The Plane-Wave String/Gauge Theory Duality

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Plan

1. Why a string/gauge duality?
2. AdS/CFT and its plane-wave limit
3. The (free) plane-wave superstring
4. $\mathcal{N} = 4$ Super Yang-Mills in a new double scaling limit
5. Effective quantum mechanical description
6. Computation of energy shifts and decay widths
7. Light Cone String Field Theory
8. The near plane wave limit
9. Conclusions

Why a String/Gauge Theory Duality?

Consider pure $U(N)$ Yang-Mills theory:

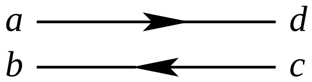
$$S_{\rm YM} = \frac{2}{{g_{\rm YM}}^2} \int d^4x\, {\rm Tr}(F_{\mu\nu}\, F^{\mu\nu}); \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

Theory has a topological expansion for $N \to \infty$: ('t Hooft '74)

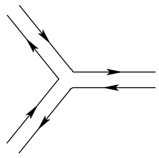
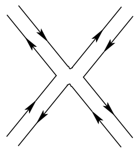
Diagrammatics: $(A_\mu)_{ab}$: $N \times N$ hermitian matrices

- Propagators:

$$\langle\, (A_\mu)_{ab}(x)\, (A_\nu)_{cd}(0)\, \rangle = \frac{{g_{\rm YM}}^2}{8\pi^2\, x^2}\, \delta_{\mu\nu}\, \delta_{ad}\, \delta_{bc}$$



$\sim {g_{\rm YM}}^2$

- Vertices:  $\sim \frac{1}{{g_{\rm YM}}^2}$  $\sim \frac{1}{{g_{\rm YM}}^2}$

- General graph with V vertices, E propagators and F index loops (faces, via $\delta^a_a = N$):

$$N^F\, ({g_{\rm YM}}^2)^{E-V} = N^{V-E+F}\, ({g_{\rm YM}}^2\, N)^{E-V}$$

Euler number: $\chi = V - E + F = 2 - 2g$ g: genus

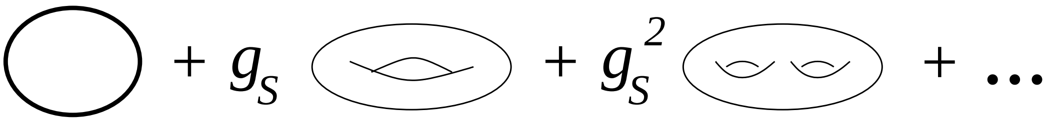
$\Rightarrow$ e.g. Free Energy:

$$\boxed{F = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \sum_{l=0}^{\infty} c_{g,l}\, \lambda^l}$$

with $\boxed{\lambda := g_{\mathrm{YM}}^2\, N}$

't Hooft coupling

- All observables have expansion in λ (loops) and $1/N$ (genus)
- 't Hooft limit: $\boxed{N \to \infty \text{ and } \lambda \text{ fixed (i.e. } g_{\mathrm{YM}} \to 0)}$
- Resembles perturbative expansion of a **String Theory**:



with string coupling constant $g_s = \exp(\langle\, \phi\, \rangle) \mathrel{\hat{=}} 1/N^2$

- strict 't Hooft limit $\Rightarrow$

$\boxed{\text{“planar” gauge theory} \stackrel{?}{=} \text{free } (g_s = 0) \text{ string theory}}$

- Dream: Dual string theory to QCD!

- 1st concrete realization of a string/gauge duality (in 4d): **AdS/CFT duality** (Maldacena)

IIB superstring on $AdS_5 \times S^5$	$\hat{=}$	$\mathcal{N} = 4$ U(N) Super Yang-Mills

- <u>The AdS String:</u>

Background metric: $ds^2_{AdS_5 \times S^5} = R^2 \left(\frac{(dx^i)^2 + du^2}{u^2} + d\Omega_5{}^2 \right)$

$$S = \frac{R^2}{4\pi\alpha'} \int d^2\xi \left(\sqrt{-\gamma}\, \gamma^{mn}\, \hat{G}_{MN}(X)\, \partial_m X^M\, \partial_n X^N + \text{ferms} \right)$$

<u>However:</u> Quantization extremely hard & unsolved!

Model so far only tractable in "stiff" string limit, where string tension $\boxed{T = \frac{R^2}{\alpha'} \to \infty}$ $\hat{=}$ supergravity approximation

Relation to dual gauge theory variables:

$$T = \frac{R^2}{\alpha'} \mathrel{\hat{=}} \sqrt{\lambda} \qquad 4\pi\, g_s \mathrel{\hat{=}} g^2_{\mathrm{YM}}$$

[CFT: $\mathcal{N} = 4$ SYM is quantum conformal theory]

- AdS/CFT conjecture predicts (among other things):

  Energies of closed AdS string excitations $\hat{=}$ Scaling dimensions of gauge invariant SYM operators

- Two perfectly incompatible regimes:

  Perturbative gauge theory: $\lambda \ll 1$
  Accessible string regime: $\lambda \gg 1$

  Strong/weak coupling duality: Beautiful if true, but hard to test/proof

- Important progress in 2002: (Berenstein, Maldacena, Nastase)

IIB superstring in **plane wave** backgrd.	$\hat{=}$	$\mathcal{N} = 4$ SYM in BMN limit

  A "corollary" of the AdS/CFT correspondence:

  Plane-wave spacetime arises as "Penrose limit" of $AdS_5 \times S^5$, translates into "BMN limit" on the gauge side.

