Loop Amplitudes and Twistor Space

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KITP,
June 11, 2004

recent work with I. Bena, V. Del Duca, L. Dixon, D. Kosower, R. Roiban

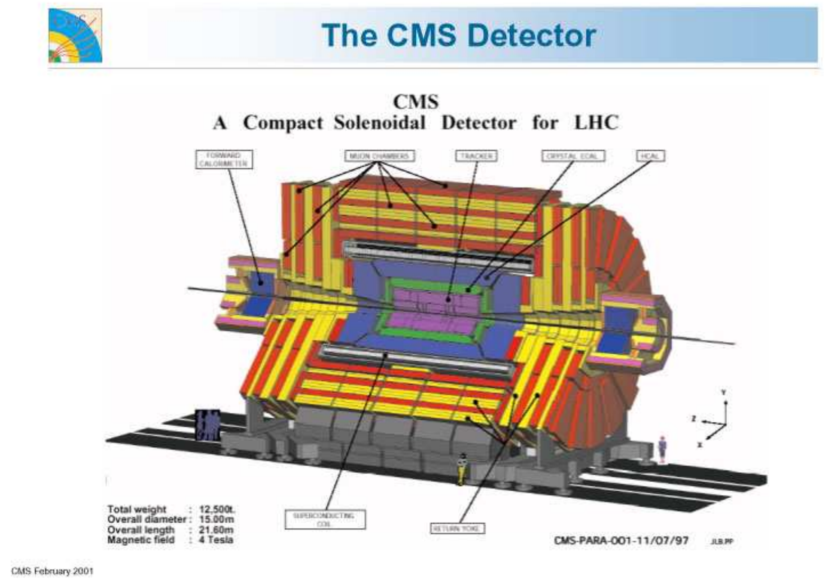
Papers:
Bern, Dixon and Kosower, hep-ph/9306240
Bern, Dixon, Dunbar and Kosower, hep-ph/9403226, hep-ph/9409265
Bern and Morgan, hep-ph/9511336
Bern, Rozowsky and Yan, hep-ph/9702424
Anastasiou, Bern, Dixon, and Kosower, hep-th/0309040
Bern, Dixon, Kosower, hep-ph/0404293
Bena, Bern and Kosower, hep-th/0406133
Bena, Bern, Kosower and Roiban, hep-th/0410054
Bern, Del Duca, Dixon and Kosower, hep-th/0410224.

Outline

- Motivation
  (a) QCD and applications to colliders, especially the LHC
  (b) Try to solve $N = 4$ maximally supersymmetric Yang-Mills theory
- Application of twistor space to QCD
  (a) CSW rules
  (b) Efficient recursive reformulation
  (c) Issues with loops
- $N = 4$ super-Yang-Mills loop amplitudes
  (a) Unitarity method
  (b) Powerful new twistor space tools
  (c) Twistor space structure
  (d) Higher loops
- Some important remaining issues
- Summary and Outlook.

Collider Physics

The issues of perturbation theory in quantum field theory are central to particle physics. Entire month of the 2004 KITP collider physics workshop was devoted to the issues of pushing QCD cross-section calculations to higher order.



Enormous resources devoted to these experiments

$N = 4$ Super-Yang-Mills

See also talks by Beisart and Staudacher

In 1974 't Hooft suggested that we could solve QCD in the planar limit.

This is too hard. We should look instead at a simpler theory.

$N = 4$ super-Yang-Mills is by far the simplest $D = 4$ gauge theory.

$N = 4$ theory is a cousin of QCD, but with specially arranged matter.

1 gluon, 4 real fermions and 6 scalars.

- $N = 4$ super-Yang-Mills is a conformal field theory (CFT). UV finite.
- It is the CFT appearing in Maldacena's AdS/CFT correspondence.
- Maldacena conjecture suggests a magical simplicity, especially in the planar limit with strong coupling – dual to weakly coupled gravity.

Can we solve $N = 4$ super-Yang-Mills theory?

This is an important question not just in string theory community.

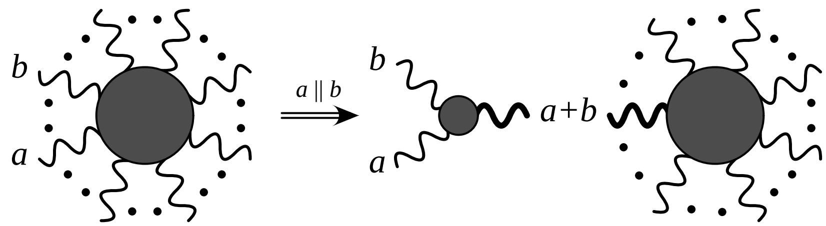
Anomalous Dimensions and Scattering Amplitudes

Two branches of recent study:

- Anomalous dimensions
- Scattering amplitudes

Can we relate these? There is at least one way to do so:

Splitting amplitudes:



Splitting amplitudes $\longrightarrow$ DGLAP splitting functions $\longrightarrow$ Anomalous dimensions of leading twist operators.

