Generalized statistical mechanics methods for pattern forming- and turbulent systems

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Theory:

C. B., Phys. Rev. Lett. 87, 180601 (2001)

C. B., E.G.D. Cohen, Physica 322 A, 267 (2003)

Application to hydrod. turbulence:

C. B., G. S. Lewis, H.L. Swinney, Phys. Rev. E 63, 035303 (R) (2001)

C. B., Europhys. Lett. 64, 151 (2003)

Application to pattern-forming systems (defect turbulence)

K. E. Daniels, C. B., E. Bodenschatz, Physica D, in press (2003) (cond-mat/0302623)

-1a-

Foundations of (ordinary) statistical mechanics:

entropy $S = -\sum_i p_i \ln p_i$

p_i: prob. of microstates

extremize subject to constraints

$$\sum p_i = 1$$

$$\sum p_i \varepsilon_i = U$$

ε_i: energies of microstates

result: $p_i = \frac{1}{Z} e^{-\beta \varepsilon_i}$ (canonical distribution)

$Z = \sum_i e^{-\beta \varepsilon_i}$ (partition function)

Q: Why this particular function $S = -\sum p_i \ln p_i$?

A1: Because it works! (physics correctly described)

A2: Because it satisfies certain nice axioms of information measures

(see, e.g., C. Beck, F. Schlögl, Thermodynamics of Chaotic Systems, Cambridge University Press 1993)

but: generalized Khinchin axioms (Abe, 2000)

↳ more general information measures possible: Tsallis entropies

Khinchin axioms

(desirable properties of an information measure)

i) $S = S(p_1, p_2, \dots, p_W)$

(function of probabilities only)

ii) $p_i = \frac{1}{W} \implies S = \max$

(maximum for equal probability distr.)

iii) $S(p_1, \dots, p_W, 0) = S(p_1, \dots, p_W)$

(no change by event with prob. zero)

iv) $S(\underbrace{I+II}_{\text{composed system}}) = S(I) + \underbrace{S(II|I)}_{\text{conditional entropy}}$

i) - iv) $\implies$ $S = -\sum p_i \ln p_i$ uniquely

But: If you allow a slightly more general form of iv)

iv*) $S(I+II) = S(I) + S(II|I) + (1-q)\, S(I) \cdot S(II|I)$

then you end up uniquely with

$$S = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right)$$

In particular, for independent subsystems I and II $\boxed{I \,|\, II}$

$$S_q(I+II) = S_q(I) + S_q(II) + (1-q)\, S_q(I)\, S_q(II)$$

entropy is not extensive any more (for $q \neq 1$)

Can now do generalized version of stat. mech. by extremizing Tsallis entropies S_q subject to constraints

$$\sum_i p_i = 1$$

$$\sum_i p_i \varepsilon_i = U_q$$

(or $\sum P_i \varepsilon_i = U_q$ with $\underbrace{P_i = \frac{p_i^q}{\sum p_i^q}}_{\text{escort distributions}}$)

$\leadsto$ $p_i = \frac{1}{Z_q}\left(1 - \beta(1-q)\varepsilon_i\right)^{\frac{1}{1-q}}$ (generalized canonical distribution)

$Z_q = \sum_i \left(1 - (1-q)\beta\varepsilon_i\right)^{\frac{1}{1-q}}$ (partition function)

Entire formalism of thermodynamics has q-generalization / q-invariance

helpful tool:
define

$$e_q^x := (1+(1-q)x)^{\frac{1}{1-q}} \quad \to e^x \quad (q\to 1)$$

q-exponential

$$\ln_q x := \frac{x^{1-q}-1}{1-q} \quad \to \ln x \quad (q\to 1)$$

q-logarithm

$$e_q^{\ln_q x} = x \quad \forall q$$

canonical distributions become

$$p(\varepsilon) \sim e_q^{-\beta\varepsilon} = (1-\beta(1-q)\varepsilon)^{\frac{1}{1-q}}$$

generalized Boltzmann factor

can also derive

$$F_q = U_q - TS_q = -\frac{1}{\beta}\ln_q Z_q$$

$$\frac{1}{T} = \frac{\partial S_q}{\partial U_q}$$

$$\frac{\partial^2 S}{\partial U_q^2} = -\frac{1}{T^2}\frac{1}{C_q} \qquad \text{etc....}$$