  Concretely:

  Perform fluctuation expansion of point-like string solution orbiting circle in S^5 with large angular momentum J.

  $\Rightarrow$ Background spacetime contracts to plane-wave metric

  $\Rightarrow$ Resulting worldsheet theory is free in light-cone-gauge!

  Novel feature: Duality appears to be perturbative on both sides $\Rightarrow$ can perform very detailed tests accessing "stringy" regime

- Philosophy:

  Use string theory as tool for studying (large N) Yang-Mills, rather than as a "theory of everything"

The Plane Wave Superstring

- Metric: $i = 1, \ldots, 8$ (transverse d.o.f.)

$$ds^2 = -4dx^+dx^- - \mu^2 \, (x^i)^2 \, (dx^+)^2 + (dx^i)^2$$

$$F_{+1234} = 4 \, \mu \qquad \text{(self dual 5-form)}$$

Is max. supersymmetric backgrd (as $\mathbb{R}^{1,9}$ and $AdS_5 \times S^5$)

- Go to light-cone gauge: $X^+(\tau, \sigma) = p^+ \, \tau$

IIB (closed) superstring action in light-cone gauge:

$$\boxed{S = \frac{1}{2\pi\alpha'} \int d^2\xi \Big(\tfrac{1}{2} \partial_a X^i \partial^a X^i - \tfrac{\mu^2}{2} \, (X^i)^2 + \text{fermions} \Big)}$$

Free massive 2d theory $\Rightarrow$ easily quantized! (Metsaev)

- Eqs. of motion: $(\partial_\tau^2 - \partial_\sigma^2 + \mu^2) \, X^i = 0$

$$\Rightarrow \quad X^i = \cos(\mu\tau) \, x_0^i + \sin(\mu\tau) \, p_0^i + \sum_{n \neq 0} \big(\, \alpha_n^i \, e^{-i(\omega_n \tau - k_n \sigma)} + \tilde{\alpha}_n^i \, e^{-i(\omega_n \tau + k_n \sigma)} \, \big)$$

with $\omega_n = \text{sign}(n) \, \sqrt{k_n^2 + \mu^2}$ and $k_n = n/(\alpha' \, p^+)$

- Quantization: Commutation relations for modes

$$[\alpha^i_m, \alpha^j_n] = \delta_{n+m,0}\,\delta^{ij} \qquad [\tilde{\alpha}^i_m, \tilde{\alpha}^j_n] = \delta_{n+m,0}\,\delta^{ij}$$

$$[p^i_0, x^j_0] = -i\delta^{ij} \qquad [\alpha^i_m, \tilde{\alpha}^j_n] = 0$$

Fock-vacuum: $\alpha^i_0|0\rangle = 0 \quad \alpha^i_n|0\rangle = \tilde{\alpha}^i_n|0\rangle = 0 \quad n \geq 1$

where $\alpha^i_0 = \frac{1}{\sqrt{2\mu}}(p^i_0 + i\,\mu\,x^i_0)$

- Hamiltonian:

$$H_0 = \mu\left(\alpha_0^{\dagger\,i}\alpha^i_0 + \sum_{n=1}^{\infty}(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n)\sqrt{1 + \frac{n^2}{(\alpha' p^+ \mu)^2}}\right)$$

- Spectrum:

$\|0\rangle$	$E_0 = 0$	groundstate
$\alpha_0^{\dagger i}\|0\rangle$	$E_0 = \mu$	supergravity state
$\alpha_0^{\dagger i_1}\ldots\alpha_0^{\dagger i_l}\|0\rangle$	$E_0 = l\cdot\mu$	supergravity state
$\alpha^i_{-n}\tilde{\alpha}^j_{-n}\|0\rangle$	$E_0 = 2\mu\sqrt{1+\frac{n^2}{(\alpha' p^+\mu)^2}}$	string state
$\vdots$		

Subject to level matching (Virasoro) constraint:

$$(N - \tilde{N})\,|\text{phys}\rangle = 0$$

The dual side: $\mathcal{N}=4$ Super Yang-Mills

- Yang-Mills + 6 real scalars + 4 gluinos (all in adjoint rep)

$$S = \frac{2}{{g_{\mathrm{YM}}}^2}\int d^4x\,\mathrm{Tr}\left[\tfrac{1}{4}F_{\mu\nu}^2 + \tfrac{1}{2}(D_\mu\phi_i)^2 - \tfrac{1}{4}[\phi_i,\phi_j][\phi_i,\phi_j] + \text{ferms}\right]$$

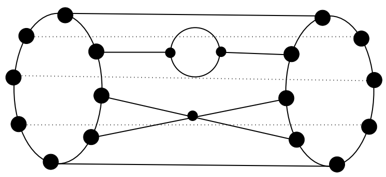
Is a "finite" theory: g_{YM} not renormalized!

