

# Symmetries and Ward Identities of Adiabatic Modes

Justin Khoury (U. Penn)

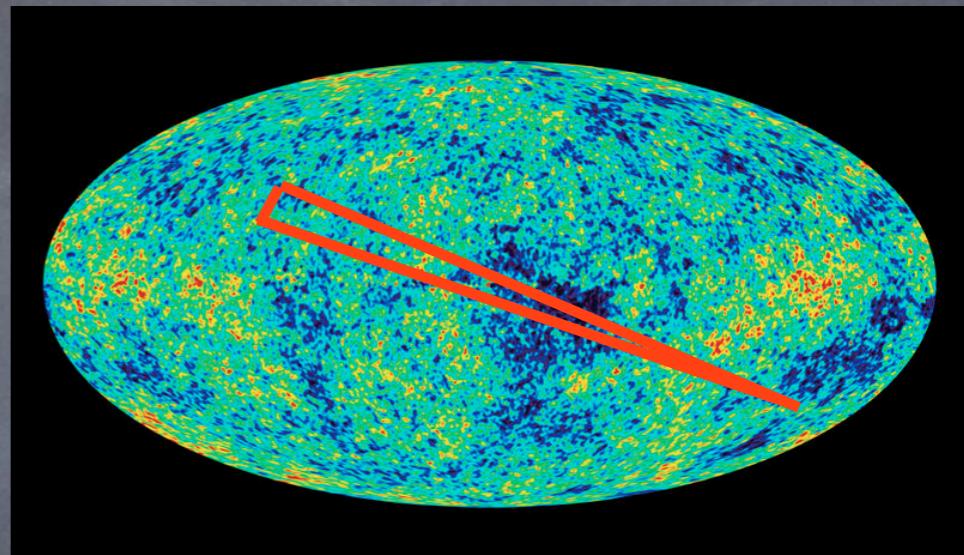
Hinterbichler, Hui & JK, 1304.5527; 1203.6351

Related work:

- Creminelli, Norena & Simonovic, 1203.4595
- Assasi, Baumann & Green, 1204.4207
- Goldberger, Hui & Nicolis, 1303.1193
- Schalm, Shiu & van der Aalst, 1211.2157
- Bzowski, McFadden & Skenderis, 1211.4550
- Mata, Raju & Trivedi, 1211.5482

# The consistency relation

Maldacena (2002);  
Creminelli & Zaldarriaga (2004);  
Cheung, Fitzpatrick, Kaplan & Senatore (2007)



$$\lim_{\vec{q} \rightarrow 0} \frac{1}{P_\zeta(q)} \langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle = -\vec{k}_1 \cdot \frac{\partial}{\partial \vec{k}_1} \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle = -(n_s - 1) \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle$$



- Holds in all single-field inflationary models (BD vacuum)
- Measuring significant 3-point function in this limit  
 $\implies$  automatically rule out all single-field models
- Proved by background-wave arguments and explicit calculations in the EFT of inflation

# The physical argument

Maldacena (2002); Creminelli & Zaldarriaga (2004)

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Background wave argument:



$$\langle \zeta_S \zeta_S \rangle_{\zeta_L} = \langle \zeta_S \zeta_S \rangle_0 + \zeta_L \left. \frac{d}{d \zeta_L} \langle \zeta_S \zeta_S \rangle \right|_0$$

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Multiply by  $\zeta_L$  and take expectation value:

$$\langle \zeta_L \langle \zeta_S \zeta_S \rangle_{\zeta_L} \rangle = \langle \zeta_L \zeta_L \rangle \frac{d}{d \ln |\vec{x}_1 - \vec{x}_2|} \langle \zeta_S \zeta_S \rangle$$

# In this talk...

- Derive consistency relation

$$\lim_{\vec{q} \rightarrow 0} \frac{1}{P_\zeta(q)} \langle \zeta(\vec{q}) \mathcal{O}(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c = - \left( 3(N-1) + \sum_{a=1}^N \vec{k}_a \cdot \frac{\partial}{\partial \vec{k}_a} \right) \langle \mathcal{O}(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c$$

from Ward identity for spontaneously broken dilation

cf. Assassi, Baumann & Green (2012)

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- Discover an infinite number of global symmetries, whose Ward identities lead to an infinite number of novel consistency relations:

$$\lim_{\vec{q} \rightarrow 0} \frac{\partial^n}{\partial q^n} \left( \frac{1}{P_\zeta(q)} \langle \zeta(\vec{q}) \mathcal{O} \rangle'_c + \frac{1}{P_\gamma(q)} \langle \gamma(\vec{q}) \mathcal{O} \rangle'_c \right) \sim - \frac{\partial^n}{\partial k^n} \langle \mathcal{O} \rangle'_c$$

- Constrains  $q^n$  behavior of correlators in soft limit
- $q^0$  and  $q$  behavior completely fixed
- $q^n, n \geq 2$ , behavior partially fixed
- These hold on any spatially-flat FRW background (no slow-roll)

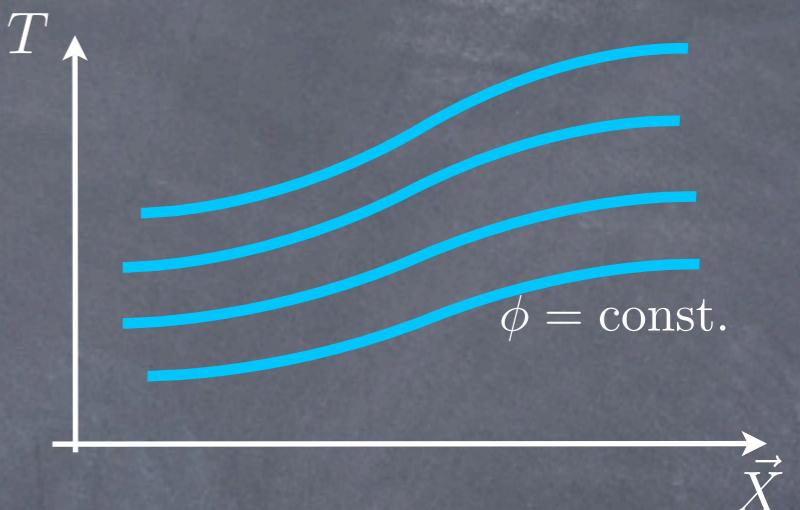
# Symmetries

# Symmetries of Scalar Pert'ns

Creminelli, Norena & Simonovic, 1203.4595  
Hinterbichler, Hui & Khoury, 1203.6351

