Geometrical measures for (mildly) non-Gaussian cosmological fields

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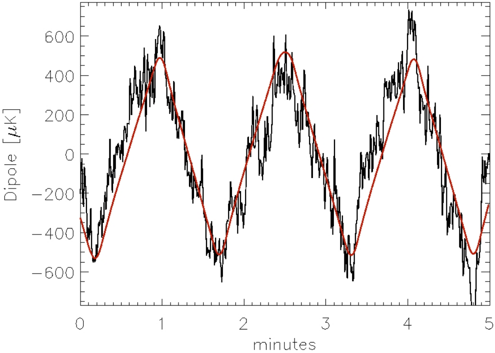
with: S. Codis, C. Gay, C. Pichon (IAP)

Examples of random fields: 1D

Radio-location signal

1940-ies.

This example - CMB time stream data from one of the bolometers of the Boomerang 2003 flight.



Geometrical questions:

- frequency of maxima
- length of high excursions

Geometry of random fields: 2D

Ocean surface waves

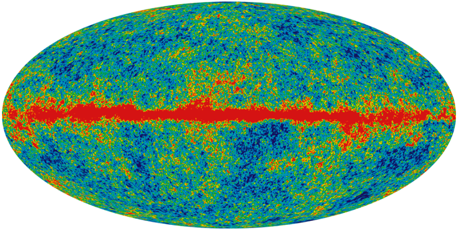
Longuet-Higgins (1957).

Geometrical questions:

- width and shape of the crests,
- distance between troughs
- anisotropy due to the wind

CMB maps (WMAP, Planck)

- Geometrical/topological descriptors as tests of non-gaussianity



Geometry of random fields: 3D

Cosmology

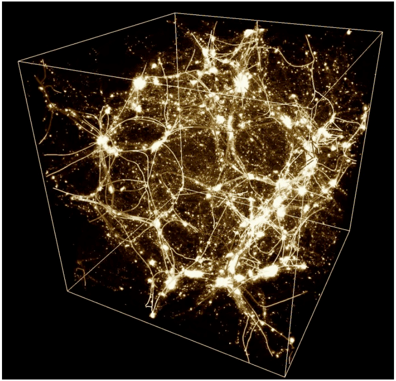
Doroshkevich, 1970

Arnold, Zeldovich, Shandarin, 1982

Bond, Bardeen, Kaiser, Szalay, 1986

Geometrical questions:

- abundance and shapes of peaks of different scale
- Length of filamentary bridges
- Connectivity of peaks by filamentary ridges
- Relation between the properties of filaments and clusters



Horizon project

Primer: how many maxima a field has ?

Field-focused description of a random field $\rho(x)$:

Joint distribution function of the field values at every point

$$\mathscr{P}\left(\rho(x_1), \rho(x_2), \ldots\right)\, d\rho_1\, d\rho_2 \ldots$$

Focus on the correlation function

$$\xi(x_1, x_2) = \langle \rho(x_1)\rho(x_2) \rangle$$

or its Fourier transform, the power spectrum.

Primer: how many maxima a field has ?

Point-based description of a random field $\rho(x)$:

Joint distribution of all the field's derivatives at a same point

$$\mathscr{P}(\rho, \nabla\rho, \nabla\nabla\rho, \ldots)\, d\rho\; d\nabla\rho\; d\nabla\nabla\rho \ldots$$

Focus on variances

$$\sigma_0{}^2 = \langle \rho^2 \rangle, \;\; \sigma_1{}^2 = \langle \nabla\rho \cdot \nabla\rho \rangle, \;\; \sigma_2{}^2 = \langle (\Delta\rho)^2 \rangle, \ldots$$

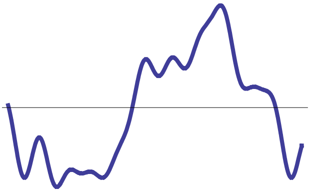
and cross-correlation coefficients

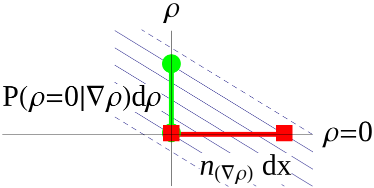
$$\langle \rho \Delta\rho \rangle = \gamma \sigma_0 \sigma_2, \;\; \langle \nabla\rho \cdot \nabla\Delta\rho \rangle = \tilde{\gamma} \sigma_1 \sigma_3, \ldots$$

