Integrable Long-Range Spin Chains and CFT_4/AdS, Part II

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The $\mathcal{N}=4$ Dilatation Generator

Two-point functions in perturbation theory:

$$\langle \mathcal{O}_i(x)\,\mathcal{O}_j(0)\rangle = \frac{1}{|x|^{2\Delta^{(0)}}}\left(S_{ij} + g_{YM}^2\, T_{ij}\, \log|x\Lambda|^{-2} + \dots \right)$$

tree level mixing →  ← one-loop mixing

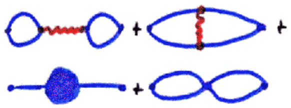
Change in philosophy: Expand dilatation generator

$$\boxed{D\,\mathcal{O}_i = \Delta_i\,\mathcal{O}_i} \qquad \boxed{D = \sum_{\ell=0}^{\infty}\left(\frac{g_{YM}^2}{16\pi^2}\right)^{\ell} D_{2\ell}}$$

Simplest example: The $su(2) \subset su(2,2|4)$ closed subsector:

$$\boxed{\mathcal{O} = \mathrm{Tr}\, Z^{L-M}\,\phi^{M} + \dots} \qquad Z = \varphi_1 + i\varphi_2\,;\ \phi = \varphi_3 + i\varphi_4, \quad \Delta^{(0)} = L$$

Tree-level: $D_0 = \mathrm{Tr}\,\phi\frac{\delta}{\delta\phi} + \mathrm{Tr}\, Z\frac{\delta}{\delta Z}$ Just counts # of legs!

One-Loop: $\boxed{D_2 = -2\,\mathrm{Tr}\,[\phi, Z][\frac{\delta}{\delta\phi}, \frac{\delta}{\delta Z}]}$ 

Protected BPS operators: $\boxed{\mathcal{O} = \mathrm{Tr}\, Z^L \quad \Delta = \Delta^{(0)} = L}$

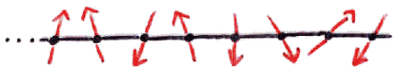
Let us see how D_2 acts; e.g. $M=2$:

$$\mathcal{O}_p = \mathrm{Tr}\,\phi Z^p \phi Z^{L-2-p} \Rightarrow D_2\cdot\mathcal{O}_p = -4\left(\mathcal{O}_{p+1} + \mathcal{O}_{p-1} - 2\mathcal{O}_p\right) +$$

+ double trace ops
Beisert, Kristjansen, Plefka, M.S. 0212269

Planar Part:
$XXX_{\frac{1}{2}}$ Heisenberg spin chain
Minahan + Zarembo 0212208

A little bit of condensed matter theory...



... L sites

The Heisenberg quantum spin-$\frac{1}{2}$ chain:

state space: $\ldots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots$

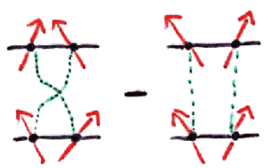
Hamiltonian: $H = \sum_{k=1}^{L} (1 - \underbrace{\vec{\sigma}_k \cdot \vec{\sigma}_{k+1}}_{\text{Pauli matrices}})$ periodic: $\vec{\sigma}_{k+L} = \vec{\sigma}_k$

spectral problem: $H \cdot \Psi = E\, \Psi$

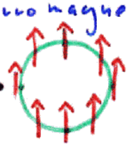
This involves diagonalizing a $2^L \times 2^L$ matrix!

simple exchange interaction:

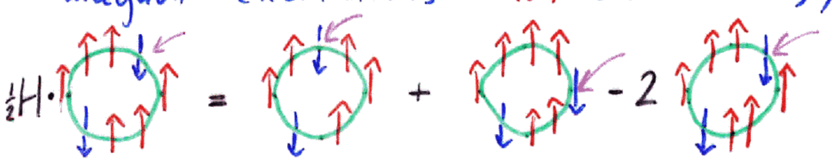
$H = 2 \sum_{k=1}^{L} (1 - \underbrace{P_{k,k+1}}_{\text{permutation operator}})$



ferromagnetic ground state:

$H \cdot$  $= 0$

"magnon" excitations: non-zero energy



Integrability and Bethe ansatz

The Heisenberg Hamiltonian is integrable.

