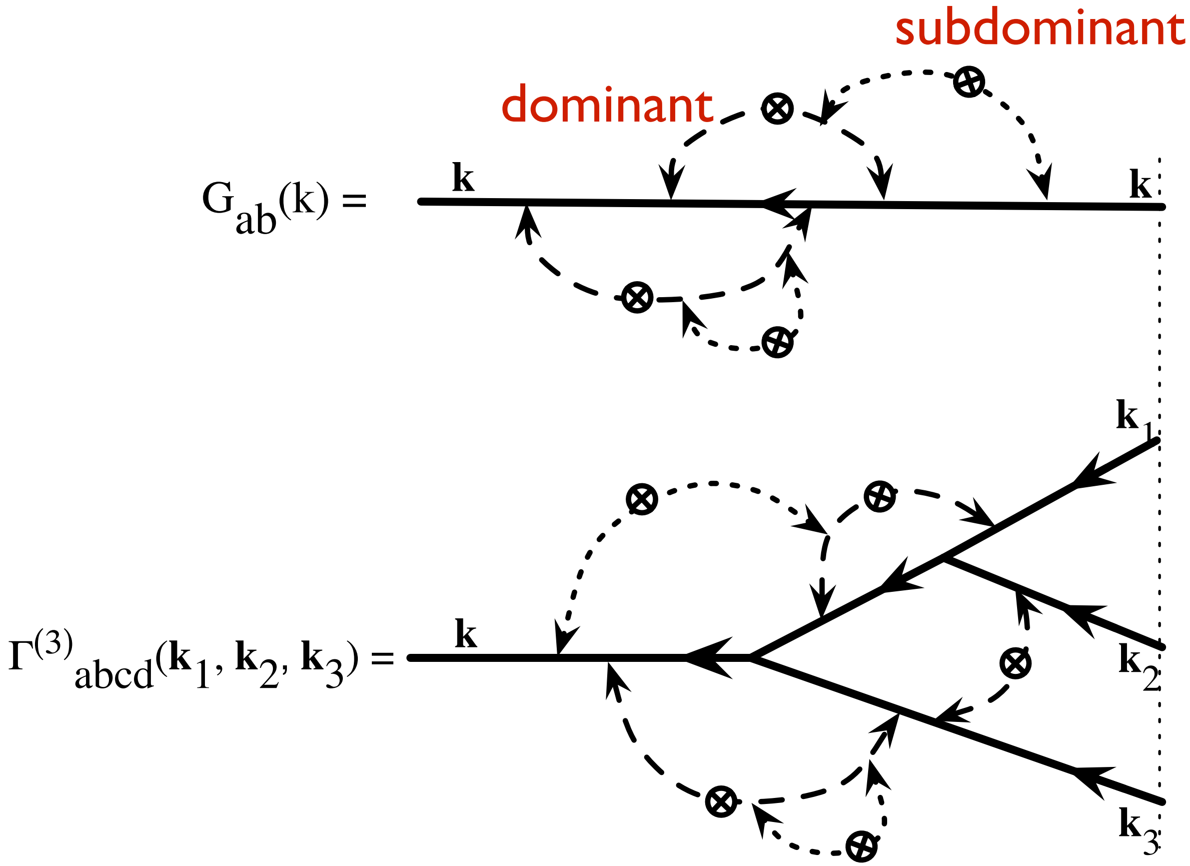
$$G_{ab}(k,\eta) \simeq g_{ab}(\eta)\ \exp\Big(-\frac{1}{2}k^2\sigma_v^2(\mathrm{e}^{\eta}-1)^2\Big)$$ (high-k limit)

with: $$\sigma_v^2 \equiv \frac{1}{3}\int d^3q\,\frac{P(q)}{q^2}$$

This is so because in the high-k limit the interaction vertex simplifies, and these diagrams have a very simple time dependence, with all propagators from initial conditions being in the purely growing mode.

