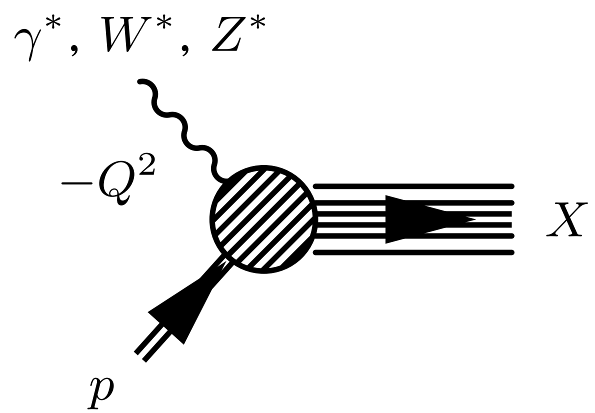
# APPROACHING THE REGGE LIMIT IN PERTURBATIVE QCD

**STEFANO FORTE**
**UNIVERSITÀ DI MILANO**

KITP, SANTA BARBARA — DECEMBER 2, 2004

# DEEP-INELASTIC SCATTERING

## Structure Functions. . .




Lepton fractional energy loss: $y = \frac{p \cdot q}{p \cdot k}$;

Bjorken $x$: $x = \frac{Q^2}{2p \cdot q}$

lepton-nucleon CM energy: $s = \frac{Q^2}{xy}$;

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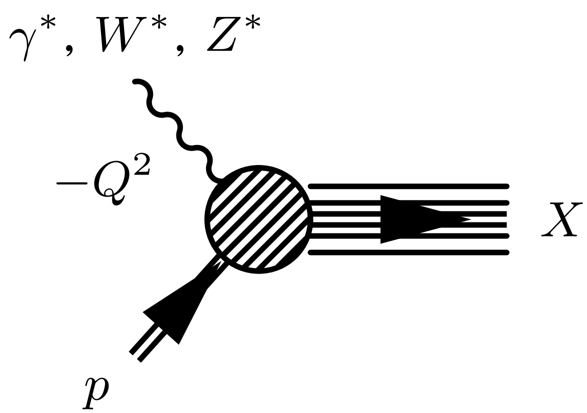


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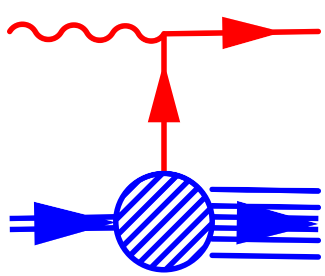
## ...AND PARTON DISTRIBUTIONS

STRUCTURE FUNCTION=HARD COEFF.⊗PARTON DISTN.



$$F_2(x,Q^2) = x \sum_{\text{flav. } i} e_i^2 (q_i + \bar{q}_i) + \alpha_s \left[C_i[\alpha_s] \otimes (q_i + \bar{q}_i) + C_g[\alpha_s] \otimes g\right]$$

$q_i$ quark, $\bar{q}_i$ antiquark, $g$ gluon

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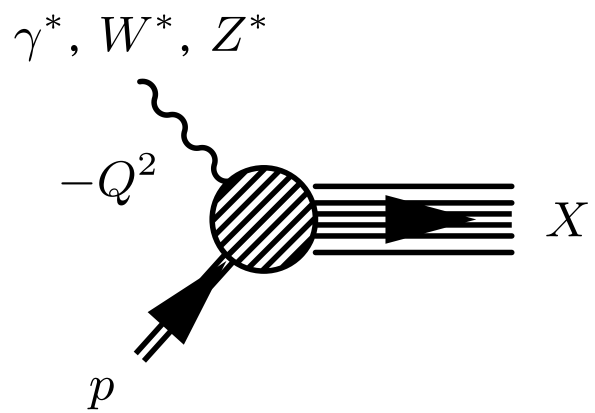


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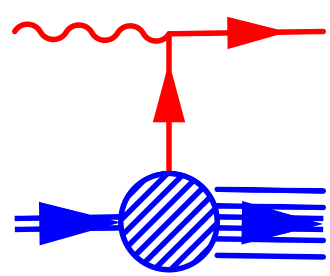

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REGGE LIMIT: $s \longrightarrow \infty \Rightarrow x \longrightarrow 0$

# EVOLUTION EQUATIONS

KNOWLEDGE OF PARTON DISTNS. AT $Q_0^2$, $x \geq x_0$

COMPLETELY DETERMINES P.D. FOR ALL $Q^2 > Q_0^2$, $x \geq x_0$

$$\begin{aligned} \frac{d}{dt} q^{\pm}_{NS}(x,Q^2) &= \frac{\alpha_s(t)}{2\pi} P_{qq}^{NS,\pm} \otimes q_{NS}, \\ \frac{d}{dt} \begin{pmatrix} \Sigma(x,Q^2) \\ g(x,Q^2) \end{pmatrix} &= \frac{\alpha_s(t)}{2\pi} \begin{pmatrix} P^S_{qq} & 2n_f P^S_{qg} \\ P^S_{gq} & P^S_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix}, \end{aligned}$$

$\Sigma(x,Q^2) \equiv \sum_{i=1}^{n_f} (q_i + \bar{q}_i)$; $\quad q^{\pm}_{NS}(x,Q^2) \equiv (q^i \pm \bar{q}^i) - (q^j \pm \bar{q}^j)$; $\quad t = \log Q^2/\Lambda^2$;

CONVOLUTION: $P \otimes f \equiv \int_x^1 \frac{dy}{y} P\left(\frac{x}{y}\right) f(y)$

$P(x,\alpha_s) = P^{(0)}(x) + \alpha_s P^{(1)} + \ldots$ KNOWN UP TO NNLO (Vogt, Vermaseren, Moch, 2004)

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$$\gamma_{ij}(N) \equiv \int_0^1 dx\, x^{N-1} P_{ij}(x,Q^2); \quad q_i(N,Q^2)) \equiv \int_0^1 dx\, x^{N-1} q_i(x,Q^2)$$

CONVOLUTIONS TURN INTO PRODUCTS:

$$\frac{d}{dt} q^{\pm}_{NS}(N,Q^2) = \frac{\alpha_s(t)}{2\pi} \gamma_{qq}^{NS,\pm}(\alpha_s,N) q_{NS}$$

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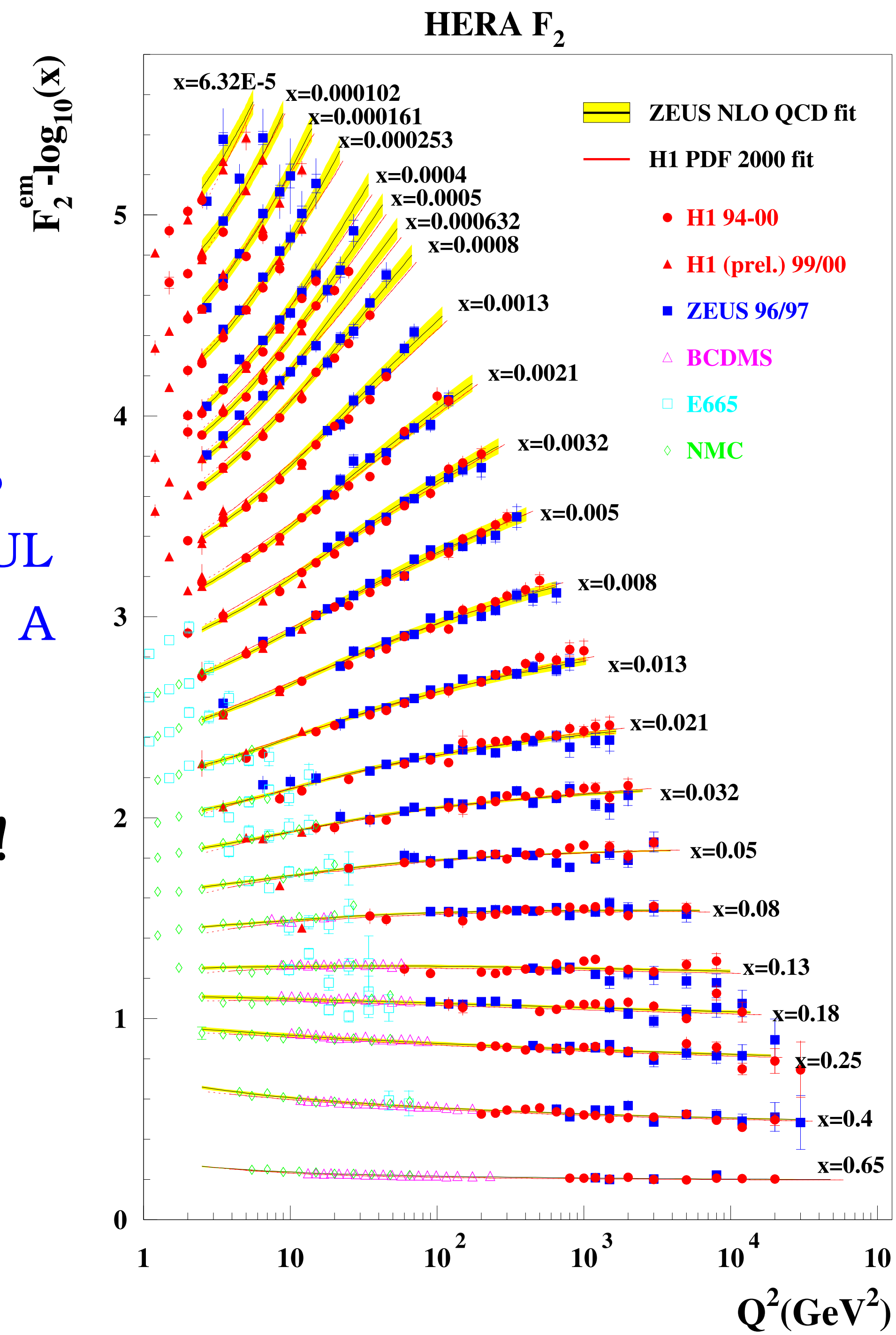
## REGGE LIMIT:

SMALL X $\Leftrightarrow$ SMALL N $\qquad \frac{1}{(k-1)!} \int_0^1 x^{N-1} (\ln 1/x)^k = \frac{1}{N^k}$

# THE PROBLEM: NLO QCD IS TOO GOOD AT SMALL $x$

FIXED–ORDER PERTURBATIVE QCD IS EXTREMELY SUCCESSFUL IN DESCRIBING DATA IN A HUGE KINEMATIC REGION…

DOWN TO VERY LOW $x$ !



# . . . INDEPENDENT OF DETAILS OF PARTON DISTRIBUTIONS:

## DOUBLE ASYMPTOTIC SCALING

$\ln F_2$ predicted in NLO QCD to rise linearly as a function of the scaling variable $\sigma$, indep. of value of scaling variable $\rho$, up to universal rescaling factor $R_F$

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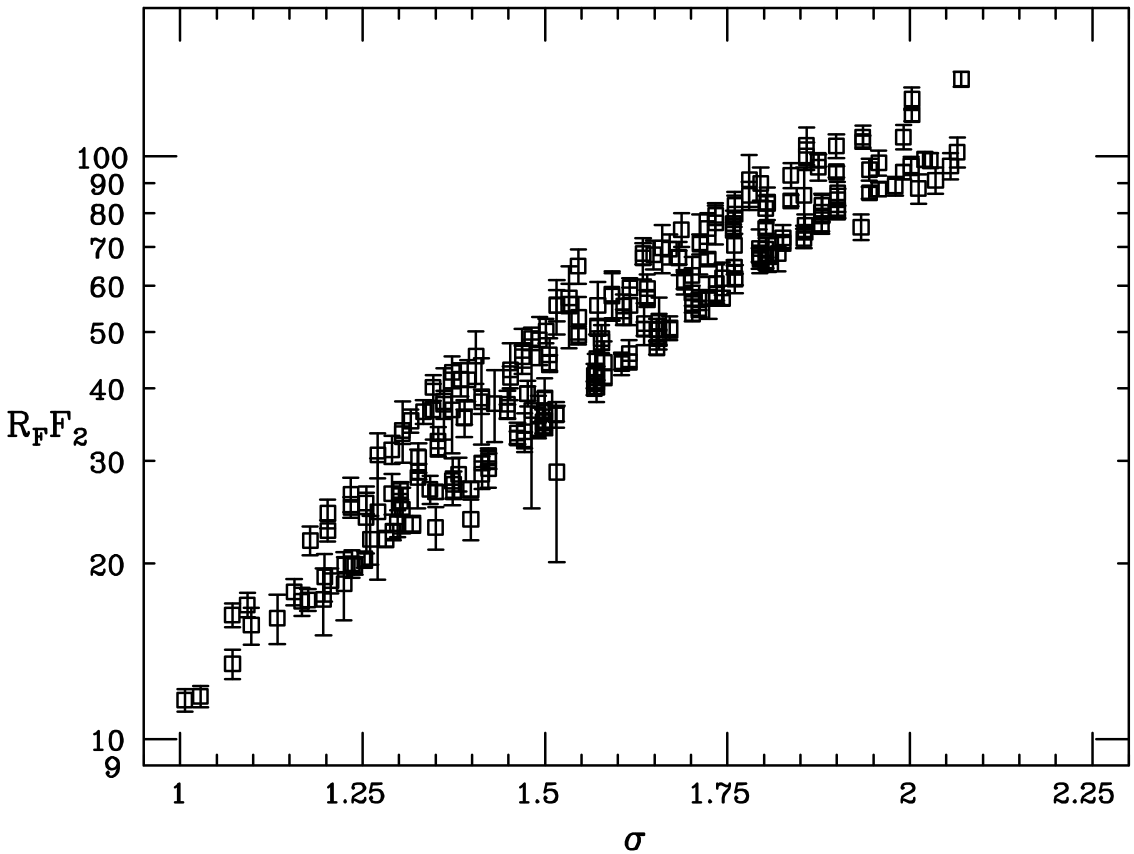


Scaling Variables

$$\sigma = \sqrt{\ln \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \ln \frac{x_0}{x}}$$

$$\rho = \sqrt{\ln \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} / \ln \frac{x_0}{x}}$$

Rescaling Factor

$$R_F = \rho\sqrt{\sigma}\exp(\delta\sigma/\rho)$$

$$\delta = (11 + 2n_f/27)/(11 - 2n_f/3)$$

Slope

$$2\gamma = 12/\sqrt{33 - 2n_f}$$

H1 94-97 data, $10^{-5} \le x \le 10^{-1}$, $1 < Q^2 < 1500$ GeV$^2$;

$\rho \ge 1$, $\sigma \ge 1$, $Q^2 \ge (1.8 \text{ GeV})^2$; $x_0 = 0.1$, $Q_0 = 1$ GeV

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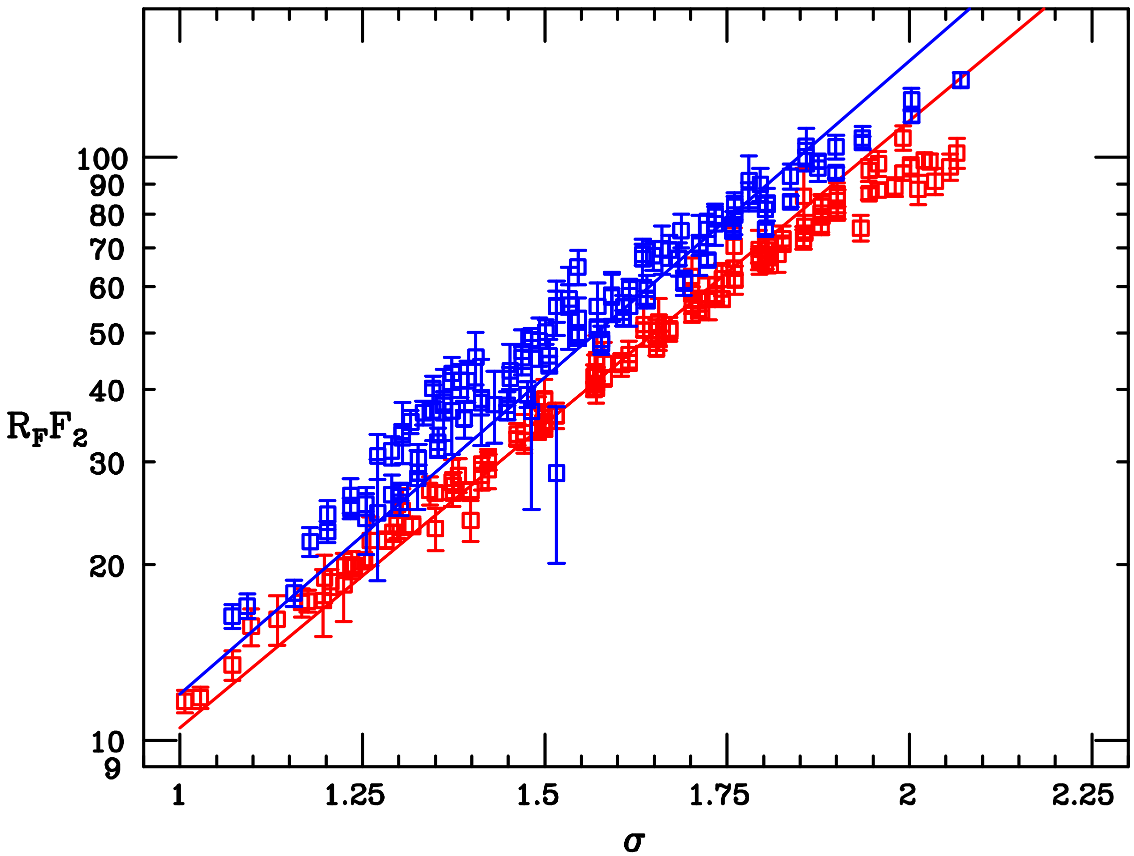


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SLOPE: ($N_f = 5$) predicted: 2.50; measured: $2.47 \pm 0.02$ (ZEUS 1997)

RISE IN THE DATA DRIVEN BY THE RESIDUE OF THE LO POLE $\gamma_{gg} \sim \frac{1}{N}$

# A PHENOMENOLOGICAL MIRACLE. . .

LARGE $\ln 1/x \Rightarrow$ EXPECT LARGE CORRECTIONS AT SMALL $x$

## SCALING VIOLATIONS

$$\frac{d}{dt}G(N,Q^2) = \gamma_N G(N,Q^2)$$

$$t \equiv \ln Q^2/\Lambda^2;\ G(N,Q^2) \equiv \int_0^1 x^{N-1} G(x,Q^2)$$

$$\gamma_N \sim \sum_{k=1}^{\infty} c_k \left(\frac{\alpha_s}{N}\right)^k \quad \Leftrightarrow P(x) \sim \frac{1}{x}\sum_{k=0}^{\infty} \frac{c_{k+1}}{k!} \left(\alpha_s \ln \frac{1}{x}\right)^k$$

SERIES FOR SPLITTING FUNCTION CONVERGES FOR ALL $x > 0$

BUT NEED $\sim 2.7 \frac{12 \ln 2}{2\pi} \alpha_s \ln \frac{1}{x} \approx 3.6 \alpha_s \ln \frac{1}{x}$ TERMS:

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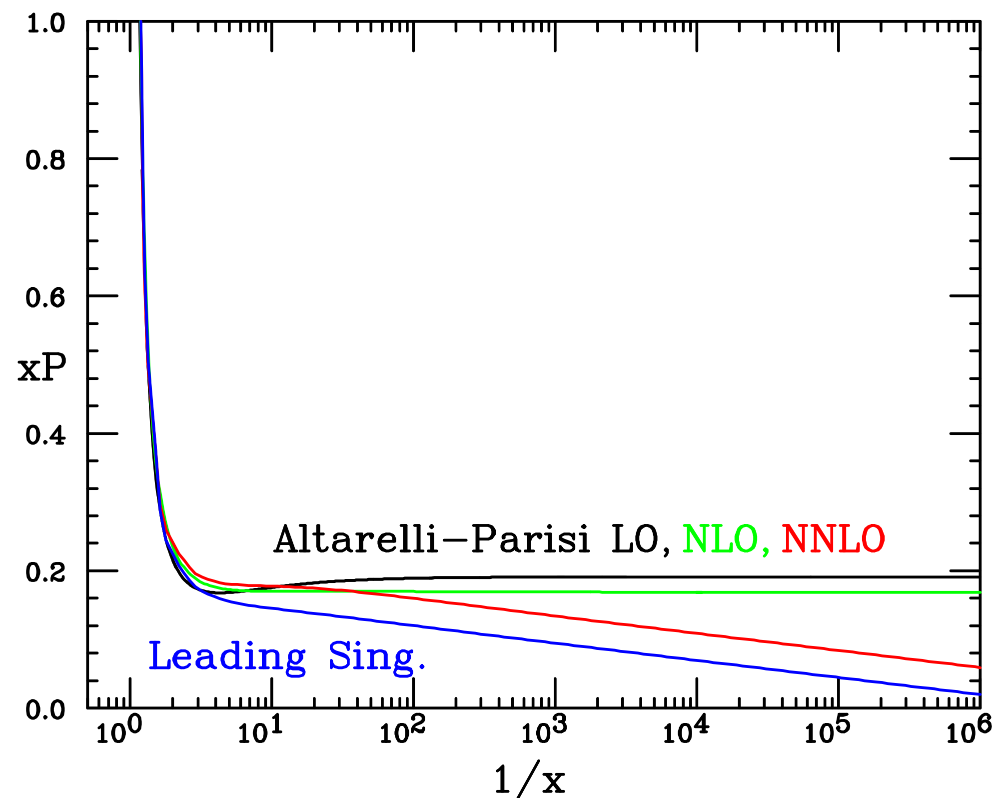
NNLO VS NLO:

- SMALL $x$ GETTING WORSE (NEGATIVE GLUON)
- LEADING LOG $1/x$ NOT A GOOD APPROX



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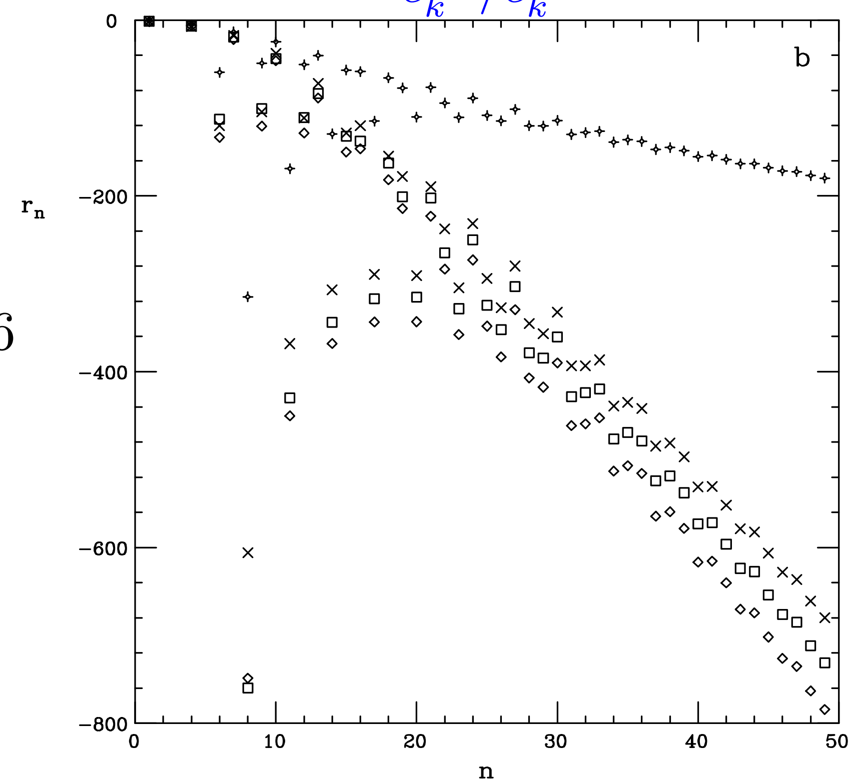
$$\downarrow \qquad \downarrow$$

BFKL FADIN-LIPATOV

$c_k^{(1)}/c_k^{(0)}$




NL SMALL $x$ CORRECTION GROWS

LINEARLY WITH PERT. ORDER:

$\lim_{k\to\infty} c_k^{(1)}/c_k^{(0)} = -Ck;\ C \sim 6 \div 16$

according to factn. scheme

@HERA, $(\alpha_s \times \text{NLO})/\text{LO} \sim -1.5 \div 15$ !

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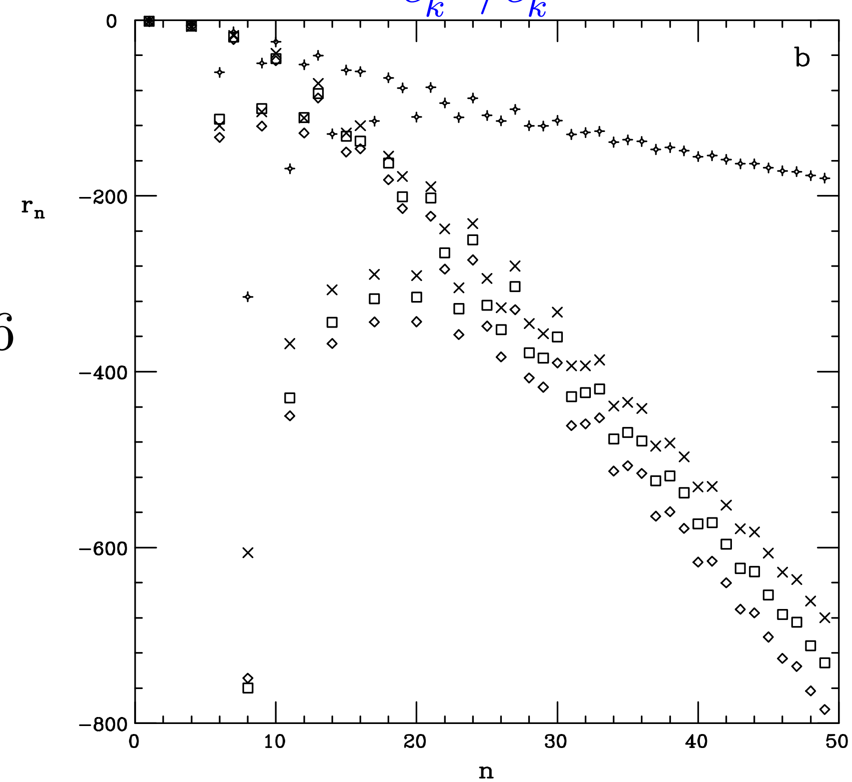
$\downarrow$ BFKL  $\downarrow$ FADIN-LIPATOV

$c_k^{(1)}/c_k^{(0)}$

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'EXACT' DETERMINATION OF SMALL $x$ NLO CORRECTION

$\Rightarrow$ COMPLEX ANOMALOUS DIMENSION (oscillating cross sect.)

# PROGRESS

G. Altarelli, R. Ball, S.F.

CONSISTENT THEORY ⇔ CONSISTENT PHENOMENOLOGY

- CONSISTENCY OF AP & BFKL (duality) →
  PERTURBATIVELY STABLE SMALL $x$ EXPANSION (momentum cons.)
  **(1999-2001)** P.L. B465 (1999) 271, N.P. B575 (2000) 313; B599 (2001) 383
- RG IMPROVEMENT (factorization) →
  SOFT LO RESUMMATION (running coupling)
  **(2001-2003)** N.P. B621 (2002) 359; B674 (2003)
- SYMMETRIZATION (anticollinear matching) →
  STABLE REGGE BEHAVIOUR (pole in an. dim.)
  **(2003-2004)** in preparation (2004)

See also work by Ciafaloni, Salam and collaborators

# PERTURBATIVE CONSISTENCY: DUALITY (fixed coupling)

THE ALTARELLI-PARISI EQN IS AN INTEGRO–DIFFERENTIAL EQUATION $\Rightarrow$ IT CAN BE EQUIVALENTLY VIEWED AS $Q^2$–EVOLUTION EQUATION FOR $x$–MOMENTS (usual RG eqn.), OR $x$–EVOLUTION EQUATION FOR $Q^2$–MOMENTS(BFKL eqn.)

| EVOLUTION IN $t = \ln Q^2$ | EVOLUTION IN $\xi = \ln 1/x$ |
|---|---|
| $\frac{d}{dt}G(N,t) = \gamma(N,\alpha_s)\ G(N,t)$ | $\frac{d}{d\xi}G(\xi,M) = \chi(M,\alpha_s)\ G(\xi,M)$ |
| MELLIN $x$–MOMENTS | MELLIN $Q^2$–MOMENTS |
| $G(N,t) = \int_0^\infty d\xi\, e^{-N\xi}\ G(\xi,t)$ | $G(\xi,M) = \int_{-\infty}^\infty dt\, e^{-Mt}\ G(\xi,t)$ |

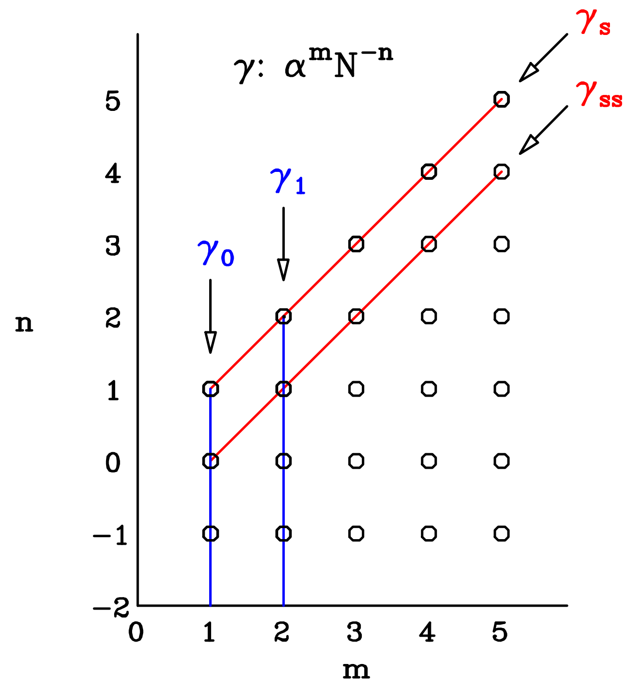
THE TWO EQUATIONS HAVE THE SAME SOLUTIONS PROVIDED THE EVOLUTION KERNELS ARE RELATED BY

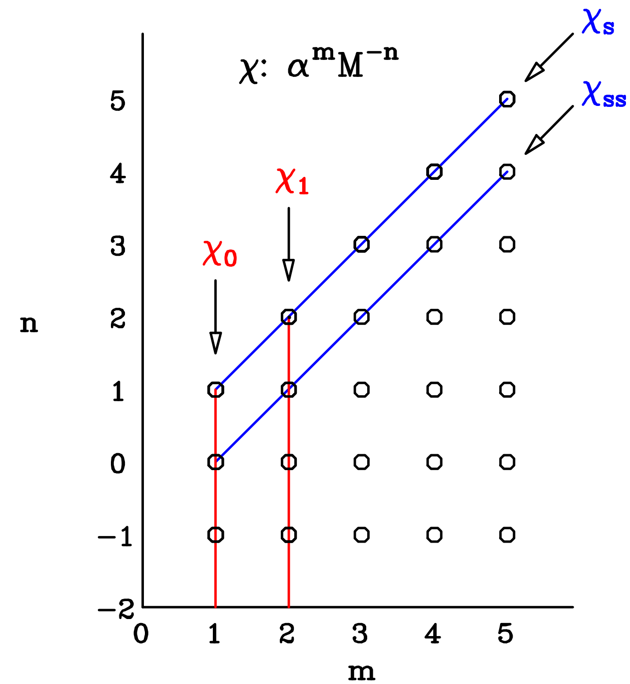
$$\chi(\gamma(N,\alpha_s),\alpha_s) = N$$
$$\gamma(\chi(M,\alpha_s),\alpha_s) = M$$

& BOUNDARY CONDITIONS RELATED BY

$$H_0[M] \to G_0(N) = H_0[\gamma(N,\alpha_s)]/\chi'(\gamma(N,\alpha_s))$$

# DUAL PERTURBATIVE EXPANSIONS




$\ln Q^2$ EVOLUTION

$\ln 1/x$ EVOLUTION

$$\gamma(N) = \alpha\left(\frac{c_{-1}^{(1)}}{N} + c_0^{(1)} + \dots\right) + \alpha^2\left(\frac{c_{-2}^{(2)}}{N^2} + \frac{c_{-1}^{(2)}}{N} + \dots\right)$$

$$\chi(M) = \alpha\left(\frac{\tilde{c}_{-1}^{(1)}}{M} + \tilde{c}_0^{(1)} + \dots\right) + \alpha^2\left(\frac{\tilde{c}_{-2}^{(2)}}{M^2} + \frac{\tilde{c}_{-1}^{(2)}}{M} + \dots\right)$$

$$\gamma_s(N) = c_{-1}^{(1)}\frac{\alpha}{N} + c_{-2}^{(2)}\frac{\alpha^2}{N^2} + \dots$$

$$\chi_s(M) = \tilde{c}_{-1}^{(1)}\frac{\alpha}{M} + \tilde{c}_{-2}^{(2)}\frac{\alpha^2}{M^2} + \dots$$

$1/N$ POLES $\Leftrightarrow \ln 1/x$

$1/M$ POLES $\Leftrightarrow \ln Q^2$

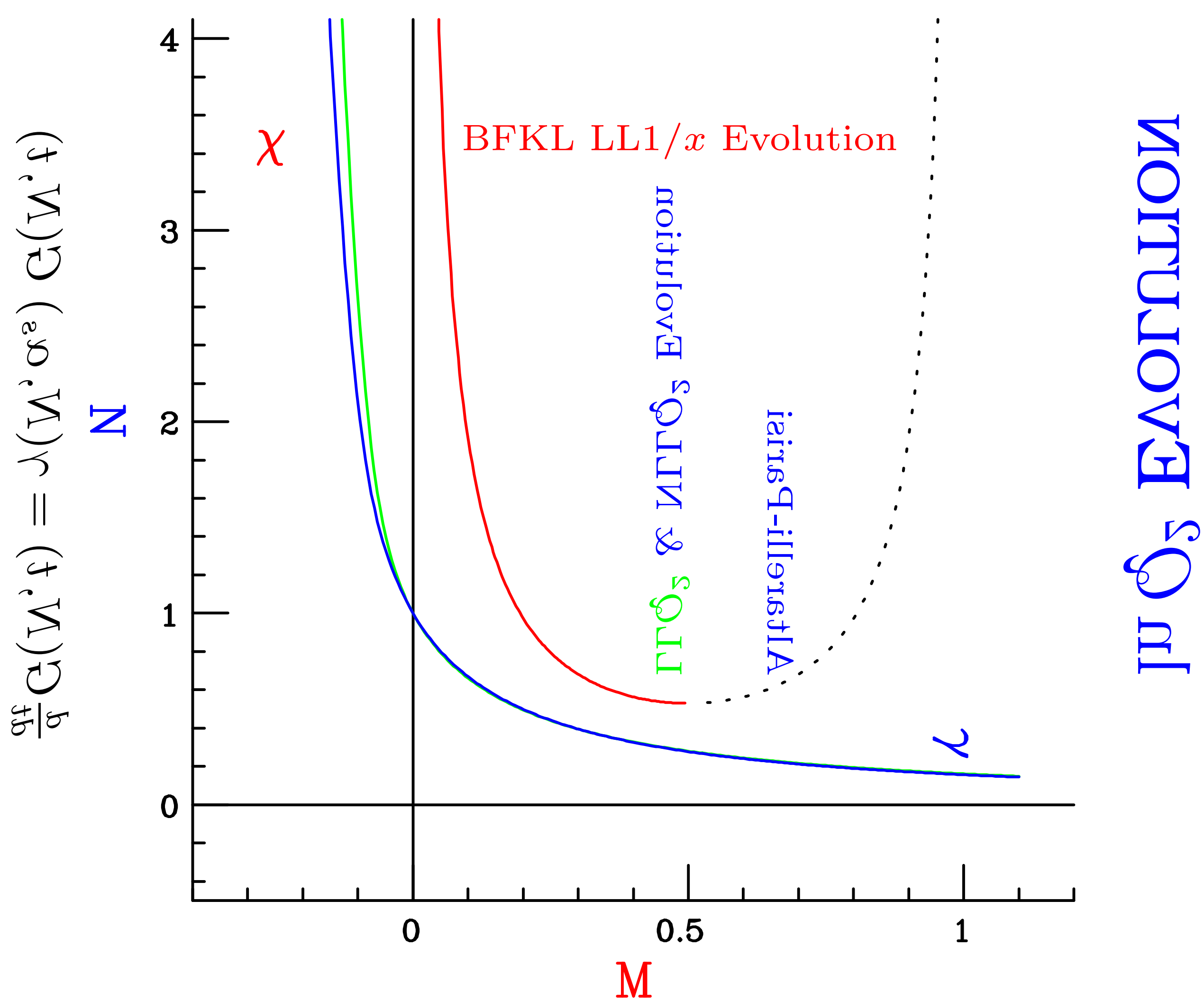
$$\gamma_0(N) \Longleftarrow\!\!\Longrightarrow \chi_s(\alpha_s/M)$$