A Yang-Baxter symmetry leads to a commuting transfer matrix:

$$[\,T(u), T(u')\,] = 0$$

depending on a spectral parameter u:

$$T(u) = 2\, u^L + \sum_{\ell=0}^{L-2} Q_\ell\, u^\ell$$

Therefore there exist commuting charges Q_ℓ:

$$\boxed{[Q_\ell, Q_{\ell'}] = 0}$$

As many as d.o.f.'s $\Rightarrow$ integrable system

The Hamiltonian H belongs to the $\{Q_\ell\}$.

This allows to diagonalize H by the Bethe ansatz:

parametric solution of the spectral problem:

$$\boxed{\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}} \;; \quad \boxed{Q_2 = \frac{1}{2} \sum_{j=1}^{M} \frac{1}{u_j^2 + \frac{1}{4}} \stackrel{!}{=} H}$$

Bethe equations — energies ~ anom. dimension

$\Rightarrow$ Exact diagonalization of quantum problem!

u_j's: Bethe roots M: # of magnons

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S Matrix for the One-Dimensional N-Body Problem with Repulsive or Attractive δ-Function Interaction

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(Received 14 December 1967)

For N particles with equal mass, interacting with repulsive or attractive δ-function interaction of the same strength, the S matrix is explicitly given and shown to be symmetrical and unitary. The incoming and outgoing states may consist of bound compounds as well as single particles. The momenta of the particles and compounds are not changed in the scattering, but particles are exchanged, such as $ABC+DE \to ADC+BE$. Only distinguishable particles are considered.

1. INTRODUCTION

FOR the one-dimensional N-body problem

$$H=-\sum_1^N \partial^2/\partial x_i^2+2c\sum_{i<j}\delta(x_i-x_j)\,, \tag{1}$$

with positive or negative c, the S matrix was discussed by McGuire[1] and by Zinn-Justine and Brezin.[2] (*Note added in proof.* K. Hepp kindly informed the author that F. A. Berezin and V. N. Sushko, Zh. Eksperim. i Teor. Fiz. 48, 1293 (1965) [English transl.: Soviet Phys.—JETP 21, 865 (1965)] have also discussed this problem.) We give in this paper a complete explicit expression for S. Only distinguishable particles are considered.

2. METHOD

The method used follows that of Sec. 1 of a recent paper.[3] We observe that all formulas there are also applicable to the case $c<0$.

If boundary conditions are not imposed, it is clear that all solutions of the Schrödinger equation are superpositions of solutions of the type ($Y2$). In other words, Bethe's hypothesis is proved in such a case.

3. INCOMING AND OUTGOING STATES

To construct scattering states, we need real values of the p's. Let us choose them so that

$$p_1<p_2<\cdots<p_N. \tag{2}$$

A term in ($Y2$) that has $P=$identity permutation$=I$, then, represents an *outgoing* wave. [A wave packet constructed out of such a term would have the left-most particle (at X_{Q1}) travel with velocity $2p_1$; the second left-most particle (at X_{Q2}) travel with velocity $2p_2$, etc. Thus the wave packet in *future* movement develops no collisions, meaning it is an outgoing wave packet.] A term in ($Y2$) that has $P=[N,N-1,\cdots,1]=I'$, i.e., the "reversed" permutation, represents an *incoming* wave.

Now each permutation Q represents a definite ordering of the coordinates and represents a scattering channel. A scattering state $Q_i\to Q_0$ is obtained if there are only incoming waves in channel Q_i:

$$[Q_i,I']=1\,,\qquad [Q,I']=0 \quad\text{for}\quad Q\neq Q_i. \tag{3}$$

In other words,

$$\langle Q_i|\xi_{I'}\rangle=1\,,\qquad \langle Q|\xi_{I'}\rangle=0 \quad\text{for}\quad Q\neq Q_i. \tag{4}$$

The amplitudes of the outgoing waves are the elements of ξ_I. Now ξ_I can be related to $\xi_{I'}$ through repeated use of ($Y2$):

$$\xi_I=[Y_{21}{}^{12}Y_{31}{}^{23}Y_{41}{}^{34}\cdots Y_{N1}{}^{(N-1)N}]\times[Y_{32}{}^{12}Y_{42}{}^{23}\cdots Y_{N2}{}^{(N-2)(N-1)}]\cdots[Y_{N(N-1)}{}^{12}]\xi_{I'}. \tag{5}$$

Thus the scattering amplitude for $Q_i\to Q_0$ is

$$\langle Q_0|S'|Q_i\rangle\,, \tag{6}$$

where S' is the right-hand side of (5) with $\xi_{I'}$ deleted.