In QCD Gross and Wilczek, and Georgi and Politzer determined DGLAP evolution by computing these anomalous dimensions.

In the $N = 4$ theory obtained to three-loop order using the QCD calculation of Moch, Vermaseren, and Vogt

A.V. Kotikov, L.N. Lipatov, A.I. Onishchenko, and V.N. Velizhanin

Helicity

Xu, Zhang and Chang
F.A.Berends, R.Kleiss, P.De Causmaecker
R. Gastmans and T. T. Wu
J.F. Gunion and Z. Kunszt
& many others

Vector polarizations

$$\varepsilon^{+}_{\mu}(k;q) = \frac{\langle q^{-}|\,\gamma_\mu\,|k^{-}\rangle}{\sqrt{2}\,\langle q\,k\rangle}, \qquad \varepsilon^{-}_{\mu}(k,q) = \frac{\langle q^{+}|\,\gamma_\mu\,|k^{+}\rangle}{\sqrt{2}\,[k\,q]}$$

More sophisticated version of circular polarization: $\varepsilon^{\pm}_{\mu} = (0,1,\pm i,0)$

All required properties of polarization vectors satisfied:

$$\varepsilon_i^2 = 0\,, \qquad k\cdot\varepsilon(k,q) = 0\,, \qquad \varepsilon^{+}\cdot\varepsilon^{-} = -1$$

Notation $\quad \epsilon^{ab}\lambda_{ja}\lambda_{lb} \longleftrightarrow \langle j\,l\rangle = \langle k_{j\,-}|k_{l+}\rangle = \sqrt{2k_j\cdot k_l}\;e^{i\phi}$

$$\epsilon_{\dot a\dot b}\tilde\lambda_j^{\dot a}\tilde\lambda_l^{\dot b} \longleftrightarrow [j\,l] = \langle k_{j\,+}|k_{l\,-}\rangle = -\sqrt{2k_j\cdot k_l}\;e^{-i\phi}$$

Changes in reference momentum q are equivalent to gauge transformations.

Twistor Space and Topological String Theory

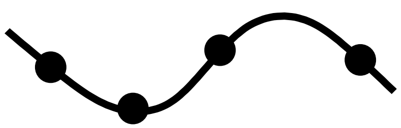
Discussed already in Freddy Cachazo's talk

In a beautiful paper Ed Witten demonstrated that "twistor space" can reveal hidden structures of scattering amplitudes. Precursor from Nair.

Link to string theory is for $N = 4$ super-Yang-Mills theory, but at tree level it might as well be QCD.

Twistor space given by Fourier transform with respect to plus helicity spinors.

$$\widetilde{A}(\lambda_i, \mu_i) = \int \prod_i \frac{d^2\widetilde{\lambda}_i}{(2\pi)^2} \, \exp\left(\sum_j \mu_j^{\dot{a}} \widetilde{\lambda}_{j\dot{a}}\right) A(\lambda_i, \widetilde{\lambda}_i)$$



Tree-level QCD scattering amplitudes $\leftrightarrow$ 'Twistor-space' $\leftrightarrow$ Topological String Theory

E. Witten; Roiban, Spradlin, and Volovich

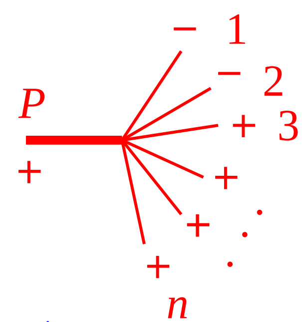
Witten observed that in twistor space external points lie on certain curves. Very constraining. Non-trivial Duality

MHV Vertices

Described already in Freddy Cachazo's talk

Motivated by twistor space structure Cachazo, Svrcek and Witten define an off-shell "MHV vertex" based on Parke-Taylor amplitudes

