PATTERNS IN THIN PLATE ELASTICITY

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Interesting issues:

* thin plates easily exhibit a large variety of folded forms

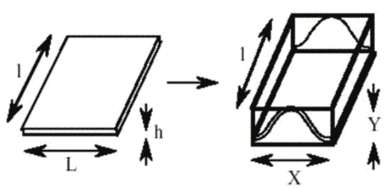
* similar kinds of patterns arise from biophysical scales to geophysical scales

* these patterns can be linked to transfer properties or to important functions

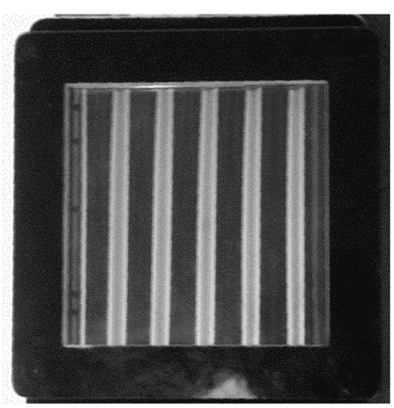
* their geometry cannot be derived as solutions of differential equations,
except for weak folding or in the absence of stretch

Can be important in real life ...

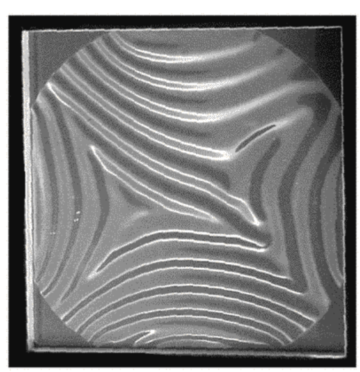
Crushing between flat plates



Y is decreased continuously

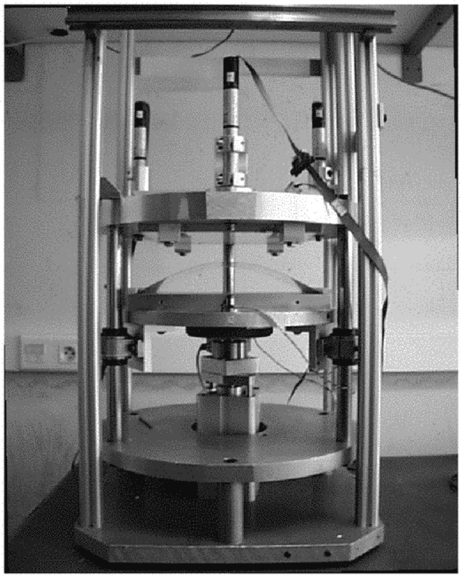


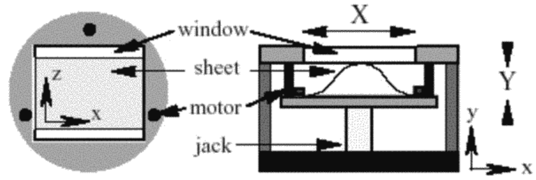
Clamping on two opposite sides



Clamping on all sides

Set-up



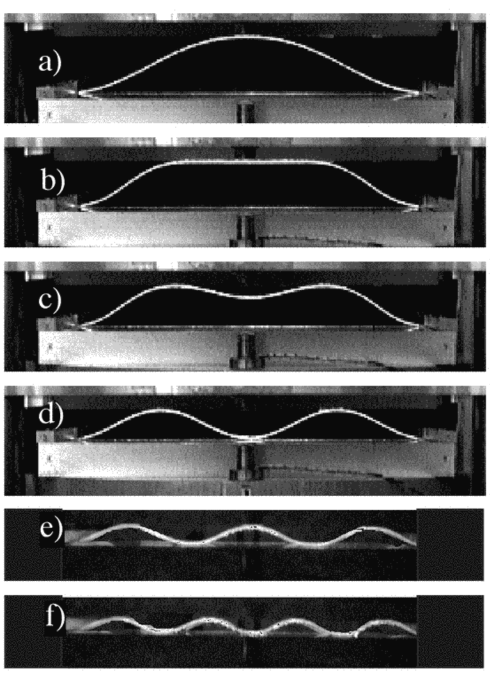


Boundary conditions:
sheets are clamped on two opposite sides

Sheets :
steel: h = 0.1 to 0.3 mm
polycarbonate: h = 100 μm
length L = 233 mm, l = 101mm
X= 220 mm