Comoving gauge:

$$\phi = \phi(t) ; \quad h_{ij} = a^2(t) e^{2\zeta(t, \vec{x})} \delta_{ij}$$



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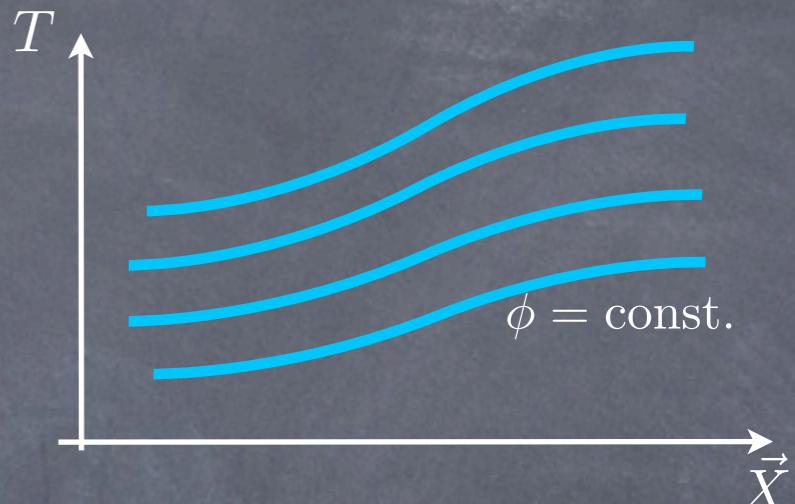
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By shifting  $\zeta$ , metric in this gauge is invariant under conformal transformations  $\delta_{ij} \rightarrow e^{2\Omega(x)} \delta_{ij}$



• Rotations + Translations

Linearly realized (unbroken)

• Dilation

$$x^i \rightarrow (1 + \lambda)x^i$$

$$\delta\zeta = \lambda$$

Non-linearly realized  
(spontaneously broken)

• Special conformal transformations (SCTs)

$$x^i \rightarrow x^i + 2\vec{x} \cdot \vec{b} x^i - b^i \vec{x}^2$$

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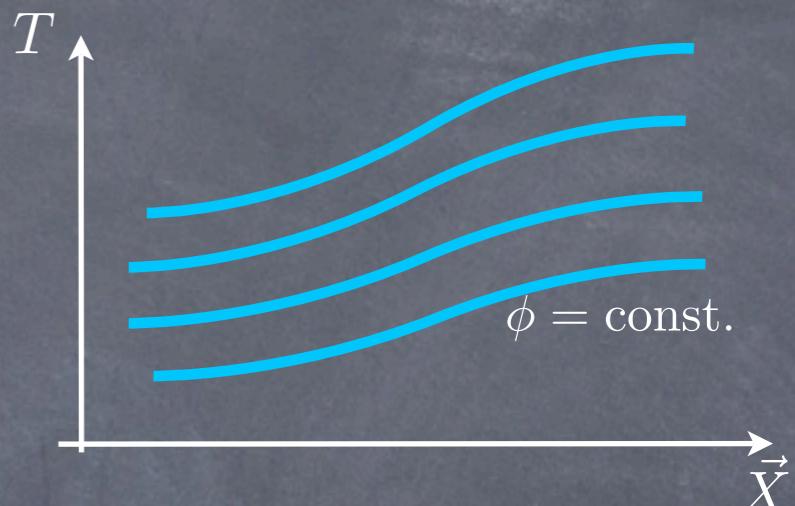
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∴  $so(4, 1) \rightarrow$  rotations + translations

$\zeta$  is Goldstone boson (dilaton) for the broken symmetries

# General Symmetries

Comoving gauge:

$$\phi = \phi(t) ; \quad h_{ij} = a^2(t) e^{2\zeta(\vec{x},t)} \left( e^{\gamma(\vec{x},t)} \right)_{ij}$$

where  $\gamma^i{}_i = 0, \quad \partial_i \gamma^i{}_j = 0$

This completely fixes the gauge, as long as we restrict to diffs that fall off at infinity.  $\implies$  Focus on diffs that do not fall off.

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Consider (time-dep.) spatial diff:  $\xi^i(\vec{x}, t)$  (leaves  $\phi = \phi(t)$  intact)

If we identify field transformations  $\delta\zeta, \delta\gamma_{ij}$  such that

$$\delta \left( e^{2\zeta} (e^\gamma)_{ij} \right) = \mathcal{L}_\xi \left( e^{2\zeta} (e^\gamma)_{ij} \right).$$

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$$\delta\gamma_{ij} = \delta\gamma_{ij}^{(\gamma^0)} + \delta\gamma_{ij}^{(\gamma^1)} + \dots$$

Solve this order by order in powers of  $\gamma$  :  $\delta\zeta = \delta\zeta^{(\gamma^0)} + \delta\zeta^{(\gamma^1)} + \dots$

$$\xi_i = \xi_i^{(\gamma^0)} + \xi_i^{(\gamma^1)} + \dots$$

# General Symmetries (cont'd)

$$\delta \left( e^{2\zeta} (e^\gamma)_{ij} \right) = \mathcal{L}_\xi \left( e^{2\zeta} (e^\gamma)_{ij} \right).$$

• **0<sup>th</sup> order:**

$$2\delta\zeta^{(\gamma^0)}\delta_{ij} + \delta\gamma_{ij}^{(\gamma^0)} = 2\xi_k^{(\gamma^0)}\partial^k\zeta\delta_{ij} + \partial_i\xi_j^{(\gamma^0)} + \partial_j\xi_i^{(\gamma^0)}$$

