# N=1 Flavor in AdS/CFT

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## Outline of Talk

1. Pep Talk
2. Review of the conifold and D3-brane CFT
3. Adding flavor to the conifold (PO, hep-th/0311084)
4. 3-form fluxes and RG flow, Duality cascade
5. IR deformation and confinement
6. Flavored cascade, towards N=1 SQCD (PO)

The standard $AdS/CFT$ correspondence relates $\mathcal{N} = 4$ $SU(N)$ super-YM at strong coupling to string theory in $AdS_5 \times S^5$ at weak coupling. if we hope to have something to say about real physics, we need to

- Break supersymmetry.
- Break conformal invariance.
- Be able to describe asymptotically free theories.
- Obtain realistic matter content (fundamental quarks.)

## Branes at Singularities

We can construct interesting gauge/gravity duals by modifying the space transverse to the branes. It is particularly interesting to consider singular spaces:

- New matter content.
- In orbifolds, can explicitly quantize string theory. String propagation is sensible despite the singular background!
- Breaks (some) supersymmetry. Can choose the singularity to preserve some SUSY and other symmetries so that the physics remains tractable.
- Some metrics are known.
- Non-trivial topology allows quantized 3-form fluxes which break conformal invariance in a controlled way (and which have a clear FT interpretation.)

## Geometry of the Conifold

Conifold singularities are generic in Calabi-Yau compactifications and have several useful properties:

- Can be written as the locus of an equation in $C^4$:

  $$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0$$

  or equivalently $z_1 z_2 - z_3 z_4 = 0$.

- The scaling isometry $w_i \rightarrow \lambda w_i$ implies that the space is conical.

- The base, known as $T^{1,1}$, is an Einstein manifold with a known(!) metric and which has as its symmetry group $SU(2) \times SU(2) \times U(1)$.

- The base $T^{1,1}$ is a sphere bundle of $S^2$ over $S^3$ (sometimes useful to think of it as $S^2 \times S^2 \times S^1$.) We will denote the corresponding two-cycle and three-cycle as $\omega_2$ and $\omega_3$. Their volumes are normalized as

$$\int_{S^2} \omega_2 = 4\pi \ , \qquad \int_{S^3} \omega_3 = 8\pi^2$$

- Can "solve" the conifold equation in terms of homogeneous coordinates:

$$\begin{aligned} z_1 &= A_1 B_1, \qquad z_2 = A_2 B_2 \\ z_3 &= A_1 B_2, \qquad z_4 = A_2 B_1. \end{aligned}$$

In terms of the $A, B$ variables there is a "baryon number" symmetry acting as $A, B \rightarrow A e^{i\beta}, B e^{-i\beta}$.

## Field Theory of the Conifold

Now suppose that we place $N$ D3-branes at the tip of the conifold. We then obtain a gauge theory $SU(N) \times SU(N)$. The positions of the branes contribute matter fields $A_1, A_2, B_1, B_2$ (not the $w_i$) to the low-energy theory which transform in the bifundamental representations $(N, \bar{N})$ and $(\bar{N}, N)$. The supersymmetry is $\mathcal{N} = 1$, consistent with the Calabi-Yau compactification.

Shifman-Vainshtein $\beta$-functions:

$$\frac{\partial}{\partial \log \Lambda} \frac{8\pi^2}{g_{YM}^2} = 3N - 2N(1 - 2\gamma_{A,B}).$$

Conformal invariance requires that $A, B$ have anomalous dimension $\gamma = -1/4$.

As described so far, the theory has too many moduli (off-diagonal components of A,B) to describe branes on the conifold. To lift the unwanted flat directions we add a superpotential (exactly marginal if $A, B$ have dimension 1-1/4=3/4):

$$W = \epsilon^{ij}\epsilon^{kl}\mathrm{Tr}A_i B_k A_j B_l \ .$$

The gauge theory was constructed by Klebanov-Witten and Morrison-Plesser, and it survives many consistency checks.