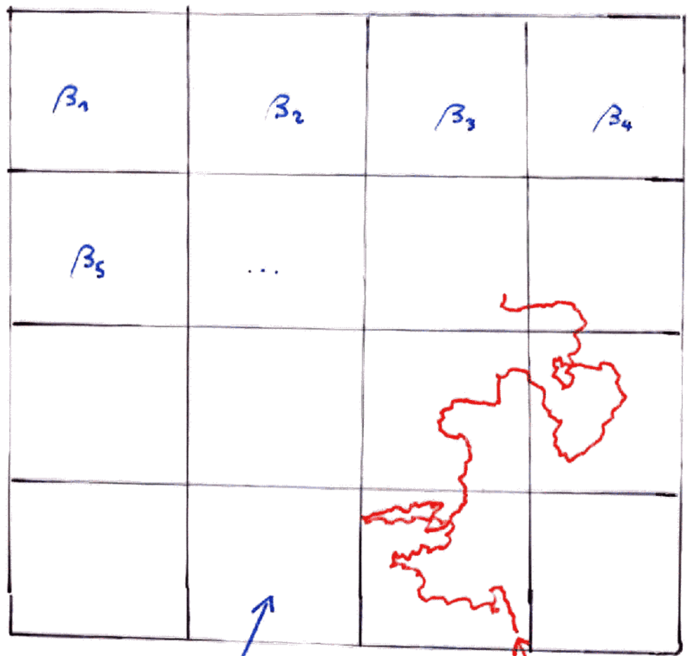
When / Why could this be physically relevant?

Basic idea:

If system (for whatever reason) cannot extremize Shannon entropy it then chooses to extremize the second best information measures. These are the Tsallis entropies.

reason could be
- system non-mixing
- long range interaction
- complicated multifractal phase space structure
- complicated networks
- external energy input (nonequilibrium system with stationary state)
- fluctuations of temp. or energy dissipation rate
- strongly inelastic collisions

Nonequilibrium system
with fluctuations of (e.g.) inverse temperature β
on long time scale
(can also be pressure, chemical potential, energy dissipation rate, ...)



test particle

simplest model $E = \frac{1}{2} u^2$

local equilibrium

$p(E) \sim e^{-\beta_i E}$

in each cell

model: choose a random configuration $\{\beta_j\}$
(β distributed according to density $f(\beta)$)
then choose next random config., and so on.

Dynamical foundation of nonextensive stat. mech. for systems with fluctuating temperature or energy dissipation rate

Brownian particle (Ornstein-Uhlenbeck process)

$$\dot{u} = -\gamma u + \sigma L(t)$$

Gaussian white noise

$$p(u|\beta) = \sqrt{\frac{\beta}{2\pi}} \exp\{-\tfrac{1}{2}\beta u^2\}$$

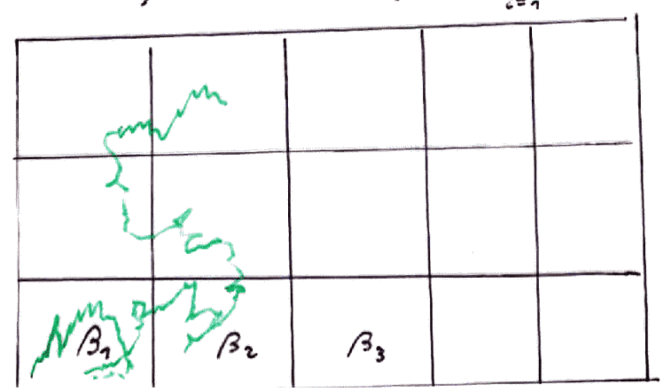
$$\beta := \frac{\gamma}{\sigma^2} \quad \text{inverse temperature}$$

Assume γ and/or σ fluctuate on large time scale s.t. $\beta = \frac{\gamma}{\sigma^2}$ is χ^2 distributed with degree n

$$f(\beta) = \frac{1}{\Gamma(\frac{n}{2})} \left\{\frac{n}{2\beta_0}\right\}^{\frac{n}{2}} \beta^{\frac{n}{2}-1} \exp\left\{-\frac{n}{2}\frac{\beta}{\beta_0}\right\}$$

prob. density

e.g. $\beta = \sum_{i=1}^{n} X_i^2 \leftarrow$ Gaussian (av. 0)



conditional prob.

$$p(u|\beta) = \sqrt{\frac{\beta}{2\pi}}\, \exp\left\{-\tfrac{1}{2}\beta u^2\right\}$$

joint prob.

$$p(u,\beta) = p(u|\beta)\cdot f(\beta)$$

marginal prob.

$$p(u) = \int p(u|\beta)\, f(\beta)\, d\beta$$

$$= \frac{1}{Z_q}\,\frac{1}{\left(1+\frac{1}{2}\tilde{\beta}(q-1)u^2\right)^{1/q-1}}$$

where

$$q = 1 + \frac{2}{n+1}$$

C.B., PRL 87, 180601 (2001)

$$\tilde{\beta} = \frac{2}{3-q}\,\beta_0$$

$$\beta_0 := \int f(\beta)\cdot\beta\, d\beta = \text{average of } \beta$$

Simple dynamical model where Tsallis statistics can be proved rigorously.

