Fast Crack Growth by Surface Diffusion

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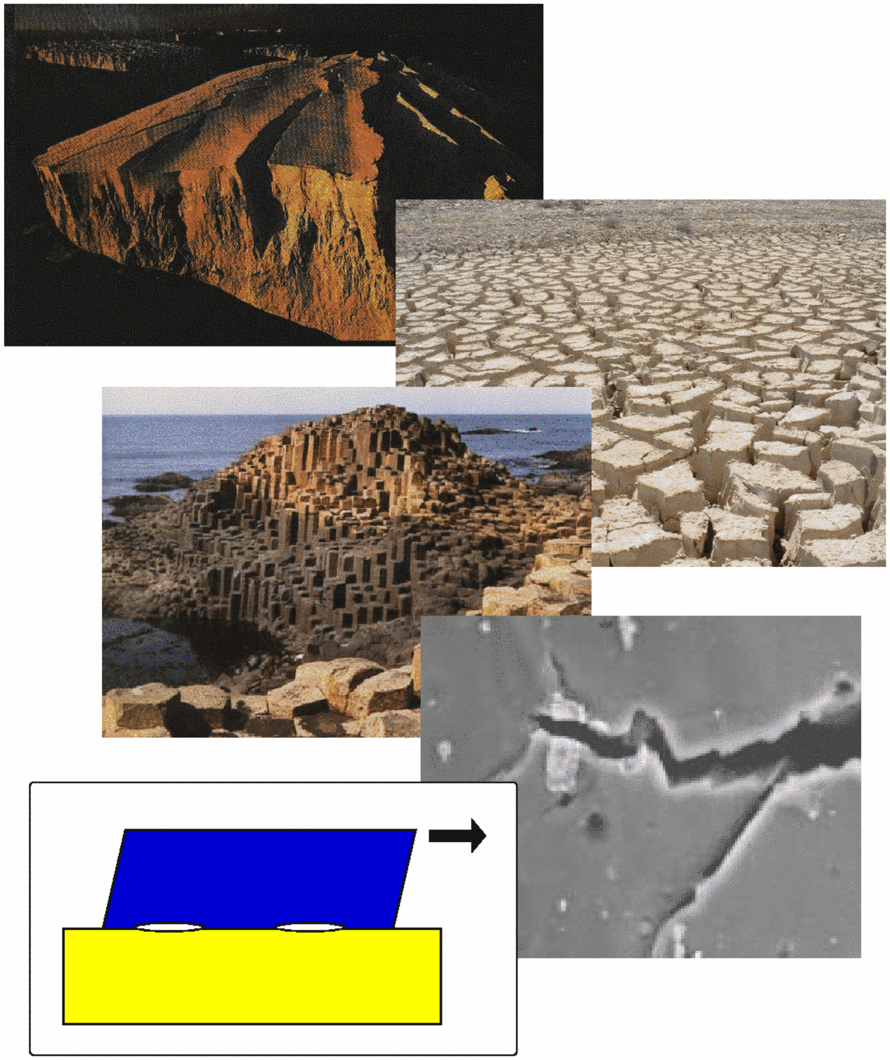
1. (Long) Introduction: two essentially independent lines of research

- Fracture Mechanics
- Asaro-Tiller-Grinfeld instability

2. Continuum Model for Fast Crack Propagation by Surface Diffusion: Synthesis of ideas

- Steady state growth
- Tip-splitting instability

The beauty of fracture...



... a story of unanswered questions

- Cracks are not always straight
- Cracks tips are not always sharp
- A tip-splitting instability can occur

Why?

