- Stochastic Fisher-Kolmogorov-Petrovsky-Piscunov Equation

$$u_t = Du_{xx} + \gamma u(1-u) + \epsilon\sqrt{u(1-u)}\,\eta(x,t)$$

Space-time white noise

- Interacting Particle Systems

$A \rightleftharpoons A+A$ reactions & diffusion

- Duality

(an exact connection)

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Interacting particles, the stochastic Fisher–Kolmogorov–Petrovsky–Piscounov equation, and duality

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Abstract

The stochastic Fisher–Kolmogorov–Petrovsky–Piscunov equation is

$$\partial_t U(x,t) = D\partial_{xx}U + \gamma U(1-U) + \varepsilon\sqrt{U(1-U)}\,\eta(x,t)$$

for $0 \leqslant U \leqslant 1$ where $\eta(x,t)$ is a Gaussian white noise process in space and time. Here D, γ and ε are parameters and the equation is interpreted as the continuum limit of a spatially discretized set of Itô equations. Solutions of this stochastic partial differential equation have an exact connection to the $A \rightleftharpoons A+A$ reaction–diffusion system at appropriate values of the rate coefficients and particles' diffusion constant. This relationship is called "duality" by the probabilists; it is *not* via some hydrodynamic description of the interacting particle system. In this paper we present a complete derivation of the duality relationship and use it to deduce some properties of solutions to the stochastic Fisher–Kolmogorov–Petrovsky–Piscunov equation.

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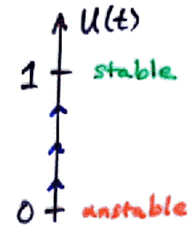
Keywords: FKPP equation; Stochastic pde; Nonlinear wavefronts

Logistic dynamics (growth & saturation)

$$\frac{d}{dt}\, u(t) = \gamma\, u(1-u)$$

growth rate

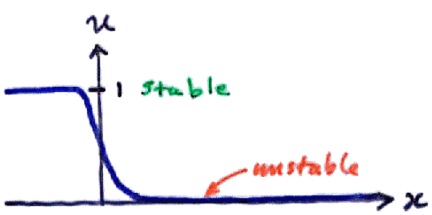


Fisher (1937); Kolmogorov, Petrovsky & Piscunov (1937):

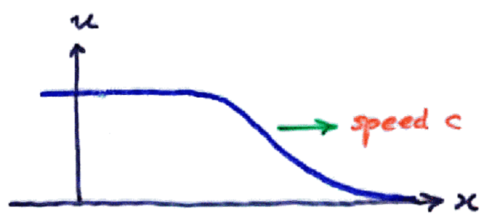
(add spatial dependence)

$$\frac{\partial u}{\partial t}(x,t) = D\frac{\partial^2 u}{\partial x^2} + \gamma\, u(1-u)$$

Wavefront solutions



$t = 0$



$t \gg \frac{1}{\gamma}$

Velocity selection

see: Ebert & van Saarloos Physic D (2000)

$$u(x,t) = F(x - ct)$$

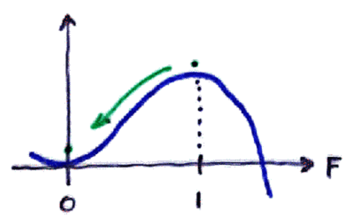
$$\Downarrow$$

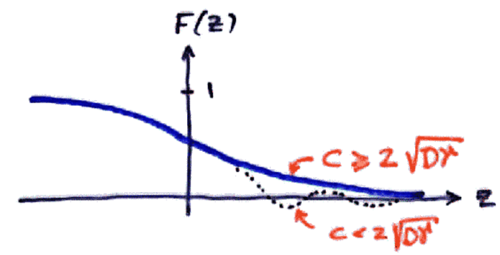
$$0 = DF''(z) + cF'(z) + \gamma F(z)(1-F(z))$$

$$\lim_{z\to-\infty} F(z) = 1 \qquad \lim_{z\to+\infty} F(z) = 0$$

Mechanical analog: c is friction coefficient for motion in potential $V(F)$

$$V(F) \sim \tfrac{1}{2}F^2 - \tfrac{1}{3}F^3$$





Minimum velocity determined by "leading tail"

"Weak velocity selection" speed $\sim c_{min} + O(\frac{1}{t})$

PROBLEM: Front speed determined by asymptotic structure where the model is most unstable — dynamically and structurally!