**Since**  $\delta\gamma_{ij}^{(\gamma^0)}$  **is traceless, this implies**

$$\delta\zeta^{(\gamma^0)} = \frac{1}{3}\partial^i\xi_i^{(\gamma^0)} + \xi_i^{(\gamma^0)}\partial^i\zeta$$

$$\delta\gamma_{ij}^{(\gamma^0)} = \partial_i\xi_j^{(\gamma^0)} + \partial_j\xi_i^{(\gamma^0)} - \frac{2}{3}\partial^k\xi_k^{(\gamma^0)}\delta_{ij}$$

**Since**  $\partial^i\delta\gamma_{ij}^{(\gamma^0)} = 0$ ,

$$\vec{\nabla}^2\xi_i^{(\gamma^0)} + \frac{1}{3}\partial_i\partial^j\xi_j^{(\gamma^0)} = 0 \quad (\star)$$

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Particular solutions:  $\xi_i^{\text{dilation}} = \lambda(t)x_i$

$$\xi_i^{\text{SCT}} = 2b^j(t)x_jx_i - \vec{x}^2b_i(t)$$

(Makes sense because  $(\star)$  is just the divergence of the conformal  
Killing equation:  $\partial_i\xi_j + \partial_j\xi_i = \frac{2}{3}\partial_k\xi^k\delta_{ij}$ )

# General Symmetries (cont'd)

$$\delta \left( e^{2\zeta} (e^\gamma)_{ij} \right) = \mathcal{L}_\xi \left( e^{2\zeta} (e^\gamma)_{ij} \right).$$

- Similarly, at 1<sup>st</sup> order:

$$\delta \zeta^{(\gamma^1)} = \xi_i^{(\gamma^1)} \partial^i \zeta + \frac{1}{3} \partial^i \xi_i^{(\gamma^1)} + \frac{1}{3} \partial_i \xi_k^{(\gamma^0)} \gamma^{ik}$$

$$\delta \gamma_{ij}^{(\gamma^1)} = \partial_i \xi_j^{(\gamma^1)} + \partial_j \xi_i^{(\gamma^1)} - \frac{2}{3} \partial^k \xi_k^{(\gamma^1)} \delta_{ij} - \frac{2}{3} \partial_l \xi_k^{(\gamma^0)} \gamma^{lk} \delta_{ij} - \frac{2}{3} \partial^k \xi_k^{(\gamma^0)} \gamma_{ij} + \mathcal{L}_{\xi^{(\gamma^0)}} \gamma_{ij}$$

$$\nabla^2 \xi_i^{(\gamma^1)} + \frac{1}{3} \partial_i \left( \partial^j \xi_j^{(\gamma^1)} \right) = \partial^j \left( \frac{2}{3} \partial_l \xi_k^{(\gamma^0)} \gamma^{lk} \delta_{ij} + \frac{2}{3} \partial^k \xi_k^{(\gamma^0)} \gamma_{ij} - \mathcal{L}_{\xi^{(\gamma^0)}} \gamma_{ij} \right)$$

known lower order pieces

Can similarly extend diffs to all orders in tensors.

- Dilation is exceptional:  $\xi_i^{\text{dilation}} = \lambda(t) x_i$  is exact
- SCTs, originally believed to be symmetry of scalars only, can be corrected to preserve transverse, traceless conditions on  $\gamma$ .

# Adiabatic Modes

Weinberg (2003); Hinterbichler, Hui & JK (2013)

The spatial diffs  $\xi^i(\vec{x}, t)$  induce

$$\boxed{\delta\zeta = \frac{1}{3}\partial_i\xi^i, \quad \delta N^i = \dot{\xi}^i, \quad \delta N = 0} \quad (\star)$$

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Assuming suitable fall-off behavior, the momentum and Hamiltonian constraints of GR imply

$$\delta N = \frac{\dot{\zeta}}{H}; \quad N_i = -\frac{\partial_i\zeta}{H} - \frac{a^2\dot{H}}{Hc_s^2}\frac{\partial_i}{\vec{\nabla}^2}\delta N$$

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To be extendible to a physical mode,  $(\star)$  must be consistent with this.

- Since  $\delta N = 0$ , require  $\frac{\partial}{\partial t}(\partial_i\xi^i) = 0$
- Sol'n for  $N^i$  requires  $\dot{\xi}^i = -\frac{1}{3H}\partial_i\partial_j\xi^j$

The physically-relevant diffs (i.e. adiabatic modes) are

$$\boxed{\xi^i = \left(1 + \int^t \frac{dt'}{H(t')}\vec{\nabla}^2\right)\bar{\xi}^i}$$

where

$$\boxed{\vec{\nabla}^2\bar{\xi}_i + \frac{1}{3}\partial_i\partial^j\bar{\xi}_j = 0}$$

# Taylor Expansion

Physical symmetries expressed in terms of time-indep. diff  $\bar{\xi}^i(\vec{x})$

Expand around the origin:

$$\bar{\xi}_i = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} M_{i\ell_0 \dots \ell_n} x^{\ell_0} \cdots x^{\ell_n}$$

where  $M_{i\ell_0 \dots \ell_n}$  is symmetric in  $\ell_0, \dots, \ell_n$  indices

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Gauge condition  $\vec{\nabla}^2 \bar{\xi}_i + \frac{1}{3} \partial_i \partial^j \bar{\xi}_j = 0 \implies$

$$M_{i\ell\ell\ell_2 \dots \ell_n} = -\frac{1}{3} M_{\ell i\ell\ell_2 \dots \ell_n}$$

(  $n \geq 1$  )

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Further “adiabatic” condition: Tensor profile induced is

$$\delta\gamma_{ij}^{(n)}(\vec{q}) = \frac{(-i)^n}{n!} \left( M_{ij\ell_1 \dots \ell_n} + M_{jil_1 \dots \ell_n} - \frac{2}{3} \delta_{ij} M_{\ell\ell\ell_1 \dots \ell_n} \right) \frac{\partial^n}{\partial q_{\ell_1} \cdots \partial q_{\ell_n}} ((2\pi)^3 \delta^3(\vec{q})) .$$

