Real-time gauge theory simulations from stochastic quantisation

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1. Complex Langevin method and real time evolution
2. Results for a scalar oscillator
3. Results for SU(2) gauge theory
4. Connection with Schwinger Dyson equations
5. Methods to improve convergence:
   Reweighting and using gauge fixing

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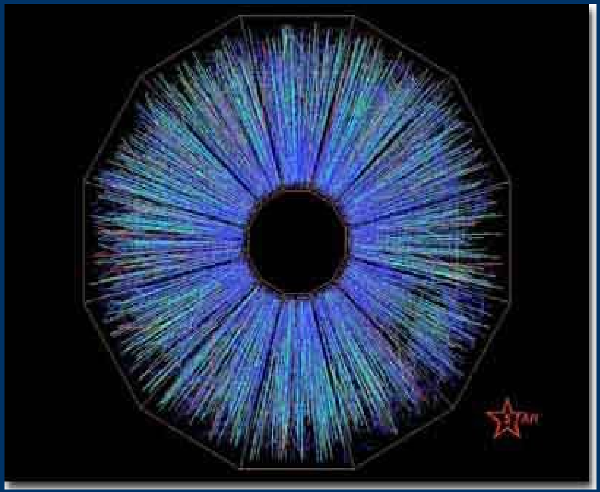
Motivations



Understanding heavy ion collisions

Not weakly coupled system

High occupation numbers prevent perturbative treatment even for weak couplings

At $n = O(1/\alpha)$ all diagrams become large

On the lattice: mainly equilibrium methods so far, static quantities with few exceptions

High occupation numbers

Classical-statistical description at earliest times

Cannot describe approach to thermal equilibrium

First principle calculations of QFT needed

Non equilibrium + Quantum fields=?

Late times approaching thermal equilibrium:

quantum effects become important

Classical approximation breaks down

Direct Method: Schödinger equation for the wave function: $\Psi[A_\mu^a(x)]$

Impossible!

Formulation with non-equilibrium generator function $Z[J]=\int D\Phi\, e^{i\int_c L(\Phi,J)dt}$

averages with complex weight is needed! e^{iS_M}

Importance sampling doesn't work

Stochastic Quantization

Parisi, Wu (1981)

Weighted, normalized average: $$\frac{\int O(x)\exp(-S(x))dx}{\int \exp(-S(x))dx}=\langle O\rangle$$

Stochastic process for x $$\frac{dx}{d\tau}=-\frac{\partial S}{\partial x}+\eta(\tau)$$

Gaussian noise $\langle \eta(\tau)\rangle=0 \quad \langle \eta(\tau)\eta(\tau')\rangle=2\delta(\tau-\tau')$

Averages are calculated along the trajectories:

$$\langle O\rangle=\frac{1}{T}\int_0^T O(x(\tau))d\tau$$

Fokker-Planck equation for the probability distribution of P(x):

$$\frac{\partial P}{\partial \tau}=\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}+P\frac{\partial S}{\partial x}\right)=-H_{FP}P$$

Real action $\rightarrow$ positive eigenvalues

for real action the Langevin method is convergent

Real-time evolution

$$\langle O(t)\rangle = \langle i|U(0,t)\,O\,U(t,0)|i\rangle$$

Schwinger-Keldysh contour

Nonequilibrium generating functional $Z[J] = \int D\Phi\, e^{i\int_C L(\Phi,J)\,dt}$

Real time= Langevin method with complex action!

$$\frac{d\phi}{d\tau} = i\frac{\partial S}{\partial \phi} + \eta(\tau)$$

Klauder '83, Parisi '83, Hueffel, Rumpf '83,
Okano, Schuelke, Zeng '91, ...
applied to nonequilibrium: Berges, Stamatescu '05, ...

5D classical langevin system → 4D quantum averages

The field is complexified

real scalar → complex scalar

link variables: SU(2) → SL(2,C)
compact → non-compact

Is it still the same theory?

Yes: real (SU(2)) averages
Schwinger-Dyson equations fulfilled

No general proof of convergence

Runaway trajectories present (supressed by small Langevin time-step)

Scalar Theory

Complex countour given by: $C_t \quad , \qquad \Delta_t = C_{t+1} - C_t \quad , \qquad C_0 = 0\,, \;\; C_{N_t} = -i\beta$

action discretised on the contour $S = \sum_t \left(\frac{(\phi_{t+1} - \phi_t)^2}{2\,\Delta_t} - \Delta_t \frac{V(\phi_t) + V(\phi_{t+1})}{2} \right)$

Langevin updating in "5th" coordinate $\frac{d\,\phi_t}{d\,\tau} = \frac{\partial S}{\partial \phi_t} + \eta_t(\tau)$

$$\langle \eta_t(\tau) \rangle = 0$$

$$\langle \eta_t(\tau)\, \eta_{t'}(\tau') \rangle = 2\,\delta(\tau - \tau')\,\delta_{t\,t'}$$

discretised: $\phi_t(\tau + \epsilon) = \phi_t(\tau) + i\,\epsilon \frac{\partial S}{\partial \phi_t} + \sqrt{\epsilon}\,\eta_t(\tau)$

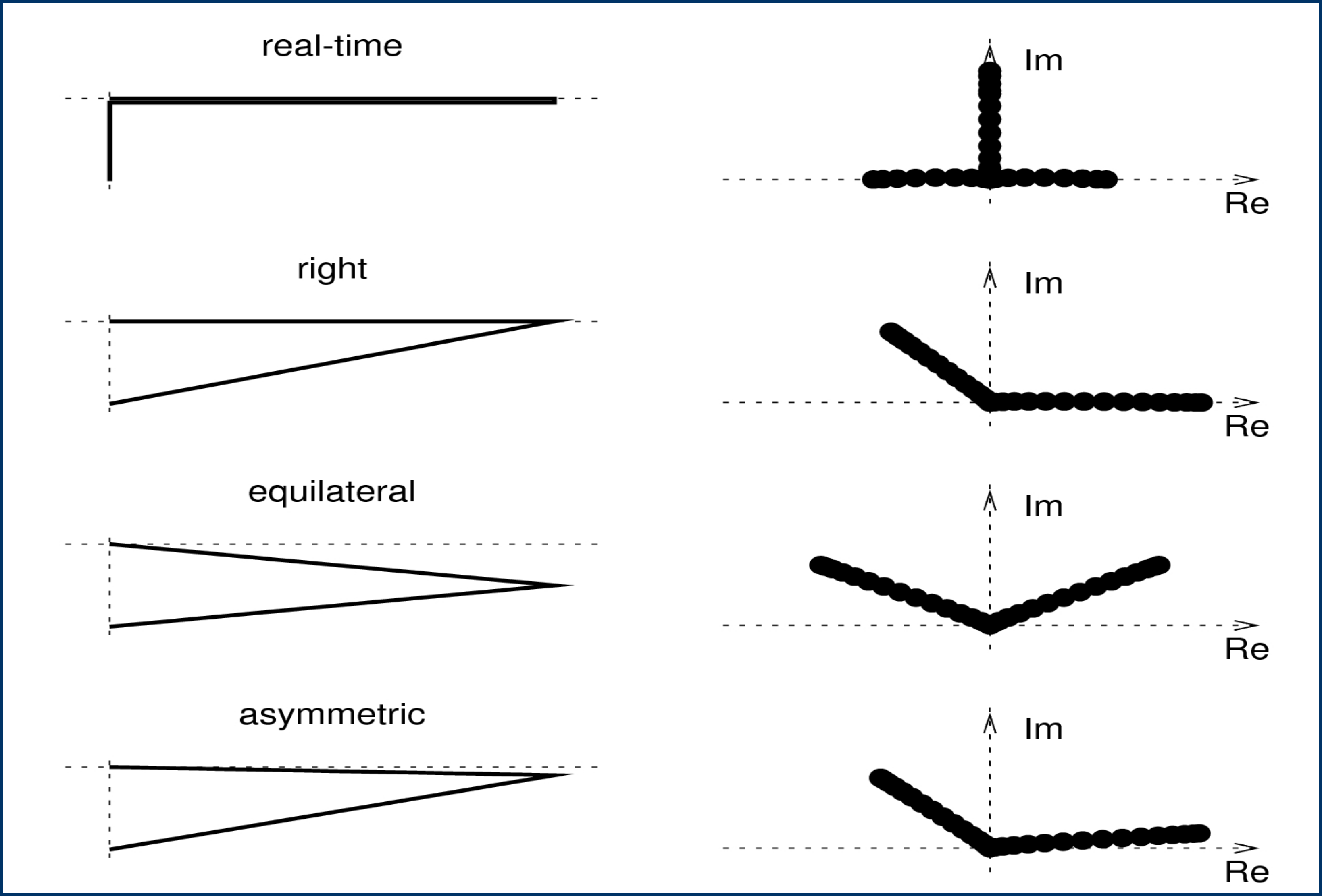
Interacting scalar oscillator $V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{24} \phi^4$