Linear Theory of Elasticity

- Displacement u_i
- Strain

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- Stress (isotropic materials)

$$\sigma_{ij} = \frac{E}{1+\nu}\left(u_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}u_{kk}\right)$$

E Young's modulus, ν Poisson ratio

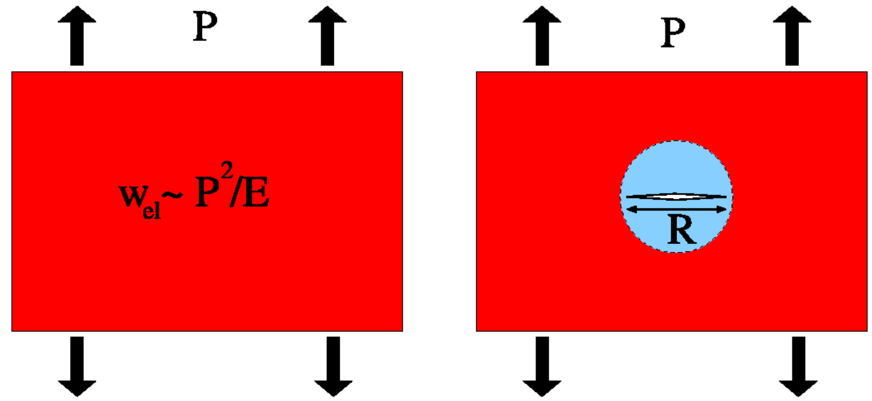
- Elastic energy density

$$w = \frac{1}{2}\sigma_{ij}u_{ij}$$

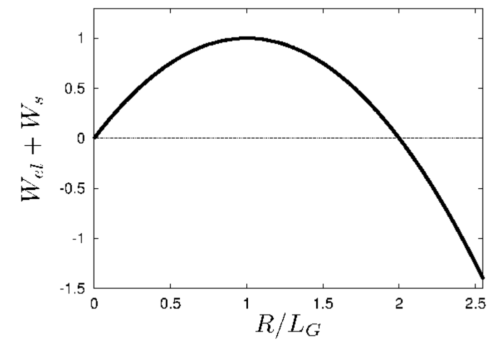
- Mechanical equilibrium

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

Theory of Cracks



- Elastic relaxation in the area $\sim R^2$: $W_{el} \sim -\dfrac{P^2R^2}{E}$
- Increase of surface energy: $W_s \sim \alpha R$



Griffith length: $L_G \sim \dfrac{E\alpha}{P^2}$

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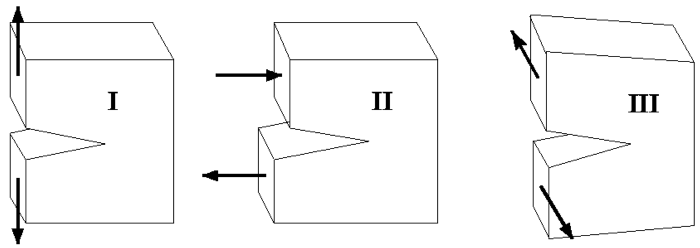
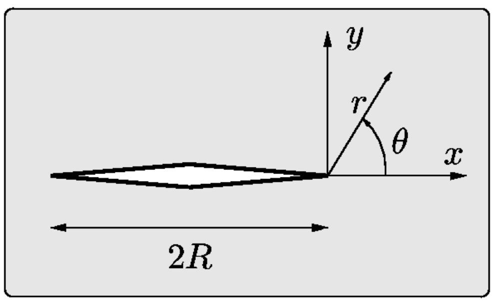
Near tip behavior





$$\sigma_{ij} = \frac{K}{r^{1/2}} f_{ij}(\theta)$$

with a *universal* function $f_{ij}(\theta)$ for each loading mode.

stress intensity factor: $K \sim PR^{1/2}$ contains the full information about the crack.