4. OPERATOR: (ij)

We did not call the matrix S' in (6) the S matrix because it differs from the usual one in that the labeling of the columns is not in accordance with the usual rules. This is so because the incoming wave in Q_i, represented by the $[Q_i,I']$ term, describes particle $Q1$ with momentum p_N, $Q2$ with momentum p_{N-1}, etc. Thus the correct S matrix is

$$\begin{aligned}S&=S'[P^{N1}P^{(N-1)2}\cdots]\\&=S'[P^{12}][P^{23}P^{12}][P^{34}P^{23}P^{12}]\cdots[P^{(N-1)N}\cdots P^{12}].\end{aligned} \tag{7}$$

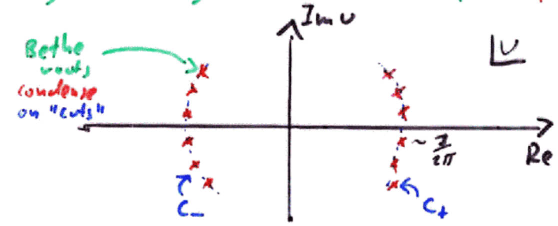
If in (7) one explicitly writes S', as given in (5), one observed that the superscripts for the Y's are the same as those for the P's, but in reverse order. One now permutes the last factor P^{12} through to just behind the first factor $Y_{21}{}^{12}$; then the new last factor P^{23} through to just behind the second factor $Y_{31}{}^{23}$, etc. The final

[1] J. B. McGuire, J. Math. Phys. 5, 622 (1964). This is a very interesting paper in which by geometrical construction many of the results of the present paper were obtained.
[2] E. Brezin and J. Zinn-Justine, Compt. Rend. Acad. Sci. Paris B263, 670 (1966).
[3] C. N. Yang, Phys. Rev. Letters, 19, 1312 (1967). Formula (m) of this paper will be called (Ym) in the present paper.

Thermodynamic Limit (Ferromagnetic)

Let us diagonalize $\mathrm{Tr}\, Z^{J_1}\phi^{J_2}$ with both J_1 and J_2 large.

Ferromagnetic ground state for fixed J_1, J_2:



$L = J \equiv J_1 + J_2$

Beisert, Minahan, M.S., Zarembo

see also: Sutherland / Shastry

filling fraction: $\alpha = \frac{J_2}{J}$ continuum variable: $u = \frac{u_j}{J}$

Bethe root density: $\rho(u) = \frac{1}{J}\sum_{j=1}^{J_2}\delta\left(u - \frac{u_j}{J}\right)$

normalization: $\int_{C=C_+ + C_-} du\, \rho(u) = \alpha$

The Bethe equations turn into integral equations:

$$\boxed{2 ⨍_{C_\pm} dv\, \rho(v)\, \frac{1}{u-v} = \frac{1}{u} \pm 2\pi n}$$

Very similar to saddle-point equations in matrix models!

energy: $E = Q_2 = \int_C du \frac{\rho(u)}{u^2}$ momentum: $P = Q = \int_C du \frac{\rho(u)}{u} = 2\pi m$
(trace cyclicity)

transfer matrix becomes resolvent: Engquist, Minahan, Zarembo; Arutyunov, M.S.

$$G(u) = \int_C dv\, \rho(v)\, \frac{1}{v-u} = \sum_{k=1}^{\infty} Q_k\, u^{k-1}$$

exactly solvable problem! $\boxed{Q_k = \int_C du \frac{\rho(u)}{u^k}}$

Two-Loop Integrability

In 0303060 we worked out the su(2) two-loop dilatation op:

$$D_4 = -2 : \mathrm{Tr}\,[[\phi, Z], \check{Z}][[\check{\phi}, \check{Z}], Z] : -2 : \mathrm{Tr}\,[[\phi, Z], \check{\phi}][[\check{\phi}, \check{Z}], \phi] : -2\, \mathrm{Tr}\,[[\phi, Z], T^a][[\check{\phi}, \check{Z}], T^a] :$$