The warped supergravity solution is $AdS_5 \times T^{1,1}$. The $S^2$ and $S^3$ which shrink to zero size in the absence of flux have finite size due to the warping.

$$\begin{aligned} ds^2 &= h(r)^{-1/2}\eta_{ij}dx^i dx^j + h(r)^{1/2}(dr^2 + r^2 d\Omega^2_{T^{1,1}}), \\ F_5 &= d^4x \wedge dh^{-1} + *(d^4x \wedge dh^{-1}) \\ h(r) &= 1 + \frac{L^4}{r^4}, \qquad L^4 = \frac{27}{4}\pi g_s N \alpha'^2. \end{aligned}$$

## Comparison with $Z_2$ Orbifold

It is instructive to compare this theory with the theory on an $S^5/Z_2$ orbifold (obtained from D3-branes transverse to $C^2/Z_2 \times C$.

- $S^5/Z_2$ is also topologically $S^3 \times S^2$.
- The orbifold can be written as the locus of an equation in $C^3$:

$$z_1 z_2 - z_3^2 = 0$$

  or equivalently $w_1^2 + w_2^2 + w_3^2 = 0$. To relate the orbifold and conifold, deform the orbifold equation by $z_1 z_2 - z_3^2 = \epsilon z_4^2$.
- The orbifold field theory contains hypermultiplet bifundamentals $A, B$ and adjoints $\Phi, \tilde{\Phi}$ (from the N=2 vector multiplet.) The deformation corresponds to adding terms in the superpotential.

In $\mathcal{N} = 1$ language, the superpotential is

$$\mathrm{Tr}\Phi(A_1B_1 + A_2B_2) + Tr\tilde{\Phi}(B_1A_1 + B_2A_2)$$

Adding masses to the adjoints,

$$\frac{m}{2}(\mathrm{Tr}\Phi^2 - \mathrm{Tr}\tilde{\Phi}^2)$$

and integrating them out, one finds the conifold superpotential

$$\frac{g^2}{2m}(\mathrm{Tr}(A_1B_1A_2B_2) - \mathrm{Tr}(B_1A_1B_2A_2))$$

## Adding Flavor, a String Perspective

QCD has quarks, so it is natural to try to describe them through the gauge/gravity correspondence. To a string theorist this means adding boundaries to the string worldsheet (D-branes):

- Must have the same worldvolume directions as the D3-branes, plus some extras – natural candidates are D7 branes (or D5 branes.)
- The D7-branes need to be embedded holomorphically so that they can preserve the some of the Killing spinors of the background (ie, $z_1 - \mu^2 = 0$, etc.)
- The 3-7 strings contribute the quarks to the FT and decouple from the 7-7 strings in the usual limit.
- Gravity side: closed strings and 7-7 strings (mesons – in progress with T.Levi)

## Adding Flavor, a Gauge Theory Perspective

In gauge theory you just add the flavors! So let's add some quark chiral superfields $q$ and $Q$ and couple them to the fields in the conifold field theory, for example:

$$W_{flavors} = hqA_1Q \qquad (????)$$

But if these are the only flavors, we have a problem – a **gauge anomaly**. To obtain an anomaly-free theory, add more flavors, charged in conjugate reps of the gauge group – call them $\tilde{q}, \tilde{Q}$, and give them couplings too

$$W_{flavors} = hqA_1Q + g\tilde{q}B_1\tilde{Q}.$$

It is also straightforward to add masses:

$$W_{masses} = \mu_1 q\tilde{q} + \mu_2 Q\tilde{Q}.$$

| Field | $SU(N_c) \times SU(N_c)$ | $SU(K) \times SU(K)$ |
|---|---|---|
| $q$ | $(\mathbf{N}, \mathbf{1})$ | $(\mathbf{K}, \mathbf{1})$ |
| $\tilde{q}$ | $(\bar{\mathbf{N}}, \mathbf{1})$ | $(\mathbf{1}, \mathbf{K})$ |
| $Q$ | $(\mathbf{1}, \mathbf{N})$ | $(\bar{\mathbf{K}}, \mathbf{1})$ |
| $\tilde{Q}$ | $(\mathbf{1}, \bar{\mathbf{N}})$ | $(\mathbf{1}, \bar{\mathbf{K}})$ |

Table 1: Representation structure of the added $\mathcal{N} = 1$ flavors.