However: Gauge invariant operators as $\mathcal{O}_\alpha = \mathrm{Tr}(\phi_{i_1}\ldots\phi_{i_k})$ are renormalized, resulting in anomalous scaling dimensions:

$$\Delta = \Delta_0 + \Delta_{\mathrm{anom}}(\lambda, 1/N^2)$$

Δ is measured by computing 2-pt functions:

$$\langle\, \mathcal{O}_\alpha(x)\, \mathcal{O}_\beta(y)\,\rangle = \frac{\delta_{\alpha\beta}}{|x-y|^{2\Delta}}$$



Δ has perturbative expansion in λ (loops) and $1/N^2$ (genus).

$\Rightarrow$ To be related to dual string spectrum

- Duality to plane wave string arises in **BMN limit**:

Complexify: $\boxed{Z := \phi_5 + i\phi_6}$ and assign unit $U(1)_R$ charge.

The BMN Limit

- **Only** consider operators of fixed $U(1)_R$ charge J, e.g.

$$\mathcal{O}_0^J = \mathrm{Tr}(Z^J), \quad \mathcal{O}_1^J = \mathrm{Tr}(\phi_i\, Z^J), \quad \mathcal{O}_p^J = \mathrm{Tr}(\phi_i\, Z^p\, \phi_j\, Z^{J-p})$$

and take limit of action **and** observables:

$$\boxed{N \to \infty \text{ and } J \to \infty \text{ with } g_{\mathrm{YM}} \text{ and } J^2/N \text{ fixed}}$$

Non 't Hooftian limit ($\lambda \to \infty$)!

- Nevertheless perturbatively sensible: In subsector of "BMN operators" two new effective parameters arise:

  ⋆ Loops: $\boxed{\lambda' := \lambda/J^2}$ (BMN; Gross,Mikhailov,Roiban; Eden,Jarczak,Sokatchev) [will see later on]

  ⋆ Genus: $\boxed{g_2 := J^2/N}$ (Potsdam; Boston)

  Can be seen already from free 2-pt function:

$$\langle \mathrm{Tr}Z^J(x)\, \mathrm{Tr}\bar{Z}^J(0)\rangle_0 = \frac{1}{x^{2J}} \int dZ\, d\bar{Z}\, \mathrm{Tr}Z^J\, \mathrm{Tr}\bar{Z}^J\, e^{-\mathrm{Tr}(Z\bar{Z})}$$

$$= \frac{J\cdot N^J}{x^{2J}} \Big[1 + \frac{1}{N^2}\Big[\binom{J}{4} + \binom{J}{3}\Big] + \mathcal{O}\big(\tfrac{J^8}{N^4}\big)\Big]$$

($\Rightarrow$ Full genus expansion in g_2 despite $N \to \infty$ limit!)

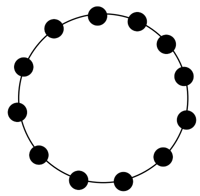
The String/Gauge Theory Dictionary

Parameters:

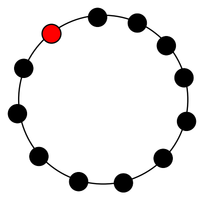
$$\boxed{\frac{1}{(\alpha' \, p^+ \, \mu)^2} \triangleq \lambda' \qquad 4\pi \, g_s \, (\alpha' \, p^+ \, \mu)^2 \triangleq g_2}$$

String States/SYM Operators: (at $g_2 = 0$)

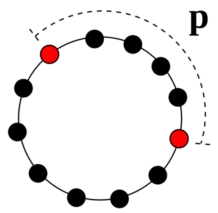
$$|0\rangle \quad \triangleq \quad \mathrm{Tr}(Z^J)$$



$$\alpha_0^\dagger |0\rangle \quad \triangleq \quad \mathrm{Tr}(\phi \, Z^J)$$



$$(\alpha_0^\dagger)^2 |0\rangle \quad \triangleq \quad \textstyle\sum_{p=0}^{J} \mathrm{Tr}(\phi \, Z^p \, \phi \, Z^{J-p})$$



$$\alpha_{-n} \, \tilde{\alpha}_{-n} |0\rangle \quad \triangleq \quad \textstyle\sum_{p=0}^{J} e^{2\pi i n p/J} \, \mathrm{Tr}(\phi \, Z^p \, \phi \, Z^{J-p})$$

$$\vdots$$

Excitations $\triangleq$ insertion of "impurities" ϕ_i and D_μ $(4+4)$

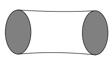
String bit picture: BMN limit $(N, J \to \infty) \triangleq$ continuum limit of discretized string model yielding smooth worldsheets of all genera.