A problem with logistic dynamics as a generic model ...

Microscopic processes $\Longrightarrow$ Logistic kinetics with noise

Example:

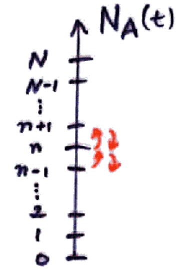
$$\left.\begin{array}{l} A+B \xrightarrow{k_1} A+A \\ A+B \xrightarrow[k_2]{} B+B \end{array}\right\}$$ in a well-stirred reactor of volume Ω.

$N_A(t)$ = # of A particles
$N_B(t)$ = # of B particles

$N = N_A(t) + N_B(t)$ Conserved by the reactions.

Natural logistic variable: $u(t) = \dfrac{N_A(t)}{N} \in [0,1]$.

Markov process description:



$$p_n(t) = \text{Prob}\{N_A(t) = n\}$$

Master equation:

$$\frac{dp_n}{dt} = -\frac{k_1}{\Omega} n(N-n)p_n + \frac{k_1}{\Omega}(n-1)(N-n+1)p_{n-1} - \frac{k_2}{\Omega} n(N-n)p_n + \frac{k_2}{\Omega}(n+1)(N-n-1)p_{n+1} \qquad \begin{pmatrix} p_{-1}=0 \\ p_{N+1}=0 \end{pmatrix}$$

Diffusion approximation: $u = \dfrac{n}{N}$; $\Delta u = \dfrac{1}{N}$

$$f(u,t) = N p_{Nu}(t) \approx \text{probability density for } u,$$

Leading orders in continuum $(N\to\infty)$ appx:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial u}\left[-\frac{(k_1-k_2)N}{\Omega} \cdot u(1-u) + \frac{1}{2}\frac{k_1+k_2}{\Omega} \cdot \frac{\partial}{\partial u} u(1-u) \right] f(u,t)$$

Fokker-Planck equation $\begin{pmatrix} f(0,t)=0 \\ f(1,t)=0 \end{pmatrix}$

Fokker-Planck equation $\Rightarrow$

Stochastic (Itô) equation

$$dU(t) = \gamma U(1-U)\,dt + \sigma\sqrt{U(1-U)}\,dW$$

growth rate: $\gamma = (k_1 - k_2)\frac{N}{\Omega}$

noise amplitude: $\sigma = \sqrt{(k_1+k_2)\frac{N}{\Omega}} \cdot \frac{1}{\sqrt{N}} \sim O\left(\frac{1}{\sqrt{N}}\right)$

with absorbing boundaries @ 0 & 1.

Leading continuum description $(N \to \infty) \Rightarrow$

deterministic logistic dynamics ... $\left(u(0) > 0 \Rightarrow u(t) \xrightarrow[t\to\infty]{} 1\right)$

Fluctuations $\Rightarrow$ possibility of extinction:

$$\mathrm{Prob}\left\{ U(t) \xrightarrow[t\to\infty]{} 0 \,\middle|\, U(0) = u_0 > 0 \right\} = \frac{e^{\frac{2\gamma}{\sigma^2}(1-u_0)} - 1}{e^{\frac{2\gamma}{\sigma^2}} - 1} > 0 \quad \forall u_0 < 1.$$

Stochastic FKPP

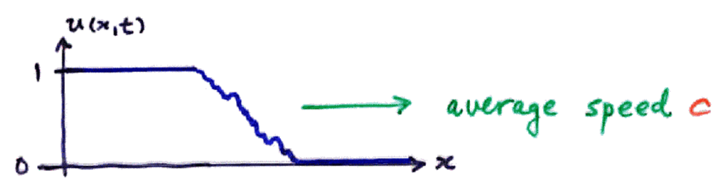
$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \gamma u(1-u) + \epsilon\sqrt{u(1-u)}\,\eta(x,t)$$

$$\langle \eta(x,t)\,\eta(y,s) \rangle = \delta(x-y)\,\delta(t-s)$$

- Itô
- L-ranged contact pr.

"Compact support property"

Theorem (Mueller & Sowers 1995): For $\epsilon > 0$ the wavefront has compact support with probability one.



What is the velocity c in the presence of noise $(\epsilon > 0)$?

