Integrable Long-Range Spin Chains and CFT_4 / AdS

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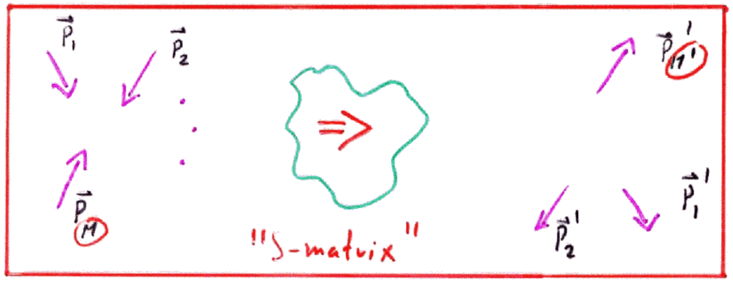
Integrability in Quantum Field Theory

- In classical mechanics, we have integrability iff # degrees of freedom = # conserved quantities ("charges")
- This definition of integrability becomes more "tricky" in (1) classical field theory and in (2) quantum mechanics.
- It becomes even trickier in quantum field theory! The "charges" are operators, and there are infinitely many degrees of freedom. Both features cause, separately, problems.

In QFT, there is a more useful alternative definition ...

$\boxed{\text{The elastic S-matrix}}$

The S-matrix and Integrability



In a generic field theory, there are just two conserved quantities: momentum and energy. (E.g. in a non-relativistic system, $\sum_i \vec{p}_i = \text{const.}$ and $\sum_i \vec{p}_i^{\,2} = \text{const.}$). Scattering is diffractive. In addition, in a relativistic system we may have particle creation + annihilation: $M \neq M'$.

In an integrable field theory scattering does not change the # of particles, and is nondiffractive:

$$\boxed{M' = M \qquad \{p_i'\} = \text{Permutation}\{p_i\}}$$

This means that particles can only exchange momenta, and, if non-identical, quantum numbers.

> Theorem: In 3+1 dimensions, the only nondiffractive S-matrix is the trivial case $S = 1$.

?

Local, Composite Operators in Gauge Theory

Consider $SU(N)$ gauge theory, and elementary fields $\{\Phi_i\}$ in the adjoint representation. Look at gauge-invariant, local, composite operators:

$$\boxed{\mathcal{O}(x) = \sum \Psi^{k_1 \dots k_n}_{i_1 \dots i_{k_n}} \mathrm{Tr}(\Phi_{i_1} \dots \Phi_{i_{k_1}})\, \mathrm{Tr}(\Phi_{i_{k_1+1}} \dots \Phi_{i_{k_2}}) \dots \mathrm{Tr}(\Phi_{i_{k_{n-1}+1}} \dots \Phi_{i_{k_n}})(x)}$$

In particular, this makes sense in conformal gauge theory. E.g. in $\mathcal{N}=4$. But only if we choose "good" combinations:

$$\boxed{\langle \mathcal{O}_I(0)\, \mathcal{O}_J(x) \rangle = \frac{\delta_{IJ}}{|x|^{2\Delta_I}}}$$

The fields $\mathcal{O}_I(x)$ are conformal operators, and Δ_I are their scaling dimensions. In general, both depend on N and the Yang-Mills coupling g_{YM}.

Furthermore, there exists a linear operator (one of the generators of the conformal symmetry), the dilatation operator, such that

$$\boxed{D \cdot \mathcal{O}_I = \Delta_I \mathcal{O}_I}$$

gives the eigensystem $\{\mathcal{O}_I ; \Delta_I\}$.

Quantum Mechanics from Conformal Gauge Theory

It turns out to be extremely useful to interpret D as a Hamiltonian, and $\mathcal{O}_I$ as a wavefunction:

$$\boxed{D \cdot \mathcal{O}_I = \Delta_I \mathcal{O}_I} \iff \boxed{H \cdot \Psi_I = E_I \Psi_I}$$

"Hidden", many-body quantum mechanics!

Now, "QM = 0+1 dimensional field theory", but (time)

"many-body QM = 1+1 dimensional field theory" (time)

discrete space, can become a smooth dimension if an appropriate continuum limit is taken.

Now: The above no-go theorem does not necessarily apply to 0+1 dim many-body QM, and to 1+1 dim QFT. There can be a non-trivial S-matrix $S \neq 1$!

> So, there is a chance to find a "hidden" integrability in conformal gauge theory.

Question: What is the discrete, one-dimensional "space" in the above analogy?