To translate these masses to the D7-brane probe picture, it is helpful to rewrite $W_{flavors} + W_{masses}$ in the following matrix form:

$$W_{flavors} + W_{masses} = \begin{pmatrix} \tilde{q} & \tilde{Q} \end{pmatrix} \begin{pmatrix} \mu_1 & hA_1 \\ gB_1 & \mu_2 \end{pmatrix} \begin{pmatrix} q \\ Q \end{pmatrix}.$$

Note that the locus of massless flavors is precisely of the form $z_1 - \mu^2 = 0$ – compare with a single D3-brane probe.

If we assume that the dimensions depend on $K/N$ as $(K/N)^2$ (natural on the gravity side), then we can predict the dimension of the $q$ fields – it is $9/8$.

$$W_{flavors} = hqA_1Q + g\tilde{q}B_1\tilde{Q}.$$

## RG Flow

It is convenient to consider the axion $C_0$ and dilaton $\Phi$ in the complex combination $\tau = C_0 + ie^{-\Phi}$. Given the embedding condition $z_1 = 0$, a natural guess for the combined dilaton-axion system is that

$$\tau \sim \log(z_1).$$

With the normalization condition $\int_{S^1} F^1 = N_{D7} = K$, we see that

$$\tau = \frac{i}{g_s} + \frac{K}{2\pi i} \log z_1,$$

which gives the correct $SL(2, Z)$ monodromy $\tau \rightarrow \tau + K$ upon circling a stack of $K$ D7-branes.

With $B_2 = 0$, the gauge couplings are equal and related to the dilaton:

$$\frac{8\pi^2}{g^2_{YM}} = \frac{\pi}{e^{\Phi}} ,$$

Making the identification of the AdS radius $r$ as the renormalization scale $\Lambda$, we find that

$$\frac{\partial}{\partial \log \Lambda} \frac{8\pi^2}{g^2_{YM}} = -\frac{3K}{4}.$$

Let us compare this RG equation with the Shifman-Vainshtein $\beta$-functions:

$$\frac{\partial}{\partial \log \Lambda} \frac{8\pi^2}{g^2_{YM}} = 3N - 2N(1 - 2\gamma_{A,B}) - K(1 - 2\gamma_q).$$

Agrees if $\gamma_A = -1/4, \gamma_q = +1/8$.

## A few comments

- Supersymmetry for the embedding $z_1 = \mu^2$ (and some but not all other holomorphic embeddings) has been checked explicitly (see hep-th/0408210, Aréan et al) so stability is guaranteed.
- The holomorphic D7-branes are topologically trivial – necessary for tadpole cancellation. Triviality is obvious by inspection, as the holomorphic variables carry no "baryon number" charge (Beasley, see hep-th/0207125)
- In principle one can consider more general holomorphic equations, motivated by adding more flavors to the field theory and giving them couplings and masses (for example, $z_1 z_2 = \mu^2$ was proposed by Karch and Katz.)

- Some of the more general holomorphic equations are trivially related to our $z_1 = \mu$ example – products of monomials, $SO(4)$ transformations (which amount to replacing $A_1 B_1$ by $(A_1 + \lambda A_2)(B_1 + \kappa B_2)$.

- Other polynomials are not so trivial – for example $z_1 + z_2 = c$. An embedding like this can be motivated from the $\mathcal{N} = 2$ orbifold. Couple flavors to the adjoints:

$$\mathrm{Tr}\Phi(A_1 B_1 + A_2 B_2) + Tr\tilde{\Phi}(B_1 A_1 + B_2 A_2) + q\Phi\tilde{q}$$

  Integrating the adjoints out as before, one obtains

$$W = \mathrm{Tr}\epsilon^{ik}\epsilon jl(A_i B_j A_k B_l) - (q\tilde{q})^2 - 2q(A_1 B_1 + A_2 B_2)\tilde{q}$$

  The locus of massless flavors is $A_1 B_1 + A_2 B_2 = z_1 + z_2 = 0$.

- Similar considerations apply to the "generalized conifolds."

## Fractional D3-branes

One simple way to break conformal invariance is to add $M$ fractional branes, which are D5-branes wrapped on the $S^2$ of $T^{1,1}$. From the perspective of the CY background the fractional branes are pinned to the singularity.

The gauge group changes to $SU(N+M) \times SU(N)$ – no longer conformal.

On string side, the D5-branes source the RR 3-form flux:

$$F_3 = \frac{M\alpha'}{2}\omega_3,$$

quantized according to $\frac{1}{4\pi^2\alpha'}\int_{S^3} F_3 = M$.