For isotropic field even and odd derivatives are uncorrelated

$$\mathscr{P}(\rho, \nabla\rho, \nabla\nabla\rho, \nabla\nabla\nabla, \ldots) = \mathscr{P}(\rho, \nabla\nabla\rho, \ldots)\, \mathscr{P}(\nabla\rho, \nabla\nabla\nabla\rho, \ldots)$$

1D level crossing primer



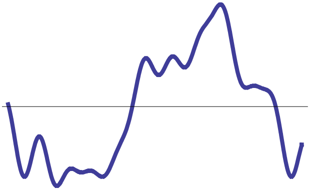


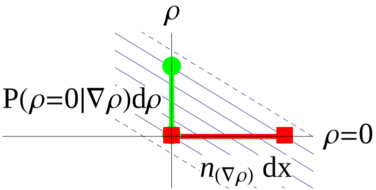
$$\left.\begin{array}{rcl} n_{\nabla\rho}\,\mathrm{dx} & = & \mathscr{P}(\rho=0|\nabla\rho)\,\mathrm{d}\rho \\ \mathrm{d}\rho & = & -\nabla\rho\,\mathrm{dx} \end{array}\right\} \Rightarrow \quad n_{\nabla\rho} = \mathscr{P}(\rho=0|\nabla\rho)\left|\nabla\rho\right|$$

$$n_{zeroes} = \int_{-\infty}^{\infty} \mathrm{d}\nabla\rho\,\left|\nabla\rho\right|\mathscr{P}(\rho=0,\nabla\rho) \propto \frac{\sigma_1}{\sigma_0}$$

$$n_{extrem} = \int_{-\infty}^{\infty} \mathrm{d}\nabla\nabla\rho\,\left|\nabla\nabla\rho\right|\mathscr{P}(\nabla\rho=0,\nabla\nabla\rho) \propto \frac{\sigma_2}{\sigma_1}$$

1D level crossing primer





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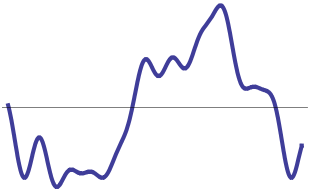

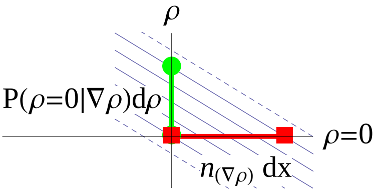

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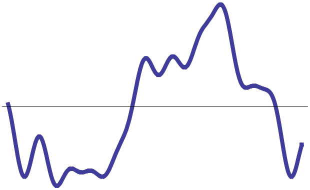

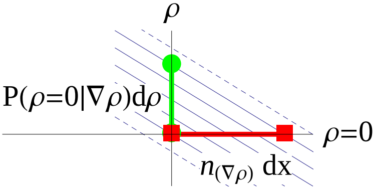

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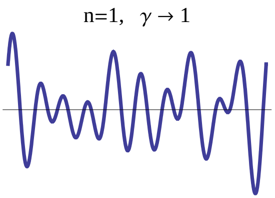
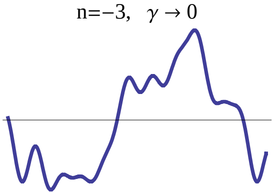
Summary of 1D Gaussian calculations

Fundamental scales R_0, R_*

$$n_{zero} \propto \frac{1}{R_0} = \frac{\sigma_1}{\sigma_0}, \quad n_{maxima} \propto \frac{1}{R_*} = \frac{\sigma_2}{\sigma_1}$$

$$\frac{n_{zero}}{n_{maxima}} = \gamma \equiv \frac{\sigma_1^2}{\sigma_0 \sigma_2} = \frac{R_*}{R_0}, \quad \gamma \in [0,1]$$