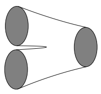
Operator identification: (Gross,Mikhailov,Roiban)

$$\boxed{\frac{\mathcal{H}_{\rm lc}}{\mu} \hat{=} \mathcal{D} - J}$$

- $\mathcal{H}_{\rm lc}$: Light-cone string field theory Hamiltonian:

$$\mathcal{H}_{\rm lc} \quad = \quad H_0 \quad + \quad g_2\, H_3 \quad + \quad g_2^2\, H_4 + \ldots$$



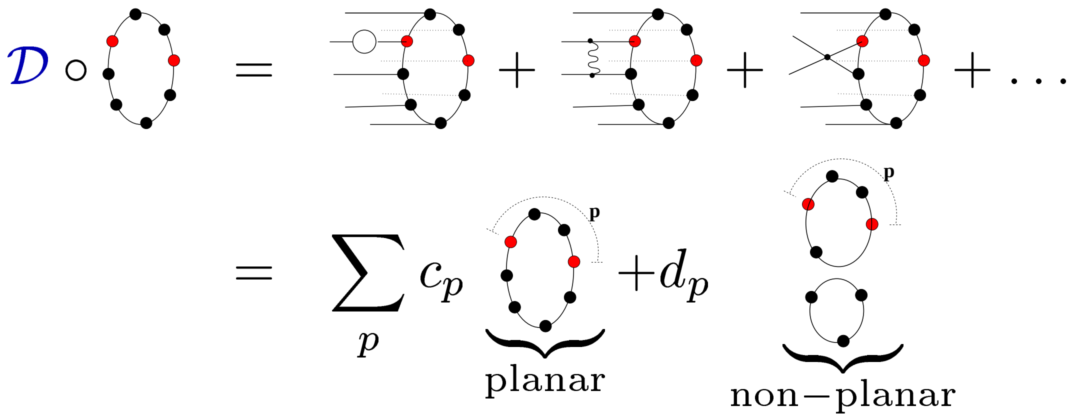


Acts in multi-string Hilbert space $\hat{=}$ Multi-trace operators.

- $\mathcal{D}$: Dilatation operator of $\mathcal{N}=4$ Super Yang-Mills measuring the scaling dimensions Δ. Has perturbative expansion:

$$\mathcal{D} = J + \#\text{impurities} + \sum_{l=1}^{\infty} D^{(l)}$$

Acts on SYM operators at the origin: (Beisert,Kristjansen,Plefka,Staudacher)



- Concretely at one-loop (in subsector of 2 complex scalars):

$$\boxed{\mathcal{D}^{(1)} = -\frac{{g_{\mathrm{YM}}}^2}{8\pi^2} :\mathrm{Tr}[Z,\phi]\,[\,\check{Z}\,,\,\check{\phi}\,]:}$$

where $(\check{Z})_{ij} = \frac{\delta}{\delta Z_{ji}}$ matrix derivative

Comment: Planar action of $\mathcal{D}^{(1)} \hat{=}$ Hamiltonian of integrable spin chain

- Yields one-loop two point functions upon insertion via:

$$\left\langle \bar{\mathcal{O}}_\alpha\Big(\bar{Z}(x),\bar{\phi}(x)\Big)\,\mathcal{O}_\beta\Big(Z(0),\phi(0)\Big)\right\rangle = \frac{1}{|x|^{2\,\Delta_0}}$$

$$\times\Big[\mathcal{O}_\alpha(\check{Z},\check{\phi})\,\mathcal{O}_\beta(Z,\phi)$$

$$+\log(\Lambda\,x)^{-2}\,\mathcal{O}_\alpha(\check{Z},\check{\phi})\,D^{(1)}\circ\mathcal{O}_\beta(Z,\phi)+\ldots\Big]\Big|_{Z=\phi=0}$$

- Compute scaling dimensions of simplest operators:

$$D^{(1)}\circ\mathrm{Tr}(Z^J) = 0 \qquad \Leftrightarrow \qquad E_{|0\rangle} = 0$$

$$D^{(1)}\circ\mathrm{Tr}(\phi Z^J) = \mathrm{Tr}[Z,\phi]\,[\check{Z}\,,Z^J] = 0 \quad \Leftrightarrow E_{\alpha_0^\dagger|0\rangle} = \mu$$

Also true for higher loop contributions to Δ (protected ops).

- First non-trivial case: 2 impurity insertion

$$\begin{aligned}
\mathcal{D}^{(1)} \circ \mathcal{O}_p^J &= g_{\mathrm{YM}}^2 N \left(2 \cdot \mathcal{O}_p^J - \mathcal{O}_{p+1}^J - \mathcal{O}_{p-1}^J\right) \\
&\quad + g_{\mathrm{YM}}^2 \sum_{l=1}^{p-1} \{\mathcal{O}_{p-l}^{J-l,l} - \mathcal{O}_{p-l-1}^{J-l,l}\} \\
&\quad + g_{\mathrm{YM}}^2 \sum_{l=1}^{J-p-1} \{\mathcal{O}_{p}^{J-l,l} - \mathcal{O}_{p+1}^{J-l,l}\}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{O}_p^J &= \mathrm{Tr}(\phi\, Z^p\, \phi\, Z^{J-p}) && \text{(single trace)} \\
\mathcal{O}_p^{J_0,J_1} &= \mathrm{Tr}(\phi\, Z^p\, \phi\, Z^{J_0-p})\, \mathrm{Tr} Z^{J_1} && \text{(double trace)}
\end{aligned}$$

Operator mixing: Needs to diagonalize $D^{(1)}$!