Demand this remains transverse when extended to a physical mode, with smooth behavior around  $\vec{q} = 0$ . That is, impose  $\hat{q}^i \delta\gamma_{ij}(\vec{q}) = 0$ :

$$\hat{q}^i \left( M_{i\ell_0 \ell_1 \dots \ell_n}(\hat{q}) + M_{\ell_0 i \ell_1 \dots \ell_n}(\hat{q}) - \frac{2}{3} \delta_{i\ell_0} M_{\ell\ell\ell_1 \dots \ell_n}(\hat{q}) \right) = 0$$

•  $n = 0$ :  $\xi_i^{(n=0)} = M_{i\ell_0} x^{\ell_0}$

**Decompose into - Trace:**  $M_{i\ell_0}^{\text{dilation}} = \lambda \delta_{i\ell_0}$  (1 dilation)

- Sym.:  $M_{i\ell_0}^{\text{aniso}} = S_{i\ell_0}$  ;  $S_{i\ell_0} = S_{\ell_0 i}$  ;  $S_i^i = 0$

and  $\hat{q}^i S_{i\ell_0}(\vec{q}) = 0$  (2 anisotropic scalings)

- Anti-sym.:  $M_{i\ell_0}^{\text{rotn}} = \omega_{i\ell_0}$  ;  $\omega_{i\ell_0} = -\omega_{\ell_0 i}$  ;  $\omega_i^i = 0$   
(3 rotations)

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**•**  $n = 1: \quad \bar{\xi}_i^{(n=1)} = \frac{1}{2} M_{i\ell_0\ell_1} x^{\ell_0} x^{\ell_1}$  **with**  $M_{i\ell\ell} = -\frac{1}{3} M_{\ell i\ell}$

**Include:**  $M_{i\ell_0\ell_1}^{\text{SCT}} = b_{\ell_1} \delta_{i\ell_0} + b_{\ell_0} \delta_{i\ell_1} - b_i \delta_{\ell_0\ell_1}$  (3 SCTs)

**and**  $M_{i\ell\ell}^T = M_{\ell i\ell}^T = 0$  ;  $\hat{q}^i (M_{i\ell_0\ell_1}(\vec{q}) + M_{\ell_0 i\ell_1}(\vec{q})) = 0$  (4 tensor syms)

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and  $M_{i\ell\ell}^T = M_{\ell i\ell}^T = 0$  ;  $\hat{q}^i (M_{i\ell_0\ell_1}(\vec{q}) + M_{\ell_0 i\ell_1}(\vec{q})) = 0$  (4 tensor syms)

•  $n \geq 2$ :

$$\underbrace{\frac{3}{2}(n+3)(n+2)}_{[M_{i\ell_0 \dots \ell_n}]} - \underbrace{\frac{3}{2}(n+1)n}_{\text{trace cond'ns}} - \underbrace{3(2n+1)}_{\text{transverse cond'ns}} = 6 \text{ syms}$$

 4 tensor syms  
 2 "mixed" syms

# Physical Interpretation

$$h_{ij} = \sum_{n=0}^{\infty} \frac{1}{n!} H_{ij\ell_1 \dots \ell_n} x^{\ell_1} \cdots x^{\ell_n}$$

$H_{ij\ell_1 \dots \ell_n}$  **sym. in**  $i, j$  **and**  $\ell_1, \dots, \ell_n$

**Gauge cond.:**  $H_{i\ell\ell\ell_2 \dots \ell_n} = \frac{1}{3} H_{\ell\ell i\ell_2 \dots \ell_n}$

**Transversality:**  $\hat{q}^i (H_{ij\ell_1 \dots \ell_n}(\hat{q}) + \dots) = 0$

$$\bar{\xi}_i = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} M_{i\ell_0 \dots \ell_n} x^{\ell_0} \cdots x^{\ell_n}$$

$M_{i\ell_0 \dots \ell_n}$  **sym. in**  $\ell_0, \dots, \ell_n$

**Gauge cond.:**  $M_{i\ell\ell\ell_2 \dots \ell_n} = -\frac{1}{3} M_{\ell i\ell\ell_2 \dots \ell_n}$

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n=0:  $[H_{i\ell_0}] = [M_{i\ell_0}] = 3$

n=1:  $[H_{i\ell_0\ell_1}] = [M_{i\ell_0\ell_1}] = 7$

n=2:  $[H_{i\ell_0\ell_1\ell_2}] = 12$        $[M_{i\ell_0\ell_1\ell_2}] = 6$

Matches geometric counting:

$$h_{ij}^{\text{Riemann}} = \delta_{ij} - \frac{1}{3} R_{ikj\ell} X^k X^\ell + \dots$$

$$\bar{\xi}_i = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} M_{i\ell_0 \dots \ell_n} x^{\ell_0} \cdots x^{\ell_n}$$

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6 indep components in D=3

# Ward Identities

# Noether Charges

The Noether charge associated with the transformation  $(\delta\zeta, \delta\gamma_{ij})$  is

$$Q = \frac{1}{2} \int d^3x \left( \{\Pi_\zeta(x), \delta\zeta(x)\} + \{\Pi_\gamma^{ij}(x), \delta\gamma_{ij}(x)\} \right).$$

where  $\Pi_\zeta \equiv \frac{\delta\mathcal{L}}{\delta\dot{\zeta}}$  and  $\Pi_\gamma^{ij} \equiv \frac{\delta\mathcal{L}}{\delta\dot{\gamma}_{ij}}$  are conjugate momenta.