$$\gamma_s(\alpha_s/N) \Longleftarrow\!\!\Longrightarrow \chi_0(M)$$

### . . . CAN SWITCH FROM LL$Q^2$ TO LL$1/x$

### CHOOSING THE EVOLUTION KERNEL

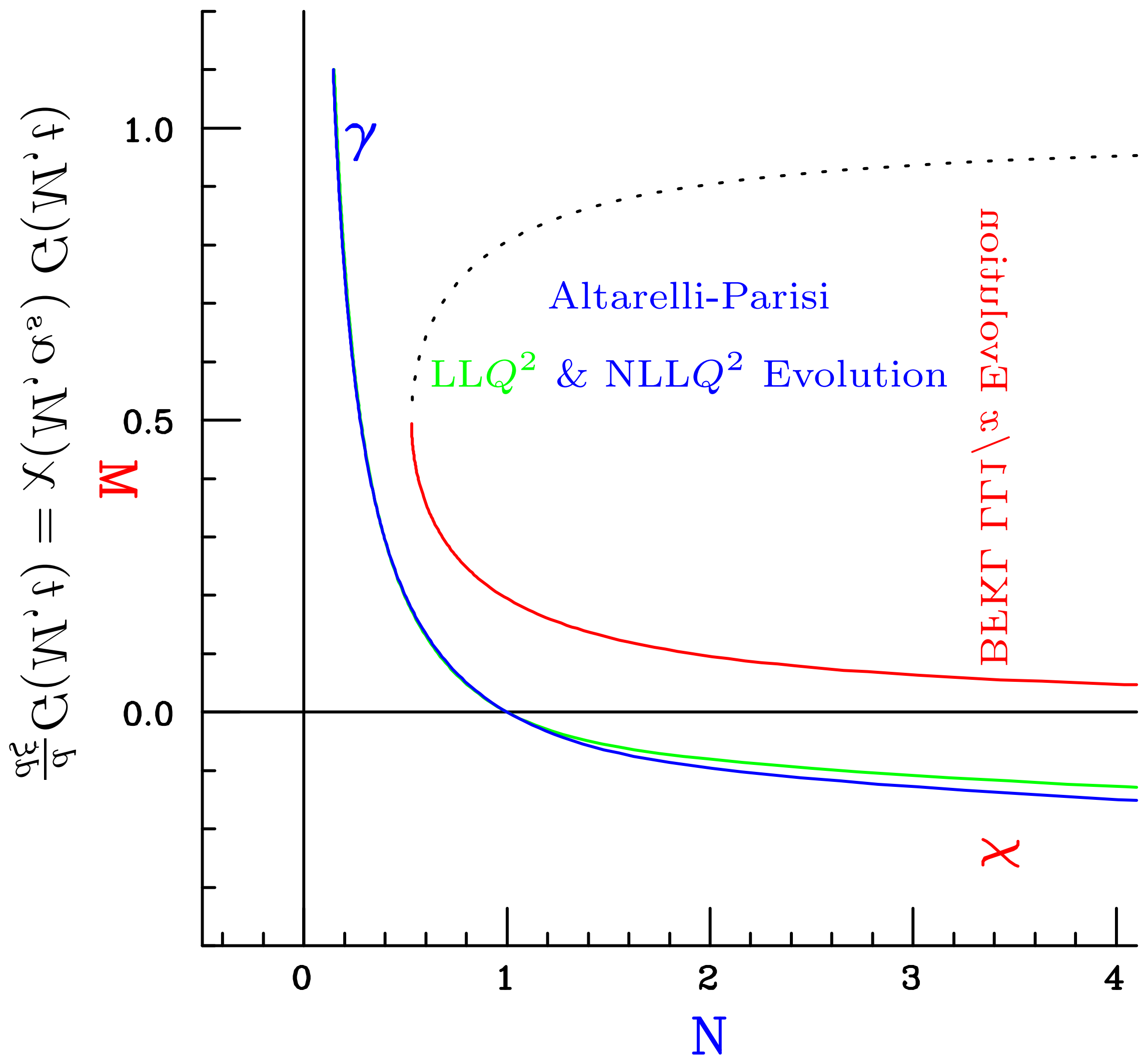
## $\ln 1/x$ EVOLUTION




$$\frac{d}{d\xi}G(M,t) = \chi(M,\alpha_s)\ G(M,t)$$

. . . IN EITHER EQUATION!

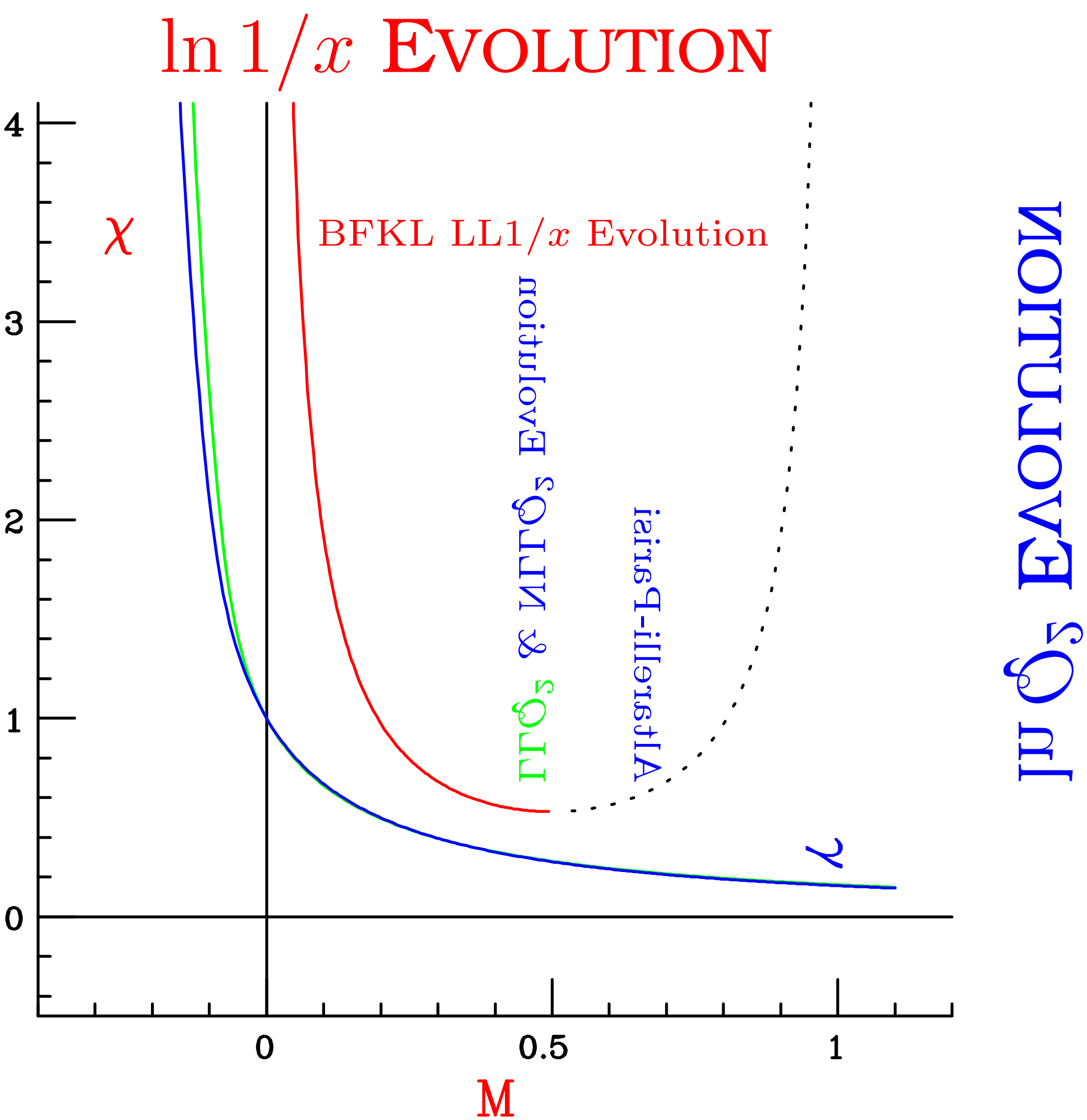
# $\ln Q^2$ Evolution




$\ln 1/x$ Evolution

$$\frac{d}{dt} G(N,t) = \gamma(N,\alpha_s)\ G(N,t)$$

## THE PROBLEM WITH LL1/$x$ EVOLUTION . . .

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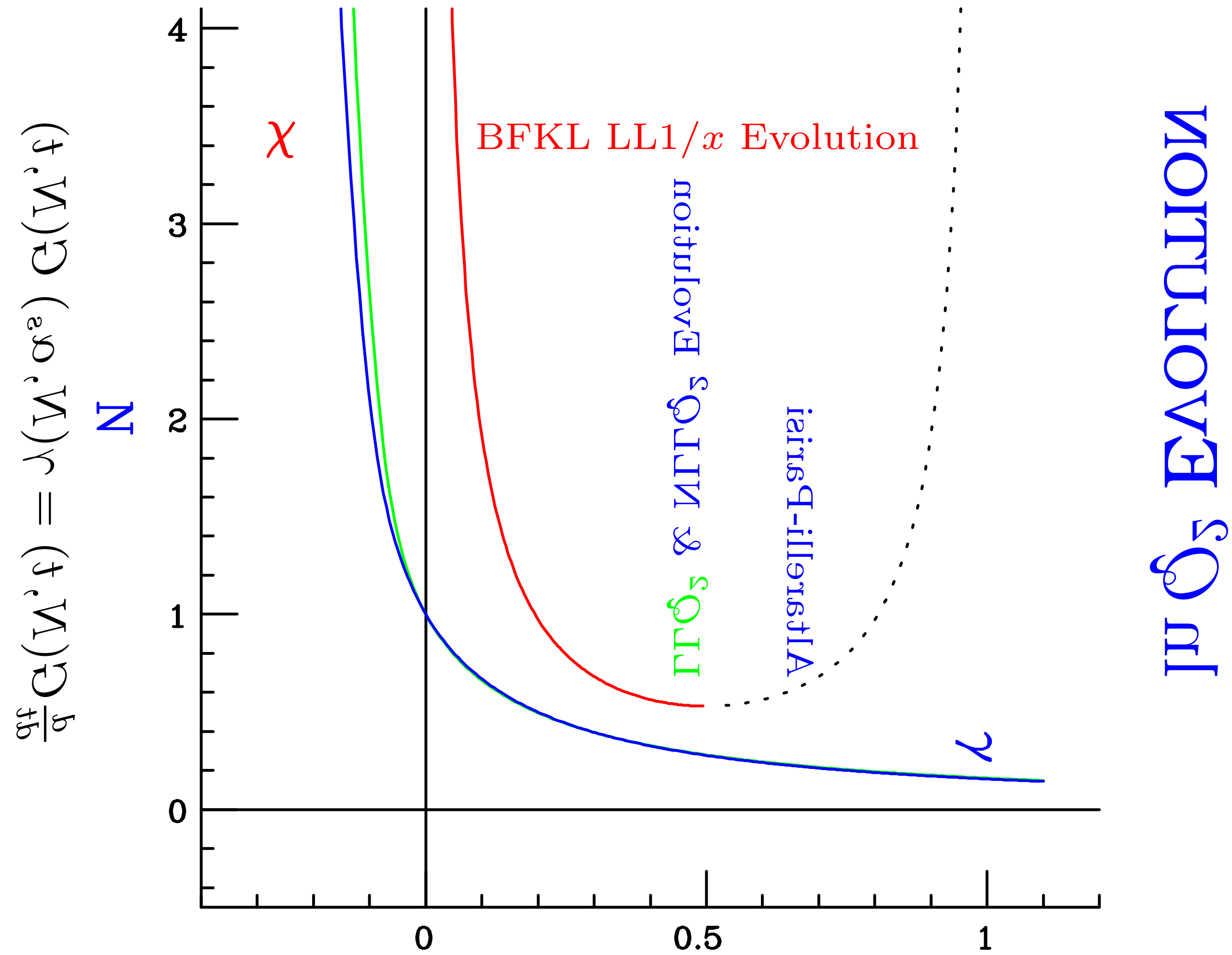
## THE PROBLEM WITH LL$1/x$ EVOLUTION . . .

# $\ln 1/x$ EVOLUTION



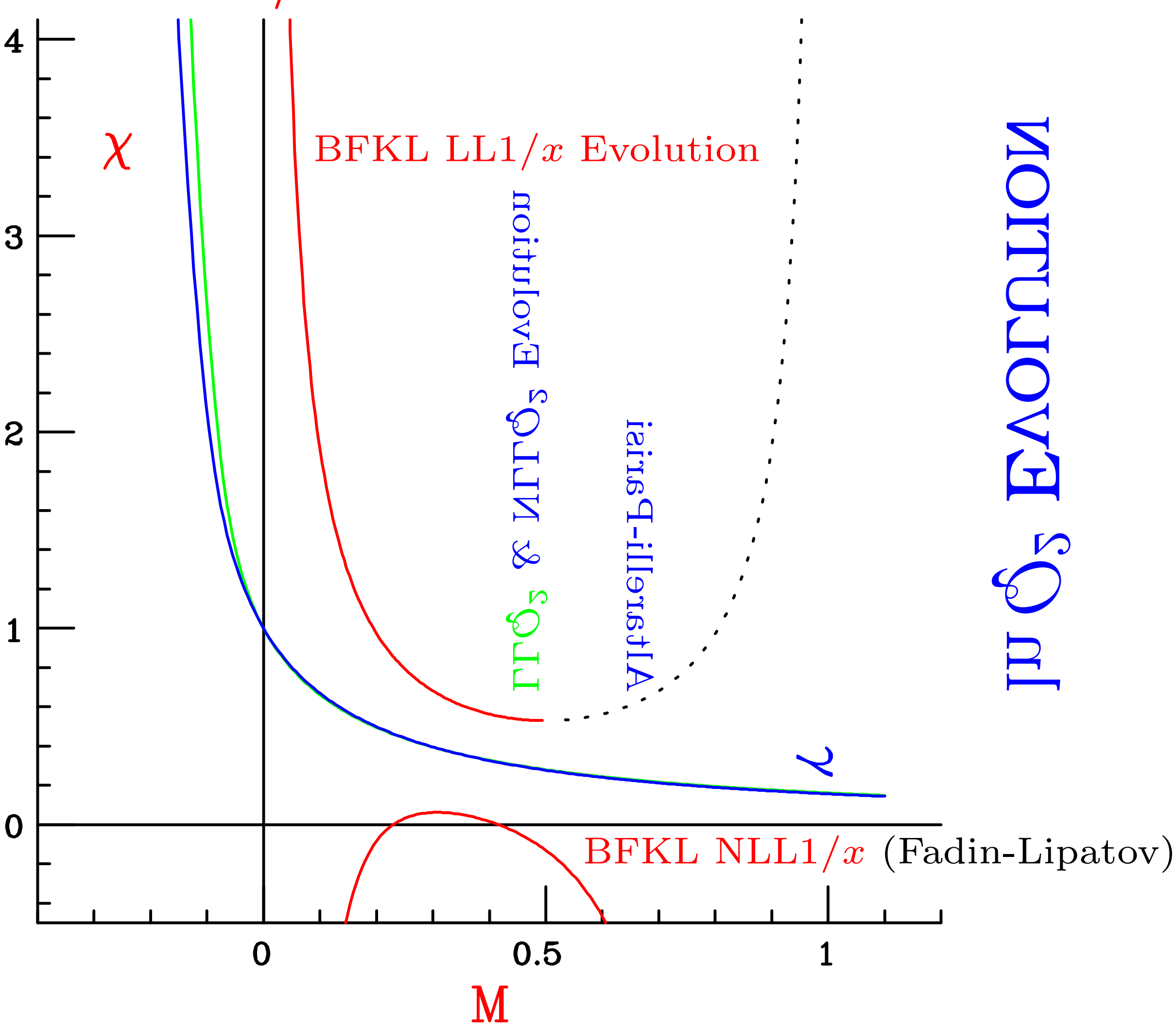

$$\frac{d}{d\xi}G(M,t) = \chi(M,\alpha_s)\ G(M,t)$$

- THE LL$Q^2$ AND LL$1/x$ KERNELS GREATLY DIFFER FROM EACH OTHER

# The problem with LL$1/x$ Evolution . . .

## NL corrections!

## $\ln 1/x$ Evolution

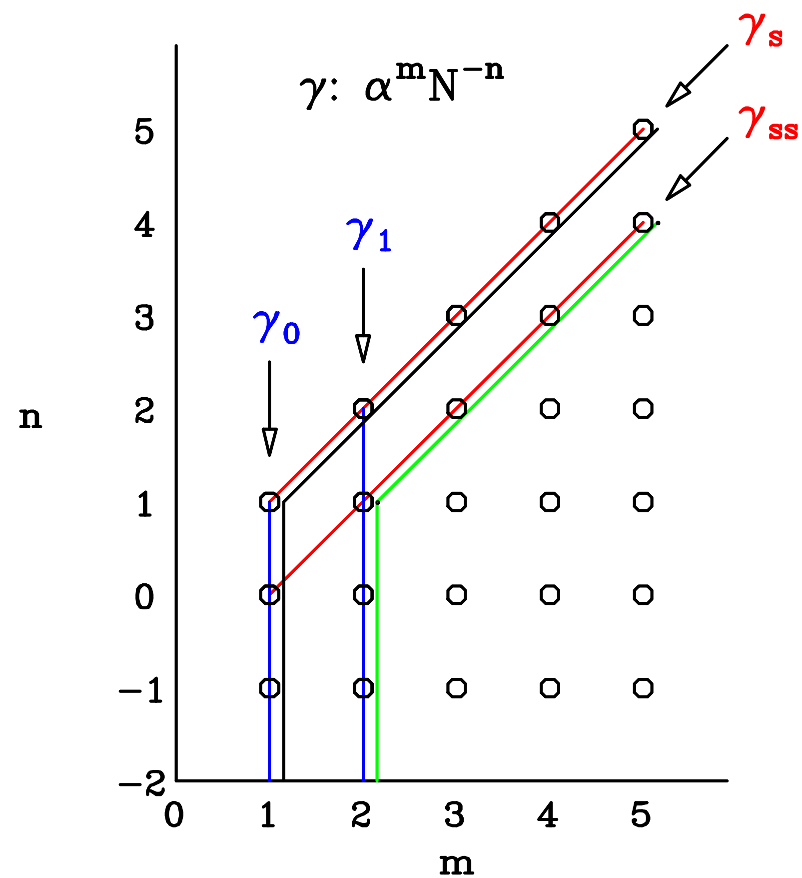


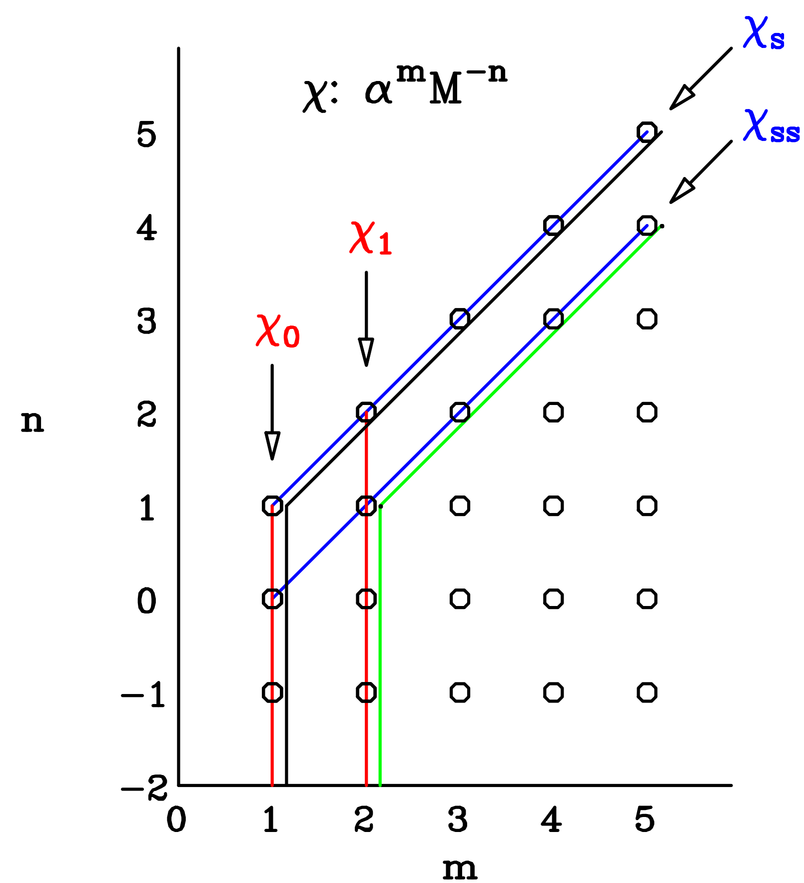

$$\frac{d}{dt}G(N,t) = \gamma(N,\alpha_s)\ G(N,t)$$

$$\frac{d}{d\xi}G(M,t) = \chi(M,\alpha_s)\ G(M,t)$$

- The LL$Q^2$ and LL$1/x$ kernels greatly differ from each other
- The expansion of the LL$1/x$ kernel looks very unstable