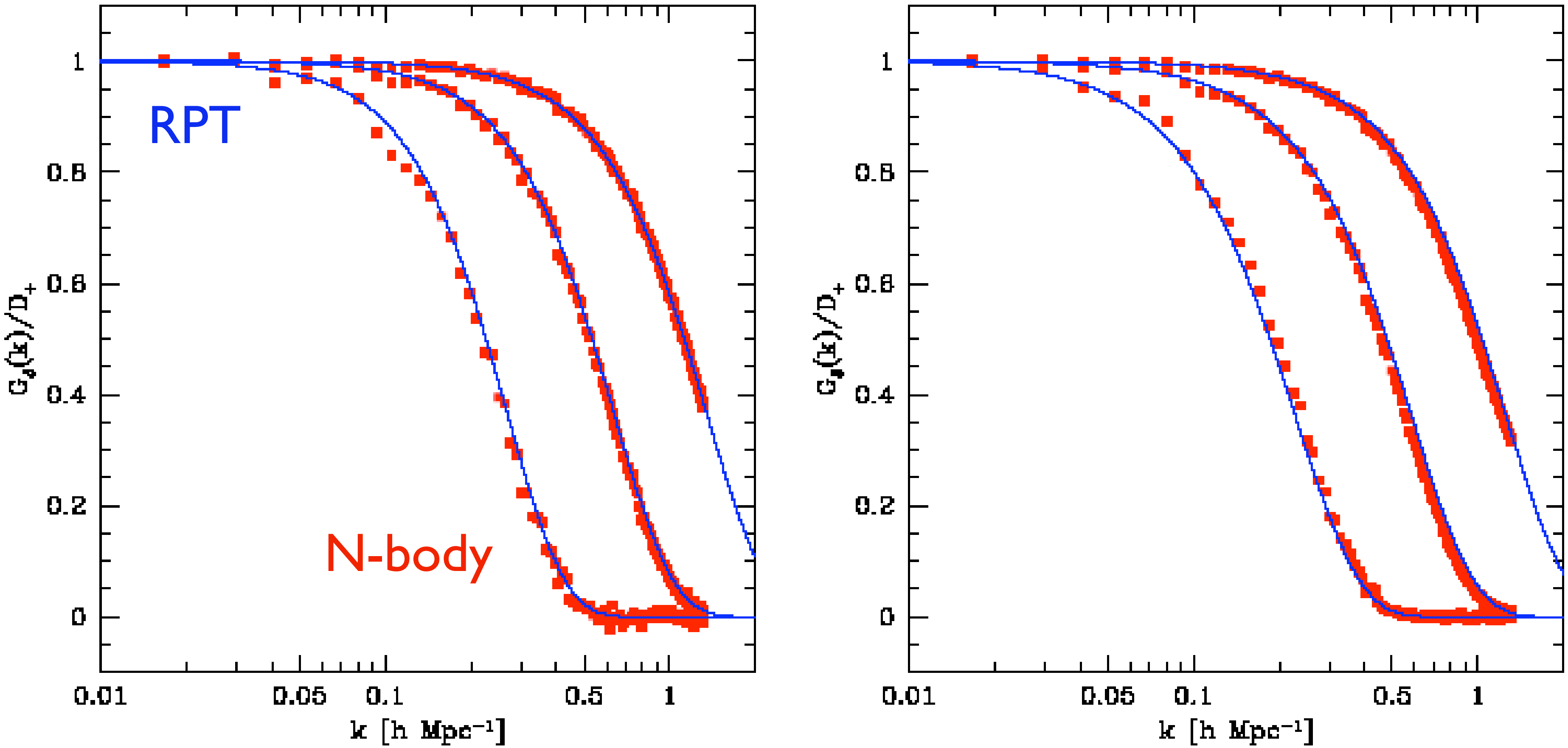
Notice the scale σ_v is rather large (thanks to the shape of CDM spectrum), so scales much smaller than this have exponentially small influence on large scales.

These results can be extended for higher-point versions of the propagator.



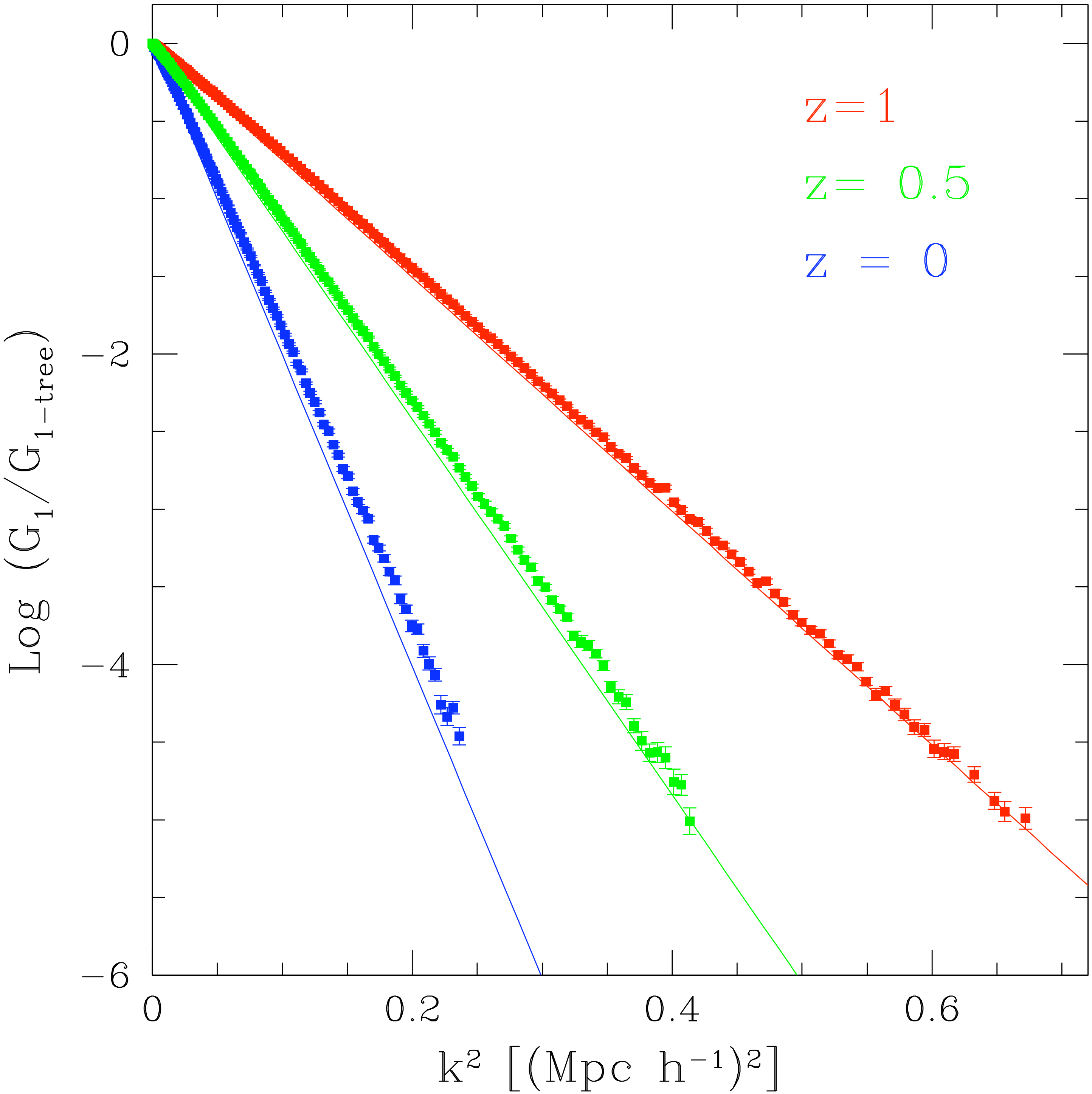
$$\Gamma^{(n)}_{a_1 \ldots a_n} = \exp\left(-\frac{k^2\sigma_v^2}{2}\left(e^{\eta}-1\right)^2\right)\ \Gamma^{(n)}_{a_1\ldots a_n,\mathrm{tree}}$$