Focus on  $Q_0$ , the part of  $Q$  which generates the non-linear transformation of the fields.

$$Q_0 = \sum_{n=0}^{\infty} \int d^3x \frac{1}{n!} M_{ij\ell_1\dots\ell_n} x^{\ell_1} \dots x^{\ell_n} \left( \frac{1}{3} \delta^{ij} \Pi_\zeta(\vec{x}) + 2 \Pi_\gamma^{ij}(\vec{x}) \right).$$

In momentum space,

$$Q_0 = \lim_{\vec{q} \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} M_{ij\ell_1\dots\ell_n} \frac{\partial^n}{\partial q_{\ell_1} \dots \partial q_{\ell_n}} \left( \frac{1}{3} \delta^{ij} \Pi_\zeta(\vec{q}) + 2 \Pi_\gamma^{ij}(\vec{q}) \right).$$

(This is a symmetry of the free theory.)

# Ward Identities

$$\langle \Omega | [Q, \mathcal{O}] | \Omega \rangle = -i \langle \Omega | \delta \mathcal{O} | \Omega \rangle$$

- Here,  $\mathcal{O}$  is an (equal-time) product of M scalars and N-M tensors:

$$\mathcal{O}(\vec{k}_1, \dots, \vec{k}_N) = \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_M) \cdot \mathcal{O}^\gamma(\vec{k}_{M+1}, \dots, \vec{k}_N)$$

- In-vacuum  $|\Omega\rangle$  is related to the free (Bunch-Davies) vacuum  $|0\rangle$  by:

$$|\Omega\rangle = \Omega(-\infty)|0\rangle$$

where  $\Omega(-\infty) \equiv U^\dagger(-\infty, 0)U_0(-\infty, 0)$

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The Left-Hand Side:

Intertwining relation:

$$Q \Omega(-\infty) = \Omega(-\infty) Q_0$$

(understood as weak operator statement)

$\therefore Q|\Omega\rangle = \Omega(-\infty)Q_0|0\rangle \implies$  It remains to calculate  $Q_0|0\rangle$ .

# The Left-Hand Side (cont'd)

Insert a complete set of free-field eigenstates  $|\zeta_0, \gamma_0\rangle$ :

$$\begin{aligned}
Q_0|0\rangle &= \int D\zeta_0 D\gamma_0 |\zeta_0, \gamma_0\rangle \langle \zeta_0, \gamma_0| Q_0 |0\rangle \\
&= \lim_{\vec{q} \rightarrow 0} \frac{(-i)^n}{n!} M_{ij\ell_1 \dots \ell_n} \frac{\partial^n}{\partial q_{\ell_1} \dots \partial q_{\ell_n}} \int D\zeta_0 D\gamma_0 |\zeta_0, \gamma_0\rangle \langle \zeta_0, \gamma_0| \left( \frac{1}{3} \delta^{ij} \Pi_\zeta(\vec{q}) + 2 \Pi_\gamma^{ij}(\vec{q}) \right) |0\rangle \\
&= \lim_{\vec{q} \rightarrow 0} \frac{(-i)^n}{n!} M_{ij\ell_1 \dots \ell_n} \frac{\partial^n}{\partial q_{\ell_1} \dots \partial q_{\ell_n}} \int D\zeta_0 D\gamma_0 |\zeta_0, \gamma_0\rangle \left( -\frac{i}{3} \delta^{ij} \frac{\delta}{\delta \zeta_0(-\vec{q})} - 2i \frac{\delta}{\delta \gamma_{0ij}(-\vec{q})} \right) \langle \zeta_0, \gamma_0| 0\rangle
\end{aligned}$$

The free-vacuum wavefunctional  $\langle \zeta_0, \gamma_0 | 0 \rangle$  is a Gaussian:

$$\langle \zeta_0, \gamma_0 | 0 \rangle \sim \exp \left[ - \int \frac{d^3 k}{(2\pi)^3} \left( \frac{1}{4P_\zeta(k)} \zeta_0(\vec{k}) \zeta_0(-\vec{k}) + \frac{1}{8P_\gamma(k)} \gamma_{0ij}(\vec{k}) \gamma_0^{ij}(-\vec{k}) \right) \right].$$

Hence:

$$\boxed{
\begin{aligned}
Q_0|0\rangle &= - \lim_{\vec{q} \rightarrow 0} \frac{(-i)^{n+1}}{2n!} M_{ij\ell_1 \dots \ell_n} \frac{\partial^n}{\partial q_{\ell_1} \dots \partial q_{\ell_n}} \int D\zeta_0 D\gamma_0 |\zeta_0, \gamma_0\rangle \left( \frac{1}{3P_\zeta(q)} \delta^{ij} \zeta_0(\vec{q}) + \frac{1}{P_\gamma(q)} \gamma_0^{ij}(\vec{q}) \right) \langle \zeta_0, \gamma_0| 0\rangle \\
&= - \lim_{\vec{q} \rightarrow 0} \frac{(-i)^{n+1}}{2n!} M_{ij\ell_1 \dots \ell_n} \frac{\partial^n}{\partial q_{\ell_1} \dots \partial q_{\ell_n}} \left( \frac{1}{3P_\zeta(q)} \delta^{ij} \zeta_0(\vec{q}) + \frac{1}{P_\gamma(q)} \gamma_0^{ij}(\vec{q}) \right) |0\rangle
\end{aligned}}$$

## The Left-Hand Side (cont'd)

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At this point would like to use intertwining rel'n,  $\Omega(-\infty)\zeta_0(\vec{q}) = \zeta(\vec{q})\Omega(-\infty)$ ,  
But this is only true in the far past, i.e.

$$\lim_{t_i \rightarrow -\infty} \Omega(t_i) \zeta_0(\vec{q}, t_i) = \lim_{t_i \rightarrow -\infty} \zeta(\vec{q}, t_i) \Omega(t_i)$$

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Assuming constant growing modes, at any given  $t_i$  the following holds:

$$\begin{aligned} \zeta(\vec{q}, t_i) &= \zeta(\vec{q}) [1 + \Delta] & ; & & \lim_{q|\tau_i| \rightarrow 0} \Delta, \Delta_0 &= 0 \\ \zeta_0(\vec{q}, t_i) &= \zeta_0(\vec{q}) [1 + \Delta_0] & & & & \end{aligned}$$