# THE DOUBLE–LEADING EXPANSION




$$\gamma(N,\alpha_s) = \left[\alpha_s\gamma_0(N) + \gamma_s\left(\frac{\alpha_s}{N}\right) - \frac{n_c\alpha_s}{\pi N}\right] \Leftarrow\Rightarrow \chi(M,\alpha_s) = \left[\alpha_s\chi_0(M) + \chi_s\left(\frac{\alpha_s}{M}\right) - \frac{n_c\alpha_s}{\pi M}\right]$$

$$+\alpha_s\left[\alpha_s\gamma_1(N) + \gamma_{ss}\left(\frac{\alpha_s}{N}\right) - \alpha_s\left(\frac{e_2}{N^2} + \frac{e_1}{N}\right) - e_0\right] \quad +\alpha_s\left[\alpha_s\chi_1(M) + \chi_{ss}\left(\frac{\alpha_s}{M}\right) - \alpha_s\left(\frac{f_2}{M^2} + \frac{f_1}{M}\right) - \right.$$

$$+\cdots \qquad\qquad +\cdots$$

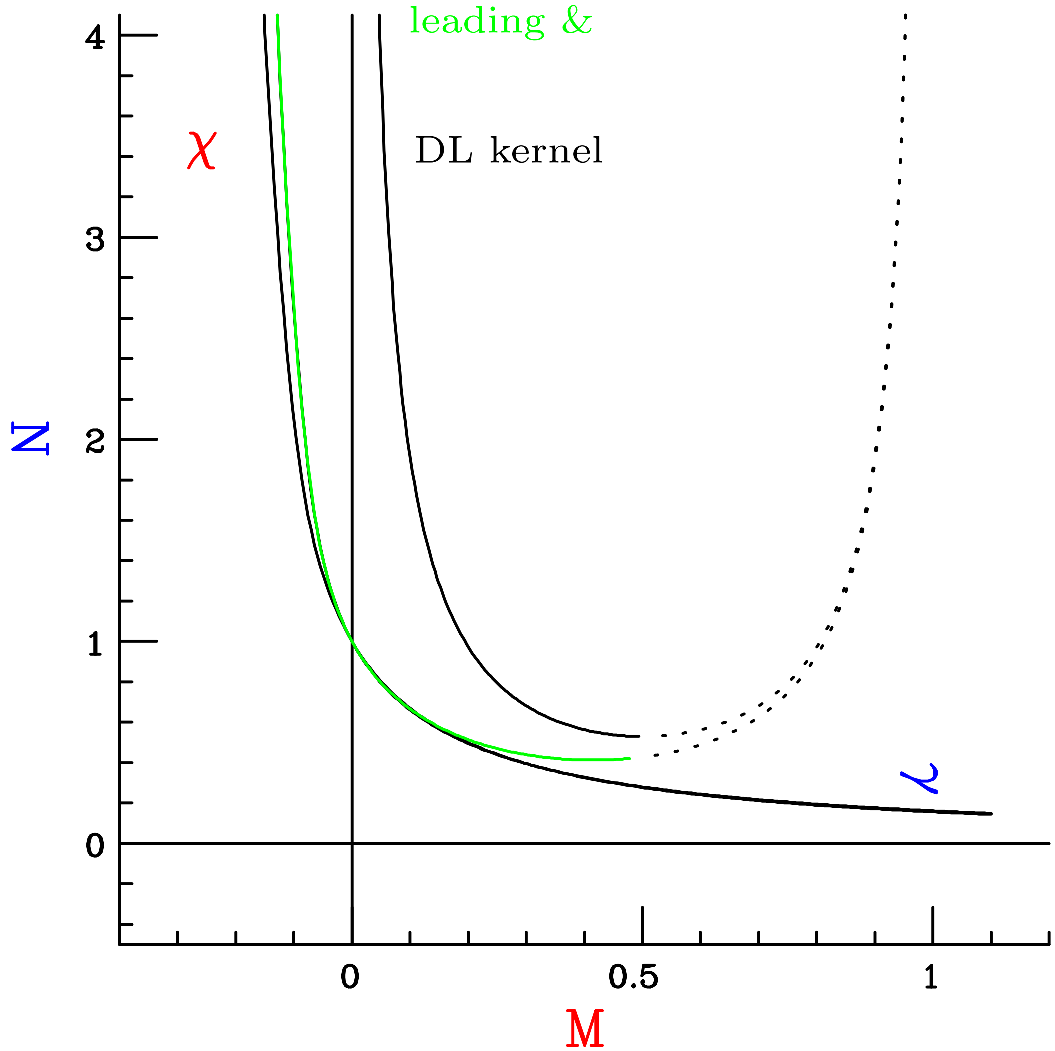
DUALITY HOLDS ORDER–BY–ORDER IN THE DOUBLE–LEADING EXPANSION:

the dual of $\chi_{\mathrm{DL}}^{\mathrm{LO}}$ is $\gamma_{\mathrm{DL}}^{\mathrm{LO}}$ up to terms of order $\gamma_{\mathrm{DL}}^{\mathrm{NLO}}$, and conversely

# Double–Leading Evolution

$$\chi_s \underset{M\to 0}{\sim} \frac{\alpha_s}{\alpha_s+M} = \frac{\alpha_s}{M} - \frac{\alpha_s^2}{M^2} + \frac{\alpha_s^3}{M^3} + \dots$$

$$\chi(\gamma) = M \to \chi(0) = 1$$




Momentum Conservation

$$\gamma(1) = 0$$

# Double–Leading Evolution

$$\chi_s \underset{M\to 0}{\sim} \frac{\alpha_s}{M+\alpha_s} = \frac{\alpha_s}{M} - \frac{\alpha_s^2}{M^2} + \frac{\alpha_s^3}{M^3} + \dots$$

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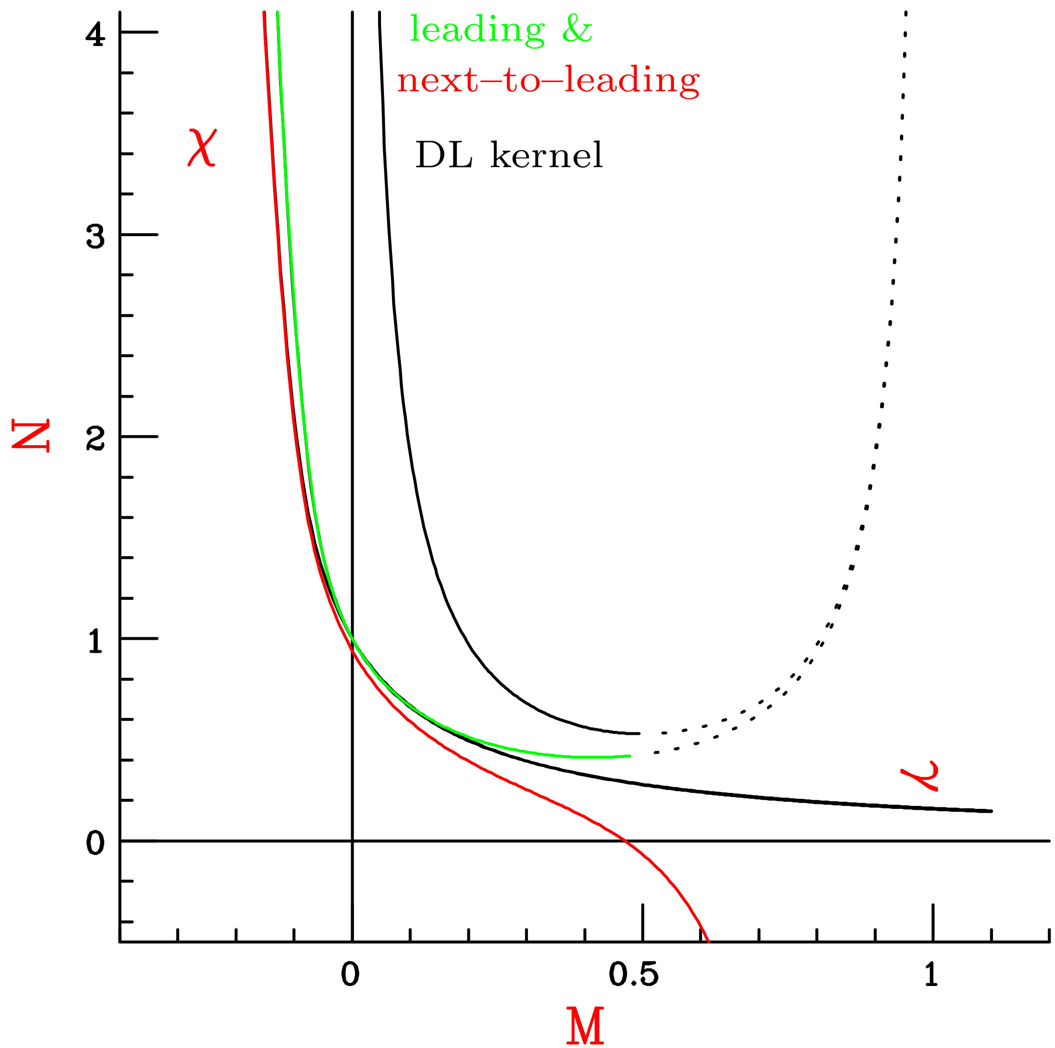


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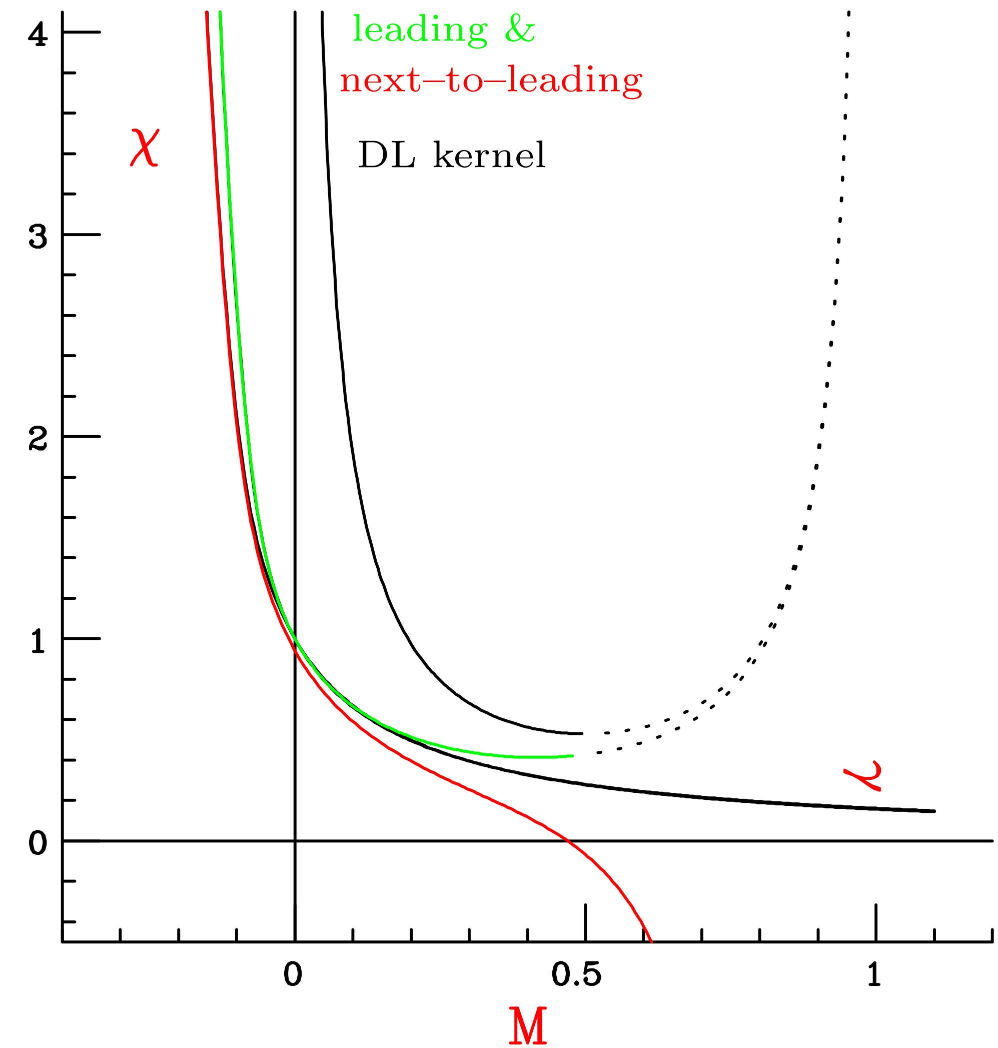


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MOMENTUM CONSERVATION

- THE DL KERNEL HAS A WELL–BEHAVED PERTURBATIVE EXPANSION

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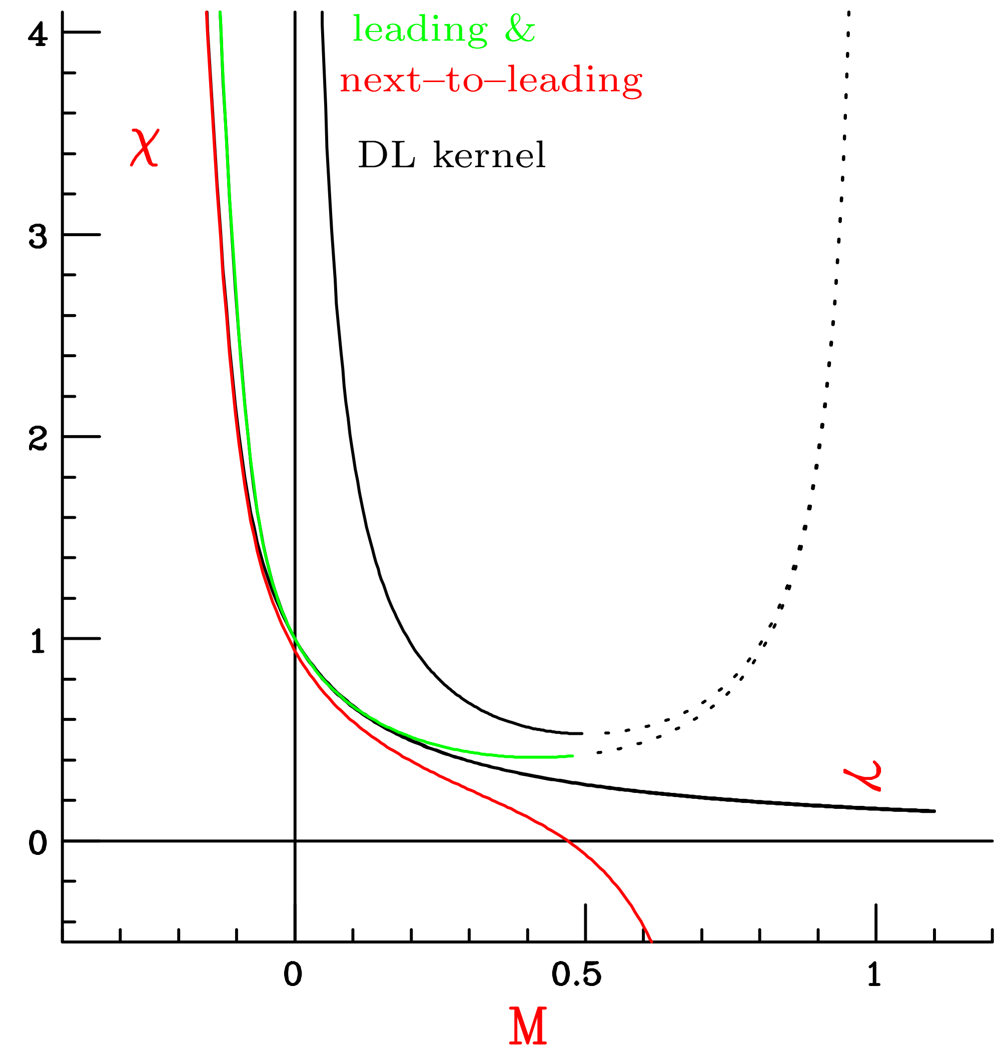