Various generalizations possible.

e.g. $\dot{u} = -\gamma F(u) + \sigma L(t)$

$$F(u) = -\frac{\partial}{\partial u}V(u) \qquad V(u) \sim |u|^{2\alpha}$$

Fluctuations of β and Tsallis statistics

integral representation of Γ function:

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt$$

substitute

$$t = \beta\left(E(u) + \frac{1}{(q-1)\beta_0}\right)$$

$$z = \frac{1}{q-1}$$

$$\Downarrow$$

any Hamiltonian or effect. energy

$$\left(1+(q-1)\beta_0 E(u)\right)^{-\frac{1}{q-1}} = \int_0^\infty e^{-\beta E(u)} f(\beta)\, d\beta$$

'generalized' Boltzmann factor

ordinary Boltzmann factor

$$f(\beta) = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)}\left\{\frac{1}{(q-1)\beta_0}\right\}^{\frac{1}{q-1}} \beta^{\frac{1}{q-1}-1} \exp\left\{-\frac{1}{q-1}\frac{\beta}{\beta_0}\right\}$$

χ^2 distribution

(occurs in many circumstances)

$$\langle\beta\rangle = \int_0^\infty \beta f(\beta)\, d\beta = \beta_0$$

$$q = \frac{\langle\beta^2\rangle}{\langle\beta\rangle^2}$$

e.g. $\beta = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

Gaussian av. 0

Wilk et. al. PRL 2000

C.B. PRL 87, 180601 (2001)

$$n = \frac{2}{q-1}$$

More generally one can consider generalized Boltzmann factors

$$B(E) = \int_0^\infty e^{-\beta E} f(\beta)\, d\beta$$

with general $f(\beta)$: "Superstatistics"
(C.B., E.G.D. Cohen, cond-mat/0205097)
Physica $\underline{322A}$, 267 (2003)

- $f(\beta) = \chi^2 \Rightarrow$ Tsallis
- $f(\beta) = \frac{1}{b}$ for $\beta \in [a, a+b]$
(uniform distribution)

$$\Rightarrow B(E) = \frac{1}{bE}\left(e^{-(\beta_0 - \frac{1}{2}b)E} - e^{-(\beta_0 + \frac{1}{2}b)E}\right)$$

$$= e^{-\beta_0 E}\left(1 + \frac{1}{24} b^2 E^2 + \frac{1}{1920} b^4 E^4 + \ldots\right)$$

- $f(\beta) =$ log-normal

$$f(\beta) = \frac{1}{\beta s \sqrt{2\pi}} \exp\left\{ - \frac{(\log \frac{\beta}{m})^2}{2s^2} \right\}$$

$w := e^{s^2}$

$$\Rightarrow B(E) = e^{-\beta_0 E}\left(1 + \frac{1}{2} m^2 w(w-1) E^2 + \frac{1}{6} m^3 w^{\frac{3}{2}} (w^3 - 3w + 2) E^3 + \ldots\right)$$

- $f(\beta) =$ F-distribution

$$f(\beta) \sim \frac{\beta^{\frac{v}{2}-1}}{(1 + c\cdot\beta)^{\frac{v+w}{2}}} \Rightarrow B(E) = \cdots$$

Main result:

For small enough variance of the fluctuations of β all superstatistics behave in a universal way

can prove

$$B(E) = e^{-\beta_0 E}\left(1 + \frac{1}{2}\sigma^2 E^2 + g(q)\, \beta_0^3 E^3 + \ldots\right)$$

variance of distribution $f(\beta)$
$\sigma^2 = \langle \beta^2 \rangle - \langle \beta \rangle^2$