Comparison between RPT and N-Body Simulations
(z = 0, 2, 5)

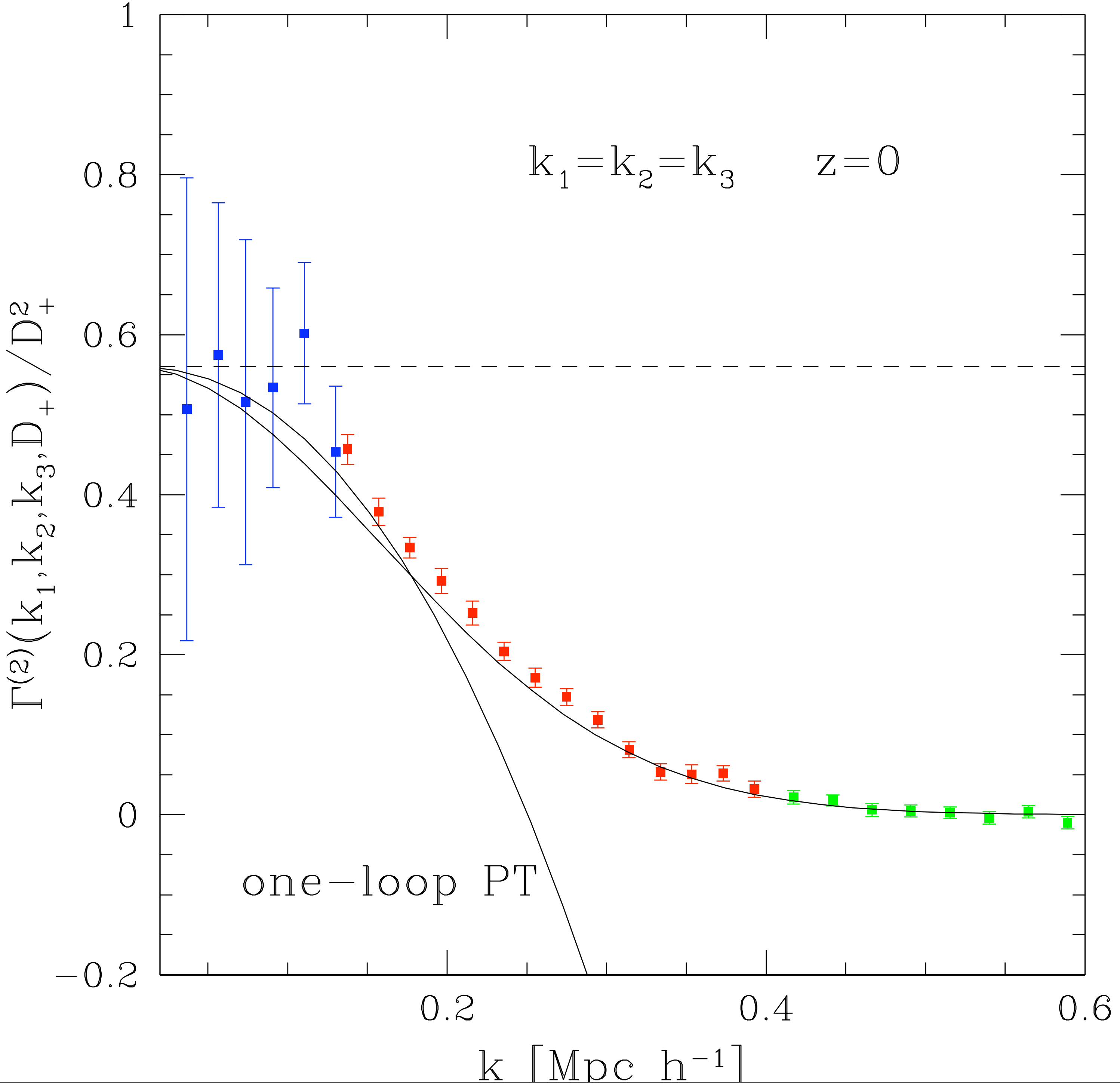


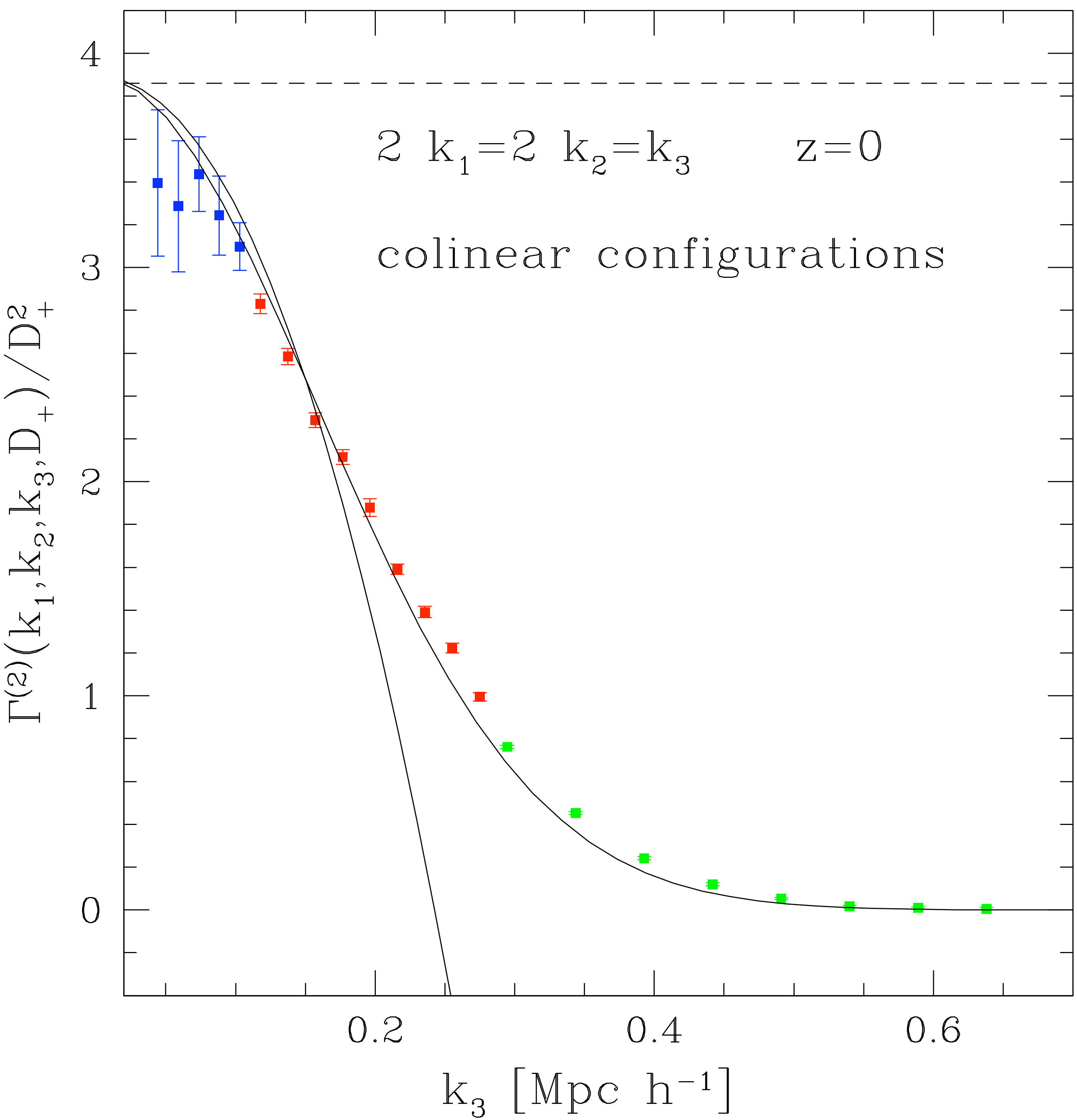
- The RPT predictions match simulations, even into the nonlinear regime, for