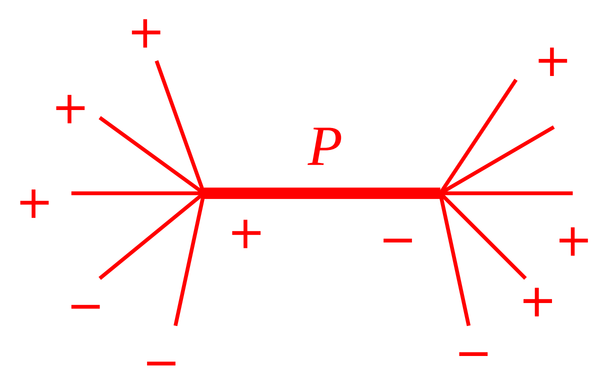
$$V(1^-, 2^-, 3^+, \ldots, n^+, P^+) = \frac{\langle 1\, 2\rangle^4}{\langle 1\, 2\rangle \cdots \langle n-1,\, n\rangle \langle n\, P\rangle \langle P\, 1\rangle}$$



Continue spinor off-shell $(P^2 \neq 0)$: $\langle i\, P\rangle = \eta \sum_{j=1}^{n} \left\langle i^- \right| \not{k}_j \left| q^- \right\rangle$
where $P = k_1 + k_2 + \cdots k_n$ and q auxiliary, satisfying $q^2 = 0$.

Non-MHV amplitudes obtained by sewing together MHV vertices.

Holds generally for any massless gauge theory, including QCD. Georgiou and Khoze



Applications to QCD Phenomenology

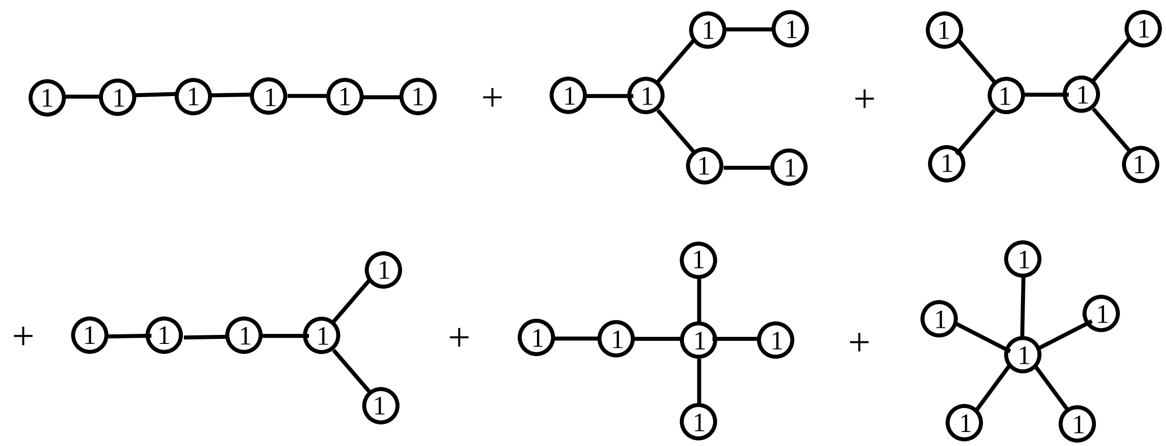
Feynman diagrams are extremely inefficient; factorial growth.

$$gg \to 8g : 10,525,900 \text{ diagrams}, \qquad gg \to 10g : 5,348,843,500 \text{ diagrams};$$

Efficient methods has been used in QCD for many years. In particular, recursive methods. Berends and Giele; Caravaglios, Mangano, Moretti and Pittau

CSW diagrams, however, also exhibit exponential growth in complexity as the number of negative helicity legs increase. Spradlin, Roiban and Volovich

CSW skeleton diagrams



There should be some way to rearrange CSW diagrams to avoid the exponential increase.