Weak noise

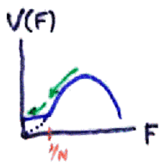
Brunet & Derrida (1997)
Kessler, Ner & Sander (1998)
see also: Pechenik & Levine (1999)

- Primary effect of noise is to extinguish the 'leading tail'.

- Model effect in deterministic FKPP by modifying growth function:

$$\gamma u(1-u) \Rightarrow \gamma u(1-u)\,\Theta\!\left(u-\frac{1}{N}\right)$$

$$= 0 \text{ for } u < \frac{1}{N} \sim \epsilon^2$$



- Matched asymptotic analysis $\Rightarrow$

(unique) $$c \sim \sqrt{D\gamma}\left(2 - \frac{\pi^2}{\left[\ln \frac{(D\gamma)^{1/2}}{\epsilon^2}\right]^2}\right)$$

as $\epsilon \to 0$.

PRELIMINARY ... no spatial dependence

DUALITY between

Stochastic ODE (Ito)

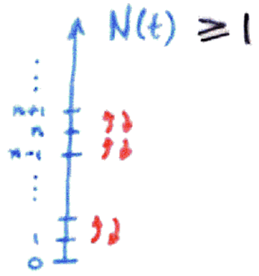
$$du = \gamma u(1-u)\,dt + \sigma\sqrt{u(1-u)}\,dW$$

and

Well-stirred Reaction Process

$$A \underset{\sigma^2}{\overset{\gamma}{\rightleftharpoons}} A + A$$

Reaction kinetics



$$P_n(t) = \text{Prob}\{N(t) = n\}$$

Kinetic coefficients:

$$A \xrightarrow{\gamma} A + A$$

$$A + A \text{ (pair)} \xrightarrow[\sigma^2]{} A$$

Master equation:

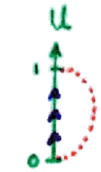
$$\frac{d}{dt} P_n(t) = \gamma(n-1)P_{n-1} - \gamma n P_n - \sigma^2 \frac{n(n-1)}{2} P_n + \sigma^2 \frac{(n+1)n}{2} P_{n+1}$$

$$= \sum_{m=1}^{\infty} M_{nm} P_m(t)$$

$$\boxed{M_{nm} = \gamma m \,\delta_{n,m+1} - \gamma m\, \delta_{n,m} - \frac{\sigma^2}{2} m(m-1)\,\delta_{n,m} + \frac{\sigma^2}{2} m(m-1)\,\delta_{n,m-1}}$$

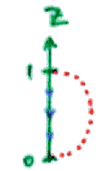
(Ito) Stochastic ODE

$$dU = \gamma U(1-U)\,dt + \sigma\sqrt{U(1-U)}\,dW$$



change variable to $Z(t) = 1 - U(t) \in [0,1]$

$$dZ = -\gamma Z(1-Z)\,dt + \sigma\sqrt{Z(1-Z)}\,dW$$



Ito formula: (for $m \geq 1$)

$$dZ^m = m Z^{m-1} dZ + \tfrac{1}{2} m(m-1) Z^{m-2} (dZ)^2$$

$$= \left(-m\gamma Z^m + m\gamma Z^{m+1} + \frac{\sigma^2}{2} m(m-1) Z^{m-1} - \frac{\sigma^2}{2} m(m-1) Z^m\right) dt + \sigma m Z^{m-1}\sqrt{Z(1-Z)}\,dW$$

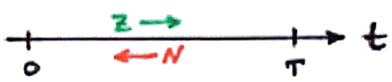
$$= \sum_{n=1}^{\infty} Z(t)^n M_{nm}\,dt + \sigma m Z^{m-1}\sqrt{Z(1-Z)}\,dW$$

$$\therefore \quad \boxed{\frac{d}{dt}\langle Z(t)^m\rangle = \sum_{n=1}^{\infty} \langle Z(t)^n\rangle M_{nm}}$$

DUALITY

Choose $T > 0$ & $0 \le t \le T$:

Define: $\mathcal{M}(t) = \sum_{m=1}^{\infty} Z(t)^m P_m(T-t)$



Claim: $\langle \mathcal{M}(t) \rangle = \langle \mathcal{M}(0) \rangle = \langle \mathcal{M}(T) \rangle$

(Actually, $\mathcal{M}(t)$ is a martingale.)

