Soret-driven thermal instability in ferrofluid binary mixtures

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http://www.mpip-mainz.mpg.de/~pleiner/instab.html

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ferrofluids/-gels: . . . . /ferro.html

rheology: . . . . /rheo.html

liquid crystals: . . . . /banana.html

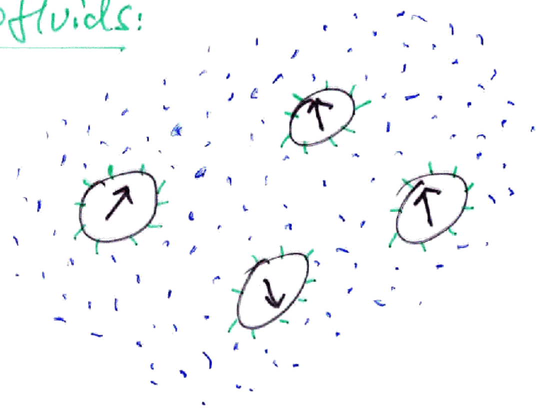
. . . . /lcpe.html

1) Introduction – ferrofluids
   – binary mixtures

2) Thermal instability without magnetic field
   – threshold
   – linear growth rate
   – nonlinear amplitude

3) Thermal instability with magnetic field
   – basic equations
   – numerical solutions
   – analytical
   – boundary layers

Ferrofluids:



magnetic particles (10 nm) in carrier fluid

sterically or charge stabilized

superparamagnetic: high M_s for low H_{ext}

black

applications: seal in hard drives
active damping
cancer therapy

science: new types of instabilities / pattern formation

developments: ferro-nematics
ferro-gels

thermal instability:

single fluid: T_0

$T_1 > T_0$

$\downarrow$ gravity (g_E)

$$Ra \sim \frac{\alpha_\theta \beta g_E}{\kappa \nu}$$

α_θ thermal expans.
β temp gradient
κ heat conduct.
ν viscosity

$Ra \geq Ra_c$ convection, pattern formation

binary fluid: 2 conserved masses, 1 velocity

concentration current $\vec{j} = \underbrace{D \vec{\nabla} c}_{\text{diffusion}} + \underbrace{D_T \vec{\nabla} T}_{\text{thermal diffusion (Soret)}}$

solutal expansion $\delta\rho / \rho_0 = \alpha_c \delta c - \alpha_\theta \delta T$

Lewis number $L \sim \frac{D}{\kappa} \ll 1$

separation ratio $\psi \sim \frac{\alpha_c}{\alpha_\theta} \frac{D_T}{D} \gg 1$

II) Thermal Instability - no field

$$\operatorname{div}\vec{v} = 0$$

$$\partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + Pr\,\Delta\vec{v} + Pr\,Ra\,[\delta T - \psi\,\delta c]\,\vec{e}_z$$

$$\partial_t T + (\vec{v}\cdot\vec{\nabla})T = \Delta T$$

$$\partial_t c + (\vec{v}\cdot\vec{\nabla})c = L(\Delta c + \Delta T)$$

$$\vec{v}\,\big|_b = 0$$

$$T\,\big|_b = \overline{T} \mp \tfrac{1}{2}$$

$$\nabla_z c + \nabla_z T\,\big|_b = 0$$

pure conduction, no convection: $\vec{v} = 0$

$$T_{cond} = \overline{T} - z$$

$$C_{cond} = \overline{C} + z$$

however: this state is never reached!

$L \ll 1$!

instead: $C = C_0$ good for $t \ll L^{-1}$ outside boundary layer

linear stability analysis, growth rate $\lambda = 0$

$$\hookrightarrow Ra_c = \frac{1}{1+\psi}\,Ra_c^0$$

$$\ll Ra_c^0$$

however: instability unobservable near Ra_c

$\lambda^{-1}(Ra) \ll L^{-1}$ required!