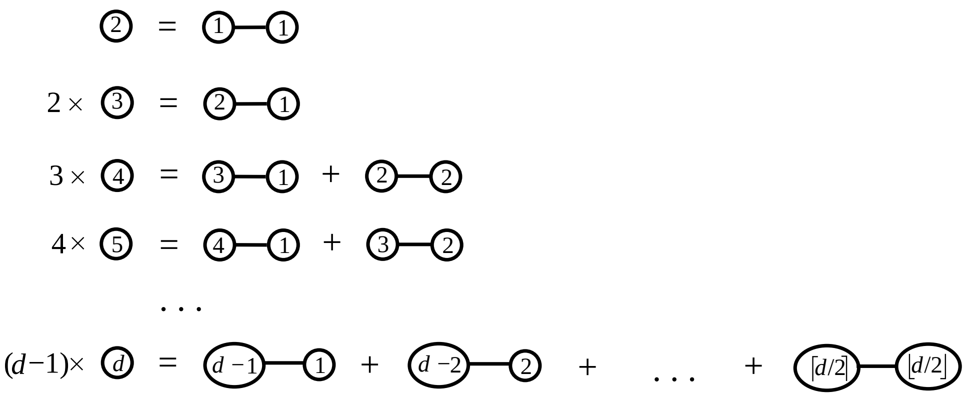
Recursive approach to CSW

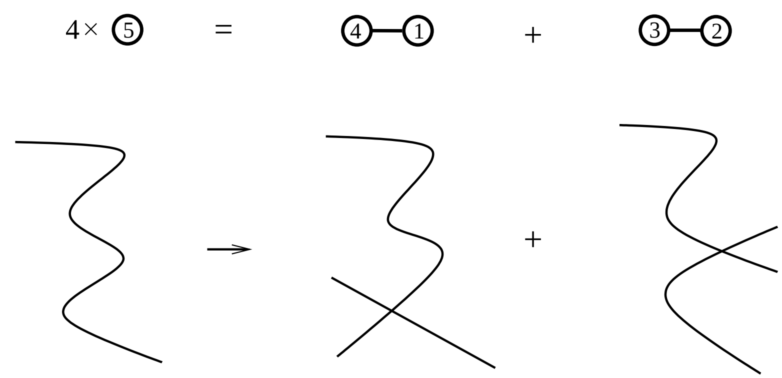
Bena, Bern and Kosower, hep-th/0406133

Can we rearrange the CSW diagrams in such a way so as to reduce the exponential growth to polynomial growth?

Idea: Introduce non-MHV vertices. Combines CSW diagrams with recursive ideas.

Define non-MHV vertices recursively.





Extremely efficient calculational method.

non-MHV vertices have a very interesting twistor space interpretation in terms of various degree curves.

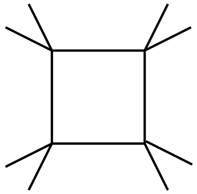
Gukov, Motl and Neitzke

Loop Amplitudes

Bern, Dixon and Kosower, hep-ph/9306240
Bern, Dixon, Dunbar, Kosower
hep-ph/9403226,9409265
Bern and Morgan, hep-ph/9511336

Summary of our early papers on the subject:

- Key Theorem: *Any* amplitude in any massless theory is fully determined from D-dimensional tree amplitudes to *all* loop orders. Off-shell formulations unnecessary. Unitarity is all that is necessary.
- Four-dimensional cut constructibility: At one-loop, any amplitude in a massless susy gauge theory is full constructible from *four-dimensional* tree amplitudes (even in the presence of IR and UV singularities).
- Basis of integrals: Any dimensionally regularized one-loop gauge theory amplitude is expressible in terms of basis of scalar integrals. For the $N = 4$ theory only scalar box integrals appear.

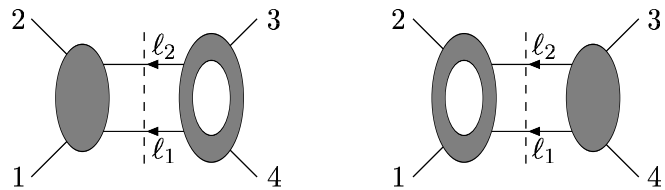


- Simplicity: The one-loop $N = 4$ amplitudes are much much simpler than they ought to be. Twistor space and toplogical string theory finally points to the origin of this simplicity.

Generalized Cuts

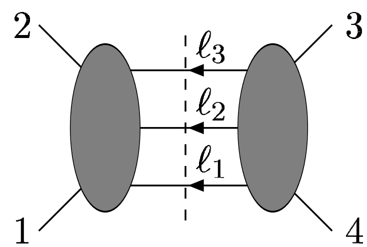
Bern, Dixon and Kosower, hep-ph/9708239
hep-ph/0404293

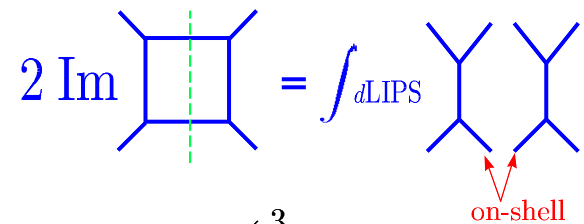
Two-particle cuts:



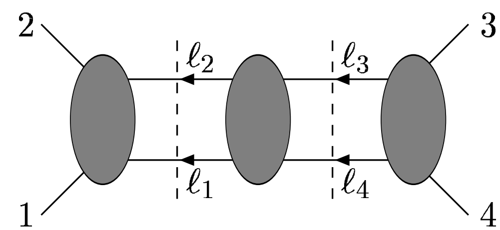
intermediate legs on shell

Three-particle cuts:





Generalized double two-particle cut:

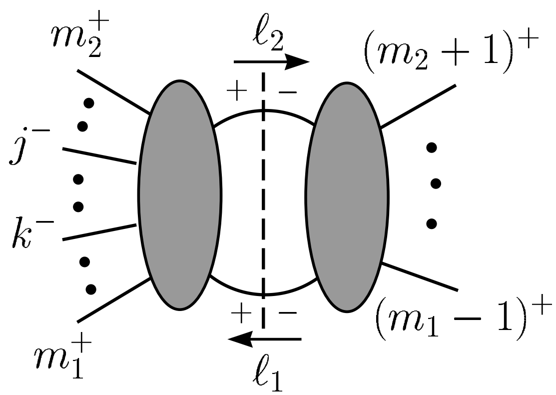


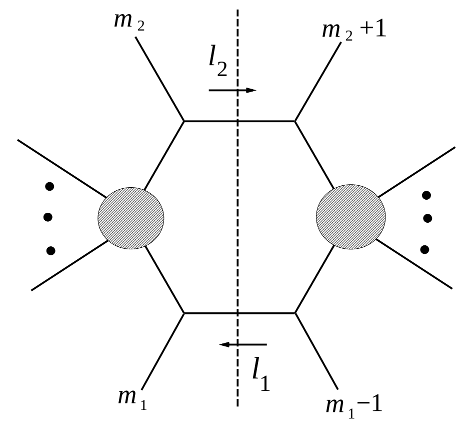
This does *not* mean "imaginary part of imaginary part". It should be interpreted as demanding that cut propagators do not cancel.

The unitarity method is a potent tool for state-of-the-art calculations. As Freddy Cachazo explained, it very effectively combines with twistor methods. Tree-level properties induce loop-level properties.

Arbitrary Number of Legs at One Loop

Consider cuts of maximally helicity violating one-loop amplitudes.





Bern, Dixon
Dunbar and Kosower

The tree-level Parke-Taylor amplitudes for n gluons have a remarkable property:

$$A^{\rm tree}(\ell_1^+, m_1^+, \cdots, k^-, \cdots, j^-, \cdots, m_2^+, \ell_2^+) = \frac{\langle k\, j\rangle^4}{\langle \ell_1\, m_1\rangle\, \langle m_1,\, m_1+1\rangle \cdots \langle m_2-1,\, m_2\rangle\, \langle m_2\, \ell_2\rangle\, \langle \ell_1\, \ell_2\rangle}$$

Only 2 denominators in each tree have non-trivial dependence on loop momentum.

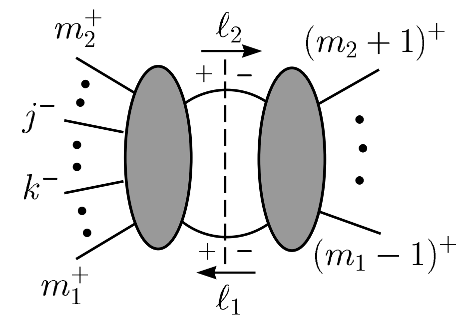
Together with 2 cut propagators the 4 denominators from the trees give at worst a hexagon integral (which simplifies in susy cases).

Examples of amplitudes obtained with unitarity sewing method:

- All MHV amplitudes in maximal $N = 4$ super-Yang-Mills theory.
- All MHV amplitudes in $N = 1$ super-Yang-Mills
- All helicities for $N = 4$ super-Yang-Mills six-points amplitudes.

$$A_5^{\text{1-loop}} = A_5^{\text{tree}} \left[-\frac{1}{\epsilon^2} \sum_{i=1}^{5} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} + \sum_{i=1}^{5} \ln\left(\frac{-s_{i,i+1}}{s_{i-2,i-1}} \right) \ln\left(\frac{-s_{i+2,i+3}}{s_{i-2,i-1}} \right) + \frac{5\pi^2}{6} \right]$$

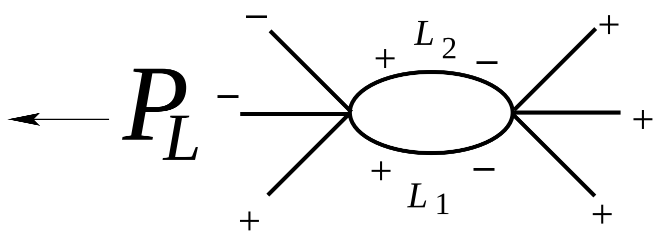
These amplitudes are the one-loop analogs of the Parke-Taylor tree-level amplitudes.