Griffith equilibrium: $K^2/E \sim \alpha \Rightarrow$ selection of R

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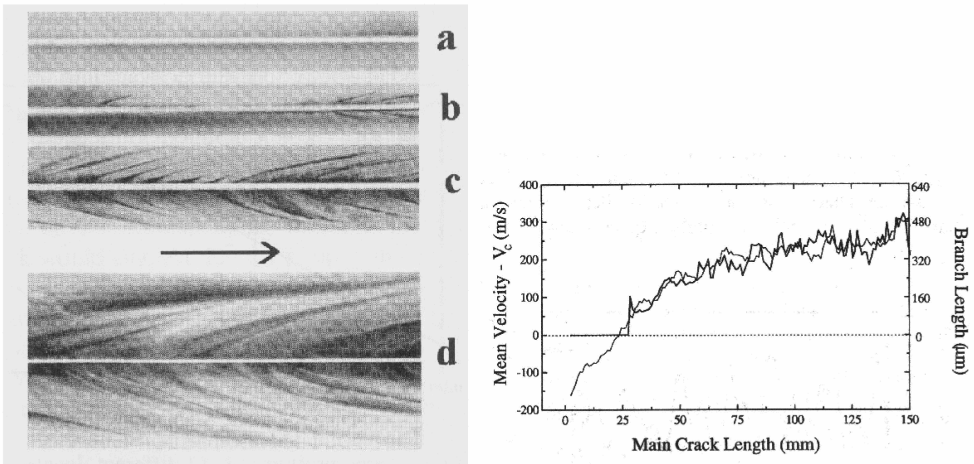
Experimental results

Expectation: maximum attained propagation velocity $v = v_R$ (Rayleigh speed = surface sound wave velocity)



[E. Sharon, S. Gross, J. Fineberg, Phys. Rev. Lett. **74**, 5096 (1995)]

- $v < v_c \approx 0.4 v_R$: straight crack growth
- $v > v_c$: *tip splitting*, strong velocity oscillations

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Dynamic fracture mechanics, Yoffe effect

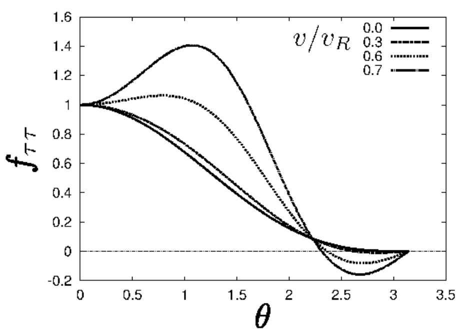
- Inertial effects are taken into account:

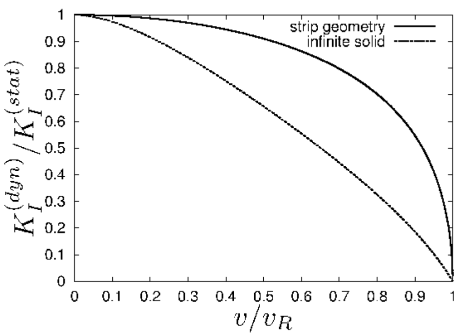
$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}$$

- Characteristic velocity: $v_R \sim (E/\rho)^{1/2}$ (Rayleigh speed)

- Modification of the near-tip behavior:

$$\sigma_{ij} = \frac{K(v/v_R)}{r^{1/2}} f_{ij}(\theta, v/v_R)$$





- Griffith equilibrium: $K^2(v)/E \sim \alpha$ selects $v \to v_R$ (integral energy balance)

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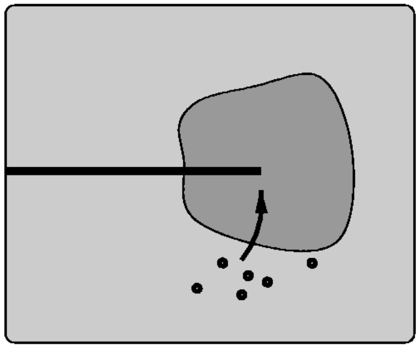
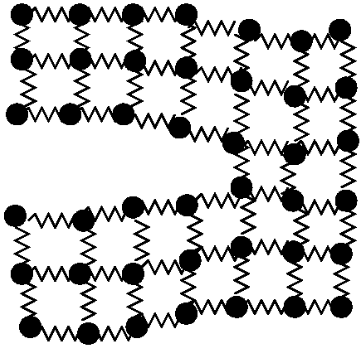
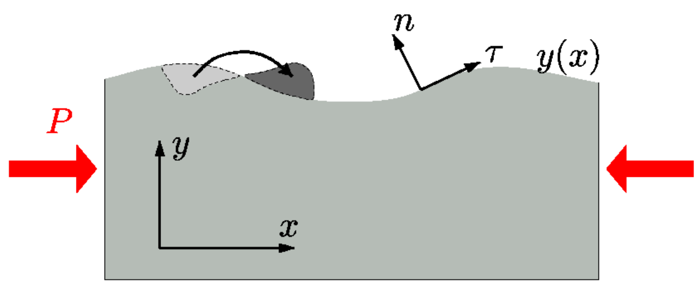
How does a crack grow?