In the planar limit this leads to (in spin chain language)

$$\boxed{D = \sum_{K=1}^{L} \Big[\underbrace{1}_{\text{tree}} + \underbrace{\left(\tfrac{\lambda}{16\pi^2}\right)}_{\text{one-loop}} (1 - \vec{\sigma}_K \cdot \vec{\sigma}_{K+1}) + \underbrace{\left(\tfrac{\lambda}{16\pi^2}\right)^2}_{\text{two-loop}} \Big((1 - \vec{\sigma}_K \cdot \vec{\sigma}_{K+2}) - 4 (1 - \vec{\sigma}_K \cdot \vec{\sigma}_{K+1}) \Big) + \ldots \Big]}$$

Superficially not an integrable Hamiltonian ... BUT a "spectroscopic survey" shows that at $N = \infty$ mysterious degeneracies appear. These are the planar pairs $\mathcal{O}_\pm$:

$$\boxed{P\, \mathcal{O}_\pm = \pm \mathcal{O}_\pm}$$

P = parity

P and D commute: $[P, D] = 0$

States of different parity do not mix, and we do not expect them to scale identically. However, at $\Delta^{(0)} = 7$ you find a $\pm$ pair with

$$\Delta_- = 7 + \frac{5}{8}\frac{\lambda}{\pi^2} - \frac{15}{128}\frac{\lambda^2}{\pi^4} \qquad \Delta_+ = 7 + \frac{9 \pm \sqrt{1 + \frac{160}{N^2}}}{16}\frac{\lambda}{\pi^2} - \frac{27 + \frac{52}{N^2} \pm \frac{3 + \frac{948}{N^2}}{\sqrt{1 + \frac{160}{N^2}}}}{256}\frac{\lambda}{\pi^4}$$

$$\Rightarrow \quad \boxed{\Delta_- \neq \Delta_+ \quad N < \infty} \qquad \boxed{\Delta_- = \Delta_+ \quad N = \infty} \sim \text{"planar degeneracy"}$$

One can show that the degeneracy is directly related to integrability:

$$[Q_3, D] = 0 \qquad \{Q_3, P\} = 0 \qquad \Rightarrow \qquad Q_3\, \mathcal{O}_+ \sim \mathcal{O}_-$$

Higher Loop integrability:

$$[Q_r(\lambda), Q_s(\lambda)] = 0$$

Three-Loop Integrability and Bethe Ansatz

In 0303060 we also proposed the su(2) three-loop dilatation operator:

$$\boxed{\begin{aligned} D_6 = \sum_{K=1}^{L} \Big[& 28 (1 - \vec{\sigma}_K \cdot \vec{\sigma}_{K+1}) - 8 (1 - \vec{\sigma}_K \cdot \vec{\sigma}_{K+2}) - \\ & - 2\, \vec{\sigma}_K \cdot \vec{\sigma}_{K+3}\, \vec{\sigma}_{K+1} \cdot \vec{\sigma}_{K+2} + 2\, \vec{\sigma}_K \cdot \vec{\sigma}_{K+2}\, \vec{\sigma}_{K+1} \cdot \vec{\sigma}_{K+3} \Big] \end{aligned}}$$

- Now proven by Beisert 0310252 and extended to su(2|3)
- Three-Loop Bethe ansatz derived (su(2) sector) Serban, M.S. 0401057 "Inozemtsev Long Range Chain"
- All-Loop "asymptotic" Bethe ansatz proposed Beisert, Dippel, M.S. 0405001 (su(2) sector)

Important Problems:

- Higher Loops in full su(2,2|4), or at least in a sector containing twist operators. (Very important for QCD connections)
- Non-asymptotic Bethe ansatz
- Bethe ansatz beyond su(2) sector

A Test of the Three-Loop Dilatation Operator

The "simplest" non-BPS state in $\mathcal{N}=4$ is the Konishi field:

$$\boxed{\mathcal{O}_K = \mathrm{Tr} \sum_{i=1}^{6} \varphi_i \varphi_i}$$

It has a descendant in the su(2) sector

$$\boxed{\mathcal{O}_K' = \mathrm{Tr}\, [\phi, Z][\phi, Z]}$$

This is an $XXX_{\frac{1}{2}}$ Heisenberg spin chain with

length $L=4$ magnon number $M=2$

$\Rightarrow$ "BMN" case with $J=2$

Our formula gives, to three loops,

$$\boxed{\Delta_K' = 4 + 3\,\frac{\lambda}{4\pi^2} - 3\left(\frac{\lambda}{4\pi^2}\right)^2 + \frac{21}{4}\left(\frac{\lambda}{4\pi^2}\right)^3 + \mathcal{O}(\lambda^4)}$$

This agrees with a recent completely independent argument due to Kotikov, Lipatov, Onishenko, Velizhanin hep-th/0404092, using hep-ph/0403192 Moch, Vermaseren, Vogt.