The SUGRA equations of motion ($d \star H_3 = F_5 \wedge F_3$, etc) also require an NS-NS flux:

$$\begin{aligned} B_2 &= \frac{3g_s M\alpha'}{2}\omega_2 \ln(r/r_0) \\ H_3 &= dB_2 = \frac{3g_s M\alpha'}{2r} dr \wedge \omega_2 \ , \end{aligned}$$

## Logarithmic Warping in SUGRA

With both $F_3$ and $H_3$ turned on, the five-form has a nontrivial backreaction $dF_5 = H_3 \wedge F_3$:

$$\begin{aligned} \tilde{F}_5 &= \mathcal{F}_5 + \star\mathcal{F}_5 \ , \qquad \mathcal{F}_5 = 27\pi\alpha'^2 N_{eff}(r)\mathrm{vol}(\mathrm{T}^{1,1}) \\ N_{eff}(r) &= N + \frac{3}{2\pi} g_s M^2 \ln(r/r_0) \ . \end{aligned}$$

The number of units of 5-form flux is now a function of $r$, suggesting that the size of the gauge group is changing! More on this later...

There is also a log-varying warp factor:

$$h(r) = \frac{27\pi(\alpha')^2[g_s N + \frac{3}{2\pi}(g_s M)^2(\ln(r/r_0) + \frac{1}{4})]}{4r^4}$$

## RG Flow, Duality Cascade

SV $\beta$-functions:

$$\begin{aligned}
\frac{d}{d\log(\Lambda/\mu)}\frac{8\pi^2}{g_1^2} &= 3(N+M) - 2N(1-2\gamma_{A,B}) \sim 3M \ , \\
\frac{d}{d\log(\Lambda/\mu)}\frac{8\pi^2}{g_2^2} &= 3N - 2(N+M)(1-2\gamma_{A,B}) \sim -3M \ ,
\end{aligned}$$

The gauge couplings flow in opposite directions, so one gauge group is strongly coupled while the other becomes weakly coupled.

Side note: $\gamma = -1/4$ above. Near a (quasi) fixed point, this is correct to order $M^2/N^2$, as guaranteed by a symmetry argument. When $M = 0$, the theory is conformal. The gauge theory is also invariant under the change $M \to -M$, $N \to N + M \sim N$ at leading order in $M/N$. So $\gamma = -1/4 + O(M^2/N^2)$.

## Seiberg Duality

- Gauge couplings flow in opposite directions, so one gauge group factor becomes strongly coupled, but this is not the end of the story! The strongly coupled gauge group factor has $N_f = 2N_c$ and undergoes **Seiberg duality**.

- $SU(N_c)$ with $N_f$ flavors is dual to $SU(N_f - N_c)$ with $N_f$ flavors (plus massive matter)

- $SU(N + M)$ with $2N$ flavors
$\rightarrow SU(2N - (N + M)) = SU(N - M)$ with $2N$ flavors.

- The other factor is unchanged, so now you have an $SU(N) \times SU(N - M)$ theory with bifundamentals (giving $2N$ flavors to each gauge group factor.) Now the RG flows are reversed and the process repeats – "duality cascade."

## Deformation and Confinement, Klebanov-Strassler

- Eventually the cascade must end when you are left with $SU(M) \times SU(p)$ with $p < M$. If $p = 0$ then naively you have $SU(M)$ SYM at the end of the cascade (though see hep-th/0405282, Gubser et al.)
- The supergravity background has a naked singularity! This is related to the presence of three-form flux through the $S^3$. As the $S^3$ shrinks, the energy density in the flux diverges.
- Expect the $S^3$ to be blown up to finite size, which can be accomplished by **deformation** of the conifold:

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = \varepsilon^2$$

- In field theory, develop an Affleck-Dine-Seiberg type superpotential in the IR which changes the moduli space of a probe D3-brane to the deformed conifold!

- Dimensional transmutation implemented by the deformation parameter $\varepsilon \sim m^{3/2}$.

- The warped deformed conifold metric may be written in the form

$$ds_{10}^2 = h^{-1/2}(\tau) m^2 dx_n dx_n + h^{1/2}(\tau) ds_6^2 \qquad (1)$$

  where $\tau$ is a radial parameter. $h(\tau \to 0) \to$ const, and the transverse part of the metric is independent of the mass scale.