- Linear theory of elasticity: stresses diverge at a sharp tip
- Finite tip radius
- Nonlinear effects
- Plastic zones
- Dislocations
- Granular media, anisotropic crystals
- Atomic bond breaking

Asaro-Tiller-Grinfeld instability



- Nonhydrostatic loading: $\sigma_{nn} \neq \sigma_{\tau\tau}$
- Morphological instability not due to elastic displacement!
- Chemical potential at the surface:

$$\Delta\mu_s = v_s \left(\frac{1-\nu^2}{2E} (\sigma_{nn} - \sigma_{\tau\tau})^2 - \alpha\kappa \right)$$

- Surface diffusion: $\Delta y = y_0 \sin(kx) e^{\lambda t}$:

$$\lambda = D v_s k^2 \left[\frac{2P^2(1-\nu^2)}{E} |k| - \alpha k^2 \right]$$

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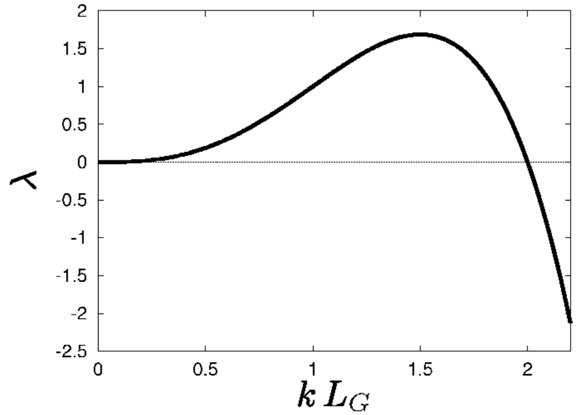
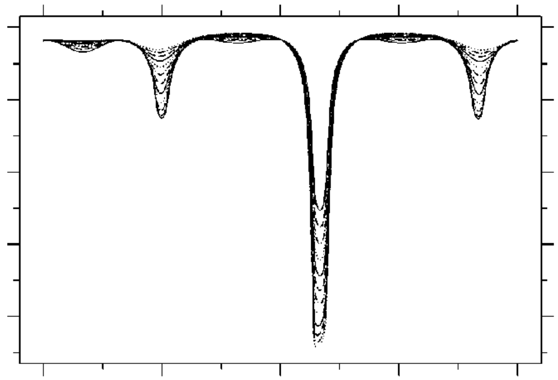
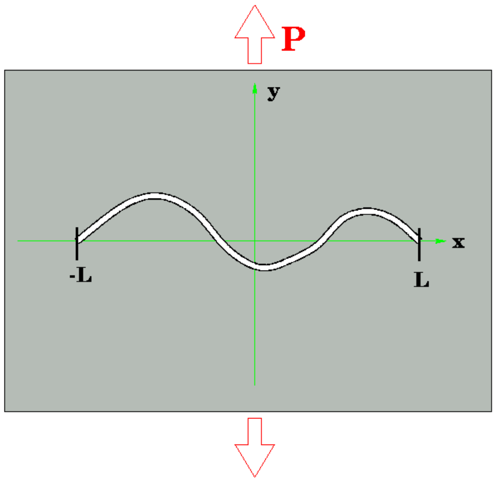




[K. Kassner et. al., Phys. Rev. E **63**, 036117 (2001)]

$\Rightarrow$ *finite time cusp singularity*

Cracks: Long Wave Grinfeld Instability



- Surface of a straight crack: $\sigma_{nn} = 0, \sigma_{\tau\tau} = -P$
  $\Rightarrow$ Long wave modes are unstable
- New degree of freedom: shape of the crack
- Fixed crack tips $\Rightarrow$ quantization
- Instability if $L > L_c = 5.18\, L_G$
- Surface diffusion slow; usually irrelevant for fast crack propagation with $L > L_G$.