The amplitudes are much much simpler than they ought to be.

Twistor space and loop level

In a very elegant paper Brandhuber, Spence and Travaglini demonstrated that CSW formalism applies also to loop level.



Reproduces our earlier MHV results.

IR divergences isolated to a few diagrams.

Bena, Bern, Kosower, Roiban.

Although CSW diagrams are rather different than unitarity cuts, BST were able to map them into the cuts.

$$\int \frac{d^4L_1}{L_1^2}\frac{d^4L_2}{L_2^2}\,\delta^{(4)}(L_2-L_2+P_L) = -4\int \frac{dz_1}{z_1}\frac{dz_2}{z_2}\,d\mathrm{LIPS}(l_2,-l_1,P_{L;z}), \qquad P_{L;z} = P_L - z\eta$$

Loop integrals converted into phase-space $\times$ dispersion integrals.

Two recent papers confirm that MHV vertices also work for one-loop $N=1$ MHV amplitudes.

Brandhuber, Spence and Travaglini; Quigley and Rozali

Little doubt that MHV vertices will work for any $D=4$ cut constructible loop amplitude.

Holomorphic Anomaly

See Cachazo's talk

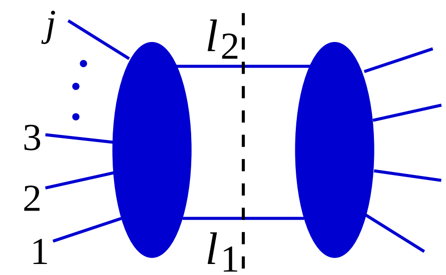
The BST results imply a simple twistor space interpretation of loop amplitudes – much simpler than previously appreciated. Cachazo, Svrcek and Witten

Twistor space collinearity operator: Witten

$$\varepsilon_{IJKL} Z_1^I Z_2^J Z_3^K = 0 \longrightarrow \left[\langle \lambda_1, \lambda_2 \rangle \frac{\partial}{\partial \tilde\lambda_3} + \langle \lambda_2, \lambda_3 \rangle \frac{\partial}{\partial \tilde\lambda_1} + \langle \lambda_3, \lambda_1 \rangle \frac{\partial}{\partial \tilde\lambda_2} \right] A_n = 0$$

The holomorphic anomaly: $\partial_{\bar z} z^{-1} = 2\pi \delta^{(2)}(z)$:

Cachazo, Svrcek and Witten



$$\tilde\eta^{\dot a} \frac{\partial}{\partial \tilde\lambda_1^{\dot a}} \frac{1}{\langle \lambda_1, \lambda_{l_1} \rangle} = 2\pi [\tilde\eta, \tilde\lambda_{l_1}] \delta(\langle \lambda_1, \lambda_{l_1} \rangle) \delta([\tilde\lambda_1, \tilde\lambda_{l_1}])$$

The anomaly delta functions freeze the phase space integrals:

Bena, Bern, Kosower, Roiban; Cachazo

$$l_1 = a k_1, \qquad l_2 = K_{1\cdots j} - \frac{K_{1\cdots j}^2}{2k_1 \cdot K_{1\cdots j}} k_1 \qquad\qquad a = \frac{K_{1\cdots j}^2}{2k_1 \cdot K_{1\cdots j}},$$

Also a jacobian which is easy to evaluate. $\quad K_{1\ldots j} = k_1 + \cdots k_j$

As shown by Cachazo the holomorphic anomaly can then be used to determine the unitarity cuts.

$$F_{123}A^{1\text{-loop}}\Big|_{\text{cut}} = F_{123}\sum_i c_i B_i\Big|_{\text{cut}} = \sum_i c_i F_{123} B_i\Big|_{\text{cut}}$$

In the NMHV case F_{123} must annihilate the c_i, otherwise logarithms would appear in the integrated anomaly, which can't happen. Cachazo

Powerful way to evaluate the unitarity cuts: obtain algebraic equations.

In the $N = 1$ super-Yang-Mills case, for NMHV amplitudes one obtains differential instead of algebraic equations. Bidder, Bjerrum-Bohr, Dixon and Dunbar

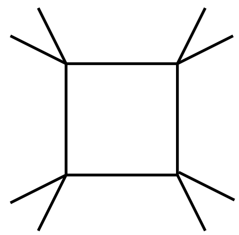
$N=4$ next-to-MHV Amplitudes

Using the holomorphic anomaly with the unitarity method Britto, Cachazo and Feng computed $A_7(1^-,2^-,3^-,4^+,5^+,6^+,7^+)$.