$\beta_0 = \langle \beta \rangle$

can define

$$q = \frac{\langle \beta^2 \rangle}{\langle \beta \rangle^2}$$

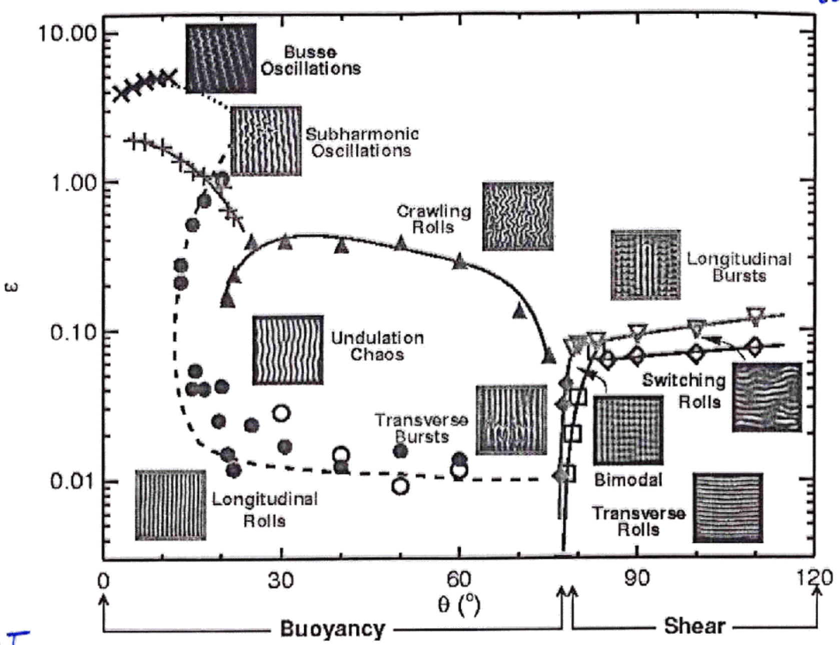
for $\underline{\text{any}}$ superstatistics

next-order term:

$$g(q) = \begin{cases} 0 & \text{(uniform)} \\ \frac{1}{3}(q-1)^2 & (\chi^2) \\ \frac{1}{6}(q^3 - 3q + 2) & \text{(log-normal)} \\ \frac{1}{3}\frac{(q-1)(5q-6)}{3-q} & (F \text{ with } v=4) \end{cases}$$

(non-universal)

Inclined layer convection experiment (Daniels, Bodenschatz 2002)



$$\varepsilon = \frac{\Delta T}{\Delta T_c} - 1$$

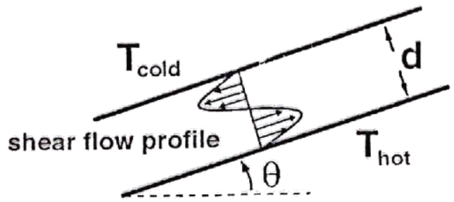
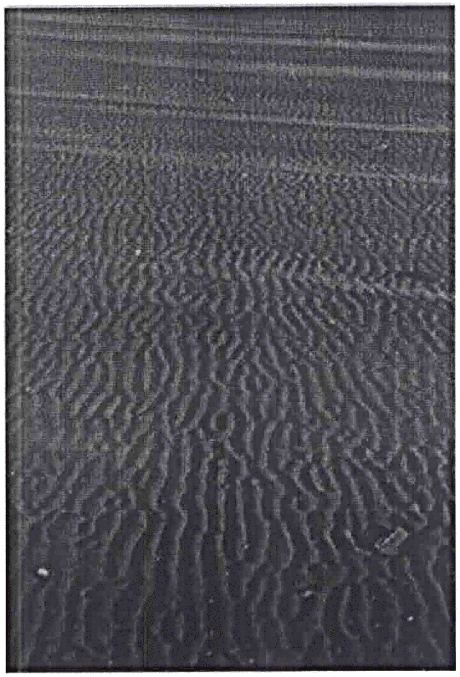


FIG. 1. Schematic drawing of inclined layer convection of cell thickness d and temperature difference $\Delta T \equiv T_{hot} - T_{cold}$.

← Undulation chaos in inclined layer convection



sand pattern
(St. Andrews, Scotland)

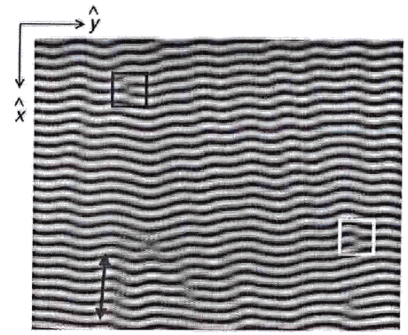


Fig. 1. Example shadowgraph image of undulation chaos in fluid layer heated from below and cooled from above, inclined by an angle $\theta = 30°$. The nondimensional driving paramter is $\epsilon = 0.08$. Black box encloses a positive defect; white, a negative. Arrow is adjacent to a tearing region of low-amplitude convection. Uphill direction is at left side of page. Region shown is the subregion of size $51d \times 63d$ used for analysis.