E. Brener and V. Marchenko, Phys. Rev. Lett. **81**, 5141 (1998)
R. Spatschek and E. Brener, Phys. Rev. E **64**, 046120 (2001)

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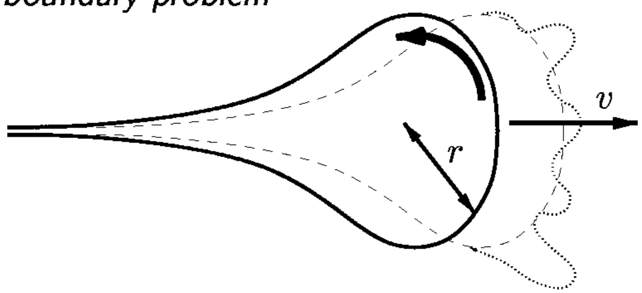
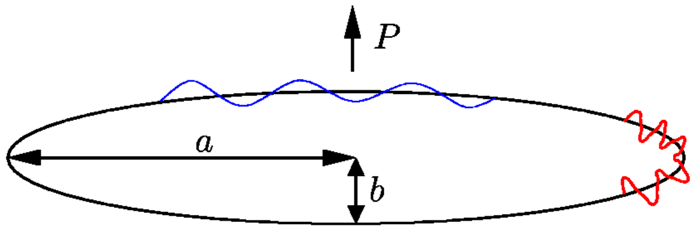
Crack growth by surface diffusion

- Only ingredients: linear theory of elasticity + surface diffusion
- Closed theory for the whole body
- Surface diffusion: $\mathbf{j} \sim D\nabla\mu_s$
- Fast crack growth due to strong gradients of chemical potential and local heating
- *Free boundary problem*



1. Is stationary growth by pure surface diffusion possible?
   - Shape of the crack?
   - How the lengthscale r is selected?
2. Do instabilities exist?
   - Is a *tip splitting* possible?

Elliptical Crack

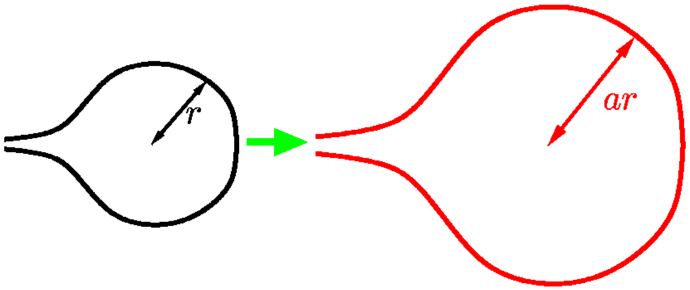


- Can be solved analytically. $a \gg b$
- Long wave instability occurs if $a \sim \frac{E\alpha}{P^2}$
- Stresses at the tip: $\sigma_{\tau\tau}^{(tip)} \sim Pa/b \to \infty$
- Grinfeld length at the tip: $\lambda_G^{(tip)} \sim \frac{E\alpha}{\sigma_{\tau\tau}^{(tip)2}}$
- Length of quantization: $r^{(tip)} = b^2/a$
- Instability at the crack tip if $r^{(tip)} \sim \lambda_G^{(tip)}$

$$\Rightarrow \boxed{a \sim \frac{E\alpha}{P^2}}$$

We expect the instability in the tip region!

Steady State Growth



- Stress: $\sigma \sim (ar)^{-1/2}$
- Curvature: $\kappa \sim (ar)^{-1}$
- Chemical potential: $\mu \sim \frac{\sigma^2}{E} - \alpha\kappa \sim (ar)^{-1}$
- Equation of motion. Normal velocity:

$$v_n \sim D\nabla^2\mu \sim (ar)^{-3}$$

Rescaling of the equation of motion possible

$\Rightarrow$ No selection of the tip radius!