Thermal equilibrium → periodic boundary conditions

$$\phi_0 = \phi_{N_t}$$

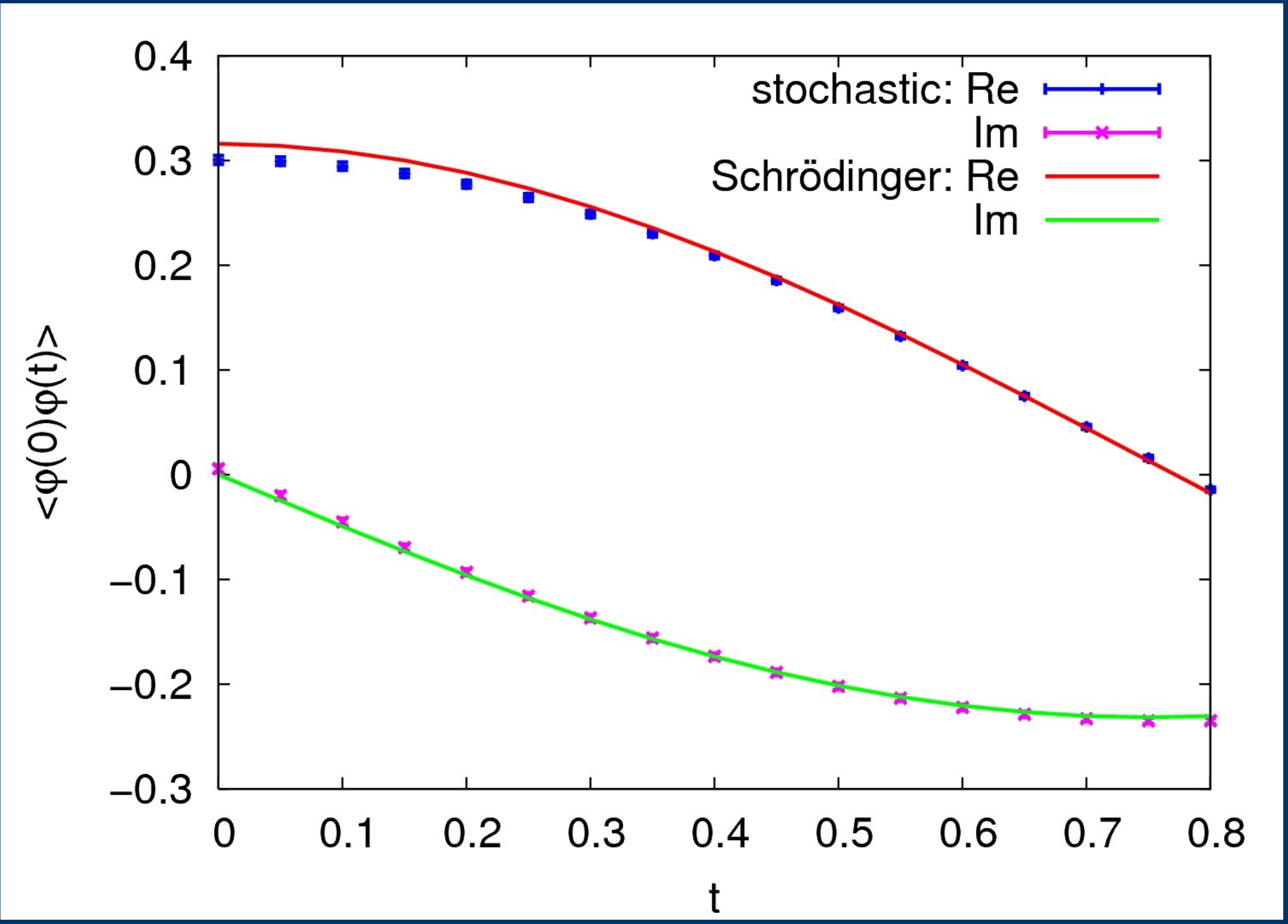
Type of contours

Eigenvalues of the free action
(positive Imaginary part = convergence)



downwards sloped countour: regulator

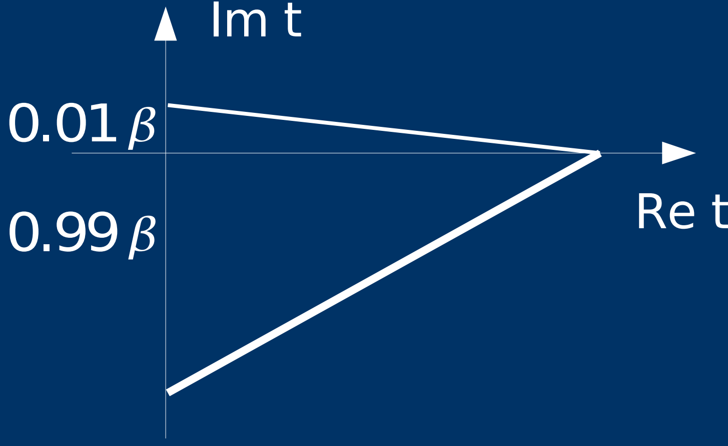
Real-time two point function



Thermal equilibrium:
periodic boundary cond.

Imaginary extent gives $\beta = \frac{1}{T}$

Asymmetric contour:

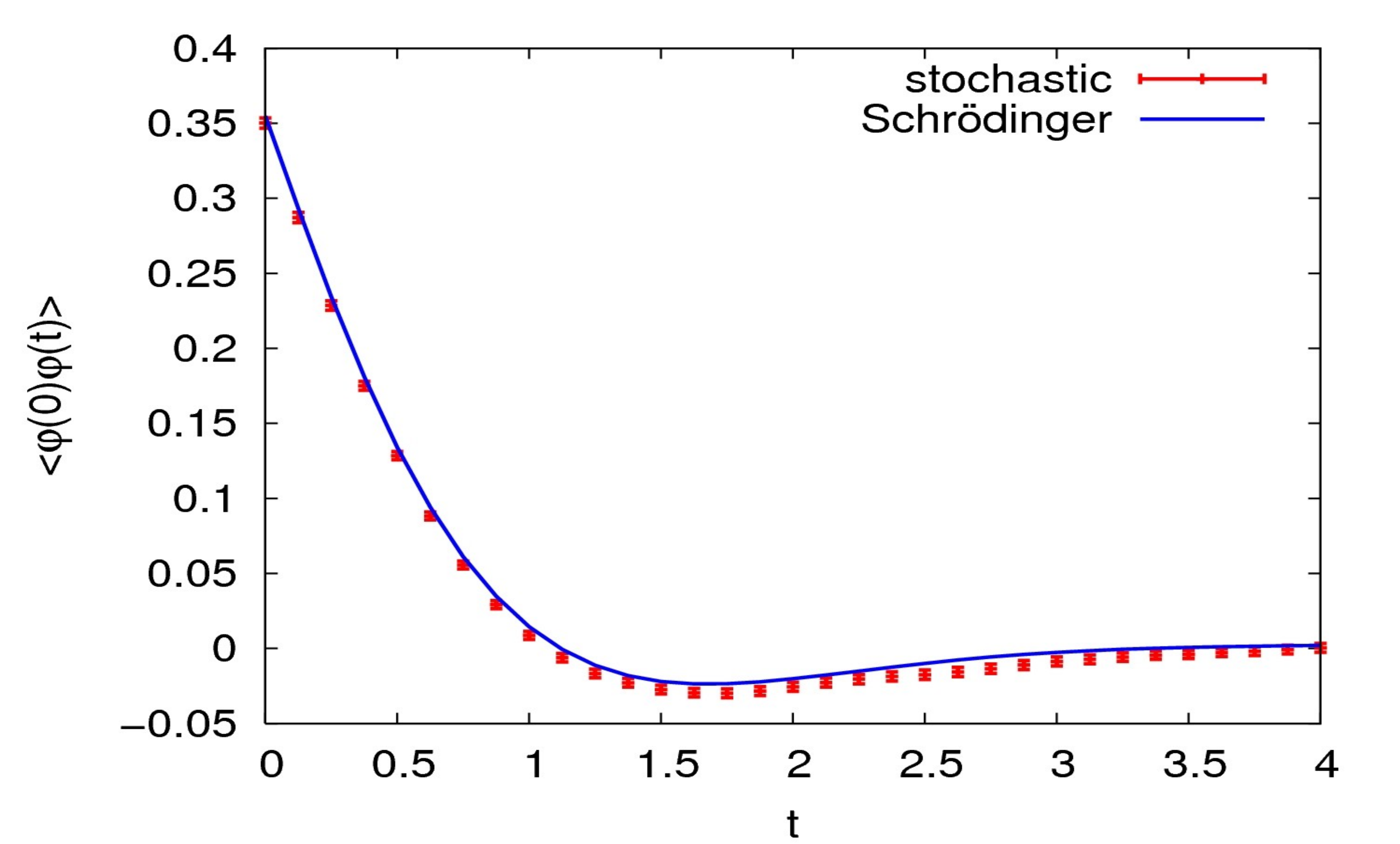


$\lambda = 24, \beta = 1$

Reproduces the Schrodinger equation result.

Two point function in thermal equilibrium with longer contour

$\lambda = 6, \beta = 8$



Non-equilibrium time evolution

Generating functional with initial density matrix:

$$Z(J,\rho)=Tr\left(\rho T_C e^{i\int_c J(x)\Phi(x)}\right)=\int d\varphi_1 d\varphi_2 \rho(\varphi_1,\varphi_2)\int_{\varphi_1}^{\varphi_2} D'\varphi e^{i\int_c L(x)+J(x)\varphi(x)}$$

Exponentializing the density matrix

Including φ_1,φ_2 in the path integral

$$\langle A(\varphi)\rangle=\int D\varphi_u D\varphi_l \exp(iS_\rho(\varphi_u,\varphi_l))A(\varphi_u)$$