Defects are created and annihilated in pairs.

They move in a spatio-temporally chaotic way, like particles.

ordinary particles would have

$$\gamma = \frac{6\pi \nu \rho a}{m} = \text{const}$$

but defects are no ordinary particles

eff. γ fluctuates $\Rightarrow$ superstatistics

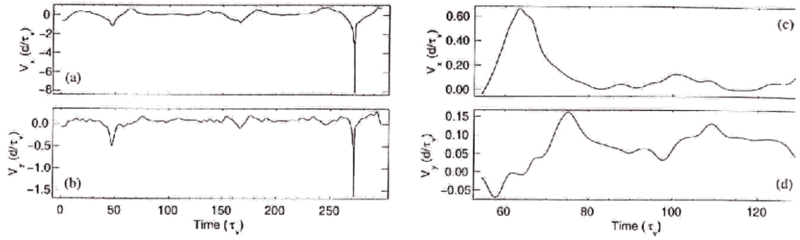


Fig. 2. Sample plots of (a,c) $v_x(t)$ and (b,d) $v_y(t)$ for $\epsilon = 0.08$ over two different tim scales (a,b) defect lifetime from creation to annihilation and (c,d) region of duration 75 τ_v

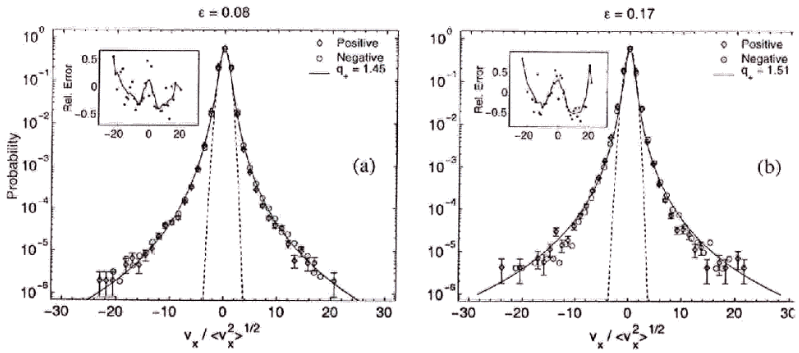


Fig. 3. Transverse velocity (v_x) distributions at (a) $\epsilon = 0.08$ and (b) $\epsilon = 0.17$ for positive and negative defects, rescaled to $\sigma = 1$. Solid lines are one-parameter fits to Eq. (6) for pos itive defects. Unrescaled standard deviations were: (a) $\sigma_+ = 0.550d/\tau_v$, $\sigma_- = 0.553d/\tau_v$ and (b) $\sigma_+ = 0.586d/\tau_v$, $\sigma_- = 0.565d/\tau_v$. Dashed line is Gaussian with $\sigma = 1d/\tau_v$. Inset Relative error of experiment and theory: $(p_{exp} - p_{theory})/p_{theory}$ for positive defects.

$$p(v) = \frac{1}{Z_q}\left(1 + \tfrac{1}{2}(q-1)\tilde{\beta}\, v^2\right)^{-\frac{1}{q-1}}$$

$$Z_q = \sqrt{\frac{2}{(q-1)\tilde{\beta}}}\;\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)}$$

$$\tilde{\beta} = \frac{2}{5-3q}$$

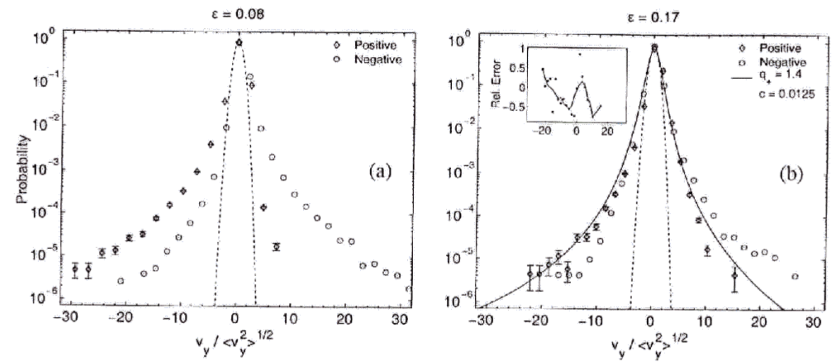


Fig. 4. Longitudinal velocity (v_y) distributions at (a) $\epsilon = 0.08$ and (b) $\epsilon = 0.17$ for positive and negative defects, rescaled to $\sigma = 1$. Solid line is a fit to Eq. (10) for positive defects. Unrescaled standard deviations were: (a) $\sigma_+ = 0.099d/\tau_v$, $\sigma_- = 0.115d/\tau_v$ and (b) $\sigma_+ = 0.141d/\tau_v$, $\sigma_- = 0.128d/\tau_v$. Dashed line is Gaussian with $\sigma = 1d/\tau_v$. Inset: Relative error of experiment and theory: $(p_{exp} - p_{theory})/p_{theory}$ for positive defects.