Answer: Trace = Space

$$\mathcal{O} = \sum \Psi^{k_1 \ldots k_n}_{i_1 \ldots i_{k_n}} \; \phi_{i_{k_1}} \phi_{i_1} \phi_{i_2} \cdots \quad \phi_{i_{k_1+1}} \cdots \quad \ldots \quad \phi_{i_{k_n}} \phi_{i_{k_{n-1}+1}} \phi \cdots$$

traces

We can think of each trace containing, say, L operators, as a one-dimensional discrete space, i.e. as a lattice with L sites and occupied by L "particles" ϕ_i.

The analogy with a quantum many-body problem becomes better if we restrict to a single trace in each term of the above sum. This is very natural in gauge theory, as

$$\boxed{\text{Large-}N \text{ Limit} \Rightarrow \mathcal{O} = \sum_{i_1 \ldots i_L} \Psi_{i_1 \ldots i_L} \operatorname{Tr}(\phi_{i_1} \ldots \phi_{i_L})}$$

subtlety, to be discussed...

So the states to be diagonalized are $\operatorname{Tr}(\phi_{i_1} \ldots \phi_{i_L})$. Let's pick one type of particle and call it, say, Z, and declare it to be a hole, or an empty site. So our vacuum is the empty lattice of length L:

$$\boxed{\text{Vacuum}: \quad \operatorname{Tr}(\underbrace{Z Z \ldots Z Z}_{L}) = \operatorname{Tr} Z^L}$$

Note that the cyclicity of the trace translates

The Emergence of Particle Scattering

For definiteness, let us mostly stick to $\mathcal{N}=4$. There it is useful (but not necessary) to pick $Z=\varphi_1+i\varphi_2$ as one of the theory's three complex scalars. Then the empty lattice is $\frac{1}{2}$ BPS. The states $\mathrm{Tr}\, Z^L$ are protected, meaning that their scaling dimension $\Delta=\Delta_0=L$ is exact (no quantum corrections). It is useful to normalize the Hamiltonian to yield the anomalous part of Δ:

$$\boxed{H=\frac{D-\Delta_0}{g^2}} \qquad \text{where} \qquad g^2=\frac{g_{YM}^2 N}{8\pi^2}.$$

Now, in the general case we have to diagonalize the states

$$\mathrm{Tr}\, Z\ldots Z\overset{\xrightarrow{p_1}}{\Phi_{i_1}} Z\ldots Z\,\overset{\xrightarrow{p_2}}{\Phi_{i_2}}Z\ldots Z\,\overset{\xleftarrow{p_{M-1}}}{\Phi_{i_{M-1}}} Z\ldots Z\,\overset{\xleftarrow{p_M}}{\Phi_{i_M}}Z\ldots Z$$

These are our particles, carrying 1-d momenta p_k!

Statement (conjecture): In $\mathcal{N}=4$, the scattering of these particles is nondiffractive. For all values of g_{YM}, and all "types" of particles, and all densities.*

*small caveat

So this is our integrability!

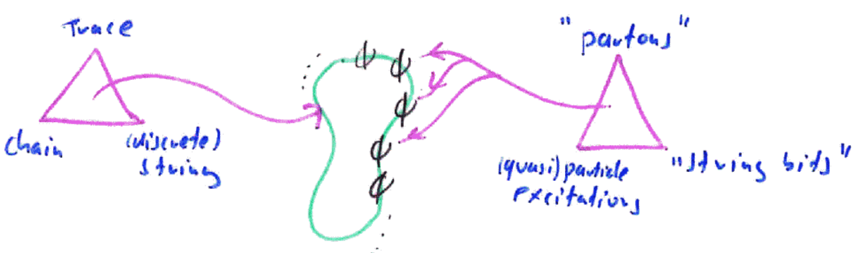
Evidence

- From QCD: work on twist 2 and twist 3 operators → confer V. Braun's talk (this workshop). Some of the results carry over ... very "short" spaces, as twist ~ length.
- $\mathcal{N}=4$, one-loop, special case where the particles (and holes) are any of the scalar fields. so(6) "spin" chain, su(2) Heisenberg XXX magnet. Minahan + Zarembo, 0212208.
- $\mathcal{N}=4$, one-loop, all fields. Particles (and holes) can be scalars φ_i, fermions $\Psi_\alpha^a, \dot\Psi_{\dot\alpha}^a$, field strengths $\mathcal{F}_{\mu\nu}$, and covariant derivatives D_μ on all of the previous fields. su(2,2|4) super spin chain. Beisert + M.S., 0307042.
- $\mathcal{N}=4$, two-loop, two complex scalars "su(2) sector". Conjecture of all-loop integrability formulated. Beisert, Kristjansen, M.S., 0303060.
- Much more evidence to follow below ...