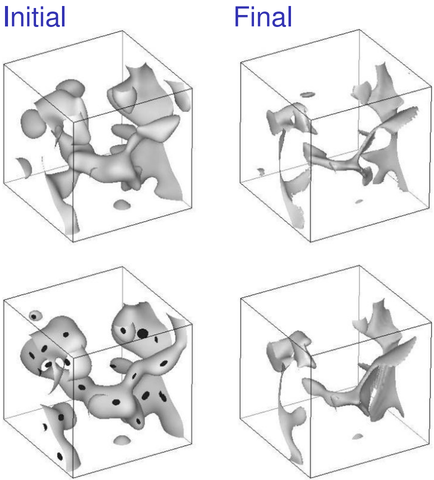


More sophisticated result:
Maxima above threshold $\nu = \rho/\sigma_0$

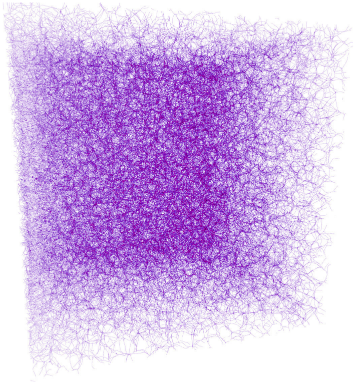
$$n_{max}(\rho > \nu\sigma_0) = \begin{cases} n_{max}\mathscr{P}(\nu) & \gamma \to 0 \\ n_{max}\mathscr{P}(\nu)(\gamma\nu) & \gamma \to 1 \end{cases}$$

From Peaks to Filaments, Skeleton of the Cosmic Web

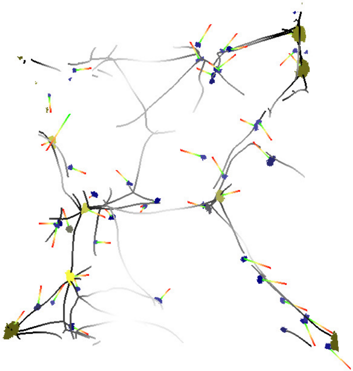
- The high rare peaks of the field largely define how large scale structure looks like.
- The Skeleton of LSS traces the filamentary bridges that form the Web between them and provides the next level of detail to LSS understanding.



Skeleton as tracer of the filaments



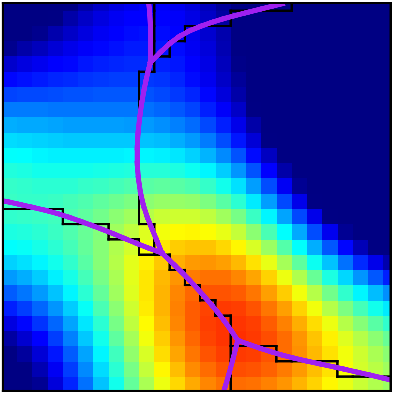
Dark matter skeleton in 4π simulation



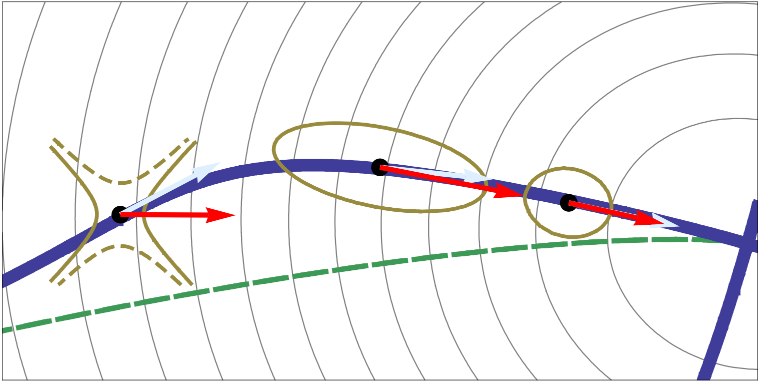
Halo spin vs the skeleton

Codis et al, 2012

Towards the local description of the Skeleton



Towards the local description of the Skeleton



$$(\nabla\nabla\rho)\cdot\nabla\rho = \lambda\nabla\rho$$

Local description of the Skeleton

Definition

Along the ridge the gradient of the field $\nabla\rho$ is aligned with the direction of the least curvature. Formally, skeleton is the set of points where (with additional restrictions)