Cusp singularity

$\Rightarrow$ Additional "physics" required!

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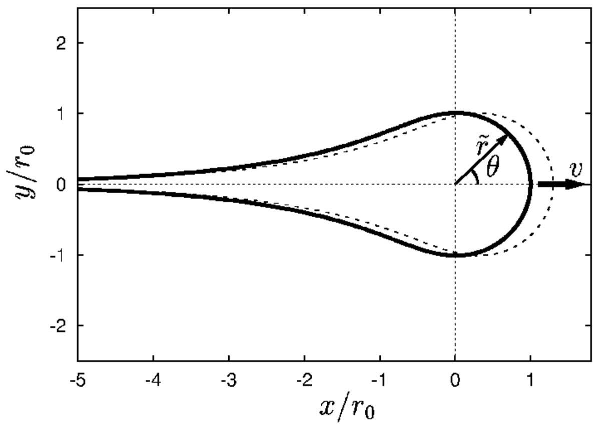
Details of the Steady State Growth



Equation of motion (steady state): $r = r(\theta)$

$$vr\sin\theta = -\frac{D}{\alpha\Omega}\frac{1}{\sqrt{r^2+r'^2}}\frac{d\mu}{d\theta}$$

Chemical potential:

$$\mu = \Omega\left(\frac{1-\nu^2}{2E}\sigma_{\tau\tau}^2 - \alpha\kappa\right)$$

Curvature:

$$\kappa = \frac{r^2+2r'^2-rr''}{(r^2+r'^2)^{3/2}}$$

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1. **Tip:** $r_0 = \kappa(\theta = 0)$

   Symmetric crack: $r'(0) = r''(0) = 0$

2. **Tail:** $Dy''' = vy$

   Suppression of two growing exponentials requires two *independent* degrees of freedom: r_0, v ???

Dimensionless rescaling

$$r = \tilde{r}r_0 \quad v = \tilde{v}D/r_0^3$$

Tip region:

$$\sigma_{ij} = \frac{K}{r^{1/2}}f_{ij}(\theta)$$

1. r_0 drops out of the equation of motion
2. Only one *independent* parameter in the problem
3. No selection of the tip radius
4. Steady-state solution does not exist!

How to overcome the cusp singularity?

- Inclusion of elastodynamic effects

$$\sigma_{ij} = \frac{K(v/v_R)}{r^{1/2}} f_{ij}(\theta, \frac{v}{v_R})$$

- Velocity appears now in the combinations vr_0^3/D and v/v_R
- Rescaling of the equation of motion is no longer possible
- Two *independent* parameters to fulfill boundary conditions
- Selection of the tip radius
- Steady state growth possible

The local crack model - Steady state growth

- Local model of the stress field (K_{stat}: static stress intensity factor)

$$\begin{aligned} \sigma_{nn} &= \sigma_{n\tau} = 0 \\ \sigma_{\tau\tau} &= \frac{K_{stat}}{r^{1/2}} \left[\sqrt{1-(v/v_R)^2} \cos(\theta/2) + (v/v_R)^2 \sin^4 \theta \right] \end{aligned}$$

- Correct qualitative behavior in the tip region:
  - Velocity dependence of the *dynamical* stress intensity factor
  - First order transition of the principal stress direction
- The steady-state equation becomes a nonlinear third order *differential* equation for the shape
- Results are robust against changes of the model