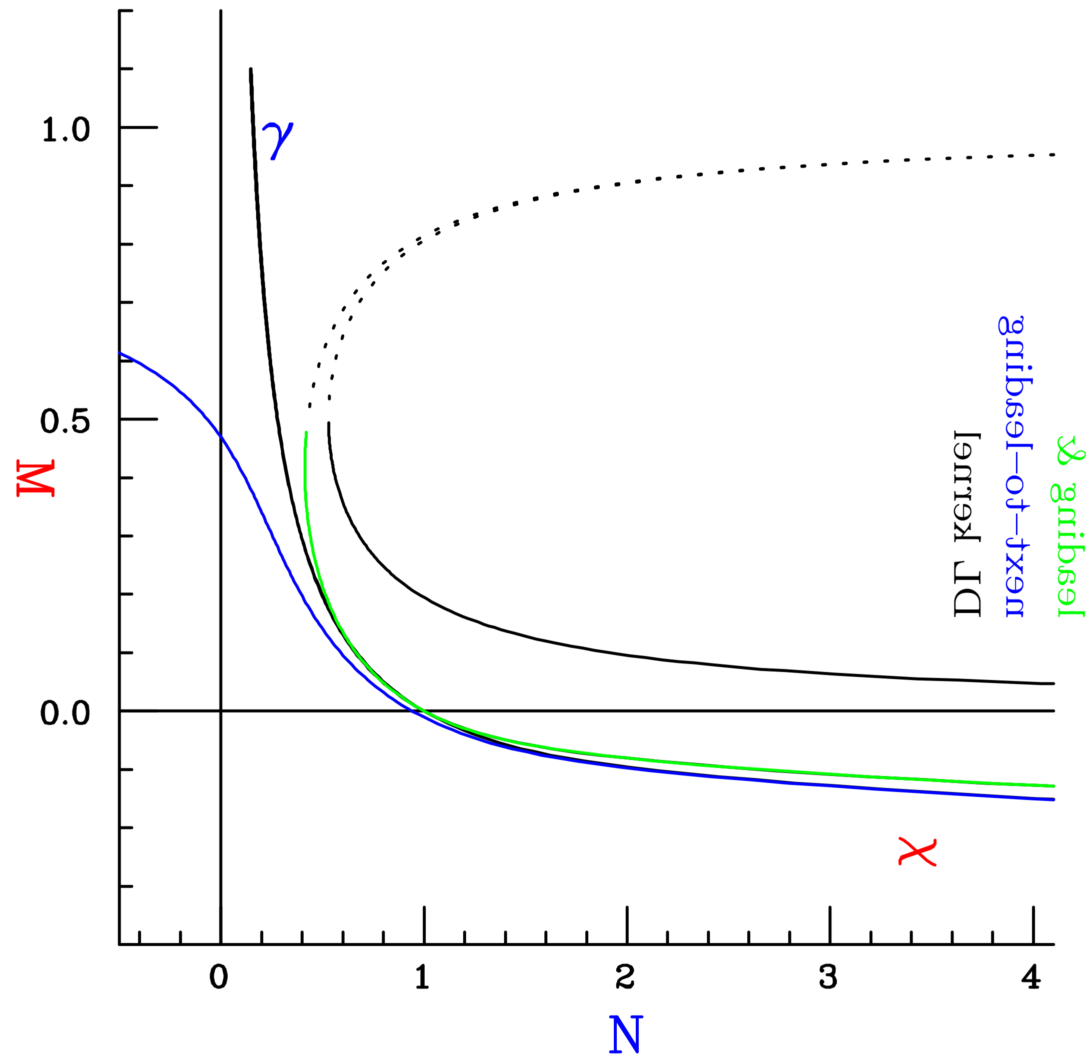

$$\gamma(1) = 0$$

MOMENTUM CONSERVATION

- THE DL KERNEL HAS A WELL–BEHAVED PERTURBATIVE EXPANSION
- DL IS CLOSE TO THE LL$Q^2$ RESULT FOR $N \gtrsim 0.3 \leftrightarrow M \lesssim 0.2$, CLOSE TO LL$1/x$ FOR $M \sim 1/2$

# Double–Leading Evolution

A robust result: Momentum Conservation

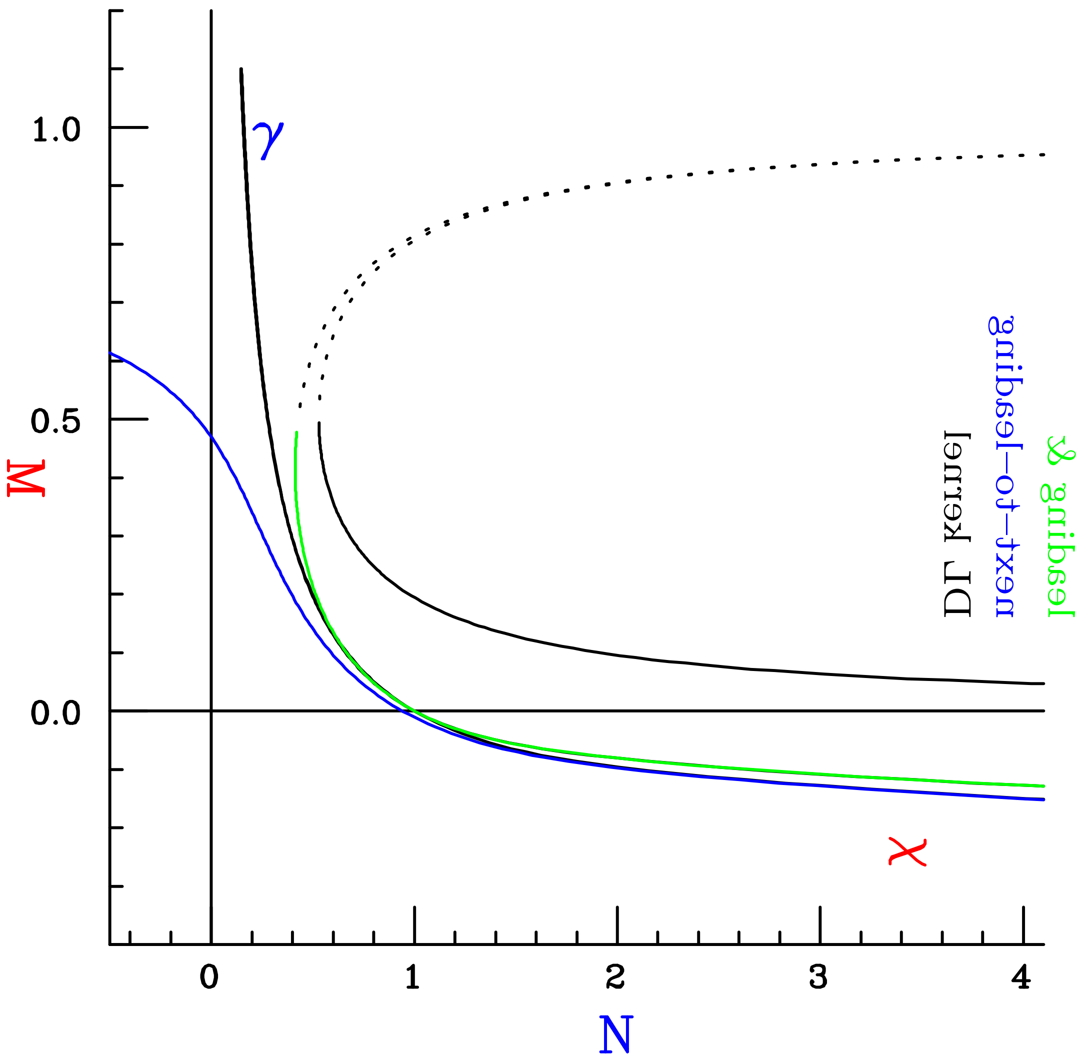



$\chi(\gamma) = N$

# DOUBLE–LEADING EVOLUTION

A ROBUST RESULT: MOMENTUM CONSERVATION

$\gamma(1) = 0$

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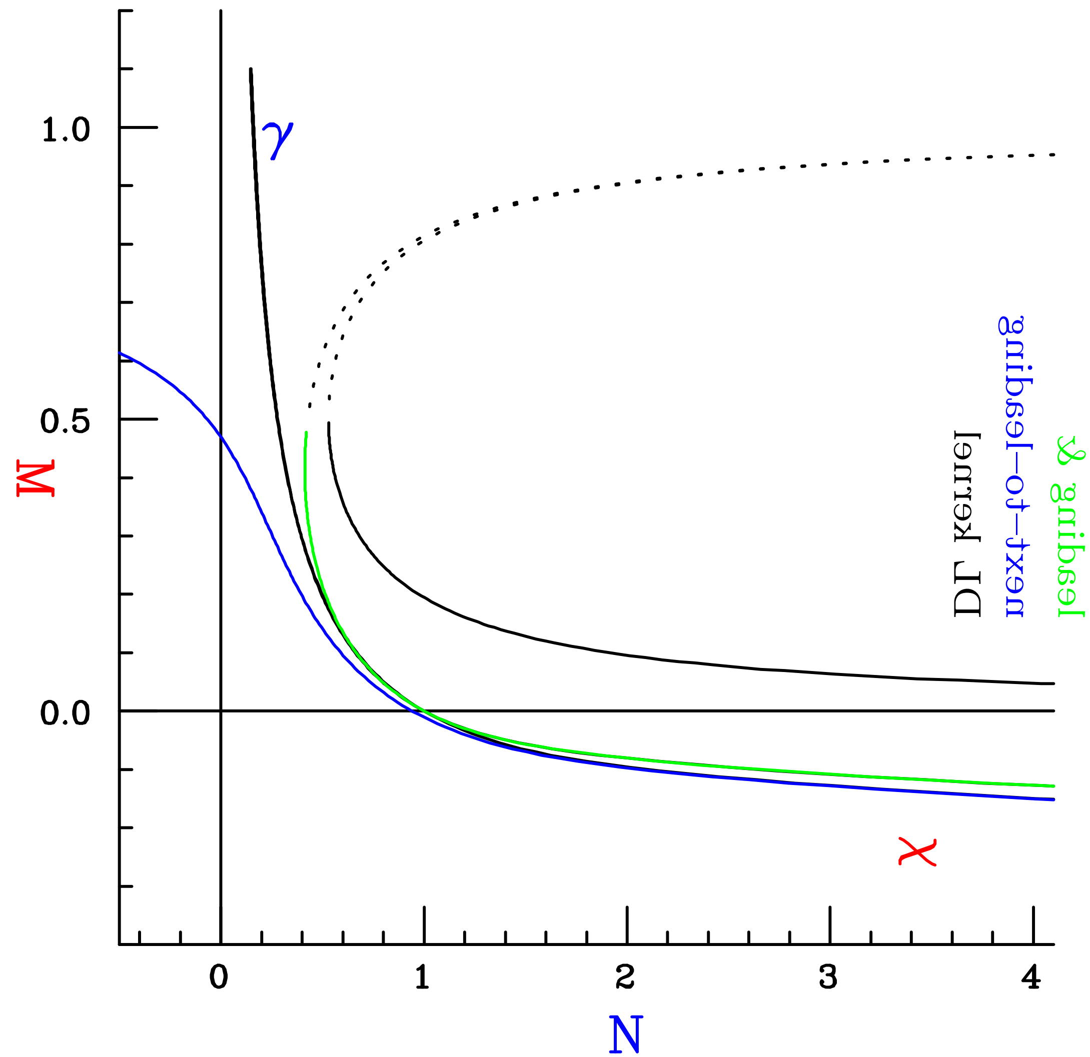


$$\chi(\gamma) = N \to \chi(0) = 1$$

$$\chi_s(M) \underset{M\to 0}{\sim} \frac{\alpha}{\alpha+M} = \frac{\alpha}{M} - \frac{\alpha^2}{M^2} + \frac{\alpha^3}{M^3} + \ldots$$

# RG CONSISTENCY: RUNNING COUPLING

- THE RUNNING OF THE COUPLING $\alpha(t) = \alpha_\mu[1 - \beta_0 \alpha_\mu t + \ldots]$
  ($t \equiv \ln \frac{Q^2}{\mu^2}$) IS LEADING LOG $Q^2$, BUT NEXT–TO–LEADING LOG $\frac{1}{x}$
- UPON M-MELLIN TRANSFORMATION ($\ln x$ EVOLUTION) $\alpha_s(t)$ BECOMES AN OPERATOR:

$$\hat{\alpha}_s(M) = \alpha_{\mu^2}[1 + \beta_0 \alpha_{\mu^2} \frac{d}{dM} + \ldots]$$

$\Rightarrow$ EVOLUTION EQUATION for $G(N, M)$ with b.c. $H_0(M)$

$$\left(1 - \frac{\alpha_\mu}{N}\right) \chi(M) G(N, M) - H_0(M) = \beta_0 \alpha_\mu \frac{d}{dM} G(N, M)$$

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- GOOD NEWS: DUALITY STILL HOLDS AT NLO& BEYOND

$$G(t,M) = G_0(N)\exp\int_{\alpha_\mu}^{\alpha(t)}\frac{d\alpha}{\beta(\alpha)}\gamma(\alpha)$$

WITH MATCHED ANOMALOUS DIMENSION AND B.C

$$\gamma(\alpha_s(t),\alpha_s(t)/N) = \gamma_s(\alpha_s(t)/N) + \alpha_s(t)\beta_0\Delta\gamma_{ss}(\alpha_s(t)/N) + $$
$$+(\alpha_s(t)\beta_0)^2\Delta\gamma_{sss}(\alpha_s(t)/N) + O(\alpha_s(t)\beta_0)^3$$
$$G_0(\alpha_s,N) = G_0(N) + \alpha_s\beta_0\Delta^{(1)}G_0(N) + (\alpha_s\beta_0)^2\Delta^{(2)}G_0(N) + O(\alpha_s\beta_0)^3,$$

$\gamma_s$, $G_0 \rightarrow$ FIXED-COUPLING DUAL

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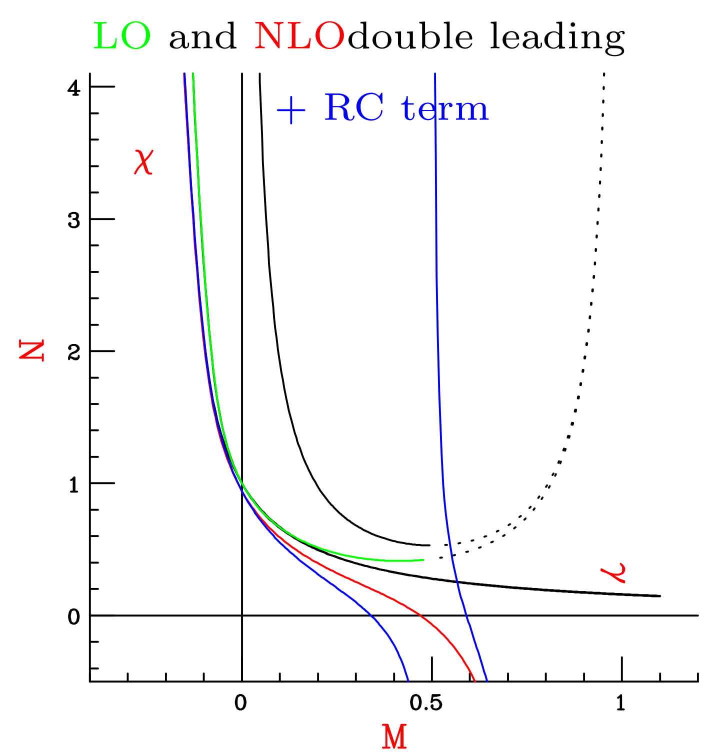
- BAD NEWS: PERTURBATIVE INSTABILITY

NLO R.C. CORRECTION

NOT UNIFORMLY SMALL AS $x \to 0$:

$$\frac{\Delta P_{ss}(\alpha_s,\xi)}{P_s(\alpha_s,\xi)} \underset{\xi\to\infty}{\sim} (\alpha_s\xi)^2$$



# RESUMMATION

- PERTURBATIVE SERIES OF UNSTABLE R.C. CORRNS. CAN BE RESUMMED

$$G(N,t) = G_0(N)\exp\left(\int_{t_0}^{t} dt'\gamma_{\rm rc}(\alpha(t'),N)\right)$$

$$\gamma_{\rm rc}(\alpha_s(t),N) = \gamma_s(\tfrac{\alpha_s(t)}{N}) + \beta_0\alpha_s(t)\Delta\gamma_{ss}(\tfrac{\alpha_s(t)}{N}) + [\beta_0\alpha_s(t)]^2\Delta\gamma_{sss}(\tfrac{\alpha_s(t)}{N}) + \dots$$

- RESULT DETERMINED BY INHOMOGENOUS SOLN. OF R.C. EVOLUTION EQUATION

$$G(N,t) = \int_{c-i\infty}^{c+i\infty}\frac{dM}{2\pi i}e^{Mt}G_I(N,M)\left[1+O\left(e^{-\frac{1}{\alpha_s(t)}}\right)\right]$$

$$= G_0(N,t_0)\int_{c-i\infty}^{c+i\infty}\frac{dM}{2\pi i}e^{\left[\frac{M-M_0}{\beta_0\alpha_s(t)}-\frac{1}{\beta_0 N}\int_{M_0}^{M}dM'\chi(M')\right]}\left[1+O\left(e^{-\frac{1}{\alpha_s(t)}}\right)\right]$$

# RESUMMATION→ FACTORIZATION

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- FACTORIZATION PRESERVED; ANOMALOUS DIMENSION

$$\gamma_{\rm res}(c,\alpha_s(t),N) \equiv \frac{d}{dt}\ln G_0(N,t)$$

# RUNNING COUPLING DUALITY

## THE OPERATOR APPROACH:

DUAL EVOLUTION EQUATIONS

$$\chi(M) = \hat{\alpha}^{-1} N$$
$$\gamma(N \hat{\alpha}^{-1}) = M$$

acting on respective boundary conditions

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DUALITY ENFORCED BY DETERMINING $\Gamma$ SUCH THAT

$$\gamma(\chi(M)) = \Gamma\left(N\hat{\alpha}^{-1}\right)$$

using $\chi = N\hat{\alpha}^{-1}$, $\quad \gamma \neq \Gamma$ because $[N\hat{\alpha}^{-1}, \chi(M)] \neq 0$

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$\Rightarrow$ CAN DETERMINE $\Gamma$ AS A FUNCTIONAL OF $\gamma$:

$$\Gamma\left(N\hat{\alpha}^{-1}\right) = \gamma\left(N\hat{\alpha}^{-1}\right) - \frac{1}{2} N\beta_0 \gamma''\left(N\hat{\alpha}^{-1}\right) / \gamma'\left(N\hat{\alpha}^{-1}\right) + \ldots$$

# EXACT ASYMPTOTIC SOLUTION

ASYMPTOTIC BEHAVIOUR CONTROLLED BY

MINIMUM OF $\chi(M) \Leftrightarrow$ RIGHTMOST SING. OF $\gamma(N)$

QUADRATIC KERNEL $\chi_q(\alpha_s, M) = \alpha_s(M)[c + \frac{1}{2}\kappa(M - M_s)^2]$

leading order: $M_s = \frac{1}{2}$; $c = \frac{4n_c}{\pi}\log 2 \approx 2.6$; $\kappa = -\frac{2n_c}{\pi}\psi''(\frac{1}{2}) \approx 32.1$

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EXACT SOLUTION!:

$G_q(N,t) \propto \mathrm{Ai}[z(\alpha_s(t), N)]$;

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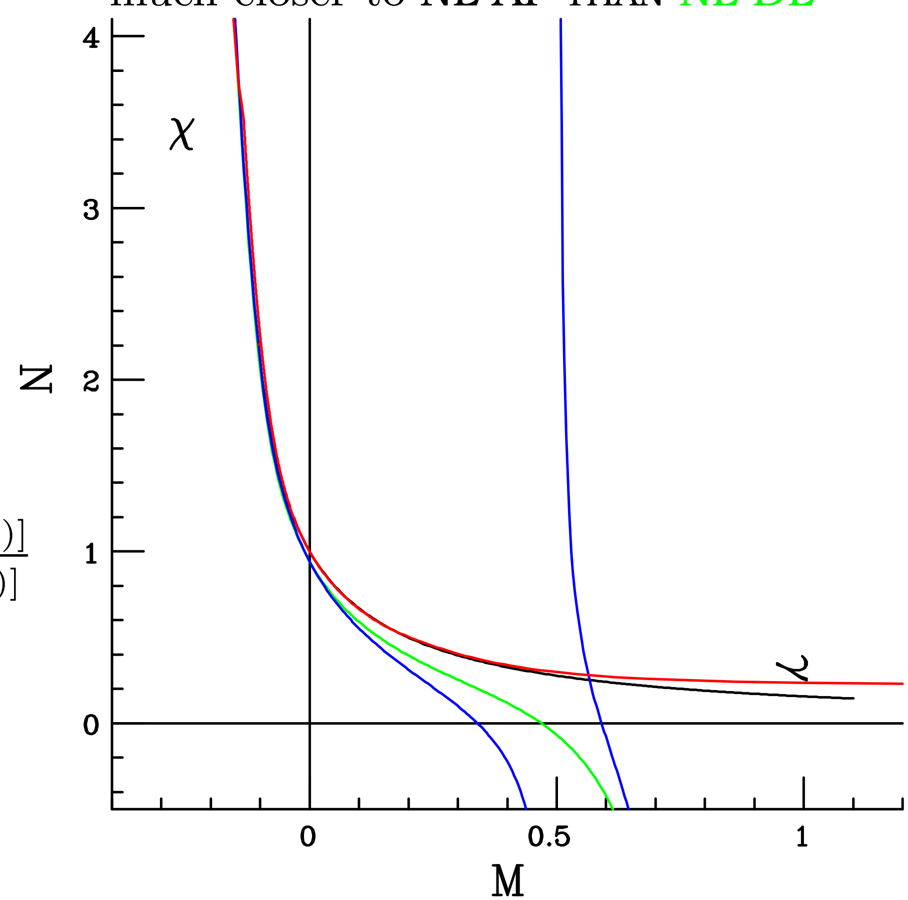
RC SINGULARITY $\Rightarrow$ RESUMMED DROP

much closer to NL AP THAN NL DL

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AIRY ANOMALOUS DIMENSION

$$\gamma_A(\alpha_s(t), N) = \frac{1}{2} + \left(\frac{2\beta_0 N}{\kappa}\right)^{1/3} \frac{\mathrm{Ai}'[z(\alpha_s(t),N)]}{\mathrm{Ai}[z(\alpha_s(t),N)]}$$

# R.C. RESUMMATION: THEORY

DOUBLE-LEADING EXPANSION OF THE AIRY AN. DIM.: (min. of $\chi(1/2) = \alpha_s c$)

$$\gamma_A(\alpha_s, N) = \frac{1}{2} - \sqrt{\frac{2}{\kappa}\left[\frac{N}{\alpha_s} - c\right]} + \frac{1}{4}\beta_0\alpha_s - \frac{1}{4}\frac{\beta_0\alpha_s^2 c}{N - \alpha_s c} + O(\alpha_s^3) + O\left(\frac{\alpha_s^3}{N}\right)$$

**LO:** $\gamma_s^A(\alpha_s/N) \;\rightarrow$ SQRT CUT AT $N = \alpha_s c$; $\gamma_0^A(\alpha_s/N) \;\rightarrow$ CONST.