Crushing route : Y→ 0

View from side :



line contact

planar contact

free-standing fold

2 wrinkles

3 wrinkles

4 wrinkles

....

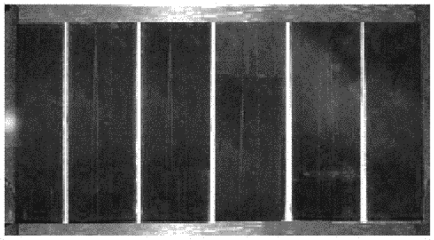
$Y=Y_1$: F=0

↓

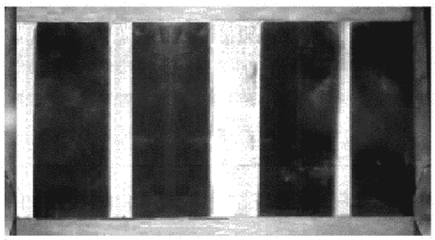
$Y=Y_1^b$: buckling

↓

View from above

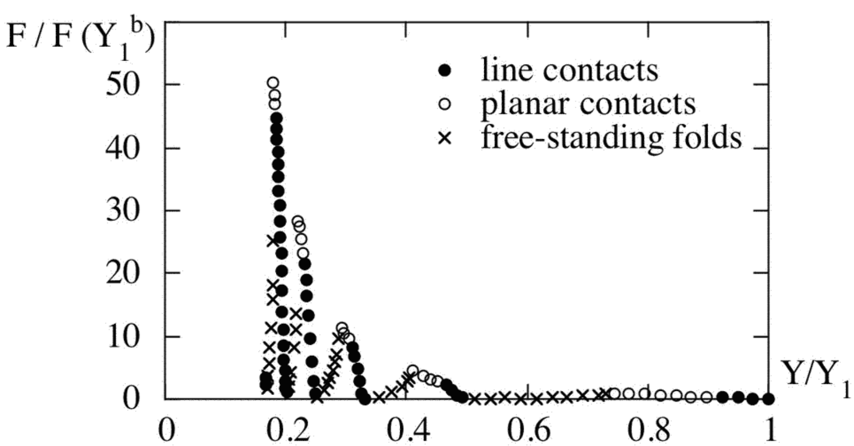


line contact



planar contact

Mechanical response : F(Y)

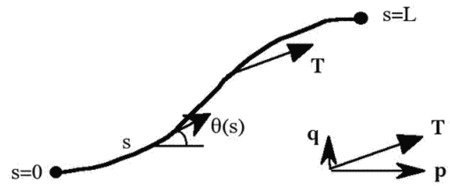


wrinkles number : 5 4 3 2 1

large non-linearity
repetitive vanishing
huge amplitude growth with the number of wrinkles

Elastica

Euler 1744

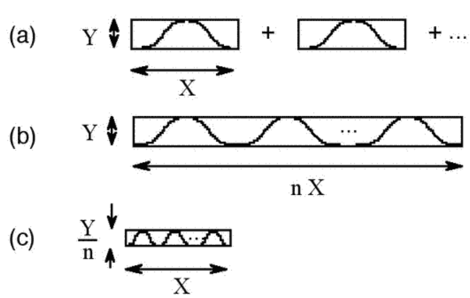


$$EI\,\frac{d^2\theta}{ds^2} = -p(s)\sin\theta + q(s)\cos\theta,$$ ⇔ pendulum dynamics (Kirchhoff 1859)