density and velocity fields, without introducing any free parameters.

Checking high-k limit for the propagator.



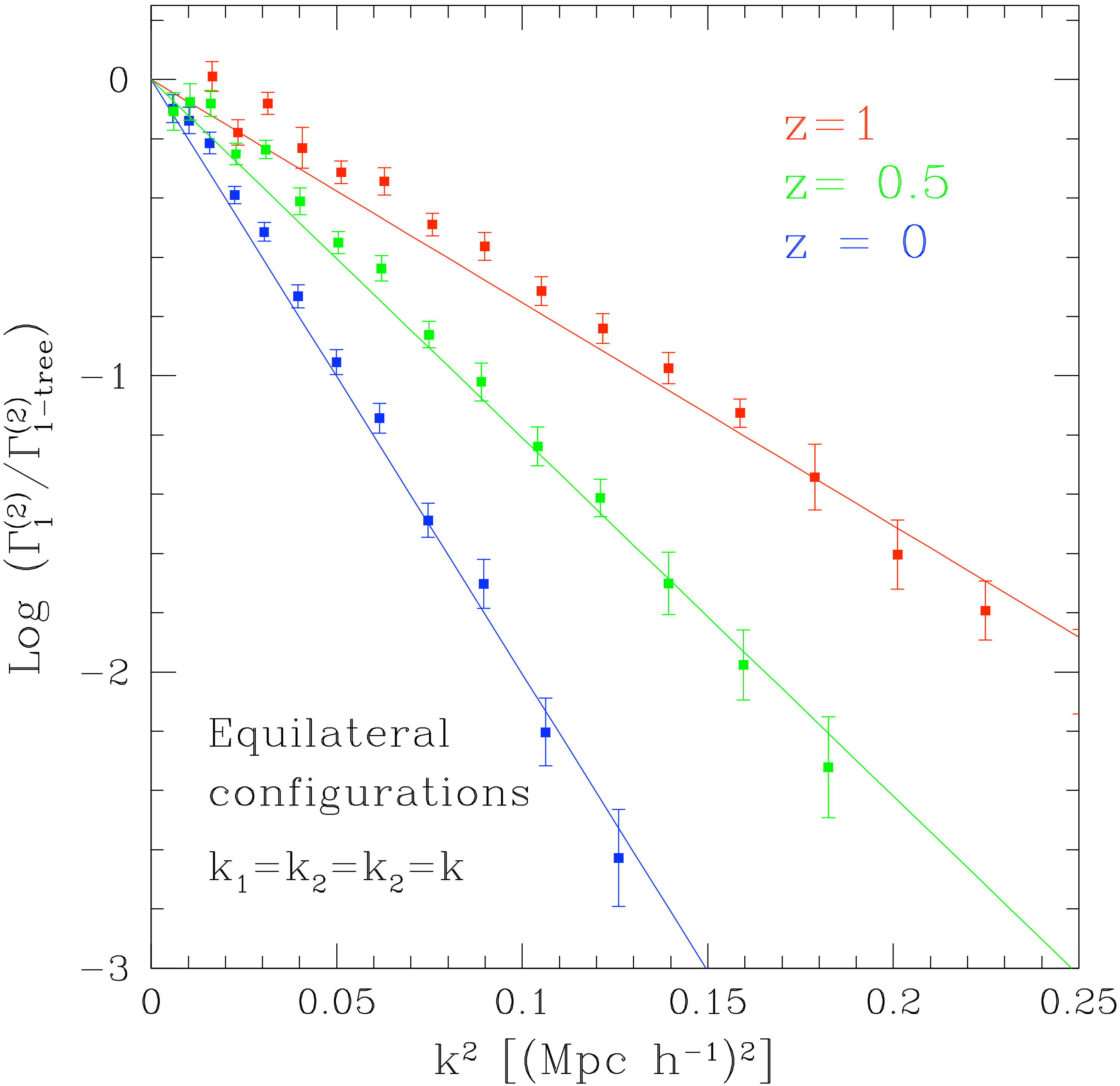
Three-point Propagator (equilateral configurations)