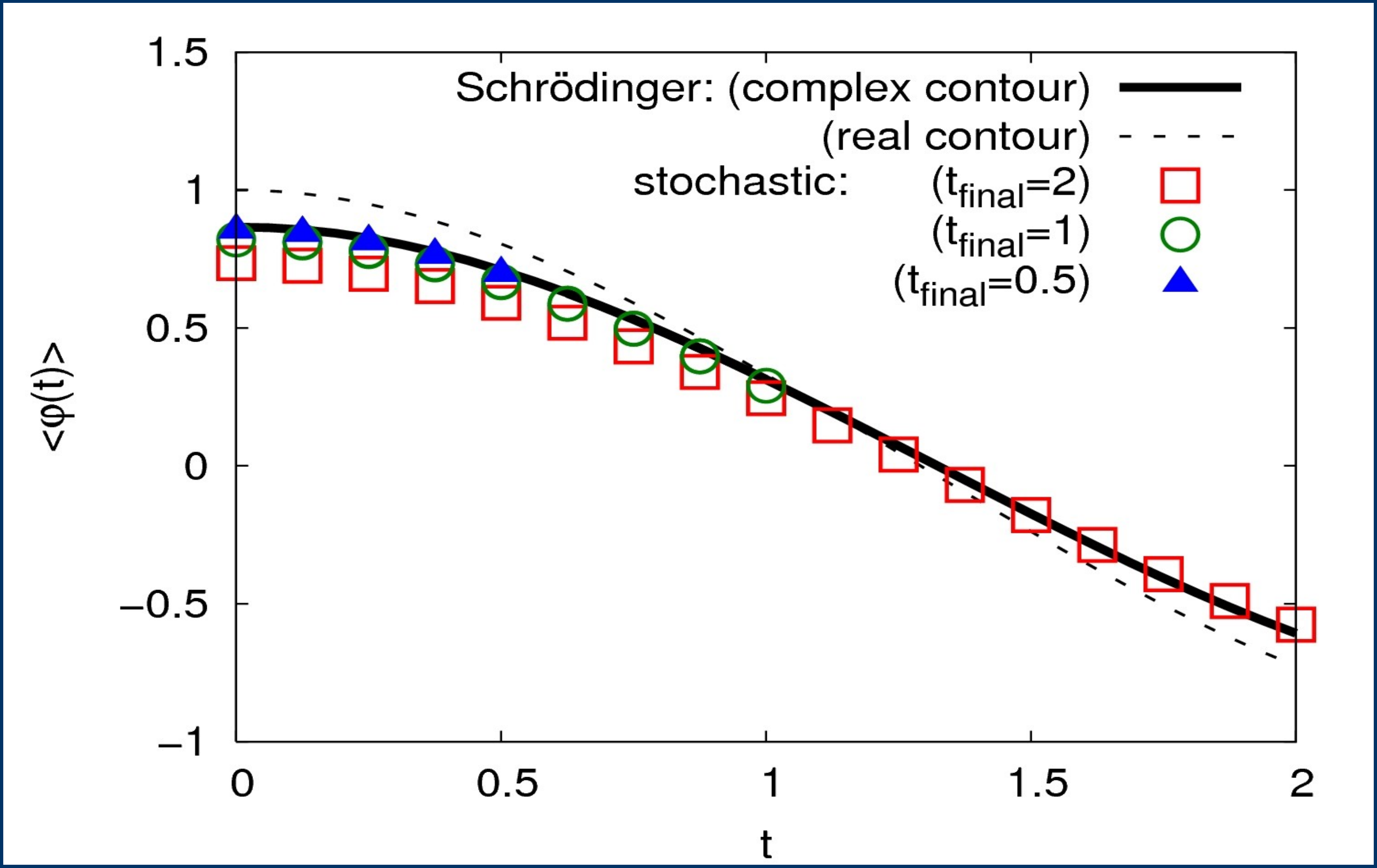
Langevin simulation with new “action”: $S_\rho[\varphi_u,\varphi_l]=S[\varphi_u]-S[\varphi_l]-\frac{i}{a_t}S_0(\varphi_u,\varphi_l)$

Most general gaussian density matrix with 5 parameters:

$$S_0(\varphi_u,\varphi_l)=i\dot{\phi}(\varphi_u-\varphi_l)-\frac{\sigma^2+1}{8\xi^2}\left((\varphi_u-\phi)^2+(\varphi_l-\phi)^2\right)$$

$$+\frac{i\eta}{2\xi}\left((\varphi_u-\phi)^2-(\varphi_l-\phi)^2\right)+\frac{\sigma^2-1}{4\xi^2}(\varphi_u-\phi)(\varphi_l-\phi)$$

Non-equilibrium time evolution



Contour with 5% slope

Bigger real time extent → worse agreement

SU(2) pure gauge theory

$$S=-\beta_0\sum_{x,i}\frac{1}{2\,Tr\,\mathbf{1}}\left(Tr\,U_{x,0i}+Tr\,U^{-1}_{x,0i}\right)-1$$
$$+\beta_s\sum_{x,i<j}\frac{1}{2\,Tr\,\mathbf{1}}\left(Tr\,U_{x,ij}+Tr\,U^{-1}_{x,ij}\right)-1$$

$$\beta_0=\frac{2\,Tr\,\mathbf{1}\,a_s}{g_0^2\,a_t}$$

$$\beta_s=\frac{2\,Tr\,\mathbf{1}\,a_t}{g_0^2\,a_s}$$

Updating the link variables:

$$U'_{x,\mu}=\exp\left(i\lambda_a\left(\epsilon\, i\, D_{x\mu a}S[U]+\sqrt{\epsilon}\,\eta_{x\mu a}\right)\right)U_{x\mu}$$

$$\langle\eta_{x\mu a}\rangle=0$$
$$\langle\eta_{x\mu a}\eta_{y\nu b}\rangle=2\,\delta_{xy}\delta_{\mu\nu}\delta_{ab}$$

Left derivative: $D_a f(U)=\left(\frac{\partial}{\partial\alpha}f\left(e^{i\lambda_a\alpha}U\right)\right)\Big|_{\alpha=0}$

$$U=\exp\left(i\frac{\varphi\,\hat{n}\,\hat{\sigma}}{2}\right)=\left(\cos\frac{\varphi}{2}\right)\mathbf{1}+i\left(\sin\frac{\varphi}{2}\right)\hat{n}\,\hat{\sigma}$$
$$U=a\,\mathbf{1}+i\,b_i\sigma_i \qquad a^2+b_i b_i=1$$

complexifed link variables

SU(2) ⟶ SL(2,C)

compact ⟶ non-compact

a, b_i become complex variables

Schwinger Dyson equations for lattice gauge theory

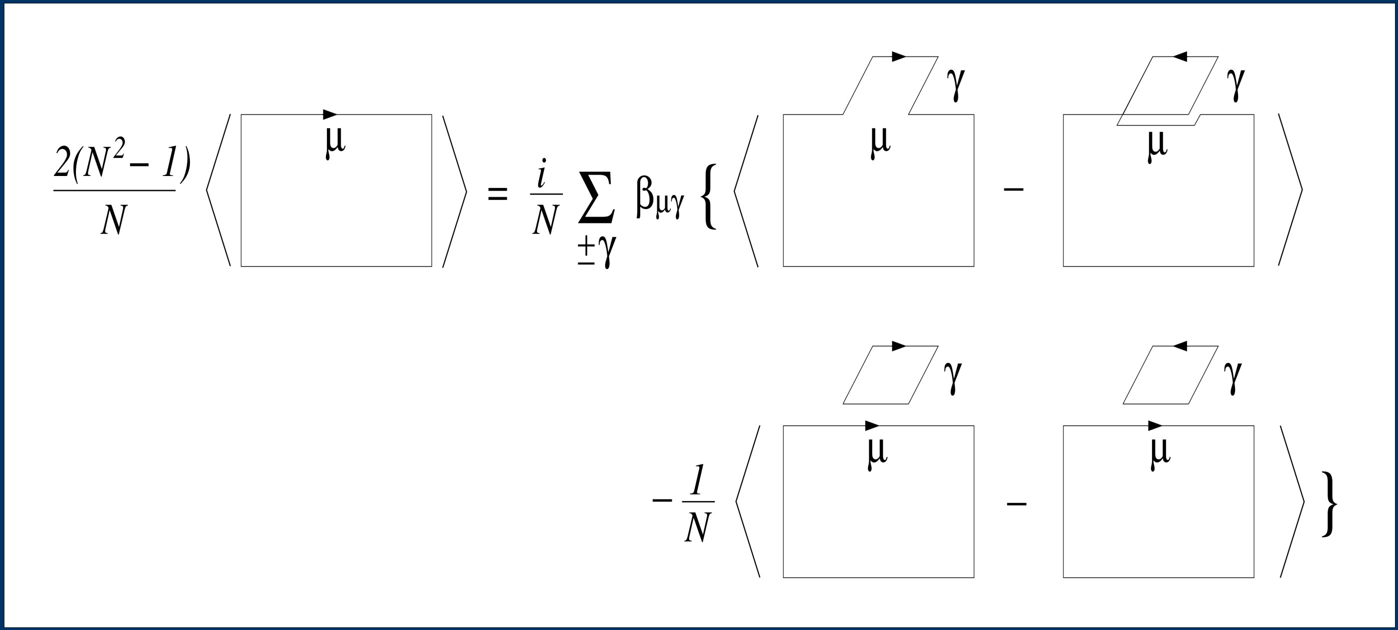
Langevin-time equilibrium reached:

$$\langle U_{x\mu a}(\tau + d\tau)\rangle = \langle U_{x\mu a}(\tau)\rangle \quad \Rightarrow \quad \langle D_{x\mu a} S\rangle = 0$$ First Schwinger Dyson equation

Plaquette average is Langevin time independent

$$\langle U_{x,\mu\nu}(\tau + d\tau)\rangle = \langle U_{x,\mu\nu}(\tau)\rangle$$ Schwinger Dyson equation for plaquette average

can also be derived using the properties of Haar integration in the original integration over group space