**NLO:** $\gamma_{ss}^A(\alpha_s/N) \;\rightarrow$ POLE AT $N = \alpha_s c \Rightarrow$ (QUADRATIC RISE IN SPL.FCTN.)

COMBINE RC (AIRY) & DL AN. DIM., SUBTRACT DOUBLE COUNTING:

$$\gamma_{\rm res} = \gamma_{\rm DL} + \gamma_A - \gamma_s^A - \gamma_{ss}^A$$

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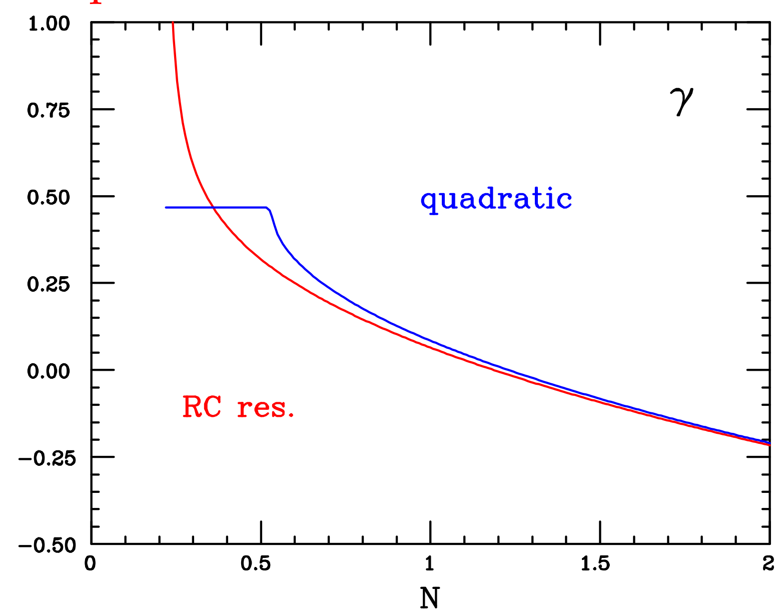
- AT SMALL $N$,
  - CUT IN $\gamma_s(\alpha_s/N)$ IS RESUMMED TO A POLE
  - POLE IN $\gamma_{ss}(\alpha_s/N)$ (PERTURBATIVE INSTABILITY) REMOVED

quadratic CUT $\Rightarrow$ RC POLE



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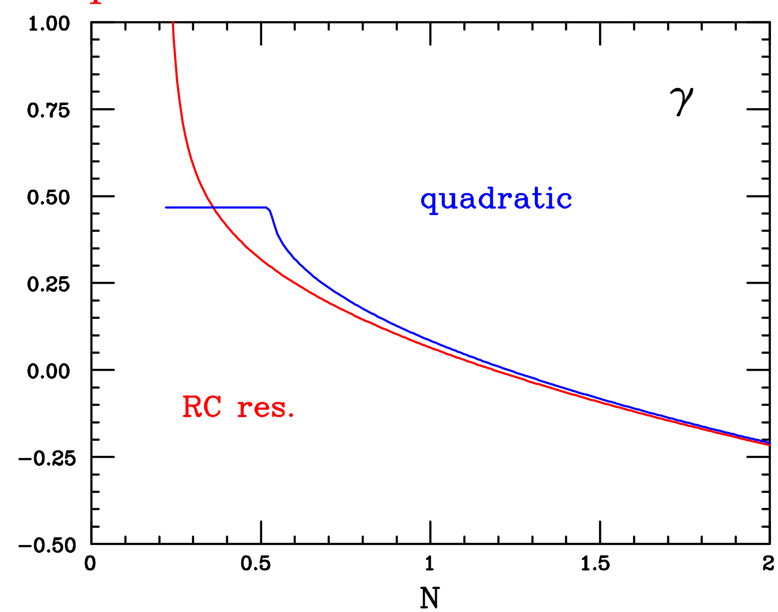


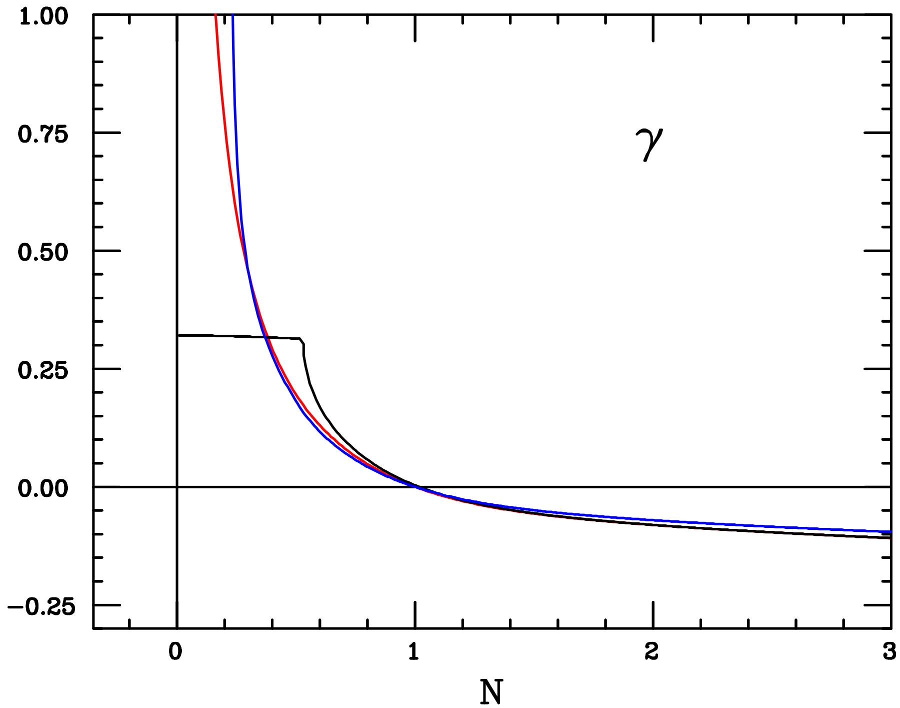

ASYMPTOTIC SMALL $x$ BEHAVIOUR $\Rightarrow$ RC (AIRY) POLE

SMALL INTERCEPT! $\alpha_s = 0.2 \Rightarrow N_0 \approx 0.21$ (LO $\alpha_s c_0 \approx 0.53$))

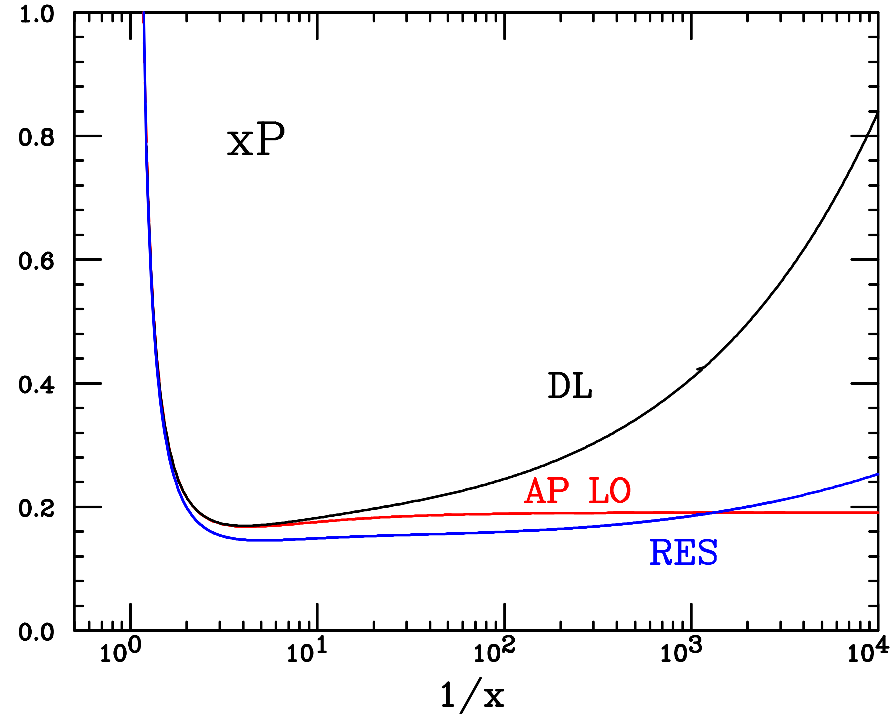
SMALL RESIDUE! $r_A = \frac{3\beta_0 N_0^2 \alpha_s(t)}{N_0 + 2c_0\alpha_s(t)}$; $\Rightarrow r_A \approx 0.014$ (LO $r_0 = \frac{3\alpha_s}{\pi} \approx 0.191$)

# R.C. RESUMMATION: PHENOMENOLOGY

ANOMALOUS DIMENSION




SPLITTING FUNCTION




$$\gamma_{\rm res}(\alpha_s, N) = \gamma_{\rm DL}(\alpha_s, N) + \gamma_A(\alpha_s, N) - \gamma_s^A(\alpha_s/N) - \alpha_s\gamma_{ss}^A(\alpha_s/N)$$

$$\gamma_{\rm DL} = \alpha_s\gamma_0(N) + \gamma_{\rm s}(\alpha_s/N)$$

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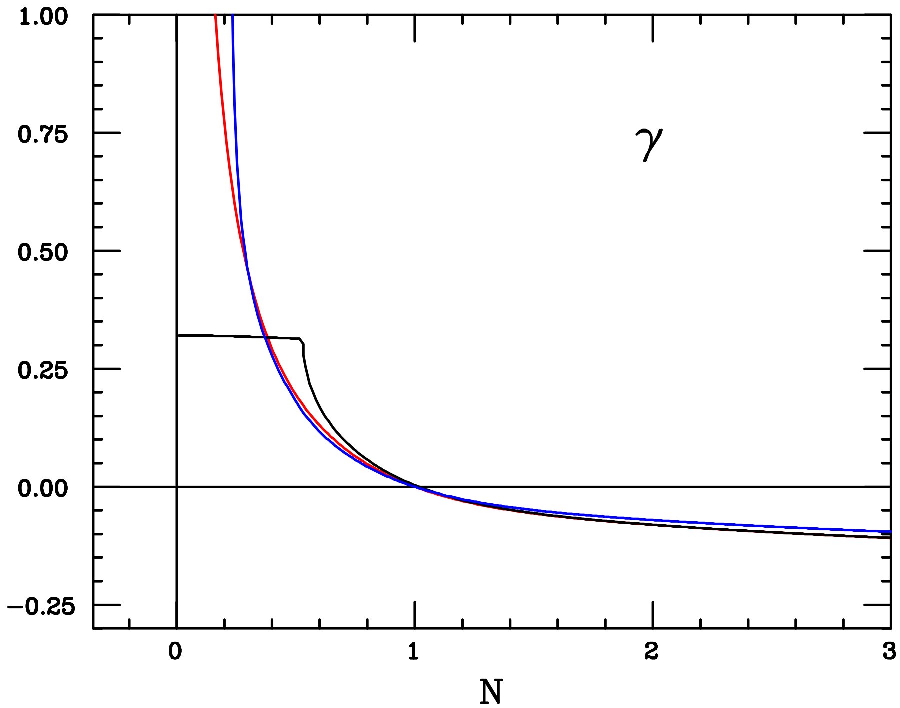


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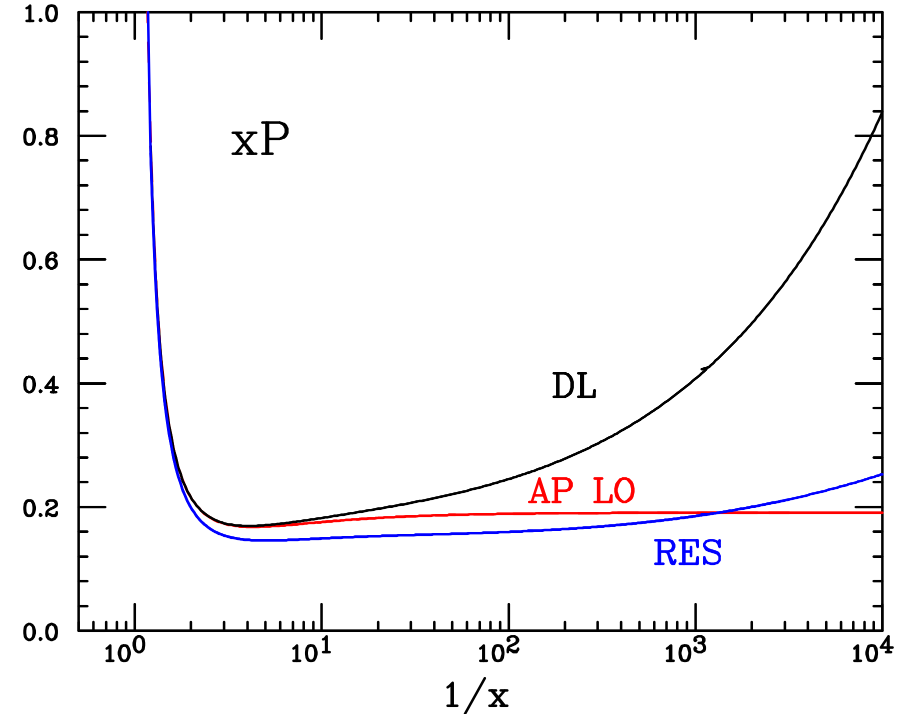
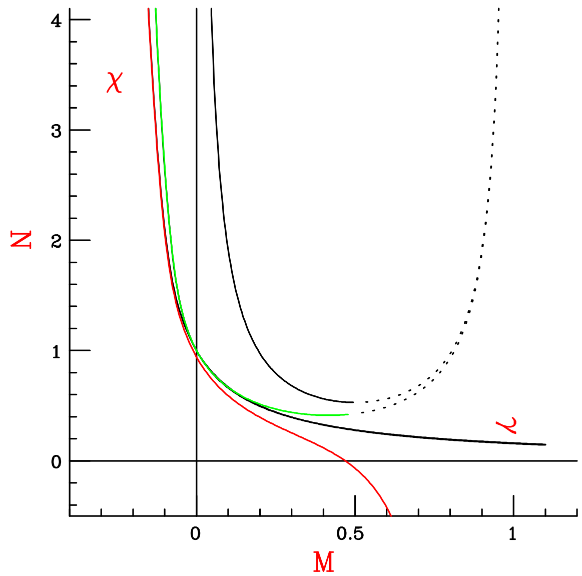



$$\gamma_{\rm res}(\alpha_s, N) = \gamma_{\rm DL}(\alpha_s, N) + \gamma_A(\alpha_s, N) - \gamma_s^A(\alpha_s/N) - \alpha_s \gamma_{ss}^A(\alpha_s/N)$$

$$\gamma_{\rm DL} = \alpha_s \gamma_0(N) + \gamma_{\rm s}(\alpha_s/N)$$

- FORMALLY SUBL. RUNN. COUPL. RESUMM. DETERMINES THE SMALL $x$ BEHAVIOUR
- AT LARGE $N$, LARGE $x$, $\gamma_{\rm res}$ ESSENTIALLY IDENTICAL TO LO ALTARELLI-PARISI $\gamma_0$
- AT SMALL $N$, SMALL $x$
  - CUT IN $\gamma_{\rm DL}(\alpha_s/N)$ (FROM $\gamma_{ss}(\alpha_s/N)$) IS RESUMMED TO A SOFTER POLE IN $\gamma_{\rm res}$
  - $\gamma_{\rm res}$ REMAINS VERY CLOSE TO LO ALTARELLI-PARISI $\gamma_0$ DOWN TO $x \sim 10^{-4}$
  - SLIGHT DEPLETION @ MEDIUM-SMALL $x$, MODERATE RISE VERY SMALL $x$

# NLO CORRECTIONS:

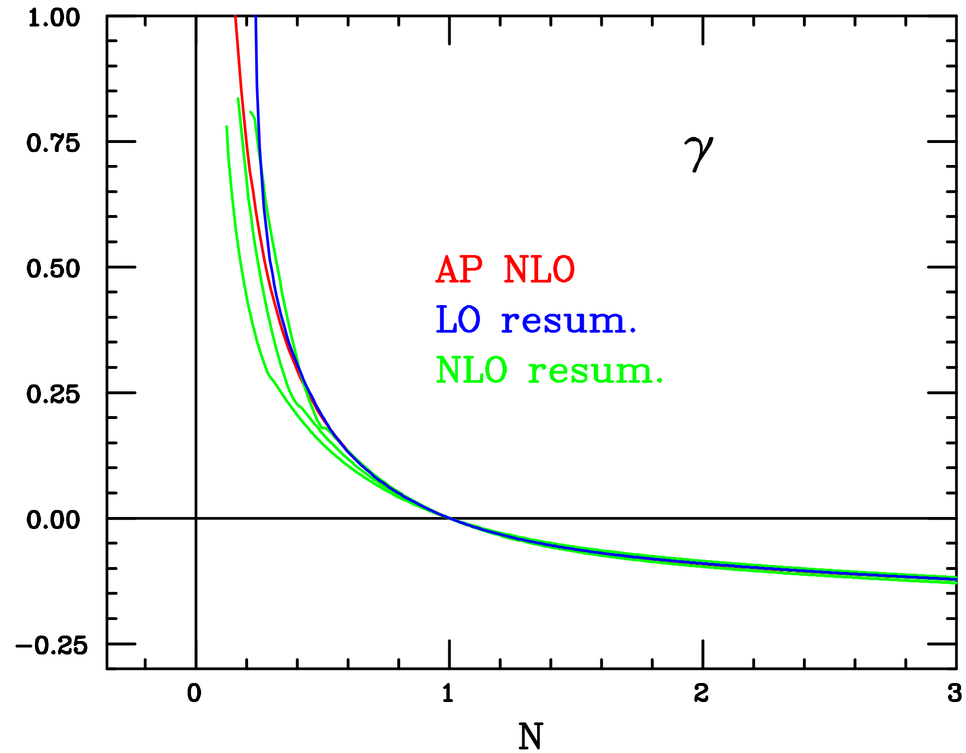



THE NL DOUBLE–LEADING KERNEL HAS NO MINIMUM...