Now proven: hep-th/0409009, a field theory computation by Eden, Jarczak, Sokatchev !

Kotikov, Lipatov, Onishenko, Velizhanin
hep-th/0404092

2.2 NNLO correction to universal anomalous dimension

The final three-loop result ‡ for the universal anomalous dimension $\gamma_{uni}(j)$ for $\mathcal{N}=4$ SYM is

$$\gamma(j) \equiv \gamma_{uni}(j) = \hat{a}\gamma_{uni}^{(0)}(j) + \hat{a}^2\gamma_{uni}^{(1)}(j) + \hat{a}^3\gamma_{uni}^{(2)}(j) + \ldots, \qquad \hat{a} = \frac{\alpha N_c}{4\pi}, \tag{9}$$

where

$$\frac{1}{4}\gamma_{uni}^{(0)}(j+2) = -S_1, \tag{10}$$

$$\frac{1}{8}\gamma_{uni}^{(1)}(j+2) = \left(S_3 + S_{-3}\right) - 2\,S_{-2,1} + 2\,S_1\left(S_2 + S_{-2}\right), \tag{11}$$

$$\begin{aligned}\frac{1}{32}\gamma_{uni}^{(2)}(j+2) = {} & 2\,S_{-3}\,S_2 - S_5 - 2\,S_{-2}\,S_3 - 3\,S_{-5} + 6(S_{-4,1} + S_{-3,2} + S_{-2,3}) + 24\,S_{-2,1,1,1} \\ & -12\left(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}\right) - (S_2 + 2\,S_1^2)(3\,S_{-3} + S_3 - 2\,S_{-2,1}) \\ & -S_1(8\,S_{-4} + S_{-2}^2 + 4\,S_2\,S_{-2} + 2\,S_2^2 + 3\,S_4 - 12\,S_{-3,1} - 10\,S_{-2,2} + 16\,S_{-2,1,1})\end{aligned} \tag{12}$$

and $S_a \equiv S_a(j)$, $S_{a,b} \equiv S_{a,b}(j)$, $S_{a,b,c} \equiv S_{a,b,c}(j)$ are harmonic sums

$$S_a(j) = \sum_{m=1}^{j} \frac{1}{m^a}, \quad S_{a,b}(j) = \sum_{m=1}^{j} \frac{1}{m^a} S_b(m), \quad S_{a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{1}{m^a} S_{b,c,\cdots}(m), \tag{13}$$

$$\begin{aligned} S_{-a}(j) &= \sum_{m=1}^{j} \frac{(-1)^m}{m^a}, \quad S_{-a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a} S_{b,c,\cdots}(m), \\ \overline{S}_{-a,b,c,\cdots}(j) &= (-1)^j\, S_{-a,b,c,\ldots}(j) + S_{-a,b,c,\cdots}(\infty)\left(1 - (-1)^j\right) \end{aligned} \tag{14}$$

The expression (14) is defined for all integer values (see [28, 7]) and can be easily analytically continued to real and complex j values by the method of Refs. [29, 7].

2.3 $j \to \infty$ limit

In the limit $j \to \infty$ the AD results (10)-(12) are simplified significantly. Note, that this limit is related to the study of the asymptotics of structure functions and cross-sections at $x \to 1$ corresponding to the quasi-elastic kinematics of the deep-inelastic ep scattering.

The great interest to this limit comes from the recent formulation of the AdS/CFT correspondence (see Refs. [30, 31, 32] and Section 3).