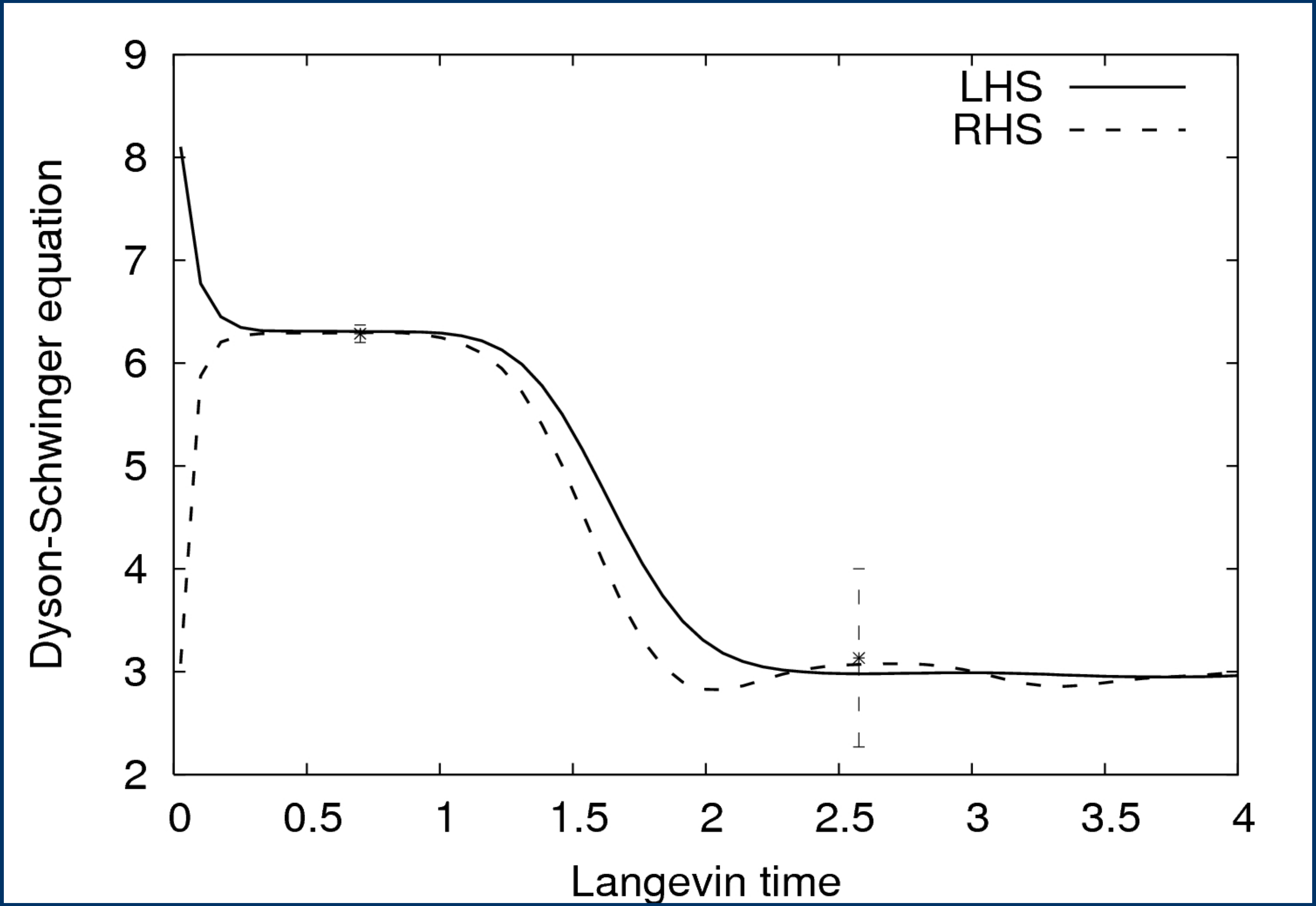


This method gives solutions of SD equations (all of them!)

(loophole: one might get unphysical solution)

SU(2) field theory

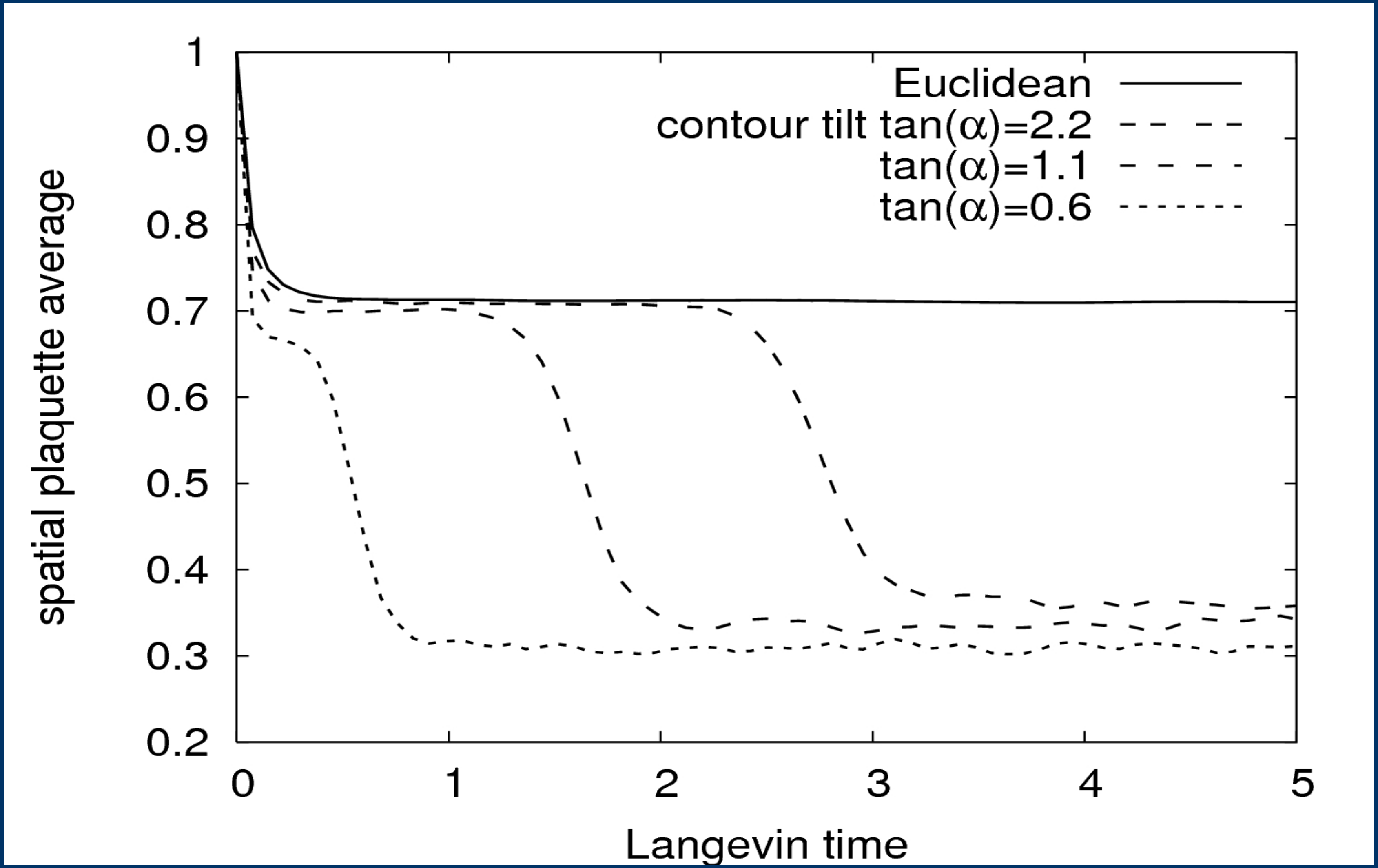
Numerical check of the Schwinger-Dyson equation



SD equations are fulfilled in both regions

SU(2) gauge theory without gaugefixing

without gauge fixing, non-physical fixpoint is always present



How to stabilize the first (physical) result?

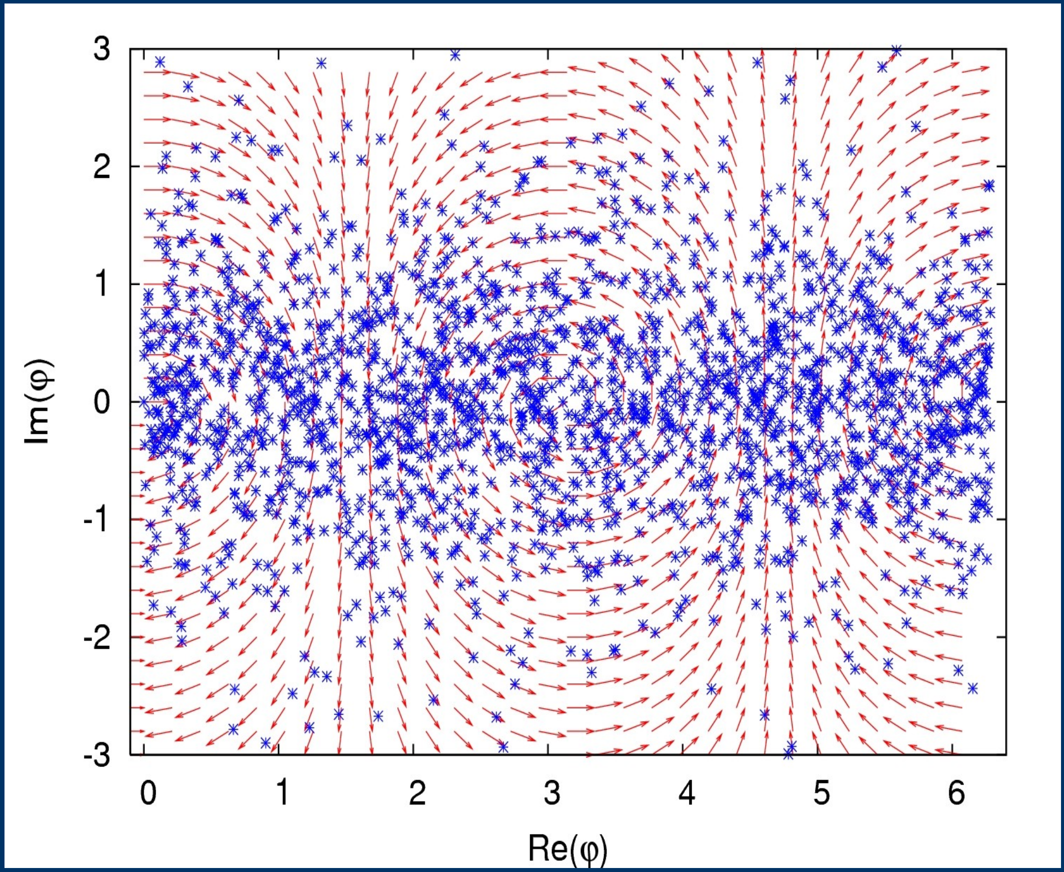
U(1) One plaquette model

$S_0 = i\beta\cos(\varphi)$

We are interested in averages: $\langle f(\varphi)\rangle = \frac{1}{Z}\int_0^{2\pi} d\varphi\, e^{i\beta\cos\varphi} f(\varphi)$

Langevin equation: $\frac{d\varphi}{d\tau} = -i\beta\sin\varphi + \eta(\tau)$

Distribution of φ on the complex plane



Failure of the naïve method

exact result: $\langle e^{i\varphi}\rangle = i\,0.575$

stochastic result:

$-0.009 \pm 0.006 + i(0.00006 \pm 0.00007)$

symmetric distribution
result compatible with zero

Stochastic reweighting

$$\langle O \rangle_p = \frac{1}{Z_p} \int_0^{2\pi} d\varphi \, e^{S_p} O(\varphi)$$

generalization: $S_p = i\beta\cos(\varphi) + i p \varphi$

Langevin equation: $\frac{d\varphi}{d\tau} = -i\beta\sin\varphi + i p + \eta(\tau)$

reweighting factor: $\omega_p = \exp(S_0 - S_p)$

Reweigting formula

averages with S_0 calculated from averages with S_p

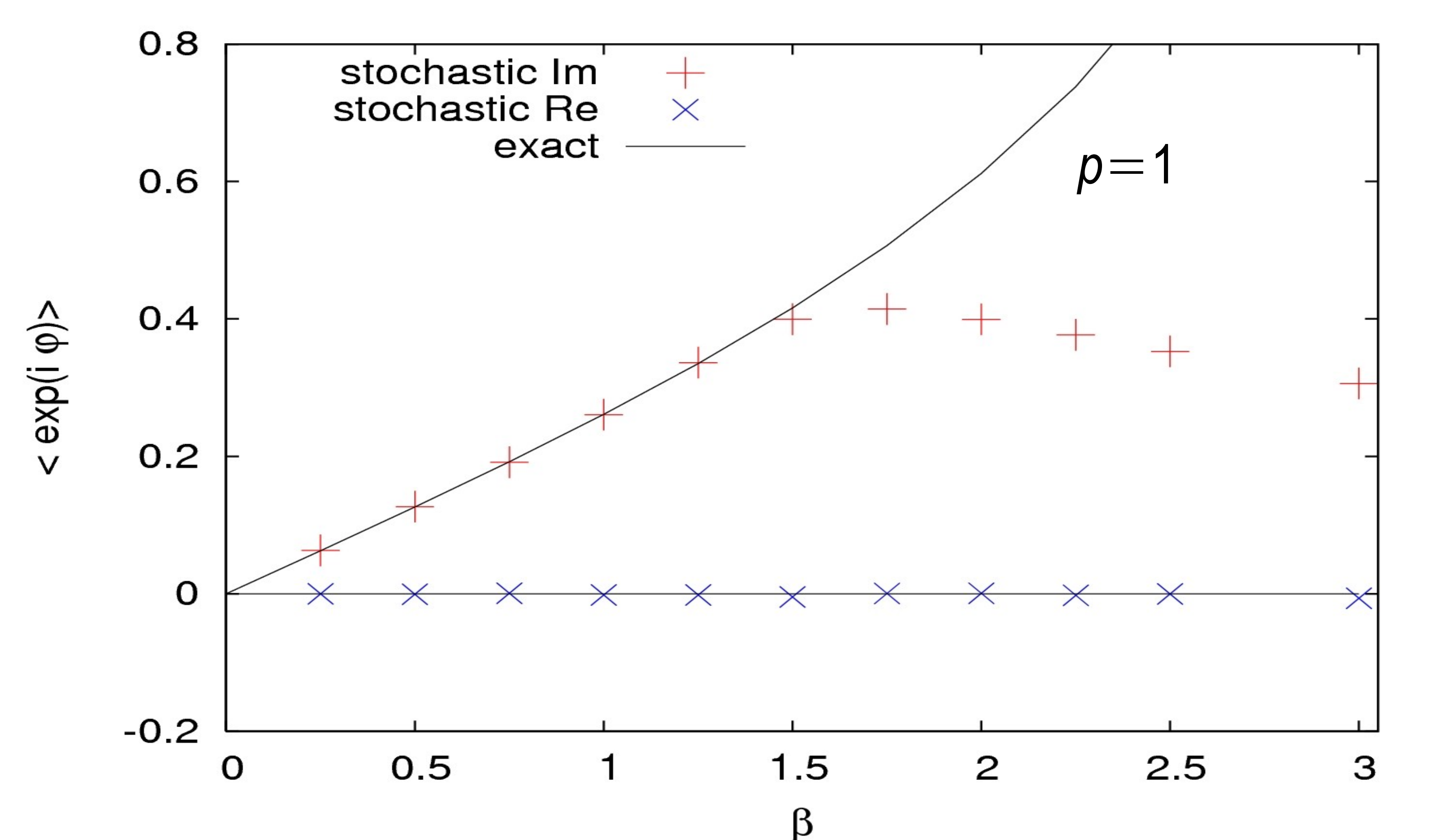
$$\langle O \rangle_0 = \frac{\int_0^{2\pi} d\varphi \, e^{i S_p} \omega_p O(p)}{\int_0^{2\pi} d\varphi \, e^{i S_p} \omega_p} = \frac{\langle \omega_p O \rangle_p}{\langle \omega_p \rangle_p}$$

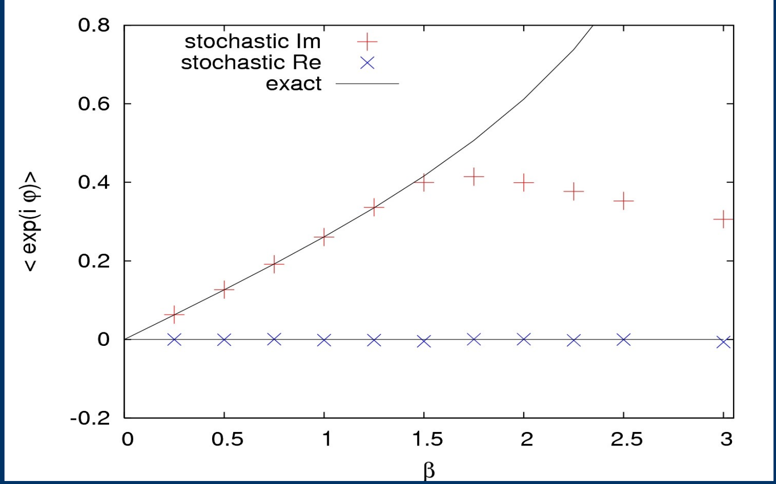
$$\langle e^{i\varphi} \rangle_0 = \frac{\langle 1 \rangle_{p=1}}{\langle e^{-i\varphi} \rangle_{p=1}} = (-0.02 \pm 0.02) + i(0.574 \pm 0.001)$$

Exact result: $\langle e^{i\varphi} \rangle_{p=0} = i\,0.575$ with reweighting it works!

Using the generalized action $S_p = i\beta\cos(\varphi) + i p \varphi$

Correct results obtained for $\langle \exp(i\varphi) \rangle$ in the region: $\beta \leq p$





Flowchart: normalized drift vectors on the complex plane

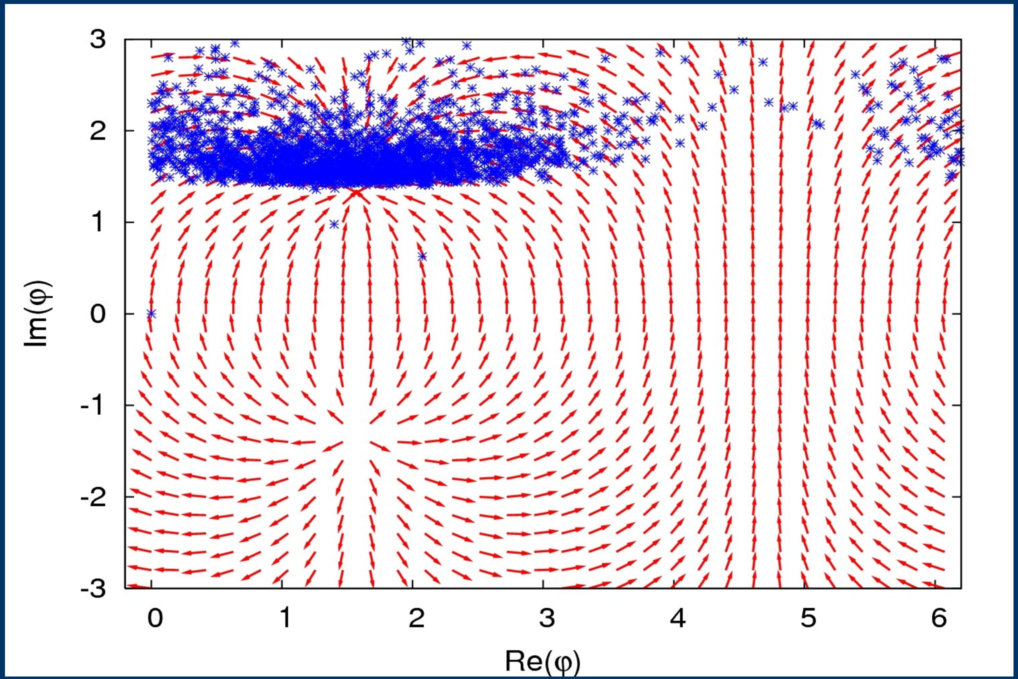
shows fixedpoint (zero drift term) structure on the complex φ plane

Attractive fixedpoint present

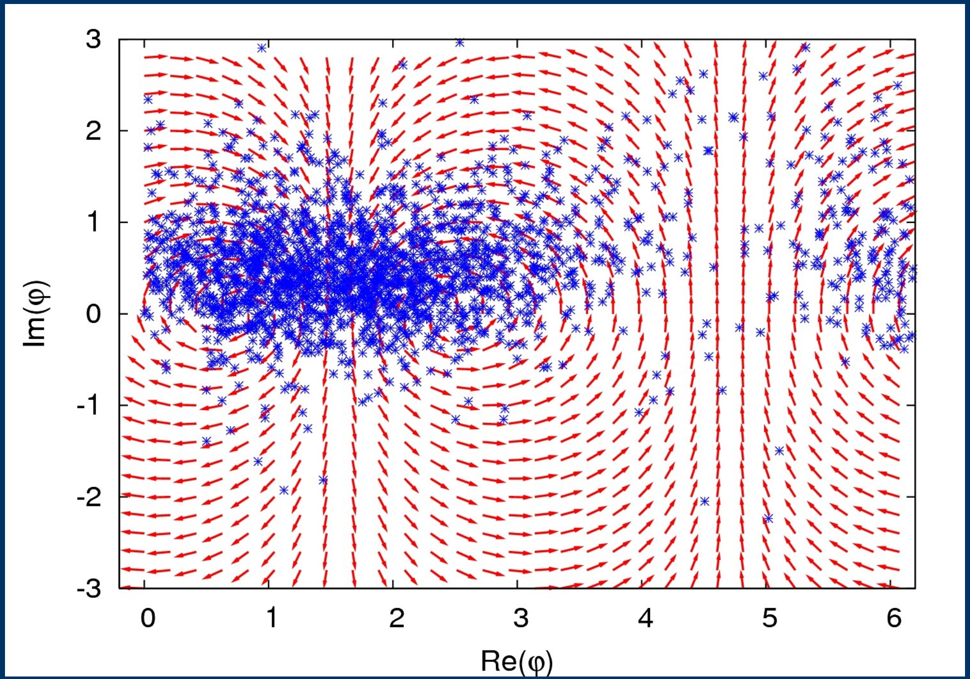
smaller distribution
correct results

No attractive fixedpoint present
(only indifferent)
larger distribution
incorrect results