$\lambda \gg L$

$\Rightarrow$ calculate linear growth rate $\lambda\,(Ra)$

calculate nonlinear (velocity) saturation amplitude linear growthrate:

deviations:

$$W = A(t) \cos kx \cos^2 \pi z$$
$$\theta = B(t) \cos kx \cos \pi z$$
$$C = -\theta + \cos kx \sum_{n=0}^{1} b_n(t) \cos 2\pi n z$$

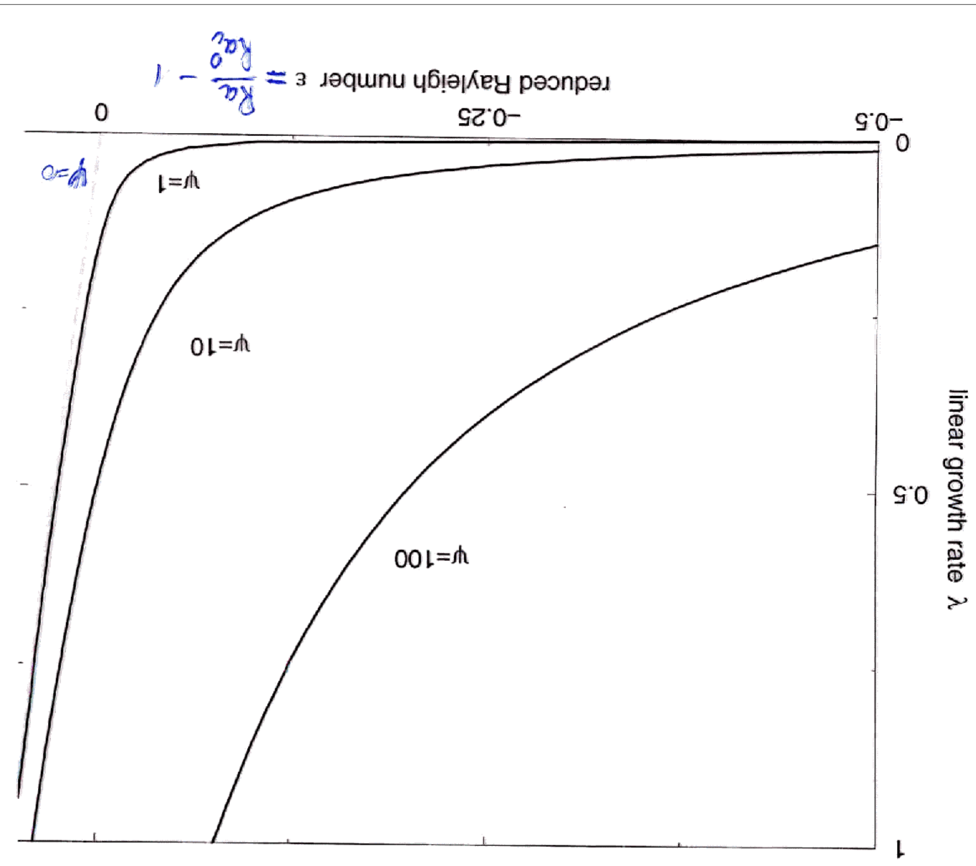
boundary conditions, symmetries ok.

Galerkin

approximate analytical solution $(\lambda \gg L, \psi \gg 1)$

$$3\, Ra\, Pr\, (\lambda + 2\pi^2 L \psi) = \lambda (2\pi^2 + \lambda)(27\pi^2 Pr + 7\lambda)$$

$$(k \approx \pi)$$



saturation amplitude:

$$C(x,z,t) = C_0(z,t) + c_1(z,t)\cos kx$$

$$T(x,z,t) = \theta_0(z,t) + \theta_1(z,t)\cos kx$$

$$v_z(x,z,t) = w_1(z,t)\cos kx$$

with $w_1(z,t) = A(t)\cos^2 \pi z$

$$\theta_1(z,t) = B(t)\cos \pi z$$

$$\theta_0(z,t) = g(t)\sin 2\pi z$$

$$C_0(z,t) = z - \theta_0(z,t) + \sum_{n=0}^{N} a_n(t)\sin(2n+1)\pi z$$

$$c_1(z,t) = -\theta_1(z,t) + \sum_{n=0}^{N} b_n(t)\cos 2n\pi z$$

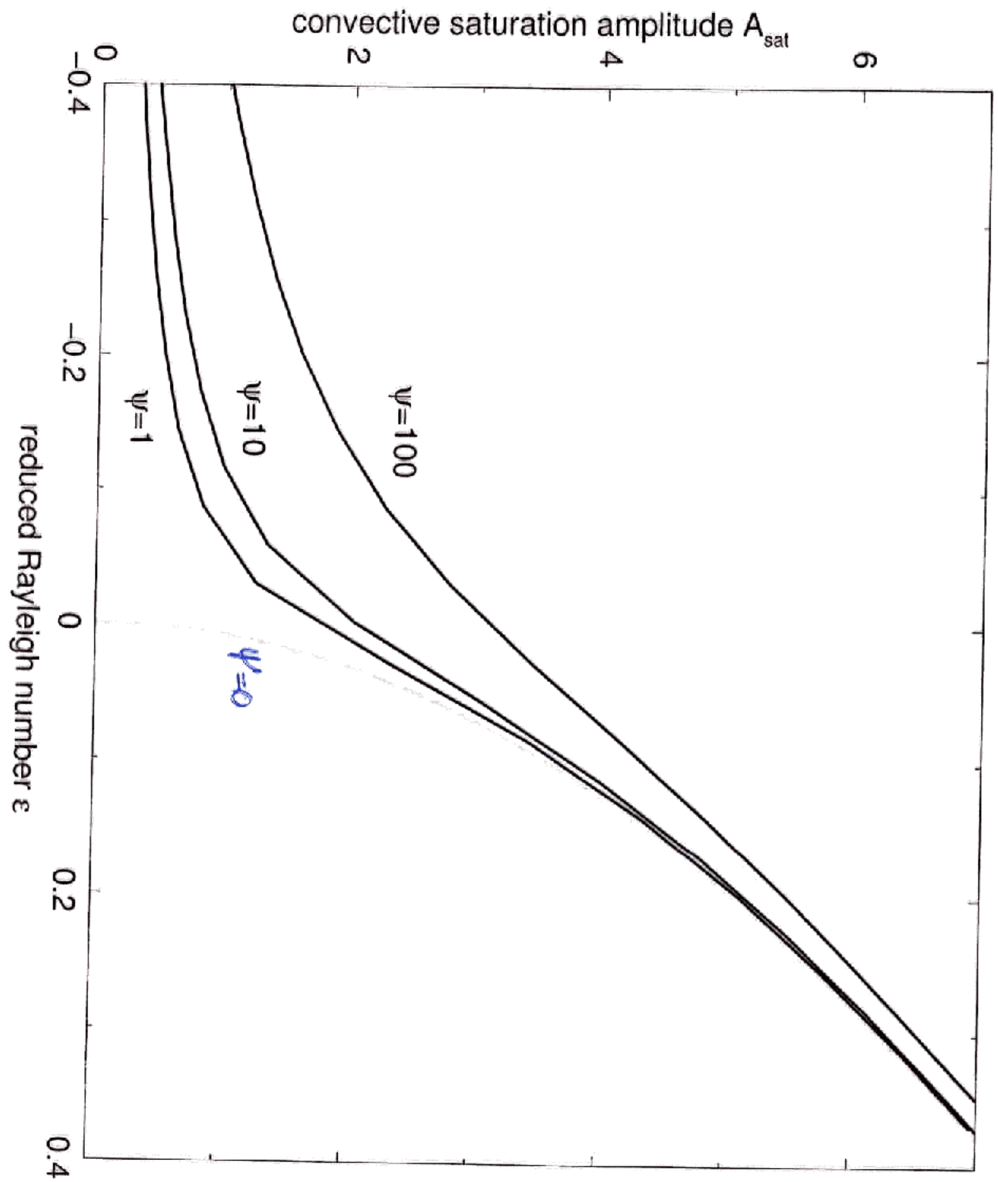
b.c. ok

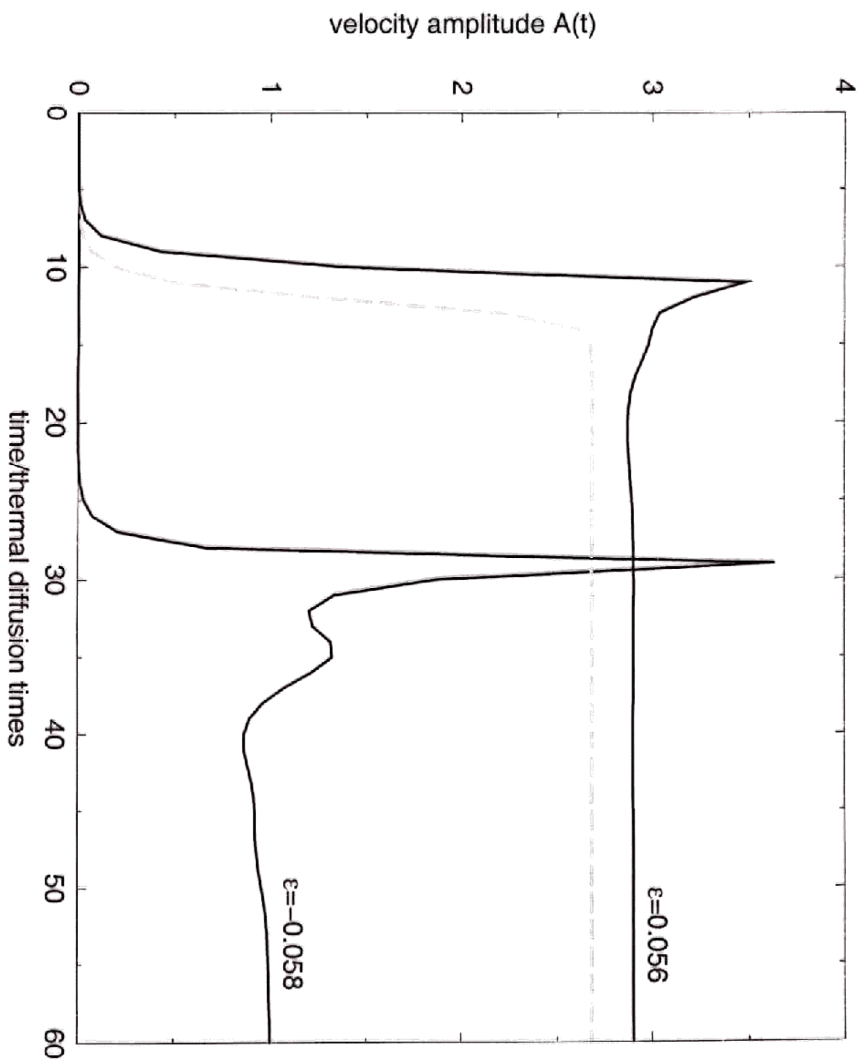
for $\psi = 0$ $\leftrightarrow$ 3 mode Lorenz model

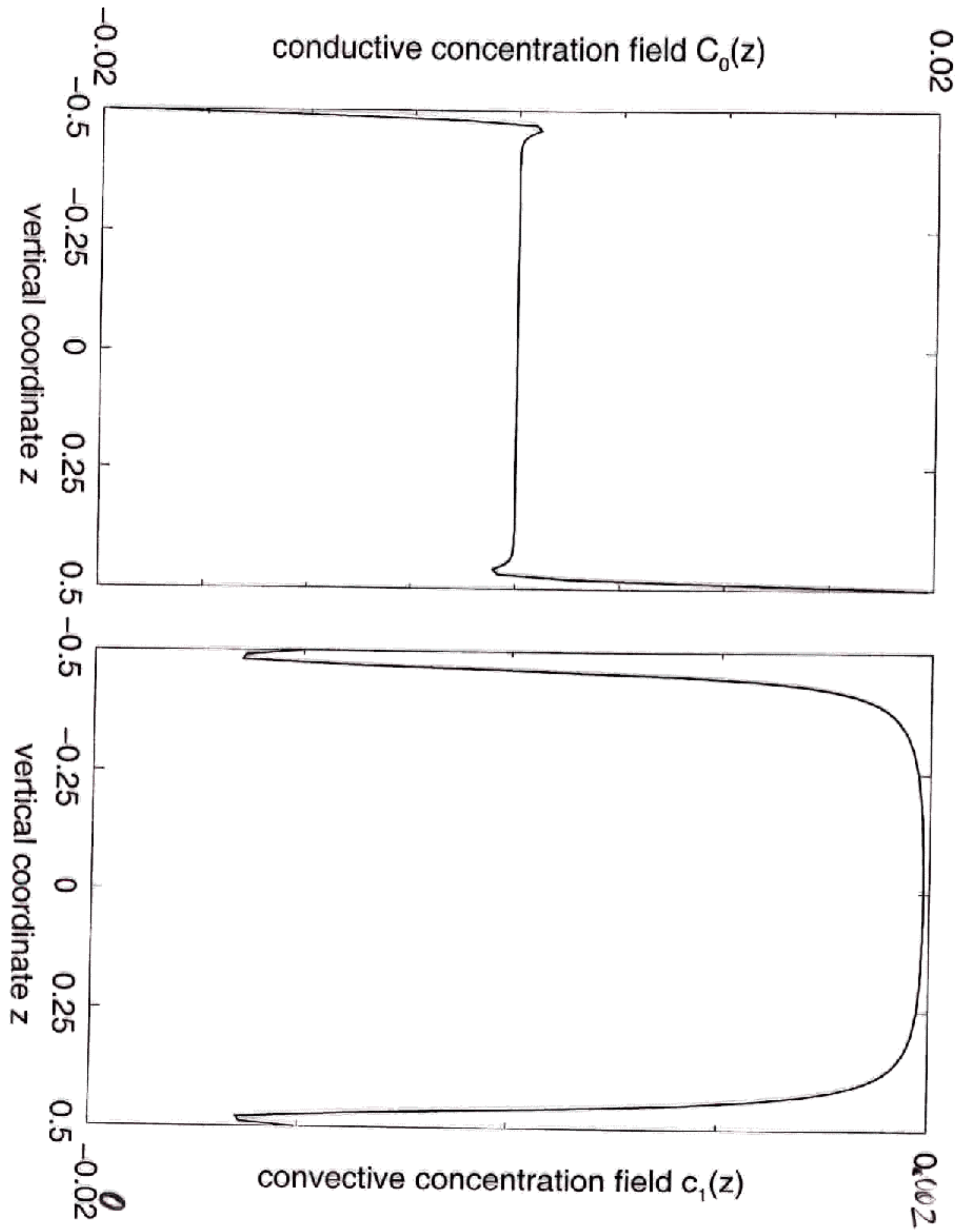
$N \approx 20$ (concentration boundary layer)