- Planar form of dilatation operator known up to three-loops in (Z, ϕ) sector. (Beisert)

Spin chain picture

For operators made of Z & ϕ only: $\mathcal{O} = \mathrm{Tr}(ZZ\phi Z\phi\ldots)$

$$\mathcal{D}_0 = \mathrm{Tr}(Z\check{Z}) + \mathrm{Tr}(\phi\,\check{\phi}) \qquad \check{Z}_{ij} := \frac{d}{dZ_{ij}}$$

$$\mathcal{D}_1 = \mathrm{Tr}\Big([Z,\phi]\,[\check{Z},\check{\phi}]\Big) \qquad \text{matrix derivative}$$

One has $\mathcal{D}_0 \circ \mathcal{O} = \big(\text{number of } Z\text{'s} + \text{number of } \phi\text{'s}\big) \cdot \mathcal{O}$

One loop piece $\mathcal{D}_1$: (1.) Nearest neighbour interactions

$$\tfrac{1}{2}\,\mathrm{Tr}\Big([Z,\phi]\,[\check{Z},\check{\phi}]\Big) \circ \mathrm{Tr}(\ldots Z\,\phi\ldots) =$$

$$N\Big(\mathrm{Tr}(\ldots\phi\,Z\ldots) - \mathrm{Tr}(\ldots Z\,\phi\ldots)\Big)$$

$$\hat{=} \qquad \mathcal{H} \circ |\ldots\uparrow\downarrow\ldots\rangle = |\ldots\downarrow\uparrow\ldots\rangle - |\ldots\uparrow\downarrow\ldots\rangle$$

$$g^2 N \sum_{n=1}^{L} (P_{n,n+1} - 1)$$ Heisenberg spin chain

- Have learned:

  One loop dilatation operator of $\mathcal{N} = 4$ Super Yang-Mills $\hat{=}$ Heisenberg spin chain $\mathcal{H} = g^2 N \sum_{n=1}^{L} \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}$ (Minahan,Zarembo)

$\vert 0\rangle$ string vacuum	$\hat{=}$	$\mathrm{Tr}(Z^J)$ gauge theory operator	$\hat{=}$	$\vert\uparrow\uparrow\ldots\uparrow\rangle$ ferromagnetic groundstate

- String excitations: "magnons"

$$|m\rangle := |\uparrow\ldots\uparrow\underbrace{\downarrow\uparrow\ldots\uparrow\downarrow}_{m}\uparrow\ldots\uparrow\rangle \qquad \hat{=} \qquad \mathrm{Tr}(\phi Z^m \phi Z^{J-m})$$

Energy eigenstate?: $\quad |\mathcal{O}_n\rangle := \sum_{m=0}^{J} \cos(\pi\, n \frac{2m+1}{J+1})\, |m\rangle$

$$\mathcal{H} \circ |\mathcal{O}_n\rangle = \frac{g^2 N}{\pi^2} \sin^2(\tfrac{\pi\, n}{J+1})\, |\mathcal{O}_n\rangle \qquad \text{(Bethe, 1931)}$$

In continuum limit:
($N, J \to \infty$, J^2/N fixed)

$$\boxed{\frac{g^2 N}{J^2} n^2}$$

- Matches the string spectrum at one loop, recall

$$\alpha^i_{-n}\tilde{\alpha}^j_{-n}|0\rangle : \qquad E/\mu = 2\sqrt{1+\lambda' n^2} = 2 + \boxed{\lambda' n^2} + \ldots$$

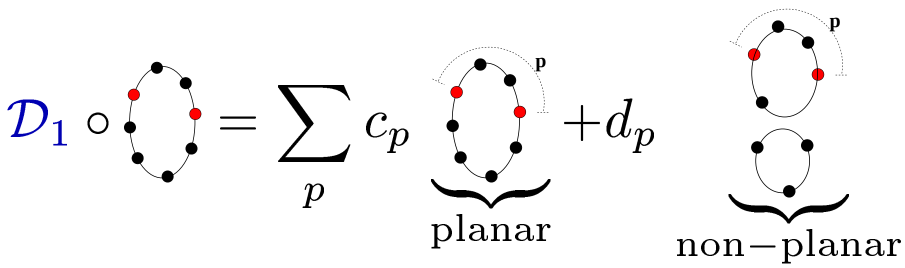
Long-range interactions: String splitting

- Contribution to $\mathcal{D}_1$ from non-nearest neighbouring "spins":

$$\mathrm{Tr}\Big([Z,\phi]\,[\check{Z},\check{\phi}]\Big)\circ \mathrm{Tr}(Z\,A\,\phi\,B) =$$
$$-\mathrm{Tr}(A)\,\mathrm{Tr}\Big([Z,\phi]\,B\Big) + (A \Leftrightarrow B)$$

Suppressed by factor of $1/N$ w.r.t. nearest-neighbour term.