It follows that:

$$\boxed{\lim_{q|\tau_i| \rightarrow 0, t_i \rightarrow -\infty} \Omega(t_i) \zeta_0(\vec{q}) = \zeta(\vec{q}) \Omega(t_i)}$$

## The Left-Hand Side (cont'd)

$$Q_0|0\rangle = -\lim_{\vec{q} \rightarrow 0} \frac{(-i)^{n+1}}{2n!} M_{ij\ell_1 \dots \ell_n} \frac{\partial^n}{\partial q_{\ell_1} \cdots \partial q_{\ell_n}} \left( \frac{1}{3P_\zeta(q)} \delta^{ij} \zeta_0(\vec{q}) + \frac{1}{P_\gamma(q)} \gamma_0^{ij}(\vec{q}) \right) |0\rangle$$

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Hence:

$$\boxed{\langle \Omega | [Q, \mathcal{O}] | \Omega \rangle = \lim_{\vec{q} \rightarrow 0} \frac{(-i)^{n+1}}{n!} M_{i\ell_0 \dots \ell_n} \frac{\partial^n}{\partial q_{\ell_1} \cdots \partial q_{\ell_n}} \left( \frac{1}{P_\gamma(q)} \langle \gamma^{i\ell_0}(\vec{q}) \mathcal{O} \rangle + \frac{\delta^{i\ell_0}}{3P_\zeta(q)} \langle \zeta(\vec{q}) \mathcal{O} \rangle \right)}$$

# The Right-Hand Side:

$$\mathcal{O}(\vec{k}_1, \dots, \vec{k}_N) = \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_M) \cdot \mathcal{O}^\gamma(\vec{k}_{M+1}, \dots, \vec{k}_N)$$

$$\begin{aligned} \langle \delta \mathcal{O} \rangle_c = & -\frac{(-i)^n}{n!} M_{i\ell_0 \dots \ell_n} \left\{ \sum_{a=1}^N \left( \delta^{i\ell_0} \frac{\partial^{n-1}}{\partial k_{\ell_1}^a \cdots \partial k_{\ell_n}^a} + \frac{k_a^i}{n+1} \frac{\partial^{n+1}}{\partial k_{\ell_0}^a \cdots \partial k_{\ell_n}^a} \right) \langle \mathcal{O}(\vec{k}_1, \dots, \vec{k}_N) \rangle_c \right. \\ & - \sum_{a=1}^M \Upsilon^{i\ell_0 i_a j_a}(\hat{k}_a) \frac{\partial^n}{\partial k_{\ell_1}^a \cdots \partial k_{\ell_n}^a} \langle \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_{a-1}, \vec{k}_{a+1}, \dots, \vec{k}_M) \gamma_{i_a j_a}(\vec{k}_a) \mathcal{O}^\gamma(\vec{k}_{M+1}, \dots, \vec{k}_N) \rangle_c \\ & \left. - \sum_{b=M+1}^N \Gamma^{i\ell_0 i_b j_b k_b \ell_b}(\hat{k}_b) \frac{\partial^n}{\partial k_{\ell_1}^b \cdots \partial k_{\ell_n}^b} \langle \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_M) \mathcal{O}_{i_{M+1} j_{M+1}, \dots, k_b \ell_b, \dots i_N j_N}^\gamma(\vec{k}_{M+1}, \dots, \vec{k}_N) \rangle_c \right\} + \dots \end{aligned}$$

where the ... indicate higher-order in  $\gamma$ , and

$$\Upsilon_{rsij}(\hat{k}) \equiv \frac{1}{2} \delta_{s(i} \delta_{j)r} - \frac{1}{4} \hat{k}_s \hat{k}_{(i} \delta_{j)r} + \frac{5}{12} \hat{k}_i \hat{k}_j \delta_{rs} ;$$

$$\Gamma_{rsijk\ell}(\hat{k}) \equiv 2 \left( \delta_{s(i} - \hat{k}_r \hat{k}_{(i} \right) \delta_{j)(k} \delta_{\ell)r} - \left( \delta_{ij} - \hat{k}_i \hat{k}_j \right) \delta_{r(k} \delta_{\ell)s} - \frac{2}{3} \delta_{i(k} \delta_{\ell)j} \delta_{rs}$$

# Consistency Relations

Hinterbicler, Hui & JK, 1304.5527

$$\begin{aligned}
 & \lim_{\vec{q} \rightarrow 0} M_{i\ell_0 \dots \ell_n}(\hat{q}) \frac{\partial^n}{\partial q_{\ell_1} \dots \partial q_{\ell_n}} \left( \frac{1}{P_\gamma(q)} \langle \gamma^{i\ell_0}(\vec{q}) \mathcal{O}(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c + \frac{\delta^{i\ell_0}}{3P_\zeta(q)} \langle \zeta(\vec{q}) \mathcal{O}(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c \right) \\
 &= -M_{i\ell_0 \dots \ell_n}(\hat{q}) \left\{ \sum_{a=1}^N \left( \delta^{i\ell_0} \frac{\partial^n}{\partial k_{\ell_1}^a \dots \partial k_{\ell_n}^a} - \frac{\delta_{n0}}{N} \delta^{i\ell_0} + \frac{k_a^i}{n+1} \frac{\partial^{n+1}}{\partial k_{\ell_0}^a \dots \partial k_{\ell_n}^a} \right) \langle \mathcal{O}(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c \right. \\
 &\quad - \sum_{a=1}^M \Upsilon^{i\ell_0 i_a j_a}(\hat{k}_a) \frac{\partial^n}{\partial k_{\ell_1}^a \dots \partial k_{\ell_n}^a} \langle \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_{a-1}, \vec{k}_{a+1}, \dots, \vec{k}_M) \gamma_{i_a j_a}(\vec{k}_a) \mathcal{O}^\gamma(\vec{k}_{M+1}, \dots, \vec{k}_N) \rangle'_c \\
 &\quad \left. - \sum_{b=M+1}^N \Gamma^{i\ell_0}_{i_b j_b} {}^{k_b \ell_b}(\hat{k}_b) \frac{\partial^n}{\partial k_{\ell_1}^b \dots \partial k_{\ell_n}^b} \langle \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_M) \mathcal{O}^\gamma_{i_{M+1} j_{M+1}, \dots, k_b \ell_b, \dots i_N j_N}(\vec{k}_{M+1}, \dots, \vec{k}_N) \rangle'_c \right\} + \dots
 \end{aligned}$$

where  $\langle \dots \rangle'$  denote correlators with  $(2\pi)^3 \delta^3(\vec{P}_{\text{tot}})$  stripped off.