asymmetric correction:

$$p(v) = \frac{1}{Z_q}\left(1 + \tilde{\beta}(q-1)\left(\tfrac{1}{2}v^2 - c\left(v - \tfrac{1}{3}v^3\right)\right)\right)^{-\frac{1}{q-1}}$$

from general dynamical systems considerations

(A. Hilgers, C.B., Phys. Rev. 60 E, 5385 (1999))

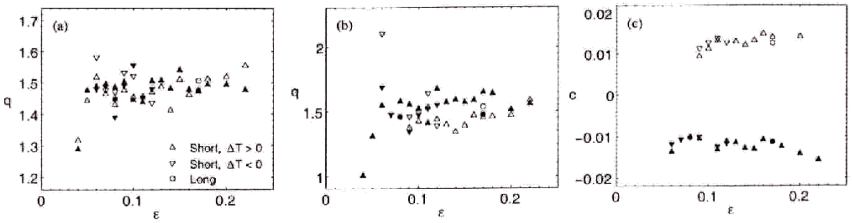


Fig. 8. Values of fit parameters (a) q for v_x (b) q for v_y, and (c) c for v_y as a function of ϵ. Open points are for positive defects; filled points are for negative. Short runs in which data were obtained by quasistatic temperature increases from below are marked with $\triangle$, and the converse with $\bigtriangledown$.

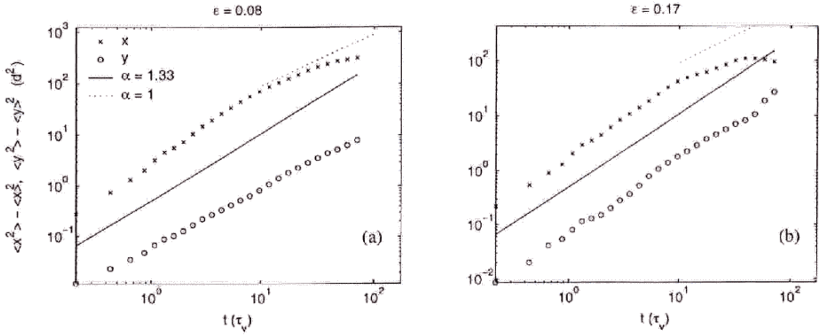


Fig. 6. Second moment of position trajectories in $\hat{x}$ and $\hat{y}$. Solid line is diffusive behavior predicted by Eq. (18) for $q = 1.5$; dotted line is normal diffusive limit ($q = 1$). Fits to the data give values ranging from $\alpha = 1.16$ to $\alpha = 1.50$ depending on the region being fit.

Position of defects exhibits anomalous diffusion

$$\langle x^2(t) \rangle \sim t^\alpha$$

$$\alpha \approx 1.3$$

Generalized Fokker-Planck eq.

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x}\left(F(x)\, p(x,t)\right) + D \frac{\partial^2}{\partial x^2}\left(p(x,t)\right)^\nu$$

$\nu \neq 1$ generates Tsallis distributions with

$q = 2-\nu$ $\qquad$ $F(x) = K_1 - K_2 x$

Plastino & Plastino, Physica A (1995)
Tsallis & Bukman, PRE (1996)

Model also generates anomalous diffusion

$$\langle x(t)^2 \rangle \sim t^\alpha$$

$$\alpha = \frac{2}{1+\nu} = \frac{2}{3-q}$$

Measured q from defect velocity distribution

$q = 1.45 \pm 0.05$

$\Rightarrow$ predicted anomalous diffusion coeff.