$\Rightarrow$ String splitting interaction (single trace $\rightarrow$ double trace)



BMN Quantum Mechanics (Beisert,Kristjansen,Plefka,Staudacher)

- Take $J \to \infty$ continuum limit: Introduce $\boxed{x := \frac{p}{J},\ r := \frac{J_1}{J}}$

  Discrete operators are replaced by continuum states:

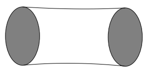
$$\mathrm{Tr}(\phi\, Z^p\, \phi\, Z^{J-p}) =: \mathcal{O}_p^J \quad \to \quad |x;1\rangle \qquad x \in [0,1]$$

$$\mathrm{Tr}(\phi\, Z^p\, \phi\, Z^{J_0-p})\, \mathrm{Tr} Z^{J_1} =: \mathcal{O}_p^{J_0,J_1} \quad \to \quad |x;1-r\rangle \otimes |r\rangle$$

- $\mathcal{D}^{(1)} \circ \mathcal{O}_p^J \;\Rightarrow\;$ Action of QM Hamiltonian $\widehat{H}$ on $|x;1\rangle$,
  e.g. $(2\,\mathcal{O}_p^J - \mathcal{O}_{p+1}^J - \mathcal{O}_{p-1}^J) \;\to\; -\partial_x^2\,|x;1\rangle$

- The full Hamiltonian: $\boxed{\widehat{H} = \widehat{H}_0 + \widehat{H}_{\mathrm{Int}}}$

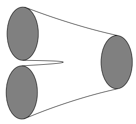
$$\widehat{H_0}\,|x;1\rangle = -\lambda'\,\partial_x^2\,|x;1\rangle$$



"free string"

$$\widehat{H}_{\mathrm{Int}}|x;1\rangle = \lambda'\,g_2 \int_0^x dr\,\partial_x\,|x-r;1-r\rangle \otimes |r\rangle$$

"string splitting"



$$-\lambda'\,g_2 \int_0^{1-x} dr\,\partial_x\,|x;1-r\rangle \otimes |r\rangle$$

A **String Field Theory** has emerged from the **Gauge Theory**!

A String Field Theory from BMN Gauge Theory

Acting on general n-trace (string) states:

$$\boxed{\widehat{H} = \widehat{H}_0 + g_2\,\widehat{H}_+ + g_2\,\widehat{H}_-}$$

with

$$H_0\,|x, r_0; r_1, \ldots, r_k\rangle = -\partial_x^2\;|x, r_0; r_1, \ldots, r_k\rangle,$$

$$H_+\,|x, r_0; r_1, \ldots, r_k\rangle = \int_0^x dr_{k+1}\;\partial_x\;|x - r_{k+1}, r_0; r_1, \ldots, r_{k+1}\rangle$$

"string splitting" $\quad - \int_0^{r_0 - x} dr_{k+1}\;\partial_x\;|x, r_0; r_1, \ldots, r_{k+1}\rangle,$

$$H_-\,|x, r_0; r_1, \ldots, r_k\rangle = \sum_{i=1}^{k} r_i\;\partial_x\;|x + r_i, r_0; r_1, \ldots, \cancel{r}_i, \ldots, r_k\rangle$$

"string joining" $\quad - \sum_{i=1}^{k} r_i\;\partial_x\;|x, r_0; r_1, \ldots, \cancel{r}_i, \ldots, r_k\rangle.$

Perturbative diagonalization of $\widehat{H}$

- Planar (free string) sector ($g_2 = 0$) easily diagonalized:

$$|n\rangle = \int_0^1 dx\, e^{2\pi i n x}\, |x\rangle \qquad \text{with} \qquad \widehat{H}_0\, |n\rangle = \lambda'\, n^2\, |n\rangle$$

$\Leftrightarrow$ matches string spectrum:

$$\alpha_{-n}\, \tilde{\alpha}_{-n} |0\rangle: \quad E_{lc}/\mu = 2\sqrt{1 + \lambda'\, n^2} = 2 + \lambda'\, n^2 + o(\lambda'^2)$$

- Higher genus corrections from standard QM perturbation theory, at genus one:

$$E^{(1)}_{|n\rangle} = \langle n|\, \widehat{H}_{\text{Int}}\, \frac{1}{E^{(0)}_{|n\rangle} - \widehat{H}_0}\, \widehat{H}_{\text{Int}}\, |n\rangle = \ldots = \frac{1}{12} + \frac{35}{32\pi^2\, n^2}$$