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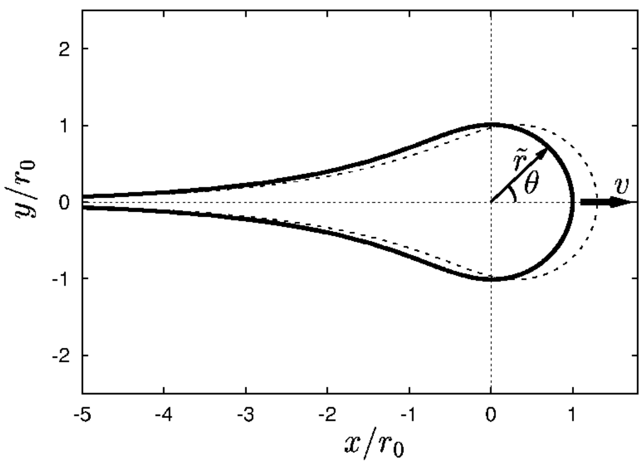
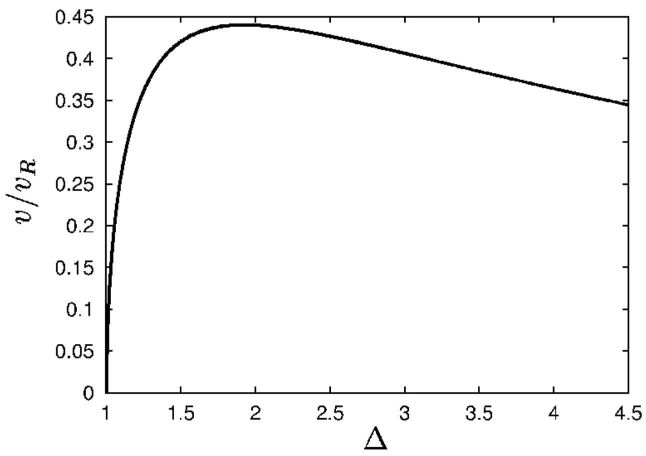


- Dimensionless driving force $\Delta = K_{stat}^2(1-\nu^2)/2E\alpha$
- Griffith point: $\Delta = 1$



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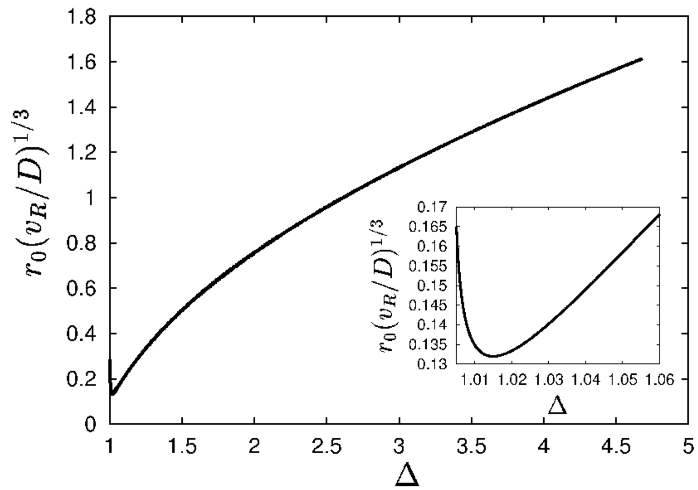

- Steady state velocity is limited to values appreciably below v_R
- Blunting of the tip
- Scale of velocity is set by v_R, independent of $D \Rightarrow$ fast crack growth
- Scale of the tip radius r_0 is set by $(D/v_R)^{1/3}$ and is of atomic scale

Grinfeld instability of the crack tip

- Decrease of the steady-state velocity with increasing driving force might be (naively) understood as a sign of instability.
- The local model itself is stable.
- – Local Grinfeld length at the crack tip: $\lambda^{(tip)} \sim r_0/\Delta$
  – Characteristic length r_0

  $\Rightarrow$ The characteristic wavelength of instability fits into the tip region, as soon as a critical driving force Δ_c is exceeded

  $\Rightarrow$ Occurrence of the Asaro-Tiller-Grinfeld instability!

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Summary and outlook

- Continuum theory to describe crack propagation by surface diffusion
- Steady state velocity is limited to values appreciably below v_R
- Blunting of the tip
- Scale of velocity is set by v_R, independent of $D \Rightarrow$ fast crack growth
- Scale of the tip radius r_0 is set by $(D/v_R)^{1/3}$ and is of atomic scale
- Instability of the tip above a critical driving force Δ
- Current investigation: exact numerical predictions by a phase field model and eigenmode expansion techniques