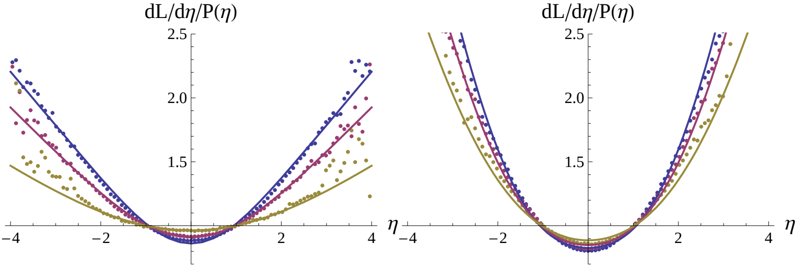
$$S \equiv (\nabla\nabla\rho \cdot \nabla\rho) \times \nabla\rho = 0$$

Features

Local definition introduces point process that allows analytical treatment similar to statistics of extrema.

Stiff Approximation: Theory fits Simulations (of course)

$$\left(\frac{\partial \mathscr{L}}{\partial \nu}\right)_{\rm stiff} \propto \frac{1}{R_*^{N-1}} \int d\lambda_1 \ldots d\lambda_N \mathscr{P}(\nu,\lambda_1,\ldots,\lambda_N) |\lambda_2 \cdots \lambda_N|$$



Notable Result:

At high densities $\nu = \rho/\sigma_0$ density of filaments is enhanced

$$2D: \propto \gamma\nu \;,\;\; 3D: \propto (\gamma\nu)^2$$

Geometrical measures for random fields

Starting point

Let us think about such properties of a random field ρ as Euler characteristic (genus), density of maxima, length of skeleton. Their computation reguire knowledge of the joint distribution

$$\mathscr{P}(\rho, \rho_i, \rho_{ij}, \ldots)$$

of the field ρ and its first ρ_i, second ρ_{ij} (Hessian matrix) and perhaps higher derivatives, for instance

$$n_{max}(\nu) = \int_{0 \geq \lambda_1 \geq \lambda_2 \geq \ldots} \mathscr{P}(\rho = \nu, \rho_i = 0, \rho_{ij}) |\rho_{ij}| d\rho_{ij}$$

Usual approach is to deal with it in the Hessian eigenvalue space, since that's where the boundary conditions are the simplest.

Non-Gaussian expansion for geometrical statistics

- To treat non-Gaussianities the idea is to expand $\mathscr{P}(\rho, \rho_i, \rho_{ij})$ into orthogonal polinomials around the Gaussian approximation like
$$P(x) = G(x)\left(1 + \sum_n \langle x^n \rangle_c H_n(x)\right)$$
- The trick to avoid difficulties is an appropriate choice of variables:
    - that are invariant wrt symmetries of the problem (isotropy)
    - that are polynomial in the field quanities (λ's are no good)
    - that simplify the Gaussian limit, being as uncorrelated as possible
- Useful set is: $I_1, \cdots, I_N,\ q^2,\ \zeta \equiv \frac{\rho + \gamma I_1}{1-\gamma^2}$
  where I_n are N polynomial rotation invariants of the Hessian matrix ρ_{ij},
$$I_1 = \mathrm{Tr}\,\rho_{ij} \quad , \ldots , \quad I_N = \det|\rho_{ij}|$$
  (and $I_2 \ldots I_{N-1}$ are built from the minors of orders 2 to N-1)
- Actually, better to use more 'irreducible' combinations J_i, in ND-space

$$J_1 = I_1 \ , \quad J_{s\geq 2} = I_1^s - \sum_{p=2}^{s} \frac{(-N)^p C_s^p}{(s-1) C_N^p} I_1^{s-p} I_p$$

Orhogonal polymonial expansion for 2D $\mathscr{P}(\rho, q^2, J_1, J_2)$

Gaussian limit JPDF

$$G_{\rm 2D}(\zeta, q^2, J_1, J_2)\,{\rm d}\zeta{\rm d}q^2{\rm d}J_1{\rm d}J_2 = \frac{1}{2\pi} e^{-\frac{1}{2}(\zeta^2+2q^2+J_1^2+2J_2)}{\rm d}\zeta{\rm d}q^2{\rm d}J_1{\rm d}J_2$$

serves as the weight for defining the expansion polynomials in

- ζ, J_1 – ($[-\infty,\infty]$, gaussian weight) – Hermite
- q^2, J_2 – ($[0,\infty]$, exponential weight) – Laguerre.