Shortly thereafter we posted our results for all remaining one-loop 7-point amplitudes. Equivalent to 227,585 Feynman loop diagrams.

Bern, Del Duca, Dixon and Kosower

$$A_7^{\text{1-loop}} = \sum_i c_i B_i$$

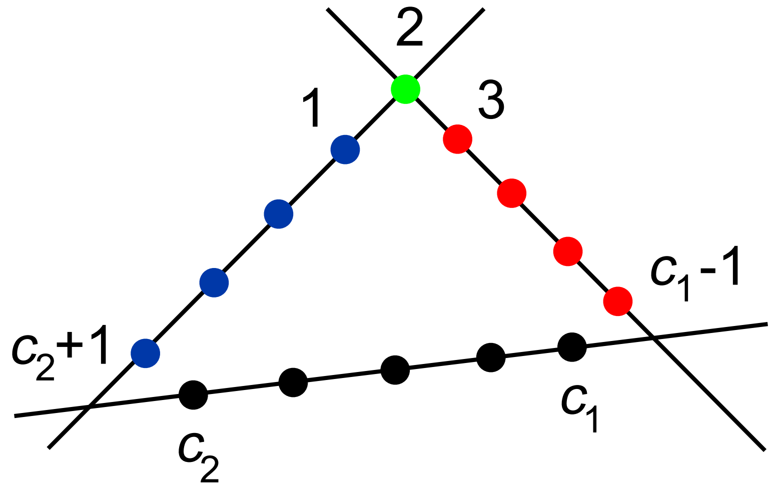


The B_i are known scalar box functions given in terms of polylogs.
Coefficients for all NMHV 7-point amplitudes are listed in our paper hep-th/0410224. Example: $(-+-+-++)$

$(1+2) \equiv \not{k}_1 + \not{k}_2$

$$c_{136} = \frac{\left(\left\langle 7^+\right| (2+4) \left|3^+\right\rangle \langle 5\,4\rangle + \left\langle 7^+\right| 6 \left|5^+\right\rangle \langle 3\,4\rangle\right)^4}{\langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,6\rangle [7\,1] \left\langle 1^+\right| (2+3) \left|4^+\right\rangle \left\langle 7^+\right| (5+6) \left|4^+\right\rangle \left\langle 4^-\right| (5+6)(7+1) \left|2^+\right\rangle \left\langle 4^-\right| (2+3)(7+1) \left|6^+\right\rangle}$$

Our key result: Beautiful twistor-space picture for terms in integral function coefficients:



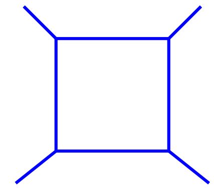
Last week two proofs of the general coplanarity of NMHV integral coefficients appeared. Bern, Del Duca, Dixon and Kosower; Britto, Cachazo and Feng

Points to further twistor space marvels awaiting discovery and exploitation.

A full understanding of the twistor space structure of loop amplitudes will surely lead to new computational advances.

Two loops in terms of one loop

The four-point one-loop $D = 4, N = 4$ amplitude:



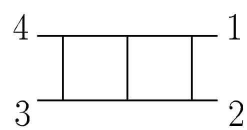
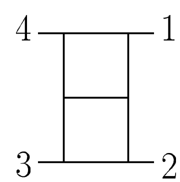
$$A_4^{\text{1-loop}}(s,t) = -st A_4^{\text{tree}} \mathcal{I}_{\text{1-loop}}(s,t)$$

$$I^{\text{1-loop}}(s,t) \sim \frac{1}{st}\left[\frac{2}{\epsilon^2}\left((-s)^{-\epsilon} + (-t)^{-\epsilon}\right) - \ln^2\left(\frac{t}{s}\right) - \pi^2\right] + \mathcal{O}(\epsilon)$$

We also have the leading color planar two-loop amplitude

Bern, Rozowsky and Yan

$$A_4^{\text{2-loop}}(1^-,2^-,3^+,4^+) = -st\, A_4^{\text{tree}}(1^-,2^-,3^+,4^+)\,\left(s\,\mathcal{I}_4^{\text{2-loop}}(s,t) + t\,\mathcal{I}_4^{\text{2-loop}}(t,s)\right)$$

$- st\, A_4^{\text{tree}} \Big\{ s$  $+ \ t$  $\Big\}$

Near $D = 4$ the double box integral is a rather complicated object involving up to 4th order polylogarithms.