2 k1=2 k2=k3 z=0
colinear configurations
Γ(2)(k1,k2,k3,D+)/D+2
k3 [Mpc h-1]
0.2
0.4
0.6
0
1
2
3
4

Checking high-k limit for the three-point propagator.



For the power spectrum, RPT reorganizes the PT expansion,

$$P(k,z) = D_+^2(z)\, P_0(k) + P_{1\text{loop}}(k,z) + P_{2\text{loop}}(k,z) + \ldots$$

into,

$$P(k,z) = G^2(k,z)\, P_0(k) + P_{\text{MC}}(k,z)$$

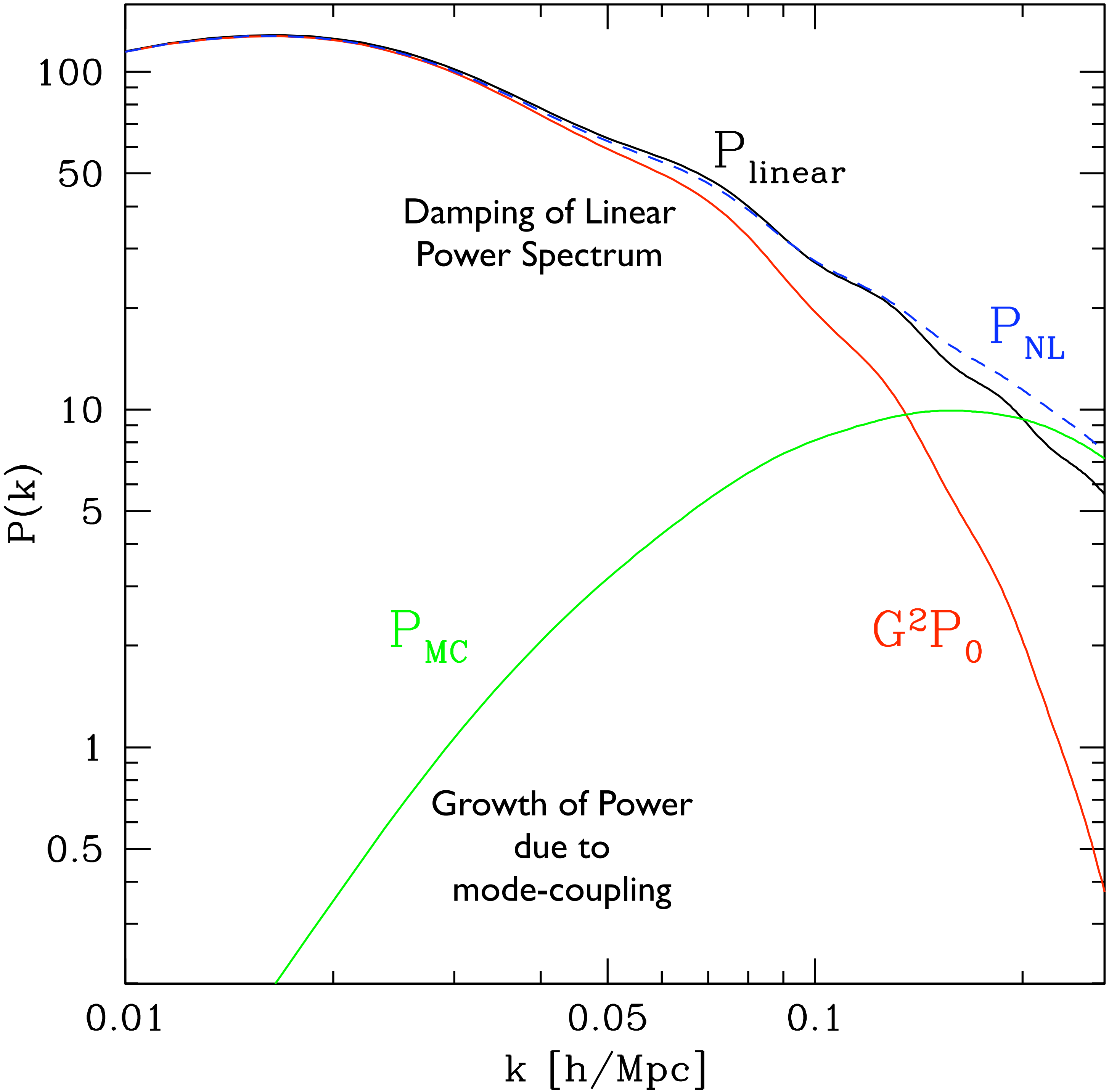
with,

$$P_{\text{MC}}(k,z) = P_{\text{MC}}^{1\text{loop}}(k,z) + P_{\text{MC}}^{2\text{loop}}(k,z) + \ldots$$