⇒ RUNNING-COUPLING RESUMM. IF MIN. RESTORED BY HIGHER ORDER CORRECTIONS

(suggested by symmetry of BFKL kernel)

LARGE AMBIGUITIES IN RESULT IF PARMS. OF ASSUMED MIN. (curvature & intercept) VARIED IN REASONABLE RANGE



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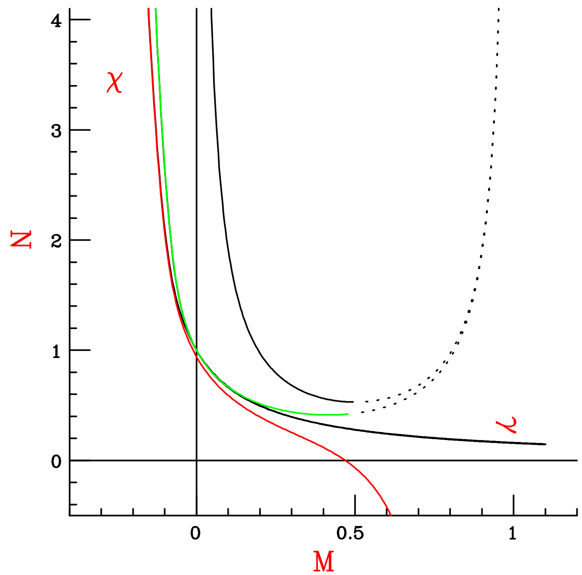


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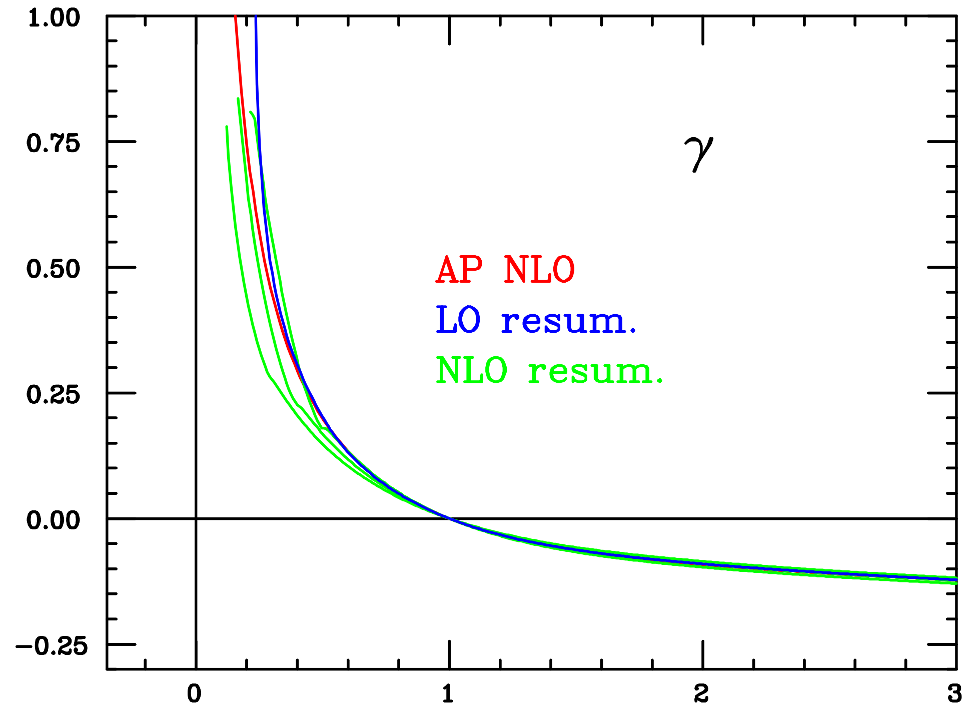


- NL RESUMM. CAN'T BE DET. WITHOUT ASSUMPTIONS ON SUBL. TERMS
- PHENOMENOLOGICAL SUCCESS OF TWO LOOPS & ITS STABILITY UPON RESUMMATION INDICATE CORRECTIONS TO $\chi_0$ MUST BE SMALL

# ANTICOLLINEAR CONSISTENCY: SYMMETRY

$\ln 1/x$ EVOLUTION KERNEL SYMMETRIC

UPON INTERCHANGE OF INITIAL AND FINAL PARTON VIRTUALITIES

$$Q^2 \leftrightarrow k^2 \Leftrightarrow M \leftrightarrow 1-M$$

$$\frac{d}{d\xi}G(\xi,M) = \chi(M,\alpha_s)\, G(\xi,M); \qquad \chi(\xi,M) = \int_{-\infty}^{\infty} \frac{dQ^2}{Q^2} \left(\frac{Q^2}{k^2}\right)^{-M} \chi(\xi,\frac{Q^2}{k^2})$$

$$\chi(M) = \chi(1-M)$$

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COLLINEAR RES. OF $\frac{1}{M}$ POLES $\leftrightarrow$ ANTICOLLINEAR RES. OF $\frac{1}{1-M}$ POLES

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CAVEATS

- SYMMETRIZATION IS DL SUBLEADING $\rightarrow$ AMBIGUITIES (can add any symmetric subl. term...)
- SYMMETRY BROKEN BY DIS CHOICE OF KIN. VARIABLES $s = \frac{Q2}{x}$
  $\rightarrow$ DIS KERNEL DETERMINED BY IMPLICIT EQN. (Fadin, Lipatov, 1998)

$$\chi^{\rm DIS}(M) = \chi^{\rm sym}(M - \frac{1}{2}\chi^{\rm DIS}(M))$$

- SYMMETRY BROKEN BY RUNNING COUPLING $\alpha_s(Q^2) \Rightarrow$ MUST
  SWITCH FROM SYM TO NONSYM SCALE IN FADIN-LIPATOV KERNEL

# SYMMETRIZED EXPANSION

## THE $\chi$ KERNEL

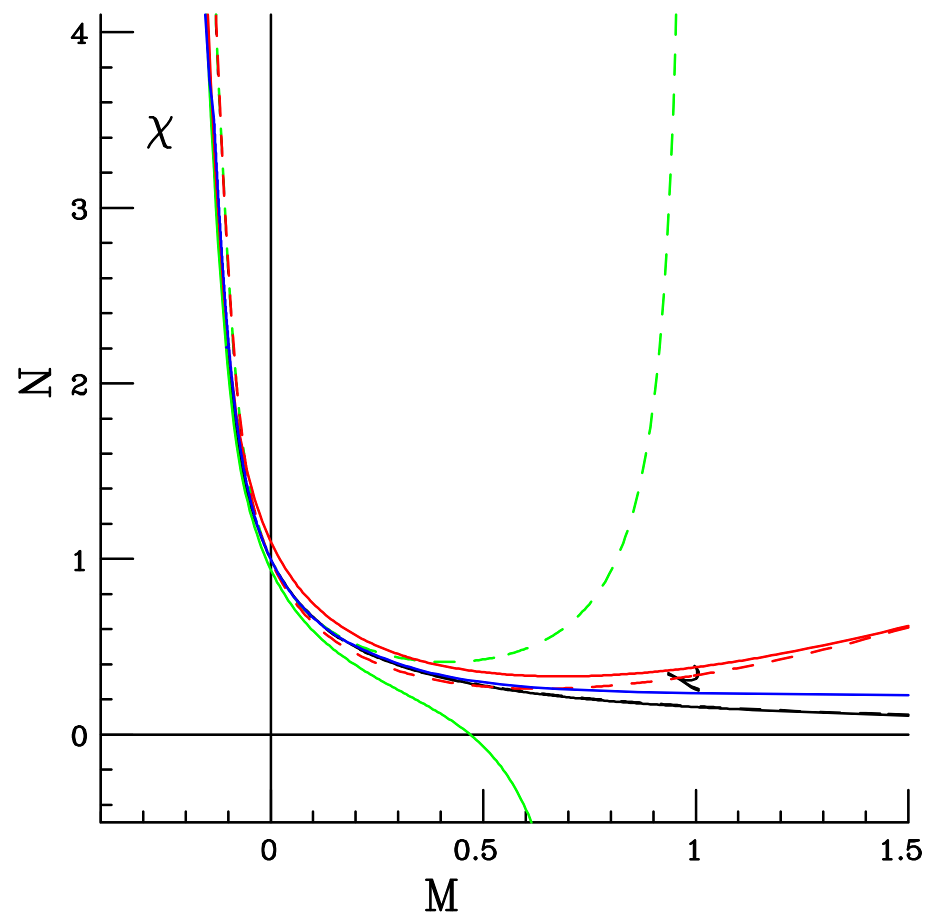
DIFFERS FROM DL EXPANSION BY SUBLEADING TERMS

LO (MINIMAL PRESCRIPTION):

$$\chi(M) = \chi_s\left(\frac{\alpha}{M}\right) + \chi_s\left(\frac{\alpha}{1-M+N}\right) + \alpha\left[2\psi(1) - \psi(M) - \psi(1-M+N) - \frac{1}{M} - \frac{1}{1-M+N}\right]$$

$\chi_0^{BFKL}(M) = 2\psi(1) - \psi(M) - \psi(1-M)$; $\chi_s\left(\frac{\alpha}{M}\right)$ dual of LO AP; $N = \chi(M)$ implicitly

- LO, NLO SYM. CLOSE TO EACH OTHER
- LO, NLO SYM. CLOSE TO AP & RC RESUMMED



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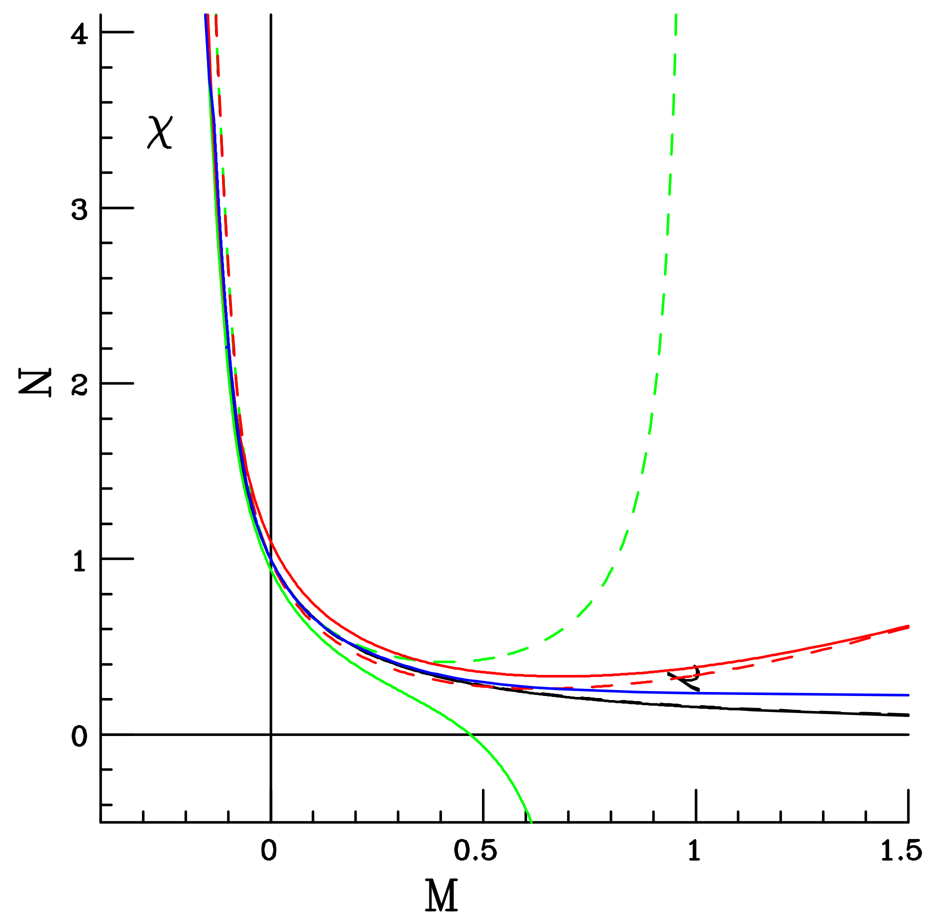

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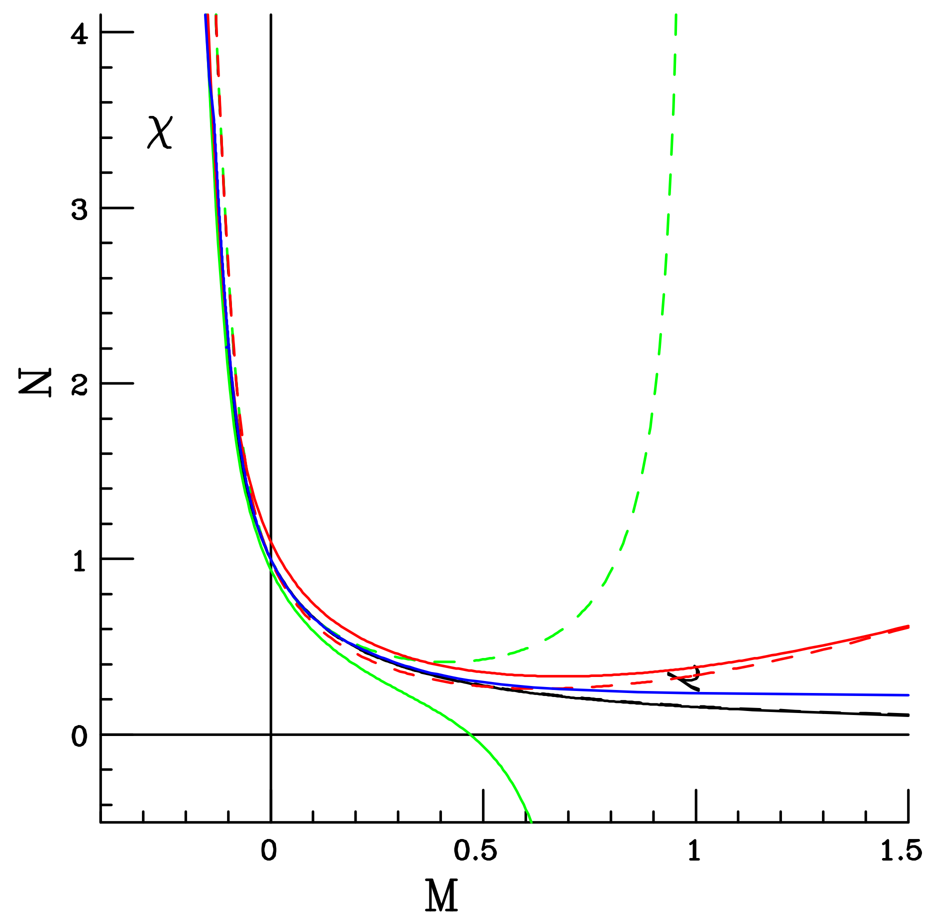
- CONSTRUCT $\gamma$ BY RC RESUMM. OF SYM. $\chi$



# SYMMETRIZED EXPANSION

## THE R.C. RESUMMED ANOMALOUS DIMENSION

COMPLICATIONS

- IN DL EXPANSION, ALL TERMS IN $\chi_s\left(\frac{\alpha}{M}\right)$ (DUAL TO LO AP) ARE SAME ORDER $\Rightarrow$
  INTEFERENCE WITH $\alpha\chi_1$ GENERATES NLO AP TERMS WHICH MUST BE REMOVED
- RC RESUMMATION MUST BE PERFORMED WITH $\alpha(Q^2)$ $\Rightarrow$
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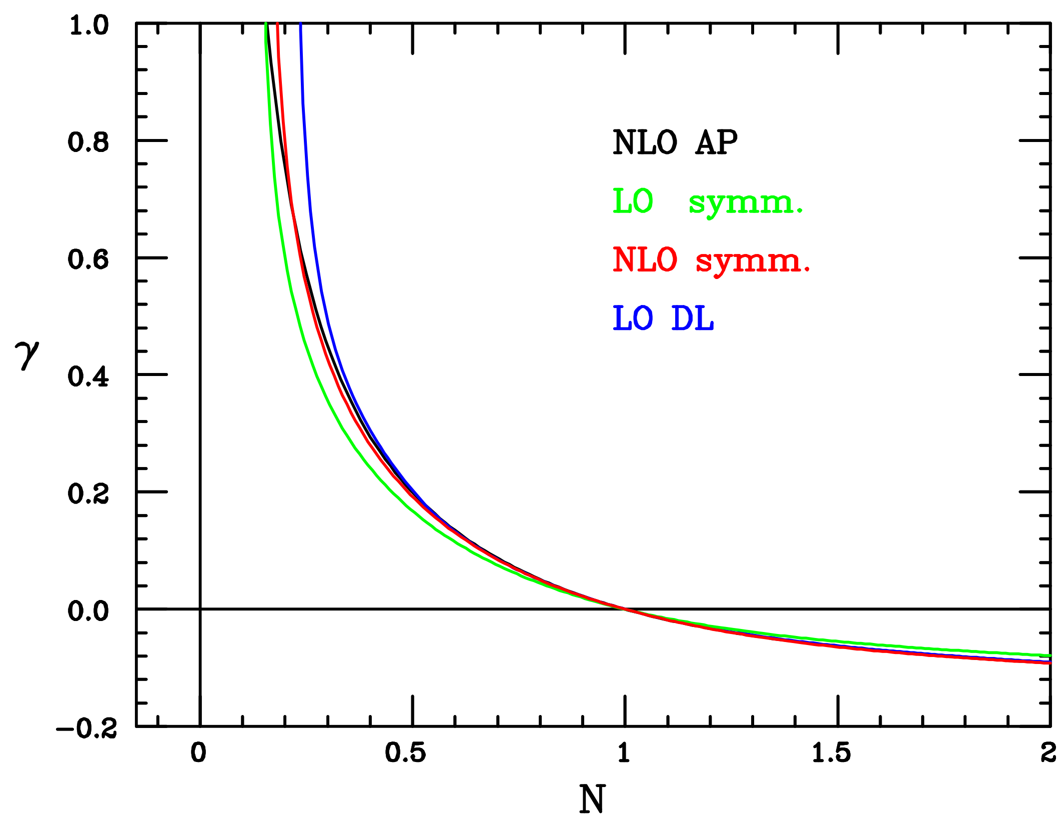
- OPERATOR ORDERING IN $\chi_s(\alpha/M)$ WHEN $\alpha$ IS AN OPERATOR (RC RESUMMATION)