$$\mathscr{P}_{\rm 2D}(\zeta, q^2, J_1, J_2) = G_{\rm 2D} \times \Big[1 + \sum_{n=3}^{\infty} \sum_{i,j,k,l=0}^{i+2j+k+2l=n} \frac{(-1)^{j+l}}{i!\,j!\,k!\,l!} \left\langle \zeta^i q^{2j} {J_1}^k {J_2}^l \right\rangle_{\rm GC} H_i(\zeta) L_j\left(q^2\right) H_k(J_1) L_l(J_2)\Big]$$

This is an expansion to all orders in powers of the field n.

"Gram-Charlier" coefficients of expansion $\langle \ldots \rangle_{\mathrm{GC}}$

are

$$\left\langle \zeta^i q^{2j} J_1^k J_2^l \right\rangle_{\mathrm{GC}} = \frac{j!\, l!}{(-1)^{j+l}} \left\langle H_i(\zeta) L_j\left(q^2\right) H_k(J_1) L_l(J_2) \right\rangle$$
$$= \left\langle \zeta^i q^{2j} J_1^k J_2^l \right\rangle + \ldots$$

which for order $n = i + k + 2j + 2l$

- $n = 3$ are equal to the simple moments, $\langle \zeta J_2 \rangle_{\mathrm{GC}} = \langle \zeta J_2 \rangle$
- $n \leq 5$ are equal to cumulants, $\langle \zeta q^2 J_1 \rangle_{\mathrm{GC}} = \langle \zeta q^2 J_1 \rangle_c$
  where in the cumulants our quadratic variables are reverted to field quantities

⇒ expansion coefficients can be predicted by pertrubation theories

Orhogonal polymonial expansion for 3D $\mathscr{P}(\rho, q^2, J_1, J_2, J_3)$

Gaussian limit JPDF

$$G_{\rm 3D}\,{\rm d}\zeta{\rm d}q^2{\rm d}J_1{\rm d}J_2{\rm d}J_3 = \frac{25\sqrt{15}}{8\pi^2}\sqrt{q^2}\,e^{-\frac{1}{2}(\zeta^2+3q^2+J_1^2+5J_2)}{\rm d}\zeta{\rm d}q^2{\rm d}J_1{\rm d}J_2{\rm d}J_3$$

serves as the weight for defining the polynomials in $\zeta, q^2, J_1, J_2, J_3$.

$$\mathscr{P}_{\rm 3D}(\zeta, q^2, J_1, J_2, J_3) = G_{\rm 3D}\times\Bigg[1+$$

$$\sum_{n=3}^{\infty}\sum_{i,j,k,l=0}^{i+2j+k+2l=n}\frac{(-1)^{j+l}3^j5^l\times 3}{i!\,(1+2j)!!\,k!\,(3+2l)!!}\left\langle\zeta^i q^{2j}J_1{}^kJ_2{}^l\right\rangle_{\rm GC}H_i(\zeta)L_j^{(1/2)}\left(\frac{3}{2}q^2\right)H_k(J_1)L_l^{(3/2)}\left(\frac{5}{2}J_2\right)$$

$$+\sum_{n=3}^{\infty}\sum_{i,j,k=0}^{i+2j+k+3=n}\frac{(-1)^j3^j\times 25}{i!\,(1+2j)!!\,k!\times 21}\left\langle\zeta^i q^{2j}J_1{}^kJ_3\right\rangle_{\rm GC}H_i(\zeta)L_j^{(1/2)}\left(\frac{3}{2}q^2\right)H_k(J_1)J_3$$

$$+\sum_{n=5}^{\infty}\sum_{i,j,k,l=0,m=1}^{i+2j+k+2l+3m=n}\frac{(-1)^j3^jc_{lm}}{i!\,(1+2j)!!\,k!}\left\langle\zeta^i q^{2j}J_1{}^kJ_2{}^lJ_3{}^m\right\rangle_{\rm GC}H_i(\zeta)L_j^{(1/2)}\left(\frac{3}{2}q^2\right)H_k(J_1)F_{lm}(J_2,J_3)\Bigg]$$