Trace ≃ Chain ≃ String Triality

So the no-go theorem in 3+1 dimensions gets circumvented by the emergence of a non-trivial S-matrix not in space-time, but in an internal 1-d space, the trace.

This is very similar to string theory, where we have a "space-time" and a "world-sheet" picture. So, borrowing the concept, we could loosely speak of "world sheet integrability". Actually, in the light of the AdS/CFT correspondence, this might possibly not be a superficial comment. Could it be that the emerging lattice models can be turned into a rigorous definition of quantum strings on $AdS_5 \times S^5$? We would have



1/N Corrections

These lead to the following effect:

Interpreting the dilatation operator as a Hamiltonian is still possible and useful (Beisert, Kristjansen, Plefka, M.S., 0212269). So this idea is independent of the integrable spin chain story.

However, the Hilbert space is vastly enlarged, i.e. "second quantized".

And unfortunately, integrability is broken. (Beisert, Kristjansen, M.S.) The partons do not preserve their momenta!

Still, integrability should still be useful in the study of such processes, as clearly some of the details of the wavefunction enters the computation of the amplitudes.

In fact, one needs information on spin chain correlation functions.
Confer Roiban + Volovich, 0407140

Scattering and Anomalous Dimensions – Three Simple One-Loop Examples

If scattering is non-diffractive, it is factorized. This means that the spectrum may be found by making a Bethe ansatz: One computes the total phaseshift of a given particle as it scatters around the ring (= trace). The sequence is irrelevant. This fact is expressed by the Yang-Baxter equation. For a two-component system the Bethe ansatz reads

$$\boxed{e^{ip_k L} = \prod_{\substack{j=1\\ j\neq k}}^{M} S(p_k, p_j)} \qquad \boxed{E = \sum_{k=1}^{M} \varepsilon(p_k)}$$

M = # of excitations. S = two-particle S-matrix

$\varepsilon(p)$ = dispersion relation = energy per excitation

In $\mathcal{N}=4$, at one loop, we have $\varepsilon(p) = 4\sin^2\frac{p}{2}$. Also, $\sum_{k=1}^{M} p_k = 0$ trace cyclicity

I One complex scalar Z, one "gluino" Ψ: in Beisert, M.S., 0307042

$$\mathrm{Tr}\ldots ZZ\overset{\to p_1}{\Psi}ZZ\ldots ZZ\overset{\leftarrow p_2}{\Psi}ZZ\ldots \qquad S(p_1,p_2)=1$$

example pointed out in Callan, Heckman, McLoughlin, Swanson, 0407096

At one loop, free fermions! Isotropic XY model [M.S., unpublished]

II Two complex scalars Z, ϕ:

$$\mathrm{Tr}\ldots ZZ\overset{\to p_1}{\phi}ZZ\ldots ZZ\overset{\leftarrow p_2}{\phi}ZZ\ldots \quad S(p_1,p_2) = \frac{\frac{1}{2}\cot\frac{p_1}{2} - \frac{1}{2}\cot\frac{p_2}{2} + i}{\frac{1}{2}\cot\frac{p_1}{2} - \frac{1}{2}\cot\frac{p_2}{2} - i} \qquad XXX_{\frac{1}{2}} \text{ chain}$$

III One complex scalar Z, one covariant "light cone" derivative $D = D_1 + iD_2$:

$$\mathrm{Tr}\ldots ZZ\overset{\to p_1}{(DZ)}ZZ\ldots ZZ\overset{\leftarrow p_2}{(DZ)}ZZ\ldots S(p_1,p_2) = \frac{\frac{1}{2}\cot\frac{p_1}{2} - \frac{1}{2}\cot\frac{p_2}{2} - i}{\frac{1}{2}\cot\frac{p_1}{2} - \frac{1}{2}\cot\frac{p_2}{2} + i} \qquad XXX_{-\frac{1}{2}} \text{ chain}$$

Scattering is not "hard core", e.g. $(D^2 Z)$ allowed

The Meaning of Integrability

From a purely one-dimensional point of view, integrability may not be all that surprising: some interactions are so simple that particles may only exchange momenta without diffraction.

What is surprising is its occurrence in gauge and string theory. Where does it come from ???

- Conformality might be necessary (even that isn't clear) but is not sufficient. Counterexample: QCD at one loop
- It's not supersymmetry (even though it might help to make integrability "complete"). Counterexample: QCD, again.
- It's not "internal" spin chain "spin" symmetry. Counterexample: XXX, XXZ, XYZ are all integrable, but the respective symmetries are $su(2)$, $u(1)$, nothing.
- For sure, planarity has something to do with it. Probably necessary, but not sufficient.

> Understanding the origin and meaning of integrability in gauge and string theory could be extremely important, in particular for further applications!