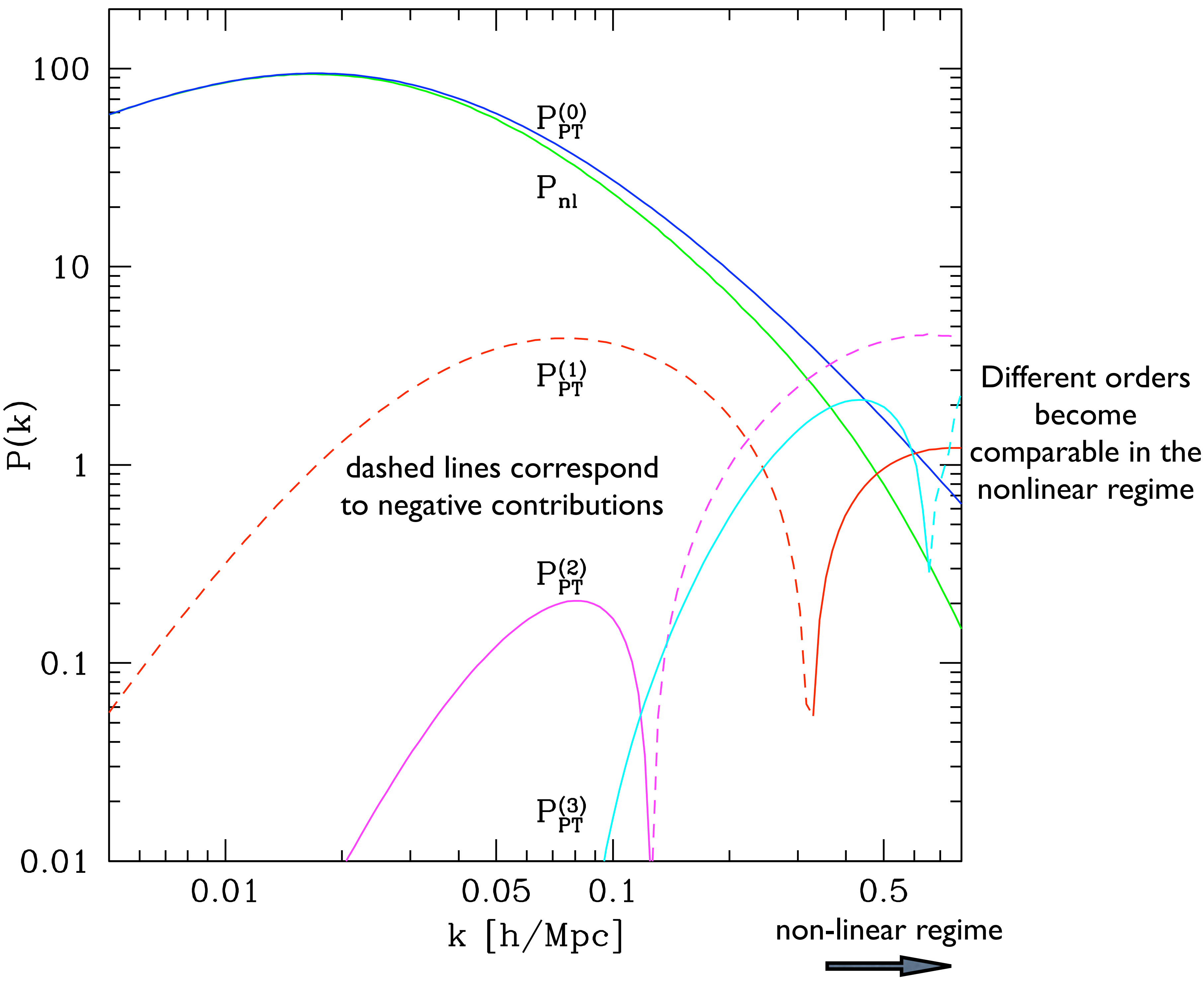
Thus, non linear effects can be divided (exactly) into two classes,

- those that are proportional to the initial power at same k.
- those that create power at k even if there was no power to begin with (mode-coupling)