$\alpha = \frac{2}{3-q} = 1.30 \pm 0.03$

measured an. diff. coeff. $\quad \alpha = 1.3 \pm 0.1$

also predicted: power law decay of vel. corr. fct

Patterns in turbulent flows

(rotating annulus)

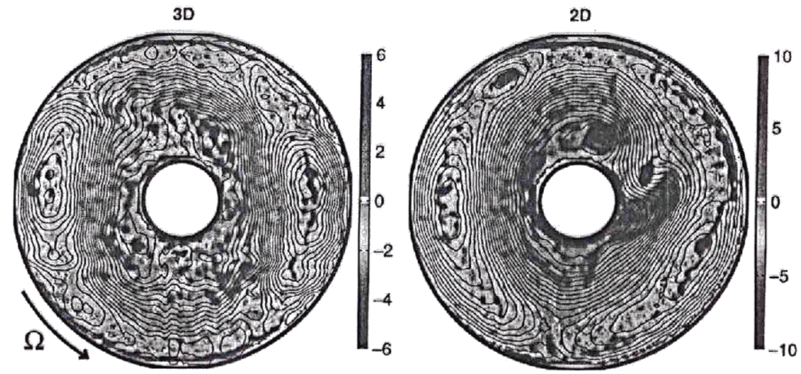


Fig. 1. Vorticity and streamfunction maps for the 3D and 2D flows, at $\Omega = 1.57$ and 11.0 rad/s, respectively. The cyclonic (red cente anti-cyclonic (blue center) vortices are advected clockwise by the mean anti-cyclonic jet, as the tank rotates counter-clockwise. The sp of the streamline contours is $12\,\mathrm{cm}^2/\mathrm{s}$ for the 3D case and $30\,\mathrm{cm}^2/\mathrm{s}$ for the 2D case, and the color bars show the vorticity values (

from:

C.N. Baroud, H.L. Swinney, Physica 184 D, 21 (2003)

Probability densities of velocity differences well approximated by 'canonical' distributions of generalized stat. mech

$q \approx 1.3$ (2D)

q scale dep. (3D)

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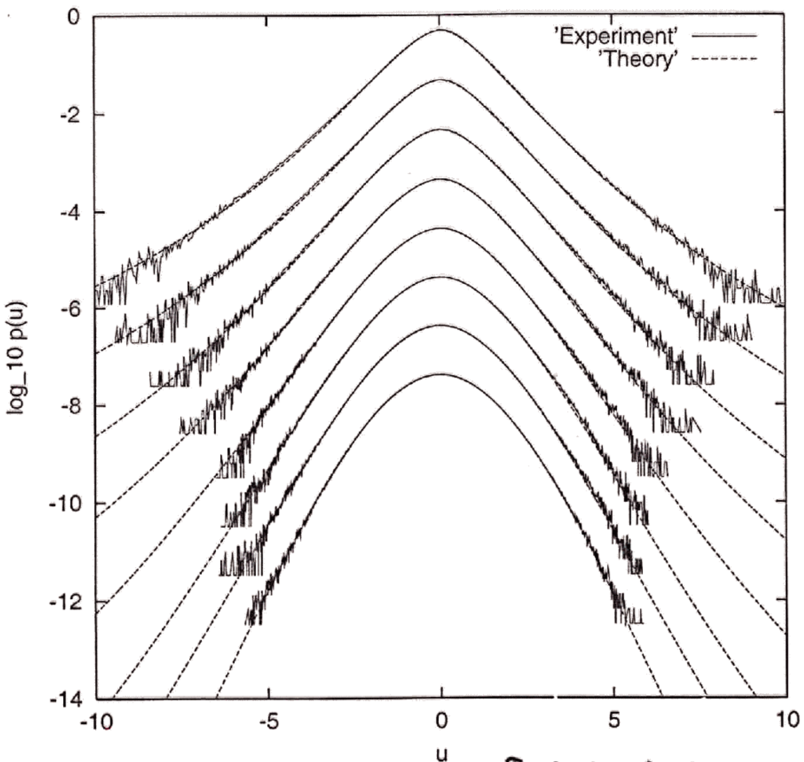
Turbulent
Couette-Taylor flow (Lewis & Swinney)

$Re = 540\,000$

Fig. 1a



Beck, Lewis, Swinney
Phys. Rev. E (2001)
63E, 035303 (R)

from top to bottom:

$\frac{r}{\eta} = 11.6, 23.1, 46.2, 92.5, 208, 399, 827, 14450$

$q = 1.168, 1.150, 1.124, 1.105, 1.084, 1.065, 1.055, 1.038$

$\alpha = 2-q$

(shift by -1 unit for better visibility)

Lagrangian test particle
in turbulent flow

Bodenschatz et al.
Nature (2001)
J. Fluid Mech. (2002)

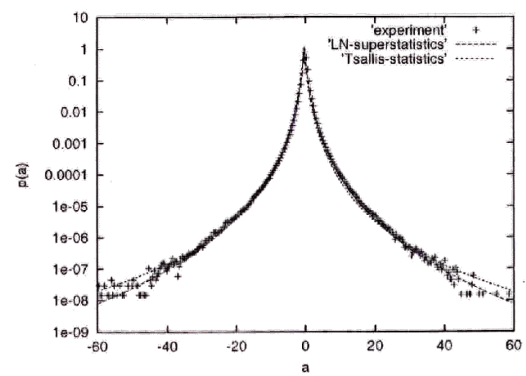


Fig. 5 Acceleration distribution as measured by Bodenschatz et al. and comparison with the log-normal superstatistics distribution (15) with $s^2 = 3.0$. Also shown is a Tsallis distribution (13) with $q = 1.2$ and $\alpha = 0.5$.