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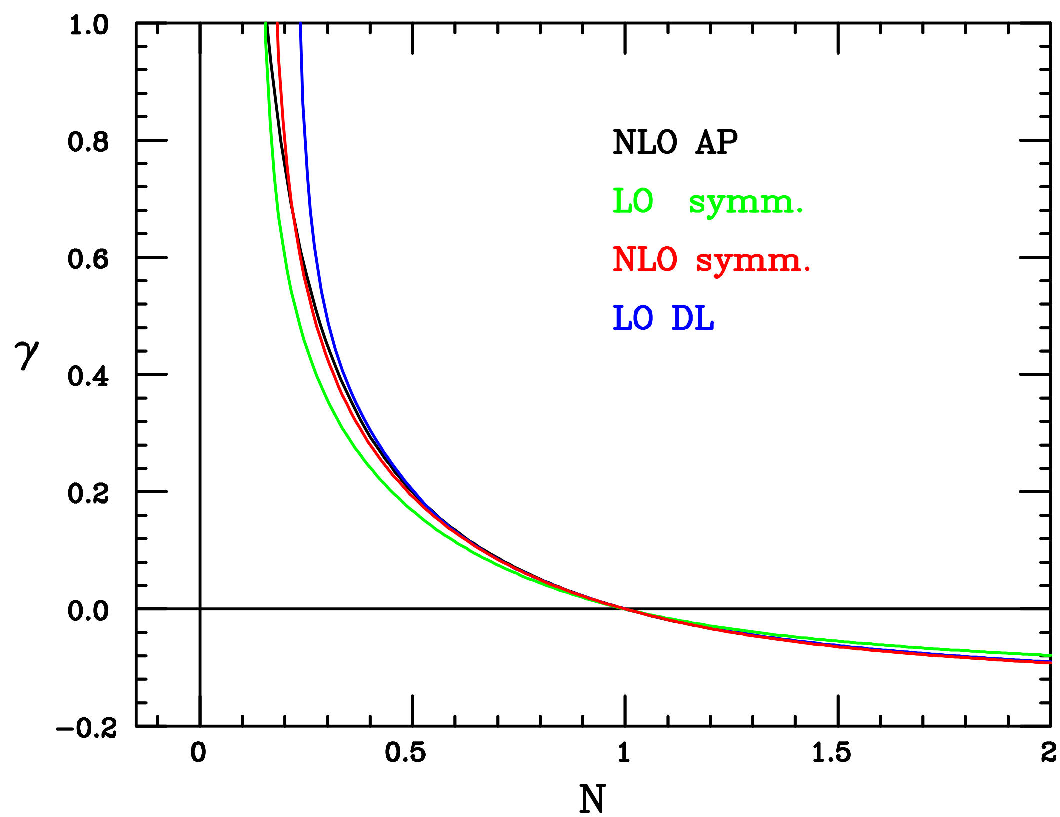


QUALITATIVE FEATURES

- MOMENTUM CONSERVATION + SYMMETRY $\Rightarrow$ MINIMUM IN $\chi$
- RUNNING COUPLING RESUMMATION $\Rightarrow$ SIMPLE POLE IN $\gamma$

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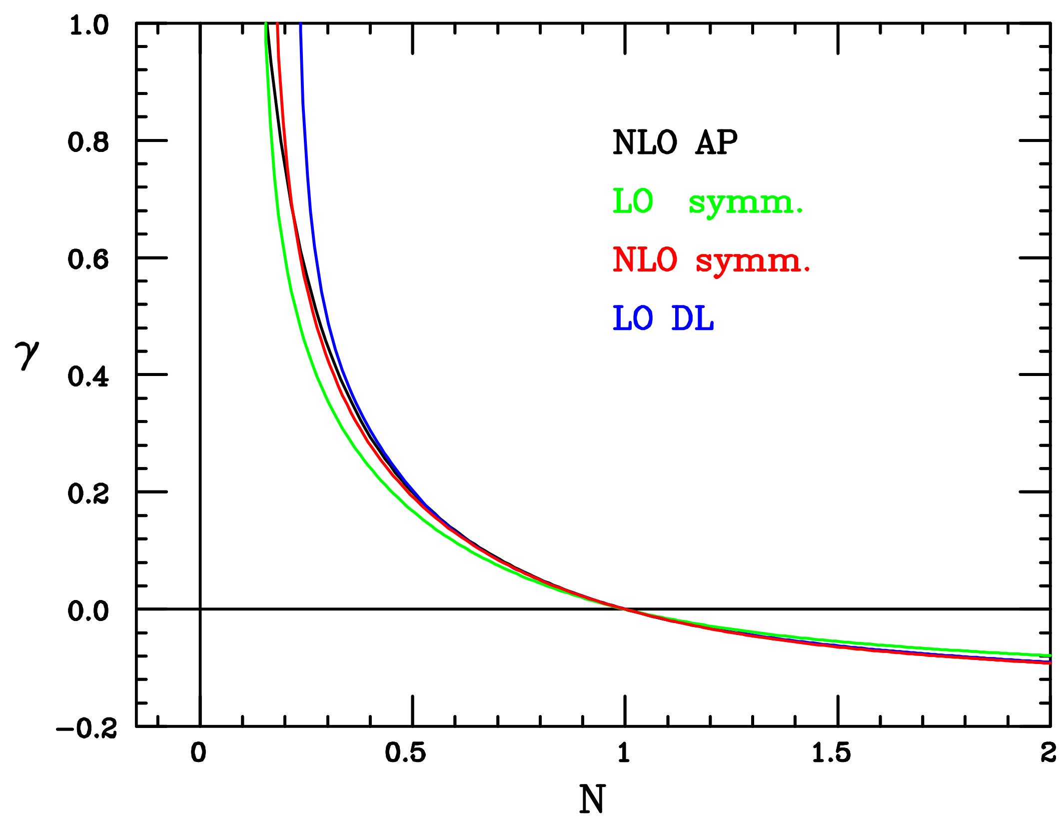


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$\Rightarrow$ REGGE BEHAVIOUR AT SMALL $x$

# SYMMETRIZED EXPANSION

## THE R.C. RESUMMED SPLITTING FUNCTION

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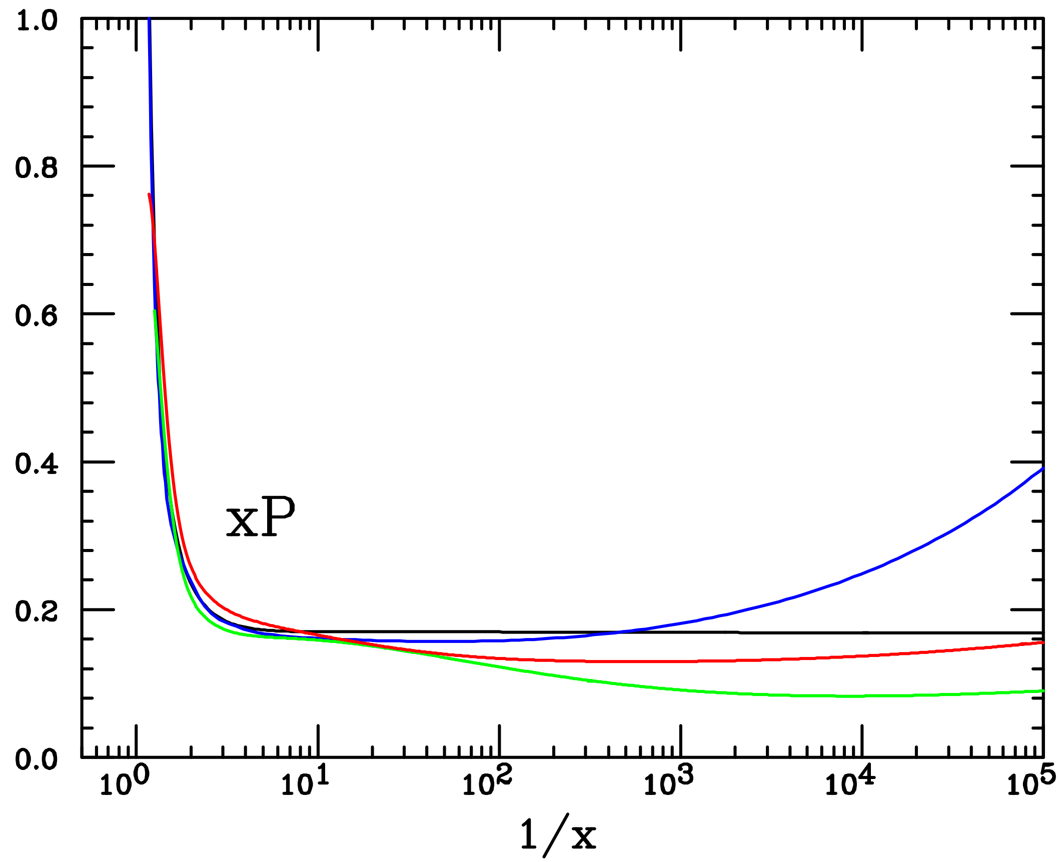


SYMMETRIZED RC RESUMMED NLO SPL. FNCT. CLOSER TO NLO AP THAN PREVIOUS DL RC RESUMMED

LO AND NLO SYMMETRIZED CLOSE TO EACH OTHER

# CONCLUSIONS

- The matching of Altarelli-Parisi and BFKL evolution is now fully understood in leading twist, large $Q^2$ factorization.
  $\Rightarrow$ When all available info on perturbative evolution is combined, the small–$x$ instabilities go away

- The Regge limit is perturbatively stable after running coupling resummation $\Rightarrow$ Resummed results are very close to LO, NLO; instability of NNLO and higher orders removed

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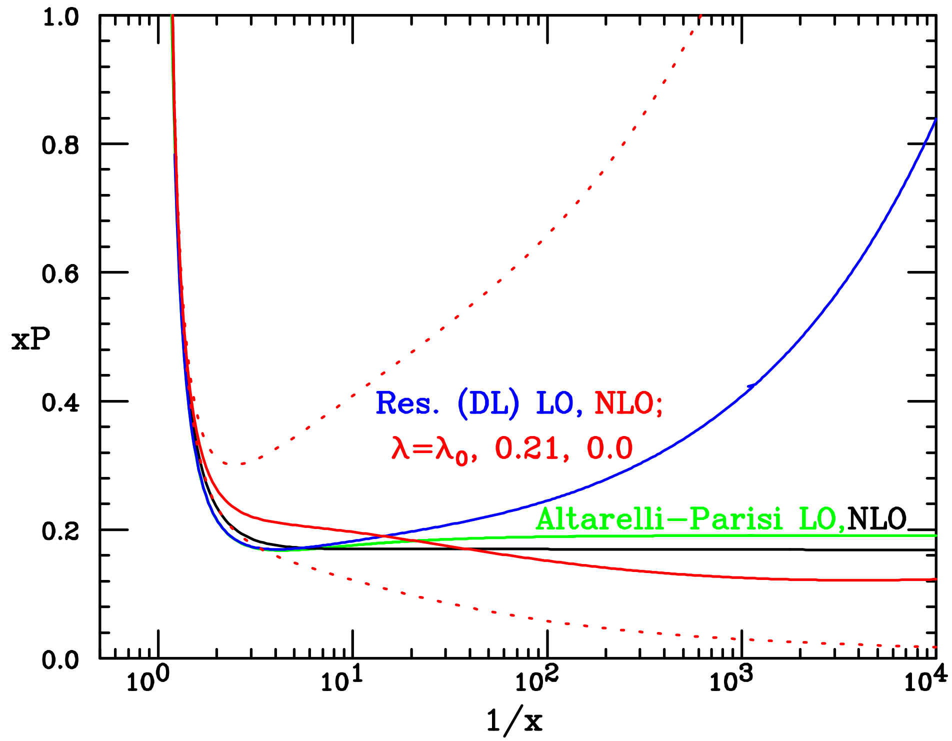
- A CONSISTENT THEORY OF SMALL-$x$ EVOLUTION
- THE REGGE LIMIT IS CONTROLLED BY QUADRATIC SYMMETRIZED COLLINEAR KERNELS

# RESUMMATION 2000

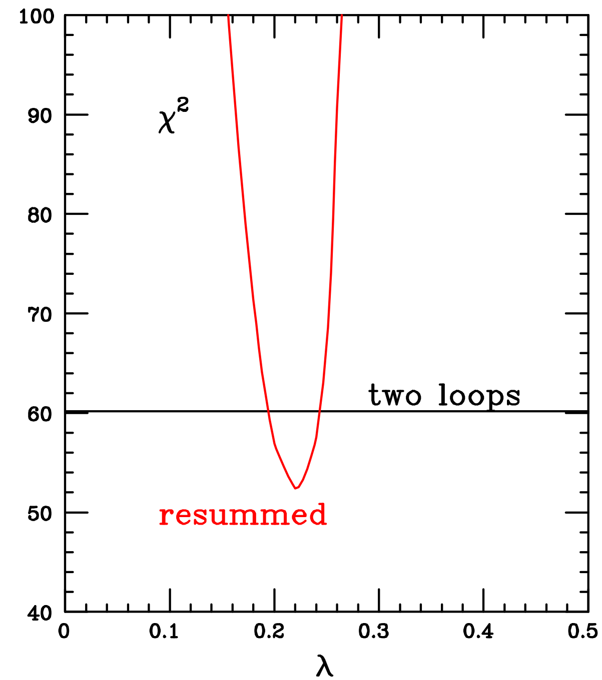
- DOUBLE-LEADING PERTURBATIVE EXPANSION STABLE
- STRONG DEPENDENCE ON NONPERTURBATIVE ALL-ORDER INTERCEPT

  $\lambda$ (value of $\chi(M)$ at min.)
- GOOD AGREEMENT WITH DATA IF $\lambda$ FITTED (FINE-TUNED)

$xP(x)$




$\chi^2$



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M. Ciafaloni, D. Colferai, G. P. Salam and A. Staśto, hep-ph/0307188 & ref. therein

SAME IDENTIFICATION OF MAIN PROBLEMS

SEVERAL TECHNICAL DIFFERENCES: e.g. “off-shell” vs. “on-shell” formalism

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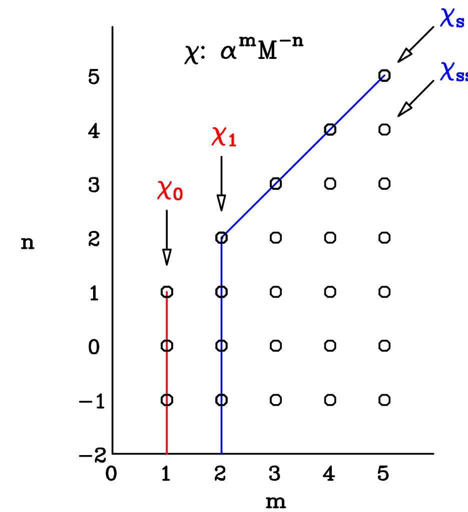
OLD: "$\omega$ EXPANSION": LET $\chi(M) \to \chi(M + A(N))$,

EXPAND IN POWERS OF $A(N)$ & ADJUST $A(N)$ TO MATCH

TO ALTARELLI-PARISI;

PROBLEM $N \sim O(\alpha_s) + O(\alpha_s/M)$, $\Rightarrow O(N^2) \sim O(1)$

NEW: "IMPROVED KERNEL": $\Rightarrow$ SMALL $x$ DL EXPANSION



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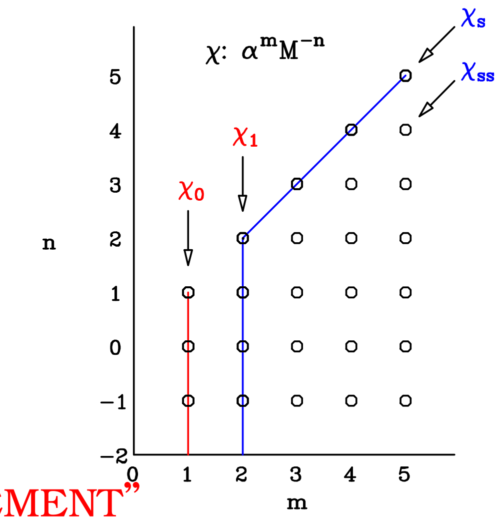


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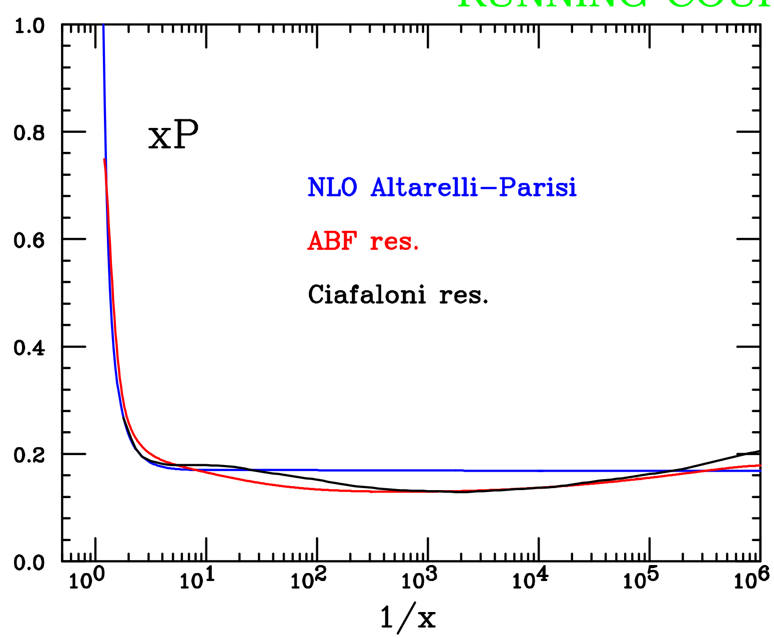


- COLLINEAR-RESUMMED $\chi$-KERNEL SYMMETRIZED
- NUMERICAL SOLUTION OF $\xi$ EVO. EQN. (NO QUADR. APPROX.)
- NUMERICAL DECONVOLUTION OF SPLITTING FCTN.