- **Result:** Gauge theory prediction for higher genus corrections to plane wave string spectrum: (Beisert,Kristjansen,Plefka,Staudacher)

$$\frac{E_{\alpha_{-n}\tilde{\alpha}_{-n}|p^+,0\rangle}}{\mu} = 2 + \lambda' \Big[n^2 + {g_2}^2\, \tfrac{1}{4\pi^2} (\tfrac{1}{12} + \tfrac{35}{32\pi^2\, n^2})$$
$$+ {g_2}^4\, \tfrac{1}{4\pi^2} (- \tfrac{11}{46080\, \pi^2\, n^2} (\tfrac{521}{12288} - \tfrac{\zeta(3)}{128})\, \tfrac{1}{\pi^4\, n^4}$$
$$+ (-\tfrac{5715}{16384} - \tfrac{45\,\zeta(3)}{512} + \tfrac{15\,\zeta(5)}{128})\, \tfrac{1}{\pi^6\, n^6} + o({g_2}^6) \Big] + o(\lambda'^2)$$

String vs. Gauge Theory Basis

- Remarkably one-loop $\widehat{H}$ terminates at order g_s:

$$\boxed{\widehat{H} = \widehat{H}_0 + g_2\,\widehat{H}_+ + g_2\,\widehat{H}_-}$$

Whereas string field theory Hamiltonian H_S is expected to have infinite series of terms in g_s !?

- Resolution: Gauge theory basis $\neq$ string basis:

$\widehat{H}$ is quasi hermitian $\widehat{H}^\dagger = S\,\widehat{H}\,S^{-1}$ where S is tree-level mixing matrix

$$\langle \bar{\mathcal{O}}_\alpha[\bar{Z},\bar{\phi}]\,\mathcal{O}_\beta[Z,\phi]\rangle_0 := S_{\alpha\beta} = 1 + g_2\,\Sigma + g_2^2\,\Sigma_{(2)} + \ldots$$

Transformation to string basis via $(S^\dagger = S)$ (Gross,Mikhailov,Roiban)

$$\boxed{H_S = S^{1/2}\,\widehat{H}\,S^{-1/2}}$$

Natural way to "Hermitize" $\widehat{H}$.

- It has been conjectured that $S = \exp[g_2\,\Sigma]$ (H. Verlinde)
$\Rightarrow$ $\exists$ nice consistency check (Spradlin,Volovich)

Corresponding string field theory computations?

- Plane-wave string field theory has been developed (Spradlin,Volovich;Pankiewicz,Stefanski,;. . .):
  Multi-string Hilbert space $||0\rangle = |0\rangle_{(1)} \otimes |0\rangle_{(2)} \otimes |0\rangle_{(3)} \ldots$ with string field theory Hamiltonian

$$\widehat{H} = \sum_r \hat{H}_{2(r)} + g_s \hat{H}_3 + g_s^2 \hat{H}_4 + \ldots$$

  where $\hat{H}_3$ (known), $\hat{H}_4$ (partially known)

- Various matrix elements of $\hat{H}_3$ (up to an arbitrary number of impurities) were shown to match after transforming GT into string basis.

  However: String field theory predicts impurity number non-conserving matrix elements w/o counterpart in GT! (Klebanov,Spradlin,Volovich)

$$\langle \text{2-imp}|\hat{H}_3|n\text{-imp}\rangle = f(\sqrt{\lambda'}) \neq 0 \qquad \text{with} \quad n \neq 2$$

  But matrix elements are no observables!

  $\Rightarrow$ Compute energy shifts and decay rates.

- To date no genus one result for energy shifts in string field theory!

  Computations so far only performed in a (unjustified) channel truncation scheme:

$$\delta E^{T}_{|n\rangle} = \langle n|\hat{H}_3 \frac{\mathcal{P}_2}{E^{(0)}_{|n\rangle} - \widehat{H}_{2(2)}} \hat{H}_3 \,|n\rangle + \langle n|\, \hat{H}_4 \,|n\rangle$$

  $\mathcal{P}_2$: Projector on two impurity states.

  ⋆ Nevertheless match to GT result reported (Roiban,Spradlin,Volovich)

  ⋆ Recently challenged by (Gutjahr,Pankiewicz):
  Observe nonanalytic $\sqrt{\lambda'}$ dependence for $\delta E^{T}_{|n\rangle}$.

  $\Rightarrow$ Open problem!

- **Complementarity of string and gauge theory:**

  ⋆ In gauge theory: λ' expansion hard, genus expansion easy.

  $\Rightarrow \widehat{H}$ known to all orders in g_2 at order λ'

  ⋆ In string theory: λ' expansion automatic, genus expansion hard.

  $\Rightarrow H_S$ known to all orders in λ' but only up to order g_2

Decay widths

Excited single-string states are unstable for decay into degenerate multi-string states.