Euler characteristic (genus) as a function of threshold

General expression in ND

$$\frac{\chi(\nu)}{2} = (-1)^N \int_\nu^\infty dx \int dq^2 q^{N-1} \delta_D^N(q^2) \int \prod_{s=1}^N dJ_s \mathscr{P}_{\rm ND}(\ldots)\, I_N$$

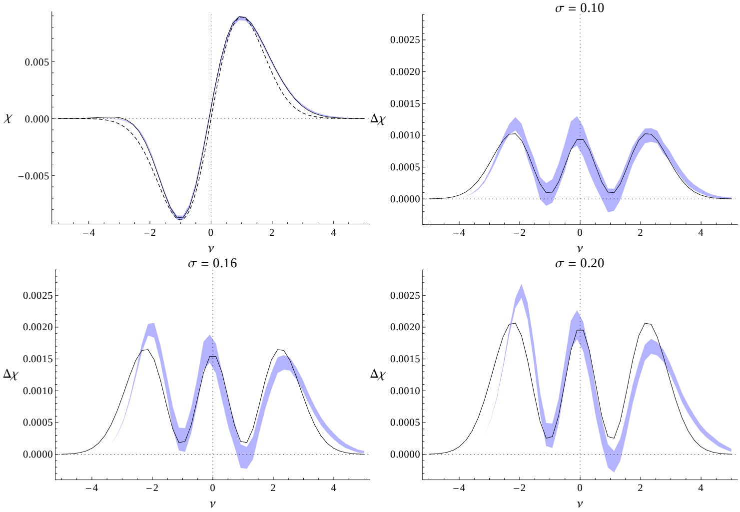
can be integrated to give "moment" expansion to all orders

$$\chi(\nu) = \frac{1}{\sqrt{2\pi}R_*} \exp\left(-\frac{\nu^2}{2}\right) \times \frac{2}{(2\pi)^{N/2}} \left(\frac{\gamma}{\sqrt{N}}\right)^N \Bigg[H_{N-1}(\nu) + \sum_{n=3}^{\infty} \sum_{s=0}^{N} \gamma^{-s} \sum_{i,j=0}^{i+2j=n-s} \frac{(-N)^{j+s}(N-2)!! L_j^{(\frac{N-2}{2})}(0)}{i!(2j+N-2)!!} \left\langle x^i q^{2j} I_s \right\rangle_{GC} H_{i+N-s-1}(\nu) \Bigg]$$

(cf first term: Matsubara 1994-2005)

with coefficients that can be predicted by perturbation theories

How non-Gaussianity in Euler charachteristic develops, 2D



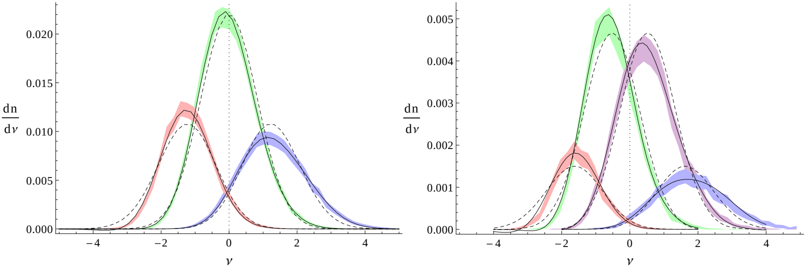
Extrema counts and more

Formalism to study popular and novel geometrical statistics in non-gaussian regime

- Genus in ND is fully described
- 2D Extrema density (CMB case) computed analytically
- 3D Extrema density (LSS)

$$n_{\mp--} = \frac{29\sqrt{15}\mp 18\sqrt{10}}{1800\pi^2 R_*^3} + \frac{5\sqrt{5}}{24\pi^2\sqrt{6\pi}R_*^3}\left(\left\langle q^2 J_1\right\rangle - \frac{8}{21}\left\langle {J_1}^3\right\rangle + \frac{10}{21}\left\langle J_1 J_2\right\rangle\right)$$