- ⦿ Constrains  $q^n$  behavior of correlators in soft limit
- ⦿  $q^0$  and  $q$  behavior completely fixed (KNOWN)
- ⦿  $q^n, n \geq 2$ , behavior partially fixed (NEW)

⇒ Complete checklist for testing single-field inflation.

## Known Consistency Relations:

For simplicity, assume product of scalars:  $\mathcal{O} = \mathcal{O}^\zeta = \prod_{a=1}^N \zeta(\vec{k}_a)$

•  $n = 0$  relations:

Maldacena (2002); Creminelli & Zaldarriaga (2004);  
Cheung, Fitzpatrick, Kaplan & Senatore (2007);  
Assassi, Baumann & Green (2012);  
Goldberger, Hui & Nicolis (2013)

Dilation consistency relation

$$M_{i\ell_0}^{\text{dilation}} = \lambda \delta_{i\ell_0}$$

$$\lim_{\vec{q} \rightarrow 0} \frac{1}{P_\zeta(q)} \langle \zeta(\vec{q}) \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c = - \left( 3(N-1) + \sum_{a=1}^N \vec{k}_a \cdot \frac{\partial}{\partial \vec{k}_a} \right) \langle \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c$$

(Exact in tensors)

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(Exact in tensors)

Anisotropic scaling consistency relation  $M_{i\ell_0}^{\text{anisotropic}} = \epsilon_{i\ell_0}^s$

$$\begin{aligned} \lim_{\vec{q} \rightarrow 0} \frac{1}{P_\gamma(q)} \langle \gamma^s(\vec{q}) \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c = & - \frac{1}{2} \epsilon_{i\ell_0}^s(\hat{q}) \sum_{a=1}^N \left\{ k_a^i \frac{\partial}{\partial k_a^{\ell_0}} \langle \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c \right. \\ & \left. - \frac{1}{2} \langle \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_{a-1}, \vec{k}_{a+1}, \dots, \vec{k}_N) \gamma_{i\ell_0}(\vec{k}_a) \rangle'_c \right\} + \dots \end{aligned}$$

## Known Consistency Relations (cont'd)

- $n = 1$  (linear-gradient) relations:

Creminelli, Norena & Simonovic, 1203.4595;  
Goldberger, Hui & Nicolis, 1303.1193

SCT consistency relation

$$M_{i\ell_0\ell_1}^{\text{SCT}} = b_{\ell_1}\delta_{i\ell_0} + b_{\ell_0}\delta_{i\ell_1} - b_i\delta_{\ell_0\ell_1}$$

$$\lim_{\vec{q} \rightarrow 0} \frac{\partial}{\partial q^i} \left( \frac{1}{P_\zeta(q)} \langle \zeta(\vec{q}) \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c \right) = -\frac{1}{2} \sum_{a=1}^N \left( 6 \frac{\partial}{\partial k_a^i} - k_a^i \frac{\partial^2}{\partial k_a^j \partial k_a^j} + 2k_a^j \frac{\partial^2}{\partial k_a^j \partial k_a^i} \right) \langle \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c + \dots$$

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Linear-gradient tensor relation

$$M_{i\ell_0\ell_1}^{\text{tensor}} = q_{\ell_1}\epsilon_{i\ell_0}^s + q_{\ell_0}\epsilon_{i\ell_1}^s - q_i\epsilon_{\ell_0\ell_1}^s$$

$$\begin{aligned} \lim_{\vec{q} \rightarrow 0} q^{\ell_1} \frac{\partial}{\partial q^{\ell_1}} \left( \frac{1}{P_\gamma(q)} \langle \gamma^s(\vec{q}) \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c \right) = & -\frac{1}{2} q^{\ell_1} \epsilon_{i\ell_0}^s(\vec{q}) \sum_{a=1}^N \left\{ \left( k_a^i \frac{\partial}{\partial k_a^{\ell_1}} - \frac{k_a^{\ell_1}}{2} \frac{\partial}{\partial k_i^a} \right) \frac{\partial}{\partial k_{\ell_0}^a} \langle \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_N) \rangle'_c \right. \\ & - \left( 2\Upsilon^{i\ell_0 i_a j_a}(\hat{k}_a) \frac{\partial}{\partial k_{\ell_1}^a} - \Upsilon^{\ell_1 i i_a j_a}(\hat{k}_a) \frac{\partial}{\partial k_{\ell_0}^a} \right) \\ & \times \left. \langle \mathcal{O}^\zeta(\vec{k}_1, \dots, \vec{k}_{a-1}, \vec{k}_{a+1}, \dots, \vec{k}_N) \gamma_{i\ell_0}(\vec{k}_a) \rangle'_c \right\} + \dots \end{aligned}$$

# Example of New Consistency Relation

•  $n = 2$  tensor relation:

$$\lim_{\vec{q} \rightarrow 0} M_{i\ell_0\ell_1\ell_2}^T(\hat{q}) \frac{\partial^2}{\partial q_{\ell_1} \partial q_{\ell_2}} \left( \frac{1}{P_\gamma(q)} \langle \gamma^{i\ell_0}(\vec{q}) \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle' \right) = -M_{i\ell_0\ell_1\ell_2}^T(\hat{q}) \sum_{a=1}^2 \frac{k_a^i}{3} \frac{\partial^3 \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'}{\partial k_{\ell_0}^a \partial k_{\ell_1}^a \partial k_{\ell_2}^a}$$