$p(s) = p \quad ; \quad q(s) = q$ ⇔autonomous equation

Geometrical similarity

Construction of similar n-fold solutions

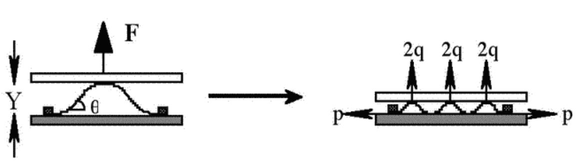


(a) replication

(b) juxtaposition
(autonomous dynamical system)

(c) geometrical similarity between (b) and (c)

Dynamical similarity

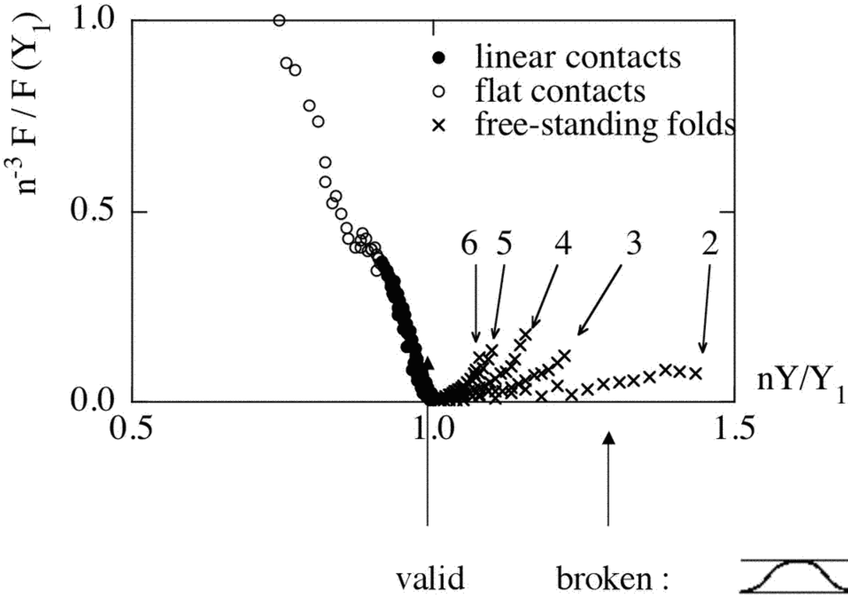


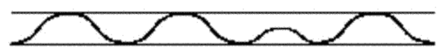
similarity transformation:
$Y \rightarrow Y' = Y/n$ with $\theta \rightarrow \theta' = \theta$
$s \rightarrow s' = s/n$

from Elastica :
$q \rightarrow q' = q\,n^2$
$F = 2q \rightarrow F' = 2q'\,n = 2q\,n^3 = n^3 F$

scaling relationship : $(Y,F) \rightarrow (Y/n, n^3 F)$

Collapse



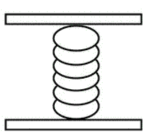


Discrete scale-invariance : Y→ Y/n , n ∈ N

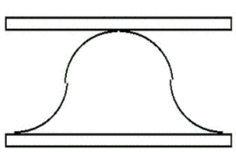
Europhys. Lett., **46** (5), pp. 602-608 (1999)

J. Mech. Phys. Solids 50 (2002) 2379–2401

Scale-invariance (SI) and morphology

Springs 

no morphology : all states are similar : continuous SI
F(Y') / F(Y) = $(Y'/Y)^e$; e=1

Sheets 

∃ morphology : states are not all similar
similarity for Y→ Y/n, n ∈ N : discrete SI

F(Y) ≠ power law
since discrete vanishing of F would imply F=0

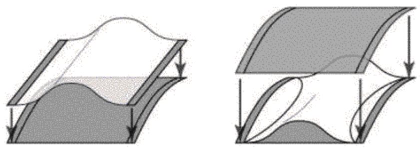
Discrete SI cannot be complete:

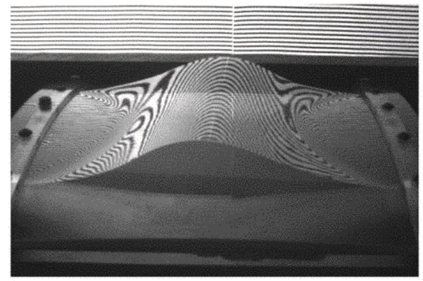
For a given Y and ∀(r,m) ∈ N^2, take Y = rY'=m Y''
and assume similarity between these states

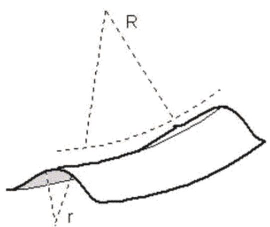
Then : F(Y)= F(Y') r^{-3} = F(Y'') m^{-3} so that F(Y') = $(r/m)^3$ F(Y''), ∀(r,m) ∈ N^2

F would then be a power law

Crushing between curved plates



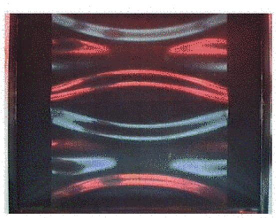




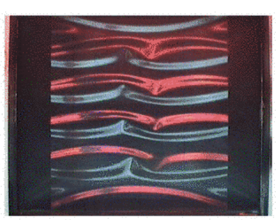
Gauss curvature :
$G \approx 1/R * 1/r \Rightarrow \exists$ stretch

Crushing route : $Y \rightarrow 0$

View from above



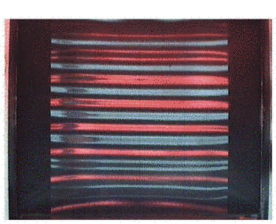
Defects (d-cones) nucleate and localize stretch



The number of defects increase

Un-crumpling transition

Still in the linear regime of elasticity



Defects suddenly disappear :

stretch is delocalized

Elastic energy E in regular states (no defects) :

$E = E_b + E_s$ with :

E_b : bending energy : $E_b = \frac{Eh^3}{24(1-\nu^2)} C^2$

E_s : stretching energy : $E_s = Eh\frac{1}{8}\left(\Delta^{-1}G\right)^2$

C: mean curvature : $C \approx 1/R + 1/r \approx 1/r$

G: Gauss curvature : $G \approx (R\,r)^{-1}$

$\Delta^{-1} G \approx G\, r^2 \approx r/R$

and $r \approx L/n$ where n is the number of folds

Thus : $E_s / E_b \approx r^4 R^{-2} h^{-2} = \gamma^4$

with $\gamma = r^2 / (Rh) = n^{-2}\ L^2 / (Rh)$

Call $n_c = L/ (Rh)^{1/2}$ so that $\gamma(n_c)=1$

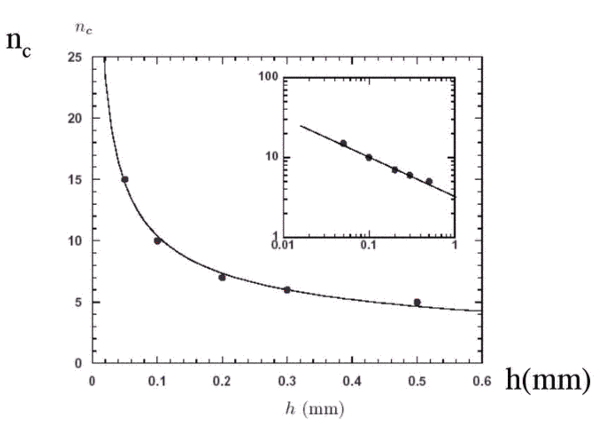
* $n < n_c$: $\gamma>1$: $E_s > E_b$

Stretch worths being localized in defects
Energy minimumcalls for defects

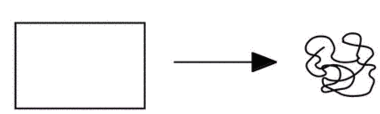
* $n > n_c$: $\gamma<1$: $E_s < E_b$

Delocalized stretch costs few energy compared to curvature
Energy can be made minimum without defect