$k = 3 \ldots 3.5$ (π!)

no oscillatory inst. found




velocity amplitude A(t)
time/thermal diffusion times
ε=0.056
ε=−0.058


conductive concentration field $C_0(z)$
vertical coordinate z
convective concentration field $c_1(z)$
vertical coordinate z

Summary part I: there is a forward bifurcation into rolls also for $Ra \lesssim Ra_c^0$, but unobservable for $Ra_c \lesssim Ra \lesssim Ra_c^0$

(exp. S. Odenbach [Bremen])

II. Thermal instability with magnetic field:

field effects:

Maxwell eqs. $\mathrm{div}\,\vec{B} = 0$, $\mathrm{curl}\,\vec{H} = 0$, + b.c. ✓

Kelvin force $M_j \vec{\nabla} H_j$ in Navier Stokes ✓

magnetoforesis $\delta\vec{B} = \varepsilon(H)\,\delta\vec{H} + H_0\,[\chi_T\,\delta T + \chi_c\,\delta c]$

$$\delta\sigma = \ldots\ldots + \chi_T\,\vec{H}_0\cdot\delta\vec{H}$$
$$\delta\mu_c = \ldots\ldots + \chi_c\,\vec{H}_0\cdot\delta\vec{H} \qquad (✓)$$

volume expansion $\delta\rho/\rho = \ldots\ldots\ \alpha_H\,\vec{H}\cdot\delta\vec{H}$ —

Hall-like effects: $D_{ij} = D\,\delta_{ij} + D^R\,\varepsilon_{ijk}\,H_k$ —

ground state:

$$\vec{v} = 0$$
$$T = T_0 - \beta z$$
$$C = C_0 \quad \longleftarrow \text{flat concentration profile}$$
$$H_z = H_0\left(1 + \frac{\chi_T}{\varepsilon}\beta z\right)$$

deviations: $\vec{v}$, θ, c, $\vec{H} = H_z\vec{e}_z - \vec{\nabla}\phi$; $\vec{H}_{ext} = \vec{H}_0 - \vec{\nabla}\phi_e$

Simple rolls $\sim \cos kx$; $M_4 = M_5 = F = 0$

sorting for different lateral dependencies yields the following system of equations

$$
\begin{aligned}
\frac{1}{Pr}\partial_t\left(\nabla_z^2-k^2\right)w_1 &= -Ra\,k^2\left[(1+M_1)\theta_1-(\psi+M_1\psi_m)c_1-M_1\nabla_z\phi_1\right] \\
&\quad +Ra\,M_1k^2\left(\theta_1-\psi_m c_1-\nabla_z\phi_1\right)\nabla_z\left(\theta_0-\psi_m c_0\right)+\left(\nabla_z^2-k^2\right)^2 w_1, && (3.58)\\
\partial_t c_0+\tfrac{1}{2}\nabla_z\left(w_1c_1\right) &= L\nabla_z^2\left[(1+M_2\psi_m)c_0+(1-M_2)\theta_0\right], && (3.59)\\
\partial_t c_1+w_1\nabla_z c_0 &= L\left(\nabla_z^2-k^2\right)\left(c_1+\theta_1-M_2\nabla_z\phi_1\right), && (3.60)\\
\partial_t\theta_0+\tfrac{1}{2}\nabla_z\left(w_1\theta_1\right) &= \nabla_z^2\theta_0, && (3.61)\\
\partial_t\theta_1+w_1\nabla_z\theta_0 &= -w_1+\left(\nabla_z^2-k^2\right)\theta_1, && (3.62)\\
\left(\nabla_z^2-M_3k^2\right)\phi_1 &= \nabla_z\left(\theta_1-\psi_m c_1\right), && (3.63)
\end{aligned}
$$