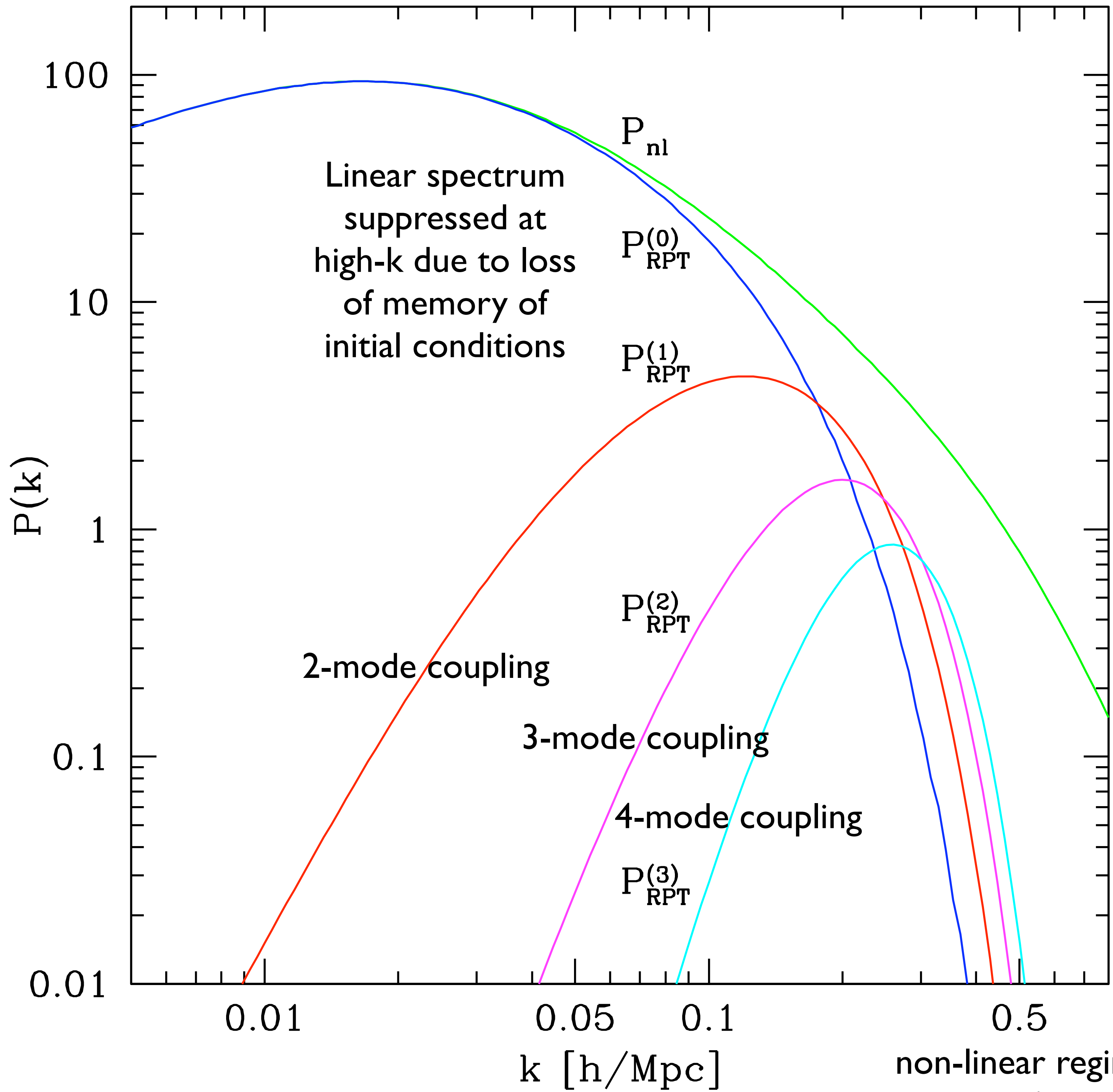
The Power Spectrum in RPT



PT expansion (schematic)



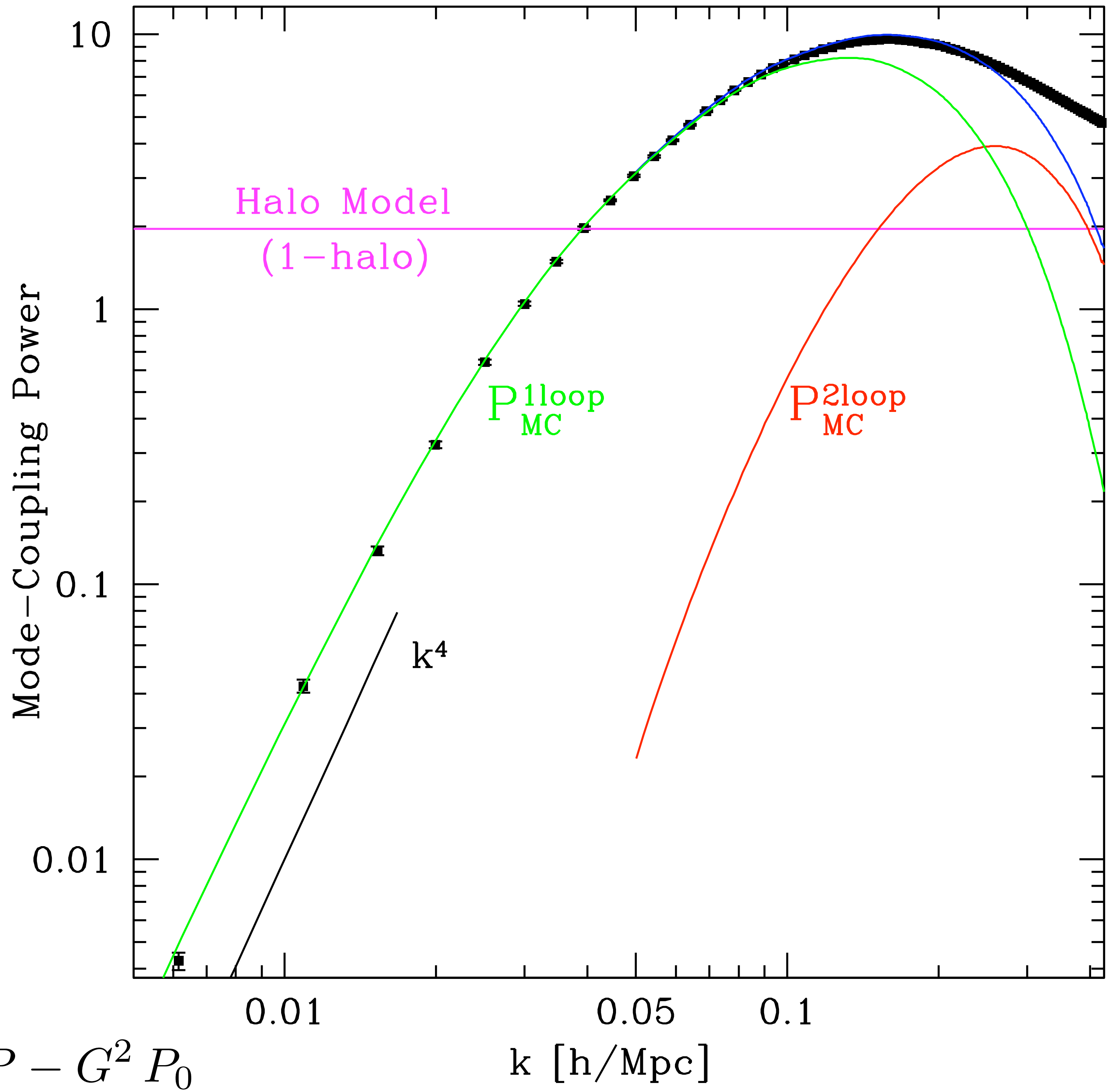
RPT expansion (schematic)