* $n \approx n_c$: transition between defect regime (weak crushing)
and regular regime (large crushing)

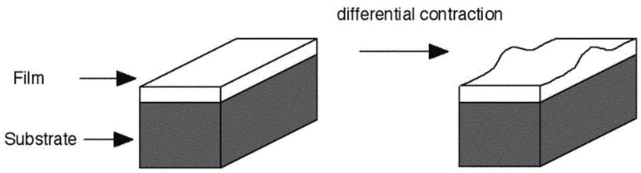


NB:
At 3d (crushing of a paper sheet with hands)

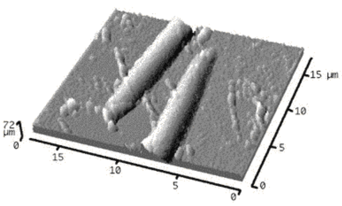
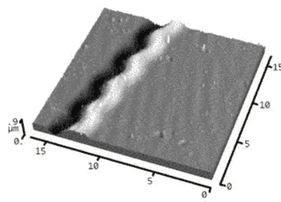
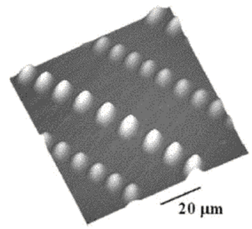


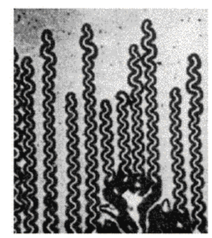
$R \approx r \approx L/n$, so that $n_c \approx L/h >>1 \Rightarrow r_c \approx h$
The transition escapes the regime of linear elasticity

Thin film delamination



Delamination patterns



Images : Coupeau, Colin, Cleymand, Grilhé

Gilles, Rau

Delamination = blister + fracture

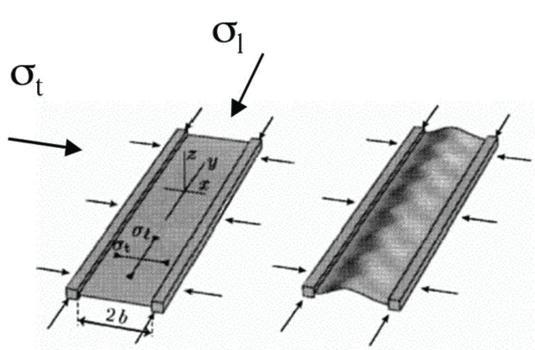
Issue :

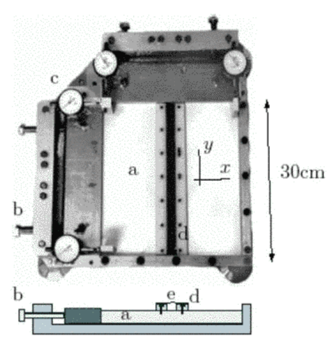
is the blister shape

dynamically generated by the fracture
or by the overall stress field ?

Modelization :

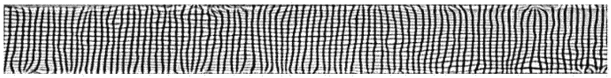
a yet formed blister in a prescribed stress field (σ_l, σ_t)

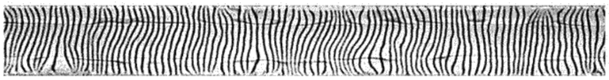


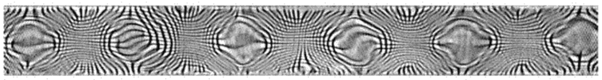


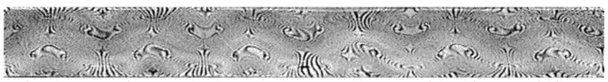
Observed patterns depending on the stress field (σ_l, σ_t)