The field ϕ_0 has already been eliminated with the help of $\nabla_z^2\phi_0 = \nabla_z(\theta_0-\psi_m c_0)$. This has also been used to write the remaining boundary conditions as

$$
\begin{aligned}
\nabla_z(c_1+\theta_1-M_2\nabla_z\phi_1)]_{z=\pm1/2} &= 0, && (3.64)\\
\nabla_z((1+M_2\psi_m)c_0+(1-M_2)\theta_0)]_{z=\pm1/2} &= 1-M_2, && (3.65)\\
\theta_1]_{z=\pm1/2} = \theta_0]_{z=\pm1/2} &= 0, && (3.66)\\
w_1]_{z=\pm1/2} = \nabla_z w_1]_{z=\pm1/2} &= 0, && (3.67)
\end{aligned}
$$

To solve this boundary-value problem we adopt vertical profiles w_1, θ_0, θ_1, c_0, c_1 and ϕ_1 in the form

$$
\begin{aligned}
w_1(z,t) &= A(t)\cos^2(\pi z), && (3.68)\\
\theta_1(z,t) &= B(t)\cos\pi z, && (3.69)\\
\theta_0(z,t) &= G(t)\sin 2\pi z, && (3.70)\\
c_0(z,t) &= \frac{1-M_2}{1+\psi_m M_2}(z-\theta_0(z,t))+\sum_{n=0}^{n=N}a_n(t)\sin(2n+1)\pi z, && (3.71)\\
c_1(z,t) &= -\theta_1(z,t)+\sum_{n=0}^{n=N}b_n(t)\cos 2n\pi z, && (3.72)\\
\phi_1(z,t) &= A_0(t)\,z+\sum_{n=0}^{n=N_1}\frac{A_n(t)\sin 2\pi nz}{2\pi n} && (3.73)
\end{aligned}
$$

multi-mode boundary layer

which satisfy the boundary conditions (3.56,3.64-3.67) identically, if $A_0(2+k)+\sum_{n=1}^{N_1}(-)^nA_n+\psi_m\sum_{n=1}^{N}(-)^nb_n = 0$ is chosen.

temperature from Eq. (3.36) is θ. Then Eqs. (3.5), and (3.22)-(3.25) lead to

$$
\begin{aligned}
\nabla\cdot\mathbf{v} &= 0 && (3.39)\\
\left[\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right](\theta-M_4\nabla_z\phi) &= w(1-M_4)+\Delta\theta+F\Delta(C-M_2\nabla_z\phi) && (3.40)\\
\left[\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right]C &= L\Delta(\theta+C-M_2\nabla_z\phi) && (3.41)\\
\frac{1}{Pr}\left[\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right](curl\mathbf{v})_i &= Ra\,\epsilon_{ikz}\nabla_k\left[(1+M_1)\theta-(\psi+\psi_mM_1)C+(M_5-M_1)\nabla_z\phi\right]\\
&\quad -Ra\,M_1\,\epsilon_{ikl}(\nabla_l\nabla_z\phi)\nabla_k(\theta-\psi_mC)+\Delta(curl\mathbf{v})_i && (3.42)\\
(\nabla_z^2+M_3\Delta_\perp)\phi &= \nabla_z(\theta-\psi_mC) && (3.43)\\
\Delta\phi_e &= 0 && (3.44)
\end{aligned}
$$