Proof:

$$\frac{d}{dt}\langle \mathcal{M}(t) \rangle = \sum_{m=1}^{\infty} \left(\frac{d}{dt}\langle Z(t)^m \rangle \right) P_m(T-t) + \langle Z(t)^m \rangle \left(\frac{d}{dt} P_m(T-t) \right)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle Z(t)^n \rangle M_{nm} P_m(T-t) - \sum_{m=1}^{\infty} \langle Z(t)^m \rangle \sum_{n=1}^{\infty} M_{mn} P_n(T-t)$$

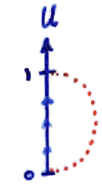
$$= 0.$$

$\therefore$ for any time $T > 0$

$$\boxed{\langle Z(0)^{N(T)} \rangle = \sum_{n=1}^{\infty} \langle Z(0)^n \rangle P_n(T) = \sum_{n=1}^{\infty} \langle Z(T)^n \rangle P_n(0) = \langle Z(T)^{N(0)} \rangle}$$

Application

$$dU = \gamma U(1-U)\,dt + \sigma\sqrt{U(1-U)}\,dW$$



Note: $U(t) \xrightarrow[t\to\infty]{} \begin{cases} 0 & \text{"extinction" with prob } p_{ext} \\ 1 & \text{"saturation" with prob } 1-p_{ext} \end{cases}$

$$\boxed{\text{What is } p_{ext}\,?}$$

Note: $N(t) \xrightarrow[t\to\infty]{}$ equilibrium with (def: $g = \frac{2\gamma}{\sigma^2}$)

$$P_n^{eq} = \frac{g^n}{n!}\,\frac{1}{e^g - 1} \quad ; \quad n = 1, 2, 3, \ldots$$

Choose: $P_n(0) = P_n(T) = P_n^{eq}$ Recall: $Z = 1 - U$

$$\langle Z(0)^{N(T)} \rangle = \left\langle \sum_{n=1}^{\infty} Z(0)^n P_n^{eq} \right\rangle = \left\langle \frac{e^{g(1-U(0))} - 1}{e^g - 1} \right\rangle$$

$$\parallel$$

$$\langle Z(T)^{N(0)} \rangle = \left\langle \sum_{n=1}^{\infty} Z(T)^n P_n^{eq} \right\rangle = \left\langle \frac{e^{g(1-U(T))} - 1}{e^g - 1} \right\rangle$$

$$\left\langle \frac{e^{\gamma(1-U(0))}-1}{e^{\gamma}-1} \right\rangle = \langle V(T) \rangle$$

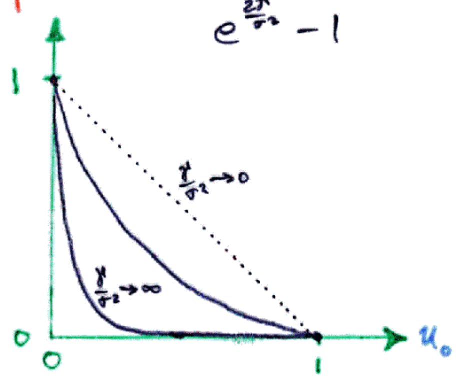
$$V(T) = \frac{e^{\gamma(1-U(T))}-1}{e^{\gamma}-1} \xrightarrow[T\to\infty]{} \begin{cases} 1 & \text{with prob } p_{ext} \\ 0 & \text{with prob } 1-p_{ext} \end{cases}$$

$\therefore$

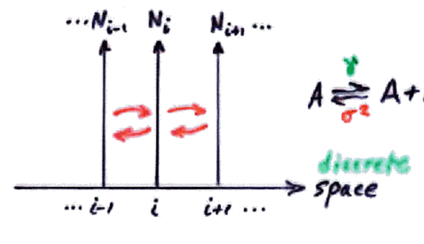
$$p_{ext} = \left\langle \frac{e^{\gamma(1-U(0))}-1}{e^{\gamma}-1} \right\rangle$$

If $U(0) = u_0$ (nonrandom) then

$$p_{ext} = \frac{e^{\frac{2\gamma}{\sigma^2}(1-u_0)}-1}{e^{\frac{2\gamma}{\sigma^2}}-1}$$