Different orders dominate only in a narrow range of scales and are always positive

non-linear regime

Explicit calculation of Mode-Coupling power to 2-loops in RPT



$$P_{\rm MC} \equiv P - G^2\, P_0$$

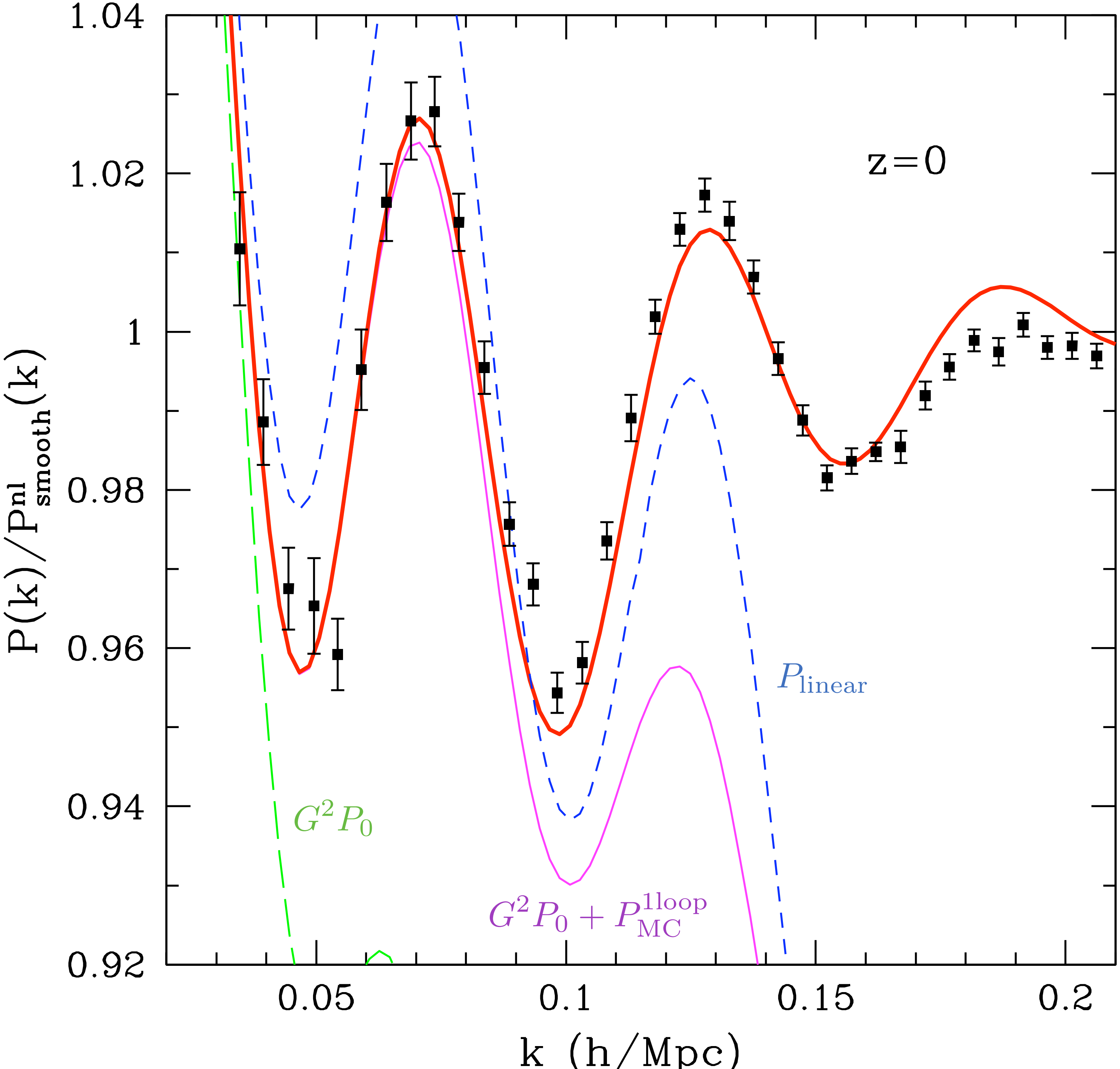
Nonlinear Evolution of the Power Spectrum and Acoustic Oscillations

- Can use acoustic oscillations imprinted in the dark matter power spectrum as a probe of expansion history (to get to dark energy / modified gravity).

- This "ruler", however, gets modified due to nonlinearities

Challenge: 1% error on sound horizon (~wiggle positions) induces about 5% error on w

Nonlinear Evolution of Acoustic Oscillations



Conclusions

- RPT gives a well-behaved perturbation theory. The power spectrum is in very good agreement with numerical simulations.

- Extensions to higher-order statistics to probe non-Gaussianity look promising.

- Understand nonlinearities from "first principles" we can model dependence on cosmological parameters.

- Useful application so far: acoustic oscillations as a probe of dark energy.

- Other applications/extensions:

neutrino mass
smaller scales for weak gravitational lensing
galaxy bias
redshift distortions