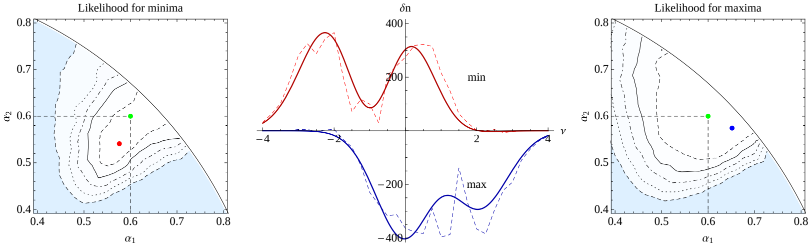
$$n_{++\pm} = \frac{29\sqrt{15}\mp 18\sqrt{10}}{1800\pi^2 R_*^3} - \frac{5\sqrt{5}}{24\pi^2\sqrt{6\pi}R_*^3}\left(\left\langle q^2 J_1\right\rangle - \frac{8}{21}\left\langle {J_1}^3\right\rangle + \frac{10}{21}\left\langle J_1 J_2\right\rangle\right)$$



- Peak shapes, skeleton length, correlation of velocity flow and filaments

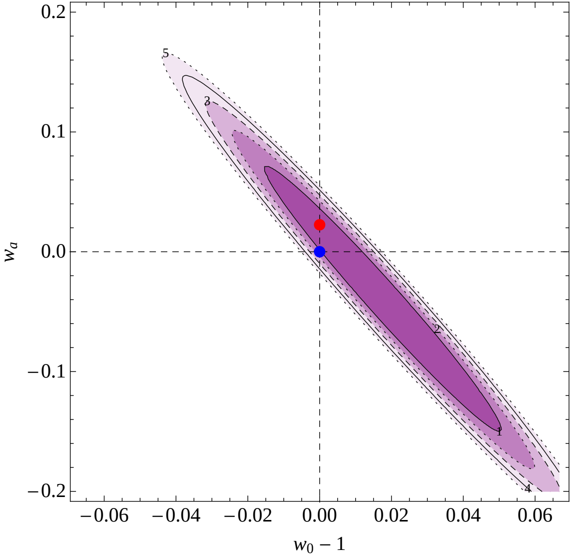
Extrema count in CMB maps

Toy Harmonic Oscillator model:



- Nside=1024, single realization
- FWHM=10' ($R_* \approx 11' \approx 3$ pixels), $\approx 50,000$ minima/maxima
- $1\sigma - 5\sigma$ contours
- Anne Ducout (IAP) works on application to Planck data.

Applications: determining $D(z)$ growth factor



Extension to anisotropic statistics

- In cosmology we do not always deal with isotropic fields, or statistics
- Obvious example - density in the redshift space
- more subtle case – skeleton statistics of the filaments are not isotropic ! (there is a angle dependence between local directions of gradient and Hessian)
- The generalization of our approach is straightforward - use symmetries that remain, define polynomial variables, find uncorrelated combinations and expansion polynomials as determined by the Gaussian limit.
- Done by Christophe Gay for the skeleton (Gay, Pichon, Pogosyan, 2011) and Sandrine Codis for the redshift space

Polymonial expansion for $\mathscr{P}_{3D}$ in redshift space

Symmetry:

Rotational around the line of sight (LOS along 3rd coordinate)

Variables:

linear(4) $x,\ x_3,\ x_{33},\ J_{1\perp} = x_{11} + x_{22}$

quadratic(3) $q^2 = x_1^2 + x_2^2,\ Q^2 = x_{13}^2 + x_{23}^2,\ J_{2\perp} = (x_{11} - x_{22})^2 + 4x_{12}^2$

cubic(1) $\Upsilon = \left(x_{13}^2 - x_{23}^2\right)(x_{11} - x_{22}) + 4x_{12}x_{13}x_{23}$

Gaussian limit JPDF

$$G(x, q_\perp^2, x_3\,\zeta, J_{2\perp}, \xi, Q^2, \Upsilon) = \frac{1}{4\pi^3\sqrt{Q^4 J_{2\perp} - \Upsilon^2}}\, e^{-\frac{1}{2}x^2 - q_\perp^2 - \frac{1}{2}x_3^2 - \frac{1}{2}\zeta^2 - J_{2\perp} - \frac{1}{2}\xi^2 - Q^2}$$

Uniformly distributed $\Upsilon \in \left[-Q^2\sqrt{J_{2\perp}}, +Q^2\sqrt{J_{2\perp}}\right]$ can be integrated over for Minkowski functionals, extrema ...