Smirnov

Nevertheless, the *planar* two-loop amplitude undergoes an amazing simplification: Anastasiou, Bern, Dixon, Kosower

$$\boxed{\begin{gathered} M_4^{\text{2-loop}}(s,t) = \frac{1}{2}\Big(M_4^{\text{1-loop}}(s,t)\Big)^2 + f(\epsilon) M_4^{\text{1-loop}}(s,t)\Big|_{\epsilon\to 2\epsilon} - \frac{5}{4}\zeta_4 \\ \text{where} \\ M_4^{\text{loop}} = A_4^{\text{loop}}/A_4^{\text{tree}}, \qquad f(\epsilon) = -\zeta_2 - \zeta_3\,\epsilon - \zeta_4\,\epsilon^2 \end{gathered}}$$

$f(\epsilon)$ is a universal IR function.

Thus, we have succeeded to express the two-loop amplitude as an iteration of the one loop amplitude together with a universal IR function.

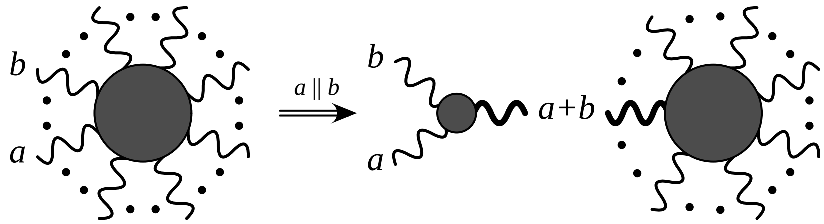
Non-trivial polylogarithm and Nielsen function identities needed to demonstrate the above.

Generalization to n-Points

Anastasiou, Bern, Dixon, Kosower

Not yet possible to explicitly evaluate $n > 4$ point two-loop integrals

But we have tools for obtaining results: Collinear behavior



Have calculated the two-loop splitting amplitudes which determine the behavior of amplitudes as momenta become collinear.

Following ansatz satisfies all collinear constraints:

$$M_n^{2\text{-loop}}(s,t) = \frac{1}{2}\Big(M_n^{1\text{-loop}}(s,t)\Big)^2 + f(\epsilon) M_n^{1\text{-loop}}(s,t)\Big|_{\epsilon \to 2\epsilon} - \frac{5}{4}\zeta_4$$

where
$$M_n^{\text{loop}} = A_n^{\text{loop}}/A_n^{\text{tree}}\,, \qquad f(\epsilon) = -\zeta_2 - \zeta_3\,\epsilon - \zeta_4\,\epsilon^2$$

Interesting quantity is finite remainder after subtracting IR divergences.

The conjecture is likely true for MHV amplitudes. Less clear for non-MHV.

Multi-loop Generalization

The above structure suggests a multi-loop generalization:

$$M_4^{n\text{-loop}}(s,t) = \frac{1}{n!}\Big(M_4^{1\text{-loop}}(s,t)\Big)^n + \text{lower powers of } M_4^{1\text{-loop}}(s,t)$$

Key part of conjecture: The only dependence on s/t is through $M_4^{1\text{-loop}}(s,t)$.

It should be possible to apply the new twistor space developments to check this.

Some Key Issues for the Future

Trees:

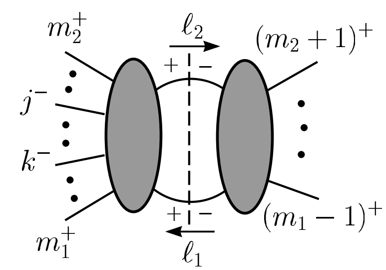
• Find a closed form solution to non-MHV vertex recursion relations.

One loop:

• Applications of twistor methods to QCD.

Key difficulty: In susy theories we can effectively ignore the distinction between $D = 4$ and $D \neq 4$, needed for dim. reg. In QCD this is not true.

Multi-loop:



• Can one prove iteration of the $N = 4$ S-matrix? Resummation?

String Theory:

• Can one find a string theory dual with conformal supergravity projected out? Topological B model is polluted by conformal gravity. Berkovits and Witten

Summary

1. Motivation.
   (a) LHC demands QCD loop calculations – new tricks necessary
   (b) Can we solve $N = 4$ super-Yang-Mills theory?
2. Efficient recursive reformulation of CSW diagrams.
3. Generalized unitarity cuts: Obtain loop amplitudes from tree amplitudes. Loop amplitude properties inherited from tree amplitudes.
4. Important new twistor space tools. Holomorphic anomaly.
5. Calculation of all one-loop 7-point helicity amplitudes. Very intriguing twistor space structure.
6. Presented non-trivial evidence that the $N = 4$ super-Yang-Mills S-matrix iterates to all loop orders.
7. There are a variety of exciting avenues for further exploration.