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•  $n = 2$  tensor relation:

$$\lim_{\vec{q} \rightarrow 0} M_{i\ell_0\ell_1\ell_2}^T(\hat{q}) \frac{\partial^2}{\partial q_{\ell_1} \partial q_{\ell_2}} \left( \frac{1}{P_\gamma(q)} \langle \gamma^{i\ell_0}(\vec{q}) \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle' \right) = -M_{i\ell_0\ell_1\ell_2}^T(\hat{q}) \sum_{a=1}^2 \frac{k_a^i}{3} \frac{\partial^3 \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'}{\partial k_{\ell_0}^a \partial k_{\ell_1}^a \partial k_{\ell_2}^a}$$

Check using Maldacena's 3-pt function:

$$\frac{1}{P_\gamma(q)} \langle \gamma_{i\ell_0}(\vec{q}) \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle' = P_{i\ell_0 j m_0}^T(\hat{q}) \frac{H^2}{4\epsilon k_1^3 k_2^3} k_1^j k_2^{m_0} \left( -K + \frac{(k_1 + k_2)q + k_1 k_2}{K} + \frac{q k_1 k_2}{K^2} \right).$$

where  $K = q + k_1 + k_2$

2-pt function:  $\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle' = \frac{H^2}{4\epsilon k_1^3}$

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LHS:  $\lim_{\vec{q} \rightarrow 0} M_{i\ell_0\ell_1\ell_2}^T(\hat{q}) \frac{\partial^2}{\partial q_{\ell_1} \partial q_{\ell_2}} \left( \frac{1}{P_\gamma(q)} \langle \gamma^{i\ell_0}(\vec{q}) \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'_c \right) = \frac{H^2}{4\epsilon k_1^3} \frac{35}{k_1^2} M_{i\ell_0\ell_1\ell_2}^T(\hat{q}) \hat{k}_1^i \hat{k}_1^{\ell_0} \hat{k}_1^{\ell_1} \hat{k}_1^{\ell_2}$

RHS:  $-M_{i\ell_0\ell_1\ell_2}^T(\hat{q}) \sum_{a=1}^2 \frac{k_a^i}{3} \frac{\partial^3 \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'_c}{\partial k_a^{\ell_0} \partial k_a^{\ell_1} \partial k_a^{\ell_2}} = \frac{H^2}{4\epsilon k_1^3} \frac{35}{k_1^2} M_{i\ell_0\ell_1\ell_2}^T(\hat{q}) \hat{k}_1^i \hat{k}_1^{\ell_0} \hat{k}_1^{\ell_1} \hat{k}_1^{\ell_2}$

# A Non-Trivial Check (suggested by Daniel Green)

Consider

$$\mathcal{L}_3 \sim \dot{\zeta}^3$$

- No corresponding  $\gamma \dot{\zeta}^2$  term in the action
- This operator is invariant under the symmetries (since  $\delta \dot{\zeta} = 0$ )

Therefore we expect, for all  $n$ ,

$$\lim_{\vec{q} \rightarrow 0} M_{ii\ell_1 \dots \ell_n}(\hat{q}) \frac{\partial^n}{\partial q_{\ell_1} \cdots \partial q_{\ell_n}} \left( \frac{1}{P_\zeta(q)} \langle \zeta(\vec{q}) \zeta(\vec{k}_1) \zeta(\vec{k}_2) \rangle'_{\dot{\zeta}^3} \right) = 0$$

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Chen et al., hep-th/0605045:

$$\frac{1}{P_\zeta(q)} \langle \zeta(\vec{q}) \zeta(\vec{k}_1) \zeta(\vec{k}_2) \rangle'_{\dot{\zeta}^3} = A \frac{q^2}{k_1 k_2 (q + k_1 + k_2)^3}$$

Checked this satisfies identity up to  $n = 3$ .

# Conclusions

- Adiabatic modes constrained by an infinite number of global symmetries, whose Ward identities lead to an infinite number of novel consistency relations.

- Master consistency relation?      Berezhiani & JK, in progress

$$\frac{1}{3}q_k \Gamma^{\zeta\zeta\zeta}(q, p, q+p) + 2q_j \Gamma_{jk}^{\gamma\zeta\zeta}(q, p, q+p) = q_k \Gamma(p) - p_k \left( \Gamma(q+p) - \Gamma(p) \right).$$

- Multiple soft limits      Creminelli, Joyce, JK & Simonovic, in progress

**Double dilation:**  $\lim_{\vec{q}_1, \vec{q}_2 \rightarrow 0} \frac{\langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'}{P_\zeta(q_1) P_\zeta(q_2)} = \delta_{\text{dil.}}^2 P_\zeta(k_1) - \frac{\langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{-\vec{q}_1 - \vec{q}_2} \rangle'}{P_\zeta(q_1) P_\zeta(q_2)} \delta_{\text{dil.}} P_\zeta(k_1)$

Senatore & Zaldarriaga, 1203.6884; Chen et al. hep-th/0610235

Dilation-SCT:

$$\lim_{\vec{q}_1, \vec{q}_2 \rightarrow 0} \frac{\partial}{\partial q_1^i} \left( \frac{\langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle'}{P_\zeta(q_1) P_\zeta(q_2)} \right) = -\frac{1}{2} \delta_{\text{SCT}} \delta_{\text{dil.}} P_\zeta(k_1) + \frac{1}{2} \frac{\langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{-\vec{q}_1 - \vec{q}_2} \rangle'}{P_\zeta(q_1) P_\zeta(q_2)} \delta_{\text{SCT}} P_\zeta(k_1) - \frac{\partial}{\partial q_1^i} \left( \frac{\langle \zeta_{\vec{q}_1} \zeta_{\vec{q}_2} \zeta_{-\vec{q}_1 - \vec{q}_2} \rangle'}{P_\zeta(q_1) P_\zeta(q_2)} \right) \delta_{\text{dil.}} P_\zeta(k_1)$$