$\beta = 0.5, p = 1$



$\beta = 1.5, p = 1$



One-plaquette model in classical limit

$$S = i\beta\cos(\varphi) + ip\varphi$$

Langevin equation: $\dfrac{d\varphi}{d\tau} = -i\beta\sin\varphi + ip + \eta(\tau)$

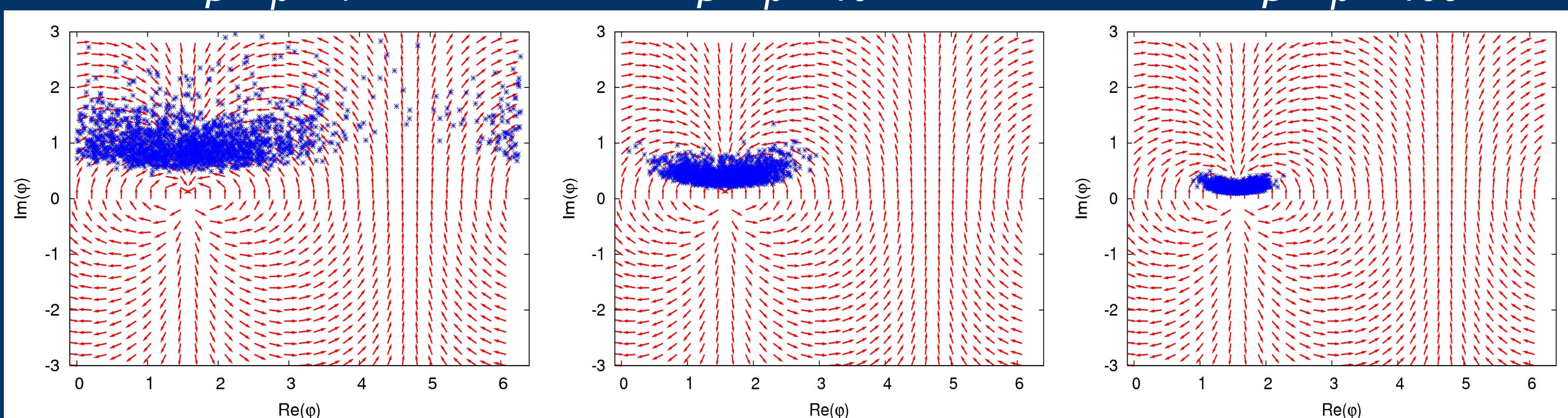
Classical limit: $(\beta = p) \to \infty$

fluctuations supressed

classical averages in the limit $\beta = p \to \infty$

Distributions of φ on the complex plane

$\beta = p = 1$ $\beta = p = 10$ $\beta = p = 100$



Gaugefixing in SU(2) one plaquette model

SU(2) one plaquette model: $S = i\beta \, Tr\, U \qquad U \in SU(2)$

“gauge” symmetry: $U \to W\, U\, W^{-1}$ complexified theory: $U, W \in SL(2, \mathbb{C})$

exact averages by numerical integration: $\langle f(U) \rangle = \frac{1}{Z} \int_0^{2\pi} d\varphi \int d\Omega \sin^2 \frac{\varphi}{2} e^{i\beta \cos\frac{\varphi}{2}} f(U(\varphi, \hat{n}))$

Langevin updating $U' = \exp\left(i \lambda_a (\epsilon\, i D_a S[U] + \sqrt{\epsilon}\, \eta_a)\right) U$

parametrized with Pauli matrices

$$U = \exp\left(i \frac{\varphi \hat{n} \hat{\sigma}}{2}\right) = \left(\cos\frac{\varphi}{2}\right) \mathbf{1} + i \left(\sin\frac{\varphi}{2}\right) \hat{n} \hat{\sigma}$$

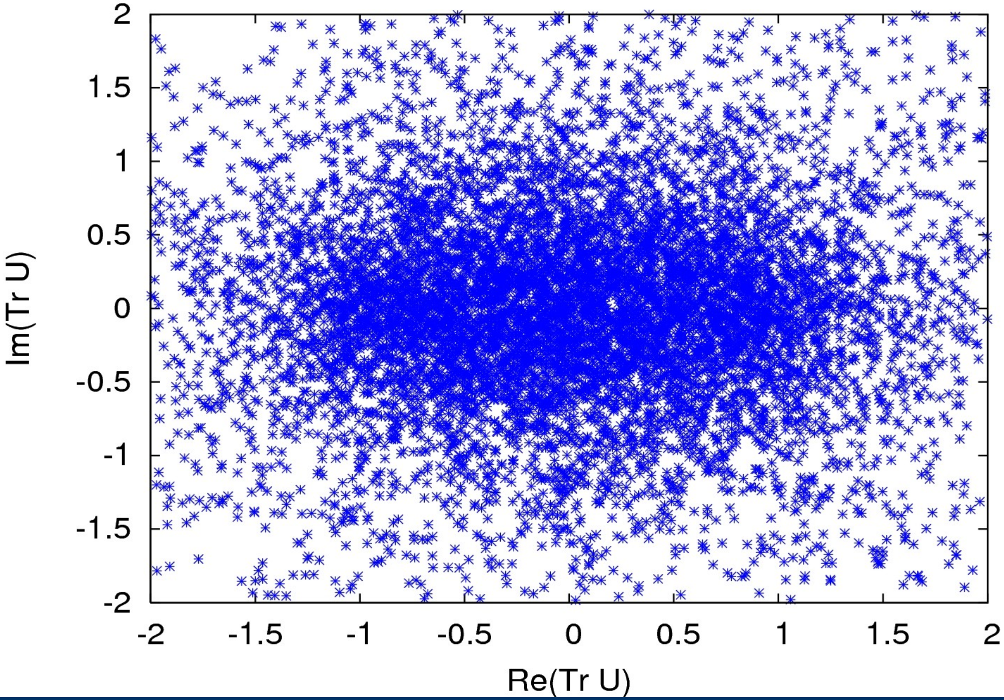
$$U = a\,\mathbf{1} + i\, b_i \sigma_i \qquad a^2 + b_i b_i = 1$$

After each Langevin timestep: fix gauge condition

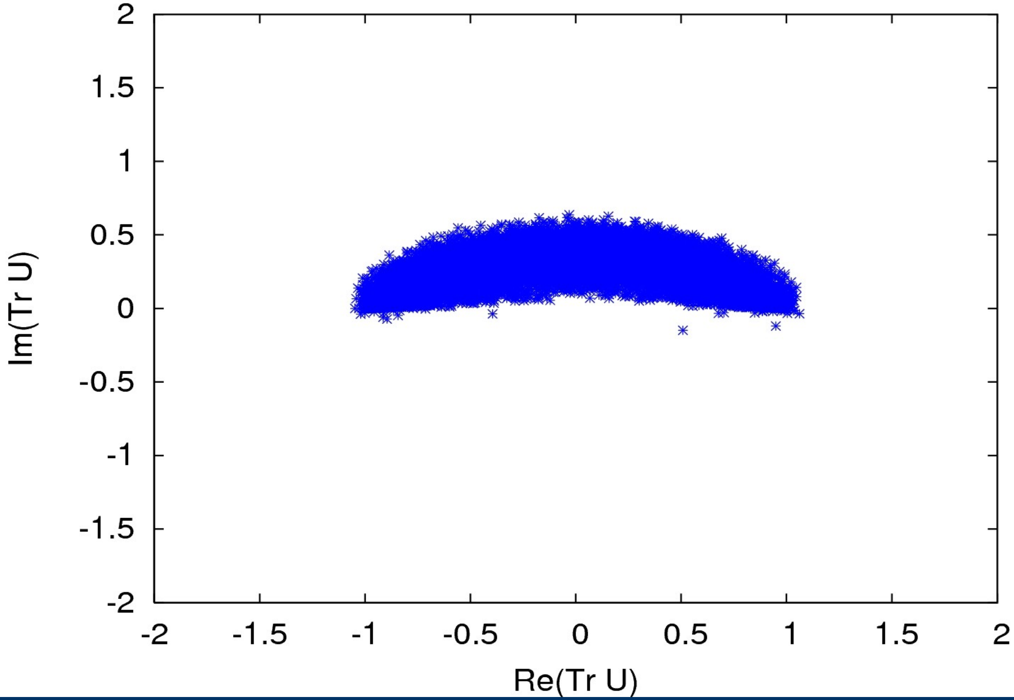
$$U = a\,\mathbf{1} + i\sqrt{1 - a^2}\, \sigma_3 \qquad b_i = (0, 0, \sqrt{1 - a^2})$$

SU(2) one-plaquette model

Distributions of Tr(U) on the complex plane



Without gaugefixing



With gaugefixing

Exact result from integration: $\langle Tr U \rangle = i\,0.2611$

From simulation:

$(-0.02 \pm 0.02) + i(-0.01 \pm 0.02)$

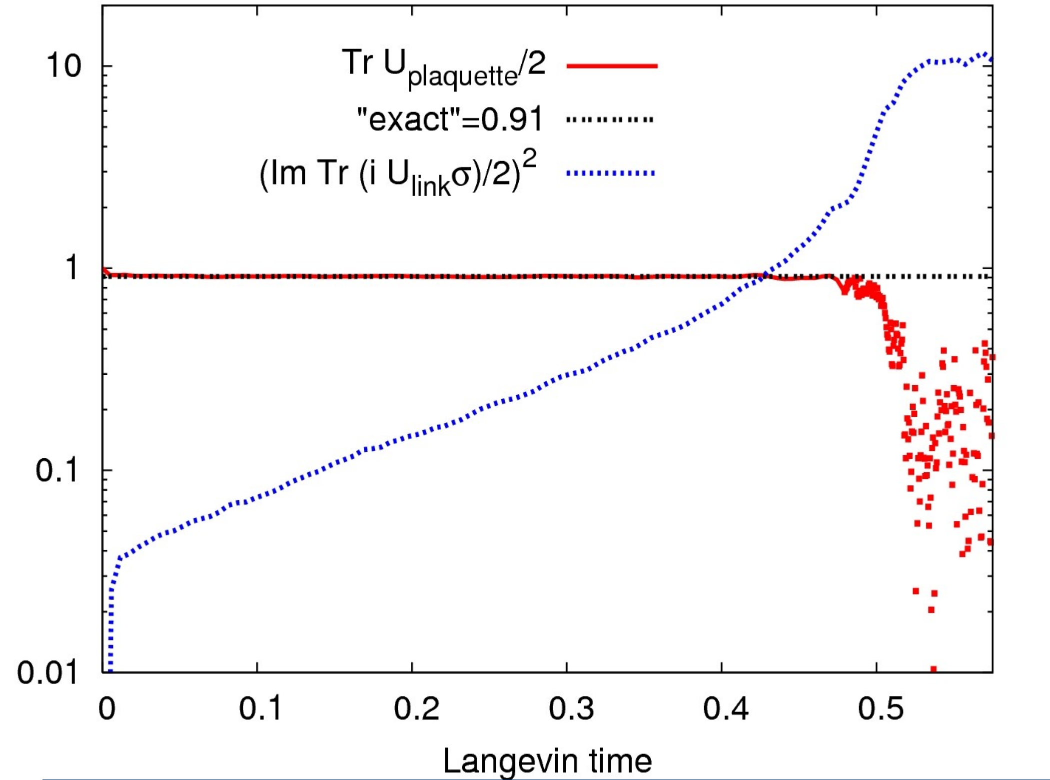
$(-0.004 \pm 0.006) + i(0.260 \pm 0.001)$

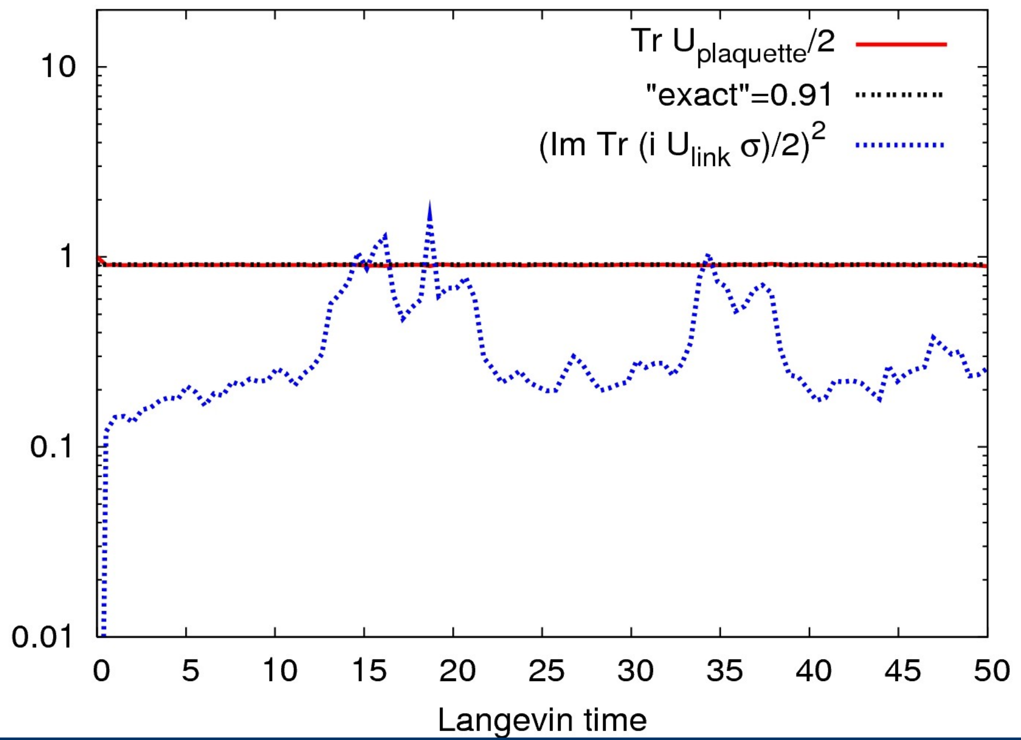
With gauge fixing, all averages are correctly reproduced

SU(2) field theory

$(Im\ Tr\,U)^2$ measures size of distribution

Without gauge fixing
non physical fixed point





Gauge fixing
small lattice coupling $\rightarrow$ large β

Correct result stabilizes

However: $a \sim \exp\left(-b_0/g^2\right)$

With the coupling $g=0.5$

$1/m_{pion} \sim 10^{15}\, a_{lat}$

Conclusions

Without optimization: short real time simulation of scalar oscillator
in equilibrium and non-equilibrium
gives correct results (Schrodinger)

Langevin method: Schwinger Dyson equation solver

Optimization methods to reduce fluctuations:

- reweighting
- gaugefixing
- using small lattice-coupling

with optimization:

Method gives physical solution for SU(2) lattice gauge theory

Scalar Theory

Complex countour given by: $C_t \quad , \qquad \Delta_t = C_{t+1} - C_t \quad , \qquad C_0 = 0 \,, \;\; C_{N_t} = -i\beta$

action discretised on the contour

$$S = \sum_t \left(\frac{(\phi_{t+1} - \phi_t)^2}{2\,\Delta_t} - \Delta_t \frac{V(\phi_t) + V(\phi_{t+1})}{2} \right)$$

Langevin updating in “5th” coordinate

$$\frac{d\,\phi_t}{d\,\tau} = \frac{\partial S}{\partial \phi_t} + \eta_t(\tau)$$

Free theory: $V(\phi) = \frac{m^2 \phi^2}{2}$

The action can be diagonalized: $S = \frac{1}{2} \sum_a c^a \chi^a \chi^a \,, \qquad \chi^a = \sum_t \psi_t^a \phi_t$

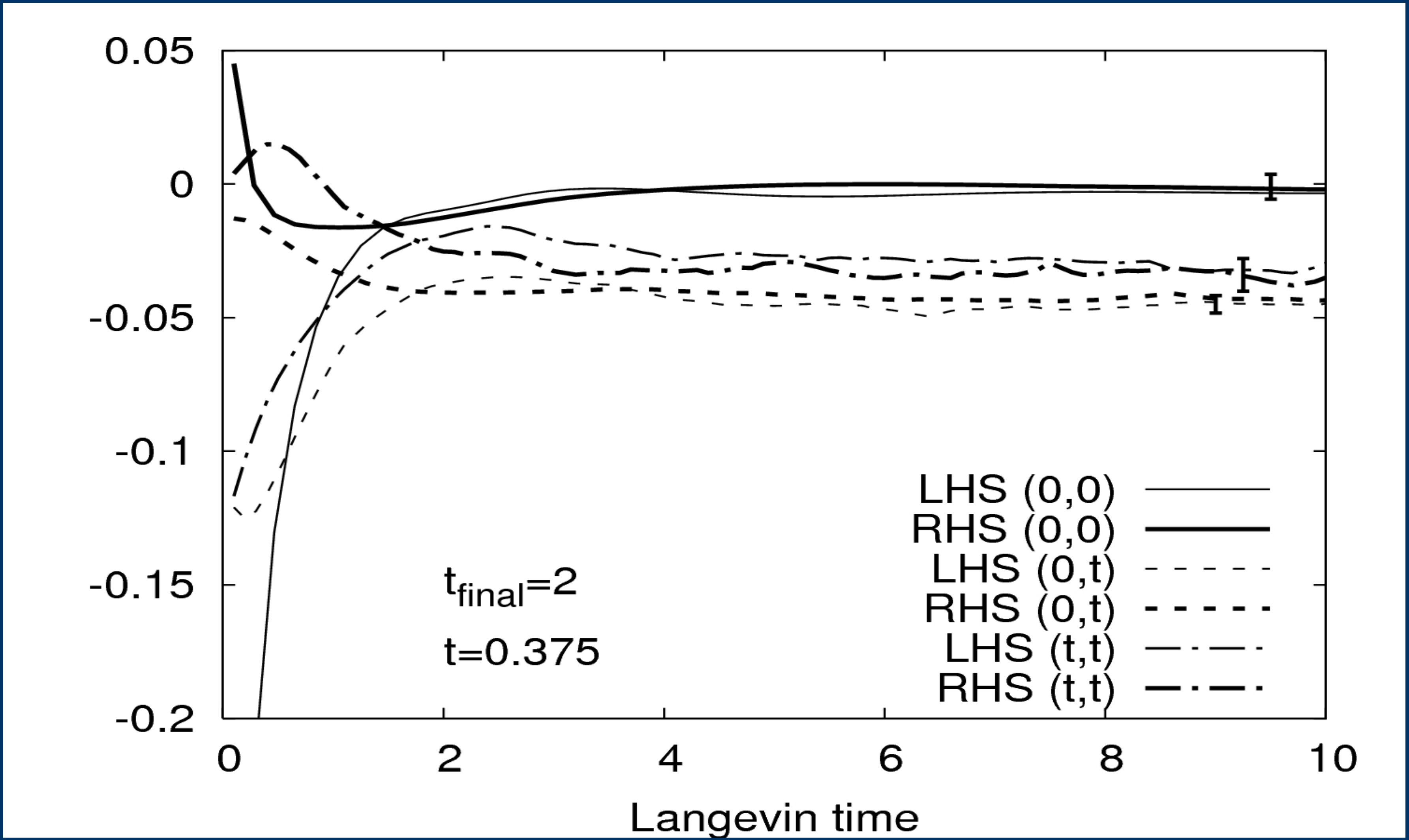
Langevin equation diagonalized coords.:

$$\frac{d\,\chi^a}{d\,\tau} = i\,c^a \chi^a + \eta^a$$

convergent if $Im(c^a) > 0$

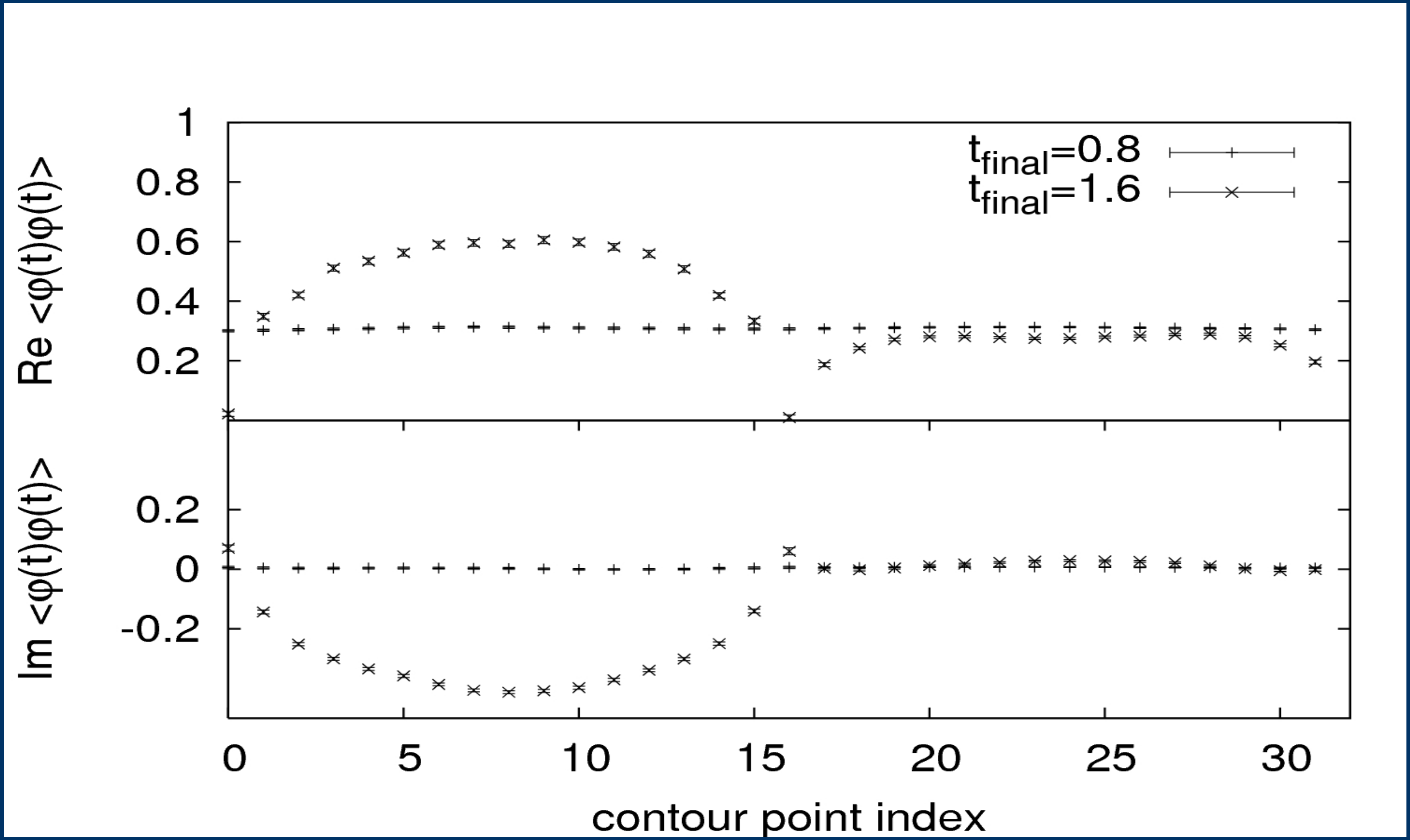
Numerical check of the Schwinger-Dyson equation

$$\sum_{\tau} G_{0,t\tau}^{-1} \langle \varphi_\tau \varphi_{t'} \rangle - \delta_{tt'} = -i\lambda \langle \varphi_t \varphi_t \varphi_t \varphi_{t'} \rangle$$



Non-Physical fixpoints

long countour ► non-time translation invariant solution

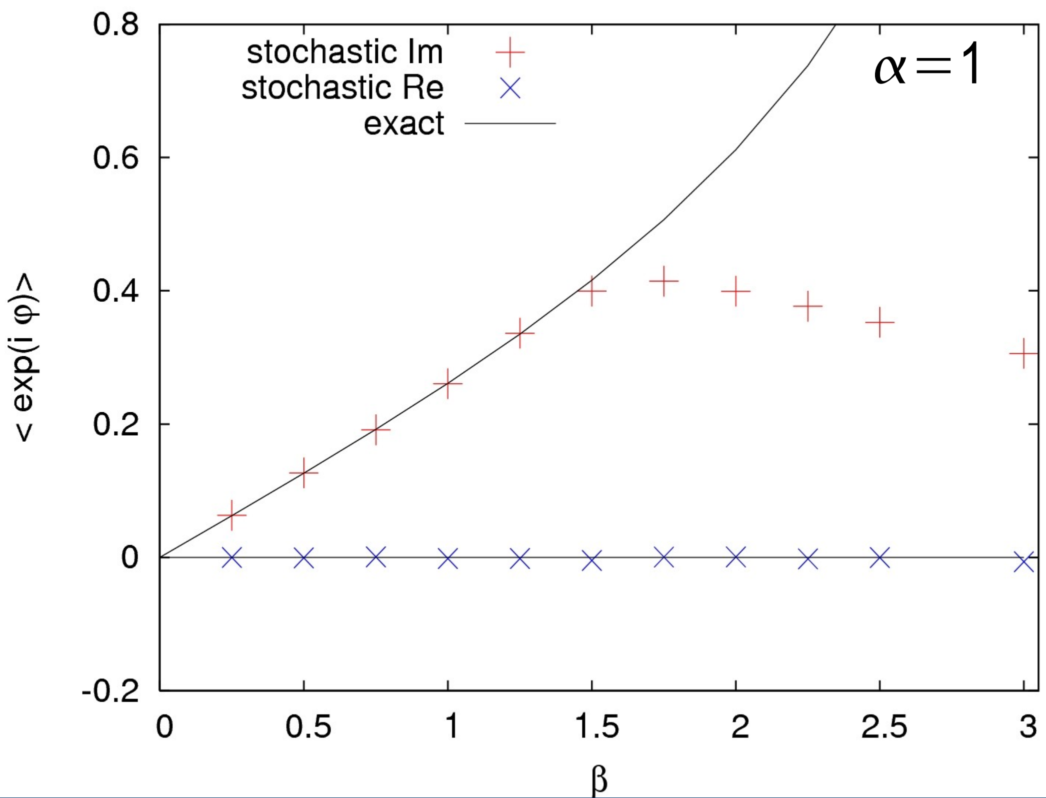


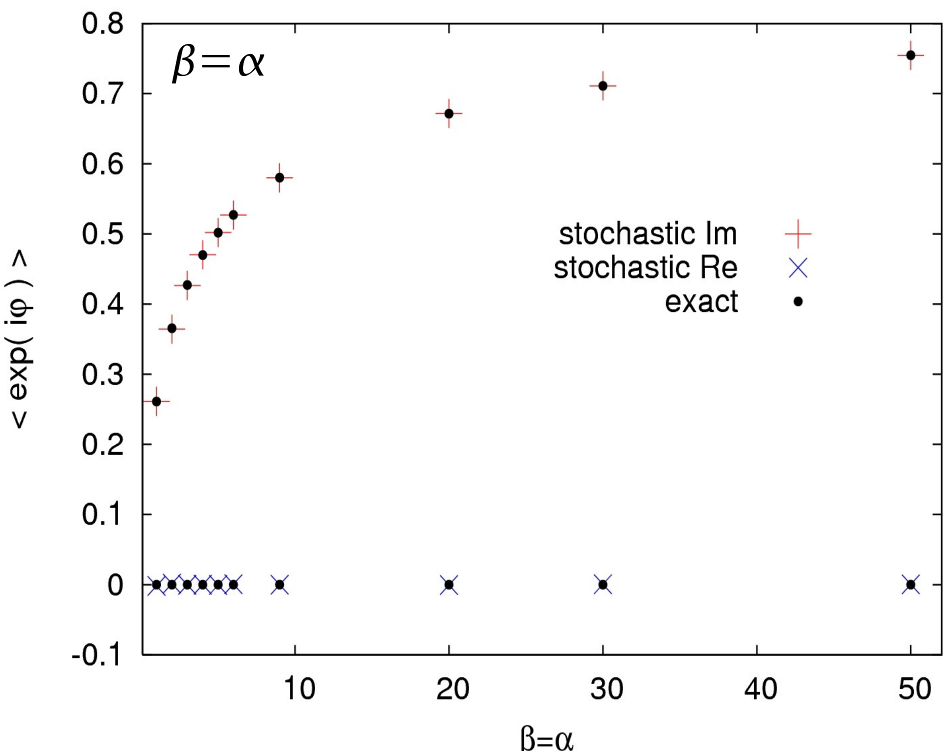
Using the generalized action S_α

Correct results obtained for

$\beta \leq \alpha$

With reweighting correct results for S_0





S_α for $\beta = \alpha$

classical fixed point (zero drift term) on the real axis

$\alpha =$ integer

action can be uniqly written as

$S(U) \qquad U \in U(1)$

Correct results for $\langle f(U) \rangle_\alpha \qquad U \in U(1)$