Reflected image of a regular grid

a)  planar sheet

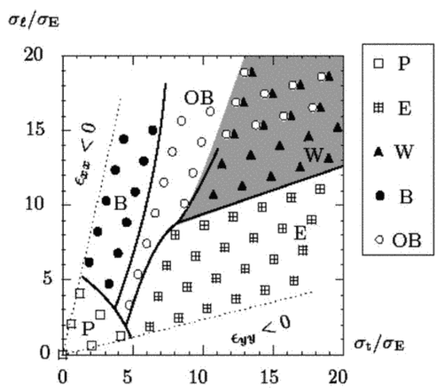
b)  Euler's column

c)  bumps

d)  worms

e)  oblique worms

Stability diagram



* isotropic compression yields worms
* multi-solutions : metastability

Modelization

Föppl- von Karman equations:

$$D\,\Delta^2\zeta - h\left(\frac{\partial^2\chi}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2} + \frac{\partial^2\chi}{\partial y^2}\frac{\partial^2\zeta}{\partial x^2} - 2\,\frac{\partial^2\chi}{\partial x\partial y}\frac{\partial^2\zeta}{\partial x\partial y}\right) = \text{h}\left(\sigma_t\frac{\partial^2\zeta}{\partial x^2} + \sigma_\ell\frac{\partial^2\zeta}{\partial y^2}\right) \quad (1a)$$

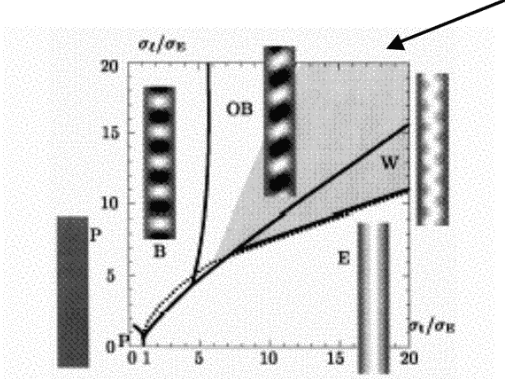
$$\Delta^2\chi + E\left\{\frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2} - \left(\frac{\partial^2\zeta}{\partial x\partial y}\right)^2\right\} = 0, \quad (1b)$$

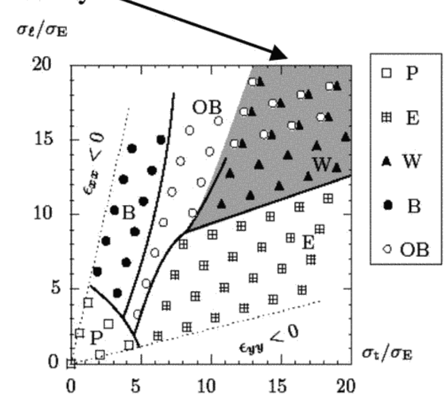
Gallerkin analysis of elastic energy:

$$\zeta(x,y) = v_E\cos^2\frac{\pi x}{2b} + v_+\cos^2\frac{\pi x}{2b}\cos(k_+ y) \quad (2)$$
$$\ldots + v_-\left(2\sin\frac{\pi x}{b} + \sin\frac{2\pi x}{b}\right)\cos(k_- y),$$

Agreement theory/experiment

metastability





Pattern formation can thus explain delamination pattern

Eur. Phys. J. B 27, p. 7-10 (2002)

Conclusion

* constrained sheets exhibit a rich variety of extended folded forms
  with a large non-linear mechanical response

* the features of their folding patterns are important regarding
  robustness, transfert, functions ...

  → challenging example of pattern formation
  with widespread applications at many scales

* patterns hardly understandable from elasticity equations

  → pattern formation modelling required