Stochastic FKPP (pde) and Diffusion-Reaction Process:



particles hop between neighboring sites at rate $\frac{D}{h^2}$

$$\underline{N}(t) = (\dots N_{i-1}(t), N_i(t), N_{i+1}(t) \dots) \qquad 0 \le N_j < \infty$$

$$\underline{n} = (\dots n_{i-1}, n_i, n_{i+1} \dots) \qquad 0 \le n_j < \infty$$

$$\underline{e}_i = (\dots 0, \underset{i^{th}\text{ slot}}{1}, 0 \dots)$$

$$P_{\underline{n}}(t) = \text{Prob}\{\underline{N}(t) = \underline{n}\}$$

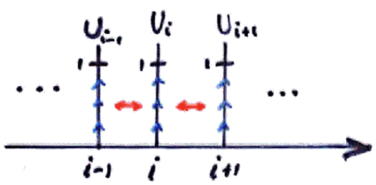
Fact: $$\frac{d}{dt} P_{\underline{n}}(t) = \sum_{\underline{m}} M_{\underline{n},\underline{m}} P_{\underline{m}}(t)$$

$$M_{\underline{n},\underline{m}} = \sum_i \Big[\gamma m_i \delta_{\underline{n},\underline{m}+\underline{e}_i} - \gamma m_i \delta_{\underline{n},\underline{m}} - \frac{\sigma^2}{2} m_i(m_i-1)\delta_{\underline{n},\underline{m}} + \frac{\sigma^2}{2} m_i(m_i-1)\delta_{\underline{n},\underline{m}-\underline{e}_i}$$
$$+ \frac{D}{h^2} m_{i-1} \delta_{\underline{n},\underline{m}+\underline{e}_i-\underline{e}_{i-1}} - \frac{2D}{h^2} m_i \delta_{\underline{n},\underline{m}} + \frac{D}{h^2} m_{i+1} \delta_{\underline{n},\underline{m}+\underline{e}_i-\underline{e}_{i+1}} \Big]$$

Discrete Stochastic FKPP (Sde)

$$dU_i = \left(\frac{D}{h^2}\left[U_{i+1}-2U_i+U_{i-1}\right] + \gamma U_i(1-U_i)\right)dt$$
$$+ \sigma\sqrt{U_i(1-U_i)}\; dW_i$$

independent at each site



- Let $Z_i(t) = 1 - U_i(t)$ and $\underline{m} = (\dots m_{j-1}, m_j, m_{j+1} \dots)$

- Then Ito (straightforward but tedious) $\Longrightarrow$

$$d\left(\prod_j Z_j(t)^{m_j}\right) = \sum_{\underline{n}} \left(\prod_i Z_i^{n_i}\right) M_{\underline{n},\underline{m}}\, dt$$
$$+ \sum_i \sigma m_i \sqrt{Z_i(1-Z_i)} \prod_j Z_j^{m_j-\delta_{ij}}\, dW_i$$

$$\therefore \quad \boxed{\frac{d}{dt}\left\langle \prod_j Z_j^{m_j}\right\rangle = \sum_{\underline{n}} \left\langle \prod_i Z_i^{n_i}\right\rangle M_{\underline{n},\underline{m}}}$$

DUALITY

(T. Shiga & K. Uchiyama, P.T.&R.F. 1986)

- Let $T > 0$ and $0 \le t \le T$.

- Define $m(t) = \sum_{\underline{n}} \left(\prod_i Z_i(t)^{n_i}\right) P_{\underline{n}}(T-t)$

- Then $\langle m(t)\rangle = \langle m(0)\rangle = \langle m(T)\rangle$

- Proof: direct computation as before

$\therefore$ The duality relation between solution of

$$dU_i = \left[\frac{D}{h^2}(U_{i+1}-2U_i+U_{i-1}) + \gamma U_i(1-U_i)\right]dt + \sigma\sqrt{U_i(1-U_i)}\; dW_i$$

and

$$A \underset{\sigma^2}{\overset{\gamma}{\rightleftharpoons}} A + A \quad \text{with hop rate } \frac{D}{h^2}$$

is, for any time $T \ge 0$,

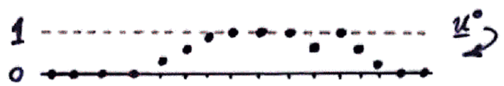
$$\boxed{\left\langle \prod_i (1-U_i(0))^{N_i(T)} \right\rangle = \left\langle \prod_i (1-U_i(T))^{N_i(0)} \right\rangle}$$