- Two impurities: (Freedman,Gursoy) (second-order process)

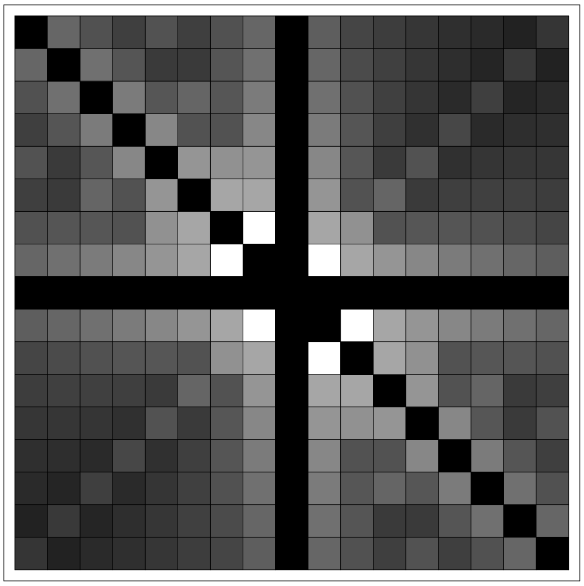
$$|n,1\rangle^{ij} \to |m, r_0; r_1, 1-r_0-r_1\rangle^{ij} \quad \text{with} \quad n^2 = \frac{m^2}{r_0^2}$$

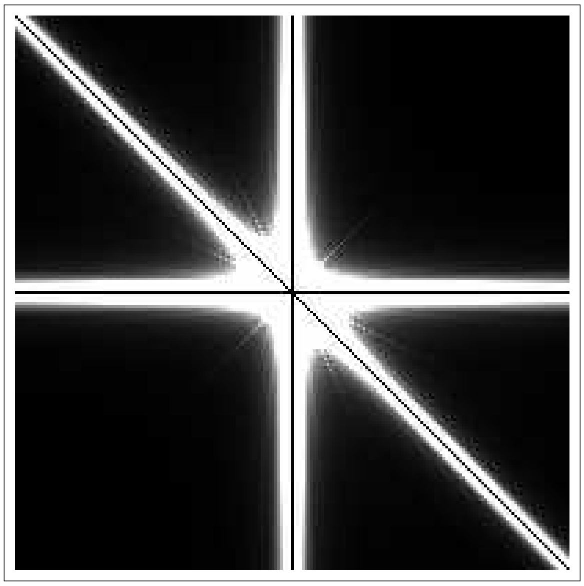
Decay width $\Gamma_n = \frac{\lambda' g_2^4}{3840\pi^3\, n^5}\,(n^2-1)\,\left(n^2+1+\frac{75}{4\pi^2}\right)$

- Three impurities: (Gutjahr,Plefka) (first order process)

1. $|n_1, n_2; 1\rangle^{123} \to |m_1, m_2; 1-r\rangle |r\rangle^{123}$
2. $|n_1, n_2; 1\rangle^{123} \to |m; 1-s\rangle^{\,ab} |s\rangle^{\,c} \qquad a,b,c \in (1,2,3)$

Decay widths:


(1.)


(2.)

The near plane wave regime

(Callan,Lee,McLoughlin,Schwarz,Swanson,Wu)

- The plane wave geometry arises in a specific $R \to \infty$ limit of $AdS_5 \times S^5$:

$$ds^2_{AdS_5\times S^5} = R^2 \Big[- \Big(\frac{1+z^2}{1-z^2}\Big)^2 dt^2 + \Big(\frac{1-y^2}{1+y^2}\Big)^2 d\phi^2 + \frac{dz_k dz_k}{(1-z^2)^2} + \frac{dy_{k'}dy_{k'}}{(1+y^2)^2}\Big]$$

take: $\quad t \to x^+ \quad \phi \to x^+ + \frac{x^-}{R^2} \quad z_k \to \frac{z_k}{R} \quad y_{k'} \to \frac{y_{k'}}{R}$

This yields

$$ds^2_{AdS_5\times S^5} = ds^2_{\text{pw}} + \frac{1}{R^2}\Big[-2y^2 dx^- dx^+ + \tfrac{1}{2}(y^4 - z^4)\,(dx^+)^2 + (dx^-)^2 + \tfrac{1}{2}z^2 dz^2 - \tfrac{1}{2}y^2 dy^2\Big] + \mathcal{O}(\frac{1}{R^4})$$

$\Rightarrow$ Treat $1/R^2$ terms as perurbation of plane wave Hamiltonian: Translates into $1/J$ corrections to the GT scaling dimensions.

- Two impurity near plane wave excitation spectrum in GT variables:

$$E(n,J) = 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}[2 + \frac{(4-L)}{\sqrt{1+n^2\lambda'}}] + \mathcal{O}(1/J^2)$$

- May be shown to agree with the exact in J one loop Bethe result for two magnons at order λ'/J!

- $1/J^3$ terms agree up to two loops with gauge theory calculations, disagreement at three loop level!

Conclusions

- Plane wave string/gauge theory duality very concrete and calculable example of a string/gauge theory duality

- The first to probe the "stringy" regime

- Established "dictionary" between states and operators

- Can extract String Field Theory Hamiltonian from BMN gauge theory

- Computation of string energy shifts and decay widths from gauge theory

- SFT computation of genus one energy shift should be done!

Problems just around the corner?

- Discrepancies between string and gauge theory matrix elements at order λ'^2 (Spradlin,Volovich)

- Discrepancies in the near plane wave limit at order $\lambda'^3 \leftrightarrow$ order of limits issue (?)

- Four-loop BMN scaling ($\sim \lambda'^4/J^8$) recently challenged from "Inozemtsev" spin-chain picture of planar dilatation operator. (Serban,Staudacher)