- To see confinement, consider a probe F-string (flux tube.) The 4-d metric is multiplied by a finite warp factor at $\tau = 0$ so the string has a finite tension, giving an area law ($T_s \sim \frac{m^2}{g_s M}$.) Contrast N=4, where the warp factor vanishes ($r^2/L^2$.)

## Flavor and the Cascade

We would like to study D7-branes on the deformed conifold (SQCD) so as a first step we consider the UV limit. With $K$ D7-branes, the number of effective flavors coupled to each gauge group increases by $K$. If we start with a gauge group $SU(N+M) \times SU(N)$, the first gauge group has $2N+K$ effective flavors, so it is Seiberg-dual in the infrared ($N_c \rightarrow N_f - N_c$) to an $SU(2N+K-(N+M)) = SU(N-M+K)$ gauge theory. The second gauge group factor remains $SU(N)$, so we see that in addition to a decrease of the overall number of colors, the *difference* in the size of the gauge groups has decreased from $M$ to $M-K$. At each step of the duality cascade, the strength of the cascade, $M$, decreases by $K$.

## Supergravity

D7-branes are tricky because of their coupling to the dilaton $e^{-\Phi} \sim \log r$. At short distances, the dilaton appears to become negative (a disaster) while at large distances the backreaction is large. These problems can be overcome by F-theory, but for simplicity we will ignore them and work at intermediate $r$ where the log is valid.

To find the SUGRA solution, a drastic simplification occurs due to SUSY – one consistent set of conditions for 4 supercharges is that the dilaton-axion is holomorphic, the complex-three form is (2,1) and primitive, and the five-form and warp factor are related as usual. The Bianchi identities imply the SUGRA field equations. (Grana & Polchinski)

For example, the RR 3-form flux is

$$\begin{aligned}\tilde{F}_3 &= \frac{M\alpha'}{2}\Bigg[\Big(1+\frac{3g_sK}{2\pi}\log r\Big)\zeta\wedge\omega_2 \\ &\quad +\frac{3g_sK}{8\pi}\Big(\frac{dr}{r}\wedge\zeta+\frac{1}{2}(\Omega_{11}+\Omega_{22})\Big)\wedge\Big((1+\cos\theta_1)d\phi_1-(1+\cos\theta_2)d\phi_2\Big)\Bigg]\end{aligned}$$

Integrating this flux over the topologically nontrivial 3-cycle of $T^{1,1}$, $\omega_3=\zeta\wedge\omega_2$, we find that the number of units of flux varies logarithmically as a function of the radius $r$:

$$M_{eff}(r)=M(1+\frac{3g_sK}{2\pi}\log r).$$

The warp factor $h$ which satisfies the equations of motion is (in the standard near-horizon limit, and at linear order in $g_sK$)

$$h(r,\theta_1,\theta_2) = \frac{L^4}{r^4}\left(1 + \frac{3g_sM^2}{2\pi N}\log r(1 + \frac{3g_sK}{2\pi}(\log r + \frac{1}{2}) + \frac{g_sK}{4\pi}\log(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}))\right)$$

To study the cascade at leading order, we may discard the angular terms. Then we find that the effective number of units of five-form flux is

$$N_{eff}(r) = N + \frac{3g_sM^2}{2\pi}(\log r + \frac{3g_sK}{2\pi}\log^2 r).$$

Under the radial rescaling $r \to e^{-\frac{2\pi}{3g_sM_{eff}}}r$, $N_{eff}$ decreases by $M_{eff} - K$ units, in agreement with the argument from Seiberg duality (keeping in mind that we are only working to linear order in $g_sK$.)

## Status of Flavor

Interesting results have been obtained is several systems (by no means an exhaustive list!!)

- Meson spectra computed in $AdS_5 \times S^5$ (Karch-Katz, Kruczenski et al)
- Several attempts at studying confining backgrounds (Babington et al, Sakai and Sonnenschein)
- Some results on backreaction (Aharony et al, Grana-Polchinski, PO, Burrington et al)
- Some explicitly supersymmetric $\mathcal{N} = 1$ duals are known and exhibit nontrivial phenomena!

More to be done...