where w is the z component of the velocity. The transverse Laplacean $\Delta_\perp = \Delta-\nabla_z^2$. The non-dimensional parameters introduced here are: the Rayleigh number $Ra = \alpha_\theta\beta_0 g_E d^4/(\kappa\nu)$, the Prandtl number $Pr = \nu/\kappa$, the separation ratio $\psi = \alpha_c\rho_0\bar{D}_T/(\alpha_\theta\gamma_H D)$, the magnetic separation ratio $\psi_m = -\chi_c\rho_0\bar{D}_T/(\chi_T\gamma_H D)$, the Lewis number $L = \gamma_H c_H D/(\rho_0^2\bar{\kappa}T_0) = D_c/\kappa$, the strength of magnetic force relative to buoyancy $M_1 = \beta_0\chi_T^2H_0^2/(\rho_0 g_E\alpha_\theta\bar{\epsilon})$, the magnetophoretic number $M_2 = D\chi_c\chi_T H_0^2/(\rho_0\bar{D}_T\bar{\epsilon})$, the nonlinearity of magnetization $M_3 = (1+\chi_0)/\bar{\epsilon} \approx 1-\chi_H H_0^2/(1+\chi_0)$, the relative strength of the temperature dependence of the magnetic susceptibility $M_4 = \chi_T^2H_0^2T_0/(c_H\bar{\epsilon})$, the ratio of magnetic to thermal buoyancy $M_5 = \alpha_H\chi_T H_0^2/(\alpha_\theta\bar{\epsilon})$, and the Dufuor number $F = \bar{D}_T^2/(D\bar{\kappa}) = D_sD_f/(\kappa D_c)$. The stability conditions (3.20) require $M_4 < 1$ and $M_2\psi_m > -1$.

5 M's $\sim H_0^2$

According to our choice of 'rigid' and ideally conducting boundaries, the boundary conditions for the deviations from the conducting state read

$$
\begin{aligned}
\theta]_{z=\pm\frac{1}{2}} &= 0 && (3.45)\\
w]_{z=\pm\frac{1}{2}} &= 0 && (3.46)\\
\nabla_z w]_{z=\pm\frac{1}{2}} &= 0 && (3.47)\\
\nabla_z(\theta+C-M_2\nabla_z\phi)]_{z=\pm\frac{1}{2}} &= 1-M_2 && (3.48)
\end{aligned}
$$

and the magnetic boundary conditions (3.28), (3.29) are

$$
\begin{aligned}
\bar{\epsilon}(\nabla_z\phi+\psi_mC)-\nabla_z\phi_e]_{z=\pm\frac{1}{2}} &= 0 && (3.49)\\
\nabla_\perp\phi-\nabla_\perp\phi_e]_{z=\pm\frac{1}{2}} &= 0 && (3.50)
\end{aligned}
$$

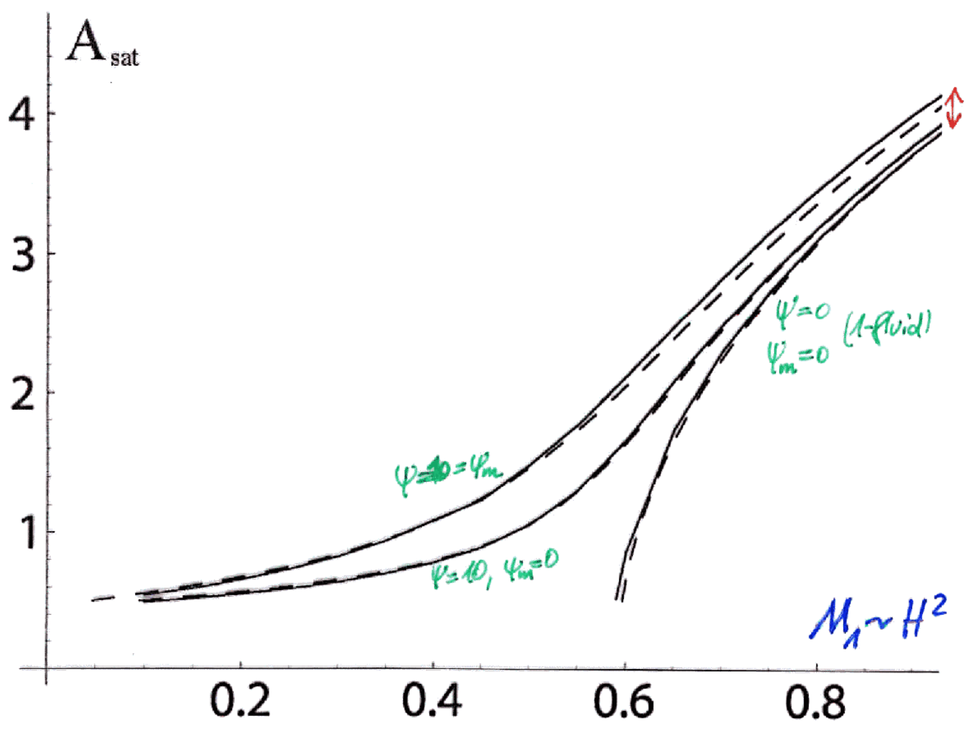
b.c. couple C with ϕ

These boundary conditions close the problem to find the fields $\mathbf{v}, \theta, C, \phi$ and ϕ_e.