APPLICATION

$\underline{U}(t) = (\dots U_{i-1}(t), U_i(t), U_{i+1}(t), \dots)$

$$\underline{U}(t) \xrightarrow[t\to\infty]{} \begin{cases} \underline{0} & \text{"extinction" with prob } p_{ext} \\ \cancel{\underline{0}} & \text{"persistence" with prob } 1 - p_{ext} \end{cases}$$

Choose initial condition $\underline{U}(0) = \underline{u}^\circ$ (nonrandom)



$\boxed{\text{What is } p_{ext}\{\underline{u}^\circ\}\text{?}}$

• Note: $\underline{N}(t) \xrightarrow[t\to\infty]{}$ equilibrium

$$P^{eq}_{\underline{n}} = \prod_i \frac{g^{n_i}}{n_i!} e^{-g} \quad \text{with} \quad g = \frac{2\gamma}{\sigma^2}$$

Use duality relation with $\underline{N}(0) \sim \underline{N}(T) \sim eq.$

$$\left\langle \prod_i (1-U_i(0))^{N_i(T)} \right\rangle = \left\langle \prod_i (1-U_i(T))^{N_i(0)} \right\rangle$$

Left side:

$$= \sum_{\underline{n}} \prod_i (1-u_i^\circ)^{n_i} P^{eq}_{\underline{n}}$$

$$= \prod_i \sum_{n=0}^{\infty} \frac{(1-u_i^\circ)^n g^n}{n!} e^{-g}$$

$$= \prod_i e^{-g u_i^\circ}$$

$$= \exp\left\{-\sum_i \frac{2\gamma}{\sigma^2} u_i^\circ\right\}$$

Right side:

$$= \left\langle \sum_{\underline{n}} \prod_i (1-U_i(T))^{n_i} P^{eq}_{\underline{n}} \right\rangle$$

$$= \left\langle \prod_i e^{-g U_i(T)} \right\rangle$$

$$= \left\langle \exp\left\{-\sum_i \frac{2\gamma}{\sigma^2} U_i(T)\right\} \right\rangle$$

$$\Big\downarrow \; T \to \infty$$

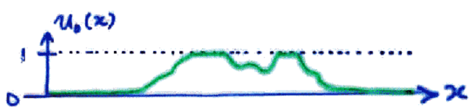
$$p_{ext}$$

In the continuum ...

$$u_t = D u_{xx} + \gamma u(1-u) + \epsilon\sqrt{u(1-u)}\,\eta(x,t)$$

noise: $\langle \eta(x,t)\eta(y,s)\rangle = \delta(x-y)\delta(t-s)$

$$u(x,0) = u_0(x)$$



Correspondence with discrete version: $(h \to 0)$

$$\epsilon = \sigma\sqrt{h} \iff \sigma^2 = \frac{\epsilon^2}{h}$$

$\therefore$

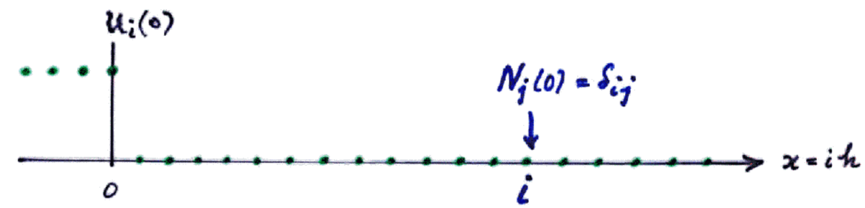
$$P_{ext} = \exp\left\{-\frac{2\gamma}{\sigma^2}\sum_i u_i^0\right\}$$

$$= \exp\left\{-\frac{2\gamma}{\epsilon^2}\int_{-\infty}^{\infty} u_0(x)\,dx\right\}$$

Duality establishes equivalence of front speeds for S-FKPP and reaction-diffusion system

$A \underset{\sigma^2}{\overset{\gamma}{\rightleftharpoons}} A + A$ with particle hop rate $\frac{D}{h^2}$

Choose initial data:



Duality $\Rightarrow$

$$\langle u_i(t)\rangle = 1 - \left\langle \prod_j \left(1 - u_j(0)\right)^{N_j(t)} \right\rangle$$

$$= \text{Prob}\{\text{there is any particle at site } j \le 0 \text{ at time } t\}$$

Starting from one particle @ site i.