Euler characteristic in redshift space

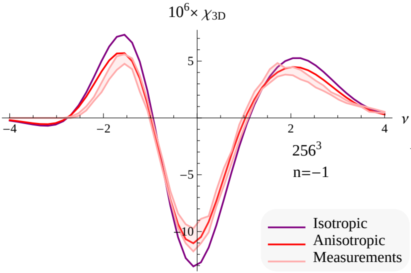
$$\chi_{2+1}(\nu) = \frac{e^{-\nu^2/2}}{8\pi^2}\left[\frac{\sigma_{1\parallel}\sigma_{1\perp}^2}{\sigma^3}H_2(\nu) + \sum_{n=3}^{\infty}\chi_{2+1}^{(n)}\right]$$

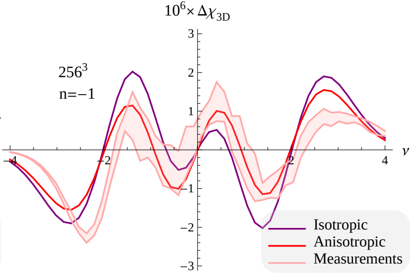
with non-Gaussian corrections $\chi_{2+1}^{(n)}$, given, to all orders, by

$$\begin{aligned}
\chi_{2+1}^{(n)}(\nu) \quad = \quad & \frac{\sigma_{2\perp}^2\sigma_{2\parallel}}{\sigma_{1\perp}^2\sigma_{1\parallel}}\Bigg[\sum_{\sigma_n}\frac{(-1)^{j+m}}{2^m\, i!\, j!\, m!}H_{i+2}(\nu)\gamma_\parallel\gamma_\perp^2\left\langle x^i q_\perp^{2j} x_3^{2m}\right\rangle_{\rm GC} \\
- & \sum_{\sigma_{n-1}}\frac{(-1)^{j+m}}{2^m\, i!\, j!\, m!}H_{i+1}(\nu)\left\langle x^i q_\perp^{2j} x_3^{2m}\left(\gamma_\perp^2 x_{33} + 2\gamma_\perp\gamma_\parallel J_{1\perp}\right)\right\rangle_{\rm GC} \\
+ & \sum_{\sigma_{n-2}}\frac{(-1)^{j+m}}{2^m\, i!\, j!\, m!}H_i(\nu)\left\langle x^i q_\perp^{2j} x_3^{2m}\left(2\gamma_\perp(J_{1\perp}x_{33} - \gamma_2 Q^2) + \gamma_\parallel(J_{1\perp}^2 - J_{2\perp})\right)\right\rangle_{GC} \\
- & \sum_{\sigma_{n-3}}\frac{(-1)^{j+m}}{2^m\, i!\, j!\, m!}H_{i-1}(\nu)\left\langle x^i q_\perp^{2j} x_3^{2m}\left(x_{33}(J_{1\perp}^2 - J_{2\perp}) - 2\gamma_2\left(Q^2 J_{1\perp} - \Upsilon\right)\right)\right\rangle_{\rm GC}\Bigg]
\end{aligned}$$

Euler characteristics in redshift space

$\sigma = 0.18, \ f = 1$





Conclusions

- We progressed beyond fixed-order non-gaussian corrections, to complete representation of geometrical statistics via the moments that are predicted by the theory. Examples are perturbation theory of the gravitational clustering or non-linear effects at inflationary stage.
- Direct utilization of the symmetries of the problem makes many previously untractable calculations possible.
- We now know the non-gaussian expansion to all orders (and in arbitrary dimension) for the Euler statistic, and isosurface area.
- Dependence of extrema count on non-gaussian corrections is understood.
- Extension of the formalism to redshift space has been developed.
- The path is open for application of geometrical descriptors to CMB data, $D(z)$ reconstruction, etc.