$N = 50$

$M_2 = 0$ (Kelvin force only)

$Ra \approx 1300 \leq Ra_c^0$



approximate analytical solutions

boundary layer problem for concentration and magnetic potential

boundary layer depth $\sim L^{1/3}$

$$\frac{18\pi^4}{Ra} = \frac{1 + M_1(\beta - 2\pi\bar{\beta}\,\xi(A^2))}{1 + \frac{3}{40\pi^2}A^2}$$

$$+ (1-M_2)\frac{32\pi^2}{3A^2}\left[L(\psi + M_1\psi_m) + \gamma(1 - \bar{\gamma}\,\xi(A^2))M_1\psi_m\left(\frac{L^2 A}{1+\psi_m M_2}\right)^{1/3}\right]$$

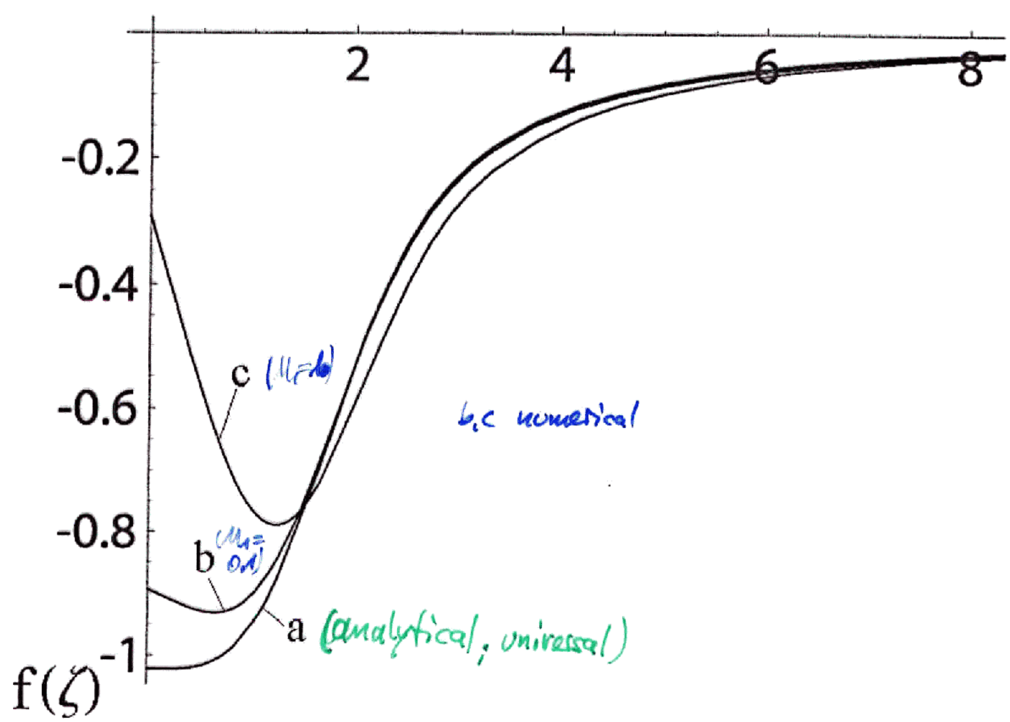
with $\xi(A^2) = \dfrac{9A^2}{160\pi^3(1 + \frac{3}{40\pi^2}A^2)}$

magnetophoresis: $M_2 < 0$; $M_2 \geq \frac{-1}{\psi_m}$ thermodynamic stability condition

(strong) influence only for $M_2\psi_m \to -1$

or $H^2 \to \gamma_H \bar{\varepsilon}\, \chi_c^{-1}$

boundary layers ($M_2=0, \psi=\psi_m=10$)



Conclusions II: external field makes the "imperfectness" of the bifurcation more pronounced

pronounced boundary layers profiles that couple to the bulk behavior

for very strong fields breakdown of 'binary mixture' description: agglomeration?

effectively $\partial_t c = L(1+\psi_m M_2)\nabla^2 c$

future: negative ψ; oscillatory instability?

patterns other than rolls?