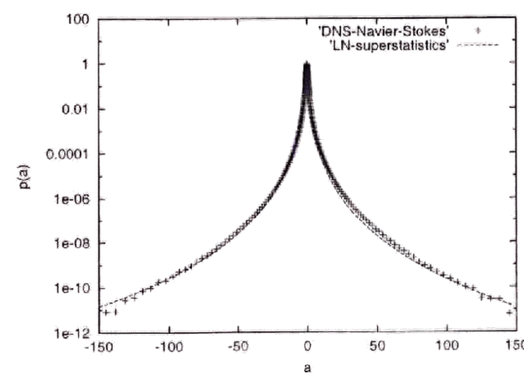


Fig. 6 Pressure statistics as obtained by Gotoh et al. in a direct numerical simulation of the Navier-Stokes equation, and comparison with log-normal superstatistics with $s^2 = 3.0$.

Sawford model

(B.L. Sawford, Phys. Fluids A3, 1577 (1991))

model for Lagrangian test particle in turbulent flow

$$(a(t), u(t), x(t))$$

accel. vel. position

$$\dot a = -(T_L^{-1} + t_\eta^{-1})a - T_L^{-1} t_\eta^{-1} u + \sqrt{2\sigma_u^2 (T_L^{-1}+t_\eta^{-1}) T_L^{-1} t_\eta^{-1}}\, L(t)$$

$$\dot u = a$$

$$\dot x = u$$

Gaussian white noise

$$T_L = \frac{2\sigma_u^2}{C_0 \bar\varepsilon} \gg t_\eta = \frac{2a_0 \nu^{\frac12}}{C_0 \bar\varepsilon^{1/2}}$$

average energy dissipation; kin. viscosity

C_0, a_0: Lagrangian structure function constants

σ_u^2: variance of velocity distribution

$$R_\lambda = \frac{\sqrt{15}\,\sigma_u^2}{\sqrt{\nu \bar\varepsilon}} \quad \text{Reynolds number}$$

predicts Gaussian acceleration distributions!

In the limit $T_L \to \infty$, Sawford model reduces to

$$\dot a = -\gamma a + \sigma L(t)$$

Gaussian for constant coeff γ, σ. How can we save this model? Let γ, σ fluctuate! $\Rightarrow$ superstatistics

with

$$\gamma = \frac{C_0}{2a_0} \nu^{-1/2} \bar\varepsilon^{1/2}$$

$$\sigma = \frac{C_0^{3/2}}{2a_0} \nu^{-1/2} \bar\varepsilon$$

Hence

$$\beta = \frac{\gamma}{\sigma^2} = \frac{2a_0}{C_0^2} \nu^{\frac12} \bar\varepsilon^{-\frac32}$$

This naturally leads to a generalized (superstatistical) Sawford model with a fluctuating β

$$\beta = \frac{\gamma}{\sigma^2} = \frac{2a_0}{C_0^2} \nu^{\frac12} \varepsilon^{-\frac32}$$

fluctuating energy dissipation

All you have to do: decide on prob. density $f(\beta)$

C.B., PRL (2001)

Europhys. Lett. (2003)

A. Reynolds

Conclusion

- Generalized stat. mech. often due to a 'superstatistics'.
- Superstatistics is a 'statistics of a statistics' relevant for driven non equilibrium systems with intensive parameter fluctuations (e.g. $\beta = (kT)^{-1}$)
- contains Tsallis statistics as a special case, but other statistics possible as well
- $q = \frac{\langle \beta^2 \rangle}{\langle \beta \rangle^2}$ can be formally defined for any superstatistics
- β sharply peaked (q close to 1) $\Rightarrow$ all superstatistics similar to Tsallis statistics
- Very good agreement with experimental data from
  - hydrodynamic turbulence
  - defect turbulence (Raleigh Benard)
  - cosmic ray statistics

Parameter predictions possible based on simple model assumptions