Strong noise limit

Strong noise $\sigma \to \infty$ or $\epsilon \to \infty$ of S-FKPP is dual to the diffusion-controlled $A \underset{\sigma^2}{\overset{\gamma}{\rightleftarrows}} A+A$ reaction!

- Exact solution for diffusion-Limited $A \rightleftarrows A+A$ (Doering, Burshka, Horsthemke '92; ben-Avraham '98):

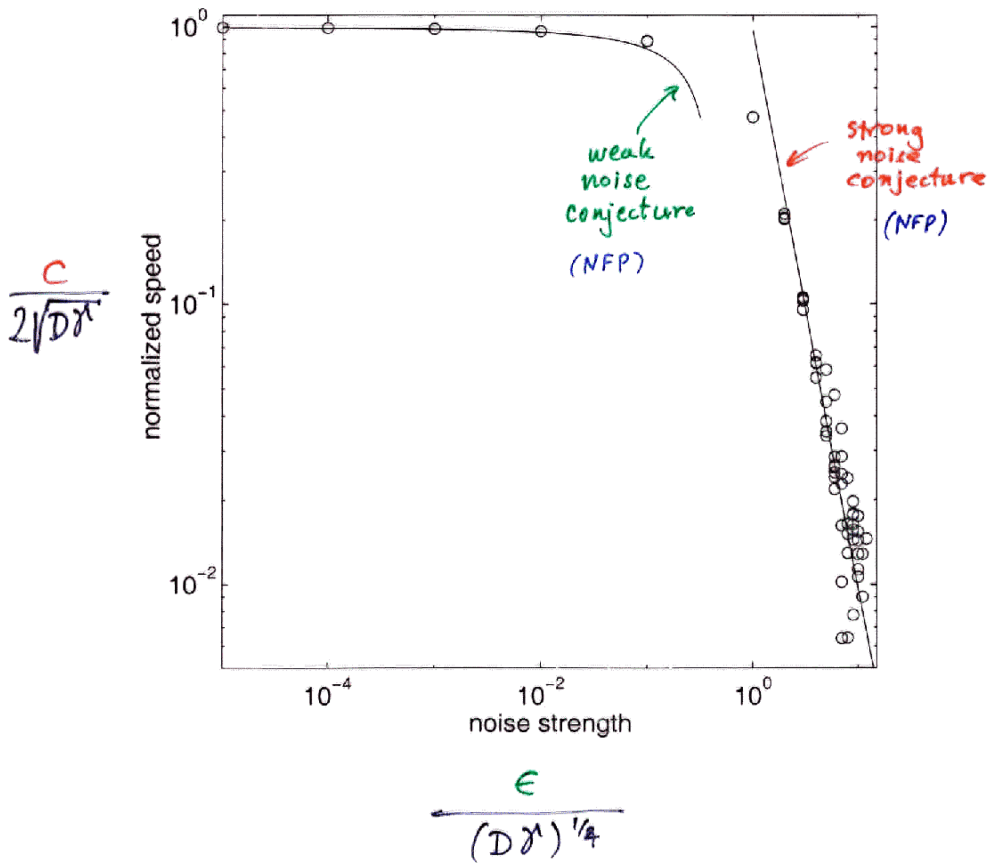
$$c = D\rho_{eq}$$

- Equilibrium density for $A \underset{\sigma^2}{\overset{\gamma}{\rightleftarrows}} A+A$:

$$\rho_{eq} = \frac{2\gamma}{h\sigma^2} = \frac{2\gamma}{\epsilon^2}$$

- Conjecture: $c \sim \frac{2D\gamma}{\epsilon^2}$ as $\epsilon \nearrow \infty$.

Direct numerical simulation of Stochastic FKPP (P. Smereka, 2002):



Conclusions:

- Fluctuations can have profound effects on the qualitative behavior of nonlinear systems.

- Duality is a useful mathematical tool for the analysis of stochastic (partial) differential equations.

Open challenges:

- Prove the weak-noise asymptotic formula for front velocity.

- Prove the strong-noise limit... is there a bridging formula?

- What is the "shape" of the stochastic wavefront?

- What is the front position's diffusion coefficient?