Using Eqs. (13) and (14), we obtain for the harmonic sums at large j with the accuracy $O(j^{-1})$

$$S_1(j \to \infty) = \ln j + \gamma_e, \quad S_2(\infty) = \zeta_2, \quad S_{-2}(\infty) = -\frac{1}{2}\zeta_2, \tag{15}$$

$$S_3(\infty) = \zeta_3, \quad S_{-3}(\infty) = -\frac{3}{4}\zeta_3, \quad S_{-2,1}(\infty) = -\frac{5}{8}\zeta_3, \tag{16}$$

‡ Note, that in an accordance with Ref. [9] our normalization of $\gamma(j)$ contains the extra factor $-1/2$ in comparison with the standard normalization (see [1, 2, 3]) and differs by sign in comparison with Vermaseren-Moch-Vogt one [10].

String Predictions for $\mathcal{N}=4$

A key proposal of AdS/CFT:

$$\boxed{\text{"}E = \Delta\text{"}}$$

energy of string state — dimension of conformal operator

Quantization of $\overline{IIB}$ superstrings on $AdS_5 \times S^5$ ill-understood. Recent, dramatic progress in certain "semiclassical" limits. These involve states with large angular momentum J_1, J_2, J_3 on the five-sphere S^5. Should correspond to operators (say $J_3 = 0$):

$$\boxed{\mathrm{Tr}\, Z^{L-M} \Phi^M} \qquad J_1 = L-M \;;\; J_2 = M$$

Two such limits were considered:

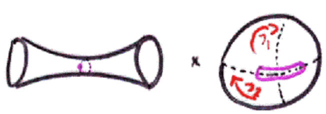
I BMN "Berenstein – Maldacena – Nastase" 0202021

$J = L - M >> 1 \;;\; M = 2, 3, \ldots$ "plane-wave limit"

II FT "Frolov – Tseytlin" 0304255

$J_1 >> 1 \;;\; J_2 >> 1$ "spinning string limit"

Semiclassical string rotations:



effective expansion parameter: $\boxed{\lambda' = \dfrac{\lambda}{L^2}}$

Breakthrough I ... ? ...

... Plane Waves, Penrose Limits, and "BMN"

At the end of 2001, a novel maximally supersymmetric background of the $\overline{IIB}$ string was discovered (Blau et. al. Metsaev). The previously known ones were flat Minkowski $\mathbb{R}^{1,9}$ and $AdS_5 \times S^5$. The plane-wave background:

$$\boxed{ds^2 = -4\, dx^+ dx^- - \mu^2 (x^i)^2 (dx^+)^2 + (dx^i)^2}$$

light cone coordinates — difference to the flat case

- may be easily quantized, like flat space
- can be obtained from $AdS_5 \times S^5$ by a Penrose limit

One focuses on the geometry seen by a tiny string moving along a great circle on S^5.
In many ways a "precursor" of what we will discuss.

Berenstein – Maldacena – Nastase "BMN" (2002) proposed the $\mathcal{N}=4$ interpretation of this background:

massless state (super-graviton)	$E = 0$	$\longleftrightarrow$	$\mathrm{Tr}\, Z^J$
simplest massive excitation (2-oscillator state)	$E = J + 2\sqrt{1 + \frac{\lambda}{J^2}}$	$\longleftrightarrow$	$\mathrm{Tr}\, Z^J \Phi^2 + \ldots$ (2-magnon)
⋮			⋮

Here it is also crucial that $J \to \infty$!

In order to get a non-trivial result, need $N \to \infty$ such that $\boxed{\lambda' = \dfrac{\lambda}{J^2} = \dfrac{N g_{YM}^2}{J^2}}$ is finite.

This looks like a planar limit, i.e. $g_s = 0$ ... ?

One- and Two-Loop Success, Three-Loop Trouble I

Using the dilatation operator techniques, we derived the finite J anomalous dimensions of BMN operators up to three loops:

$$\boxed{\mathcal{O}_n^J = \frac{1}{J+1} \sum_{p=0}^{J} \cos\left(\frac{\pi n(2p+1)}{J+1}\right) \operatorname{Tr} \phi Z^p \phi Z^{J-p} + \mathcal{O}(\lambda)}$$

$$\boxed{\begin{aligned}\Delta_n^J = J+2 &+ \frac{\lambda}{\pi^2} \sin^2 \frac{\pi n}{J+1} - \frac{\lambda^2}{\pi^4} \sin^4 \frac{\pi n}{J+1}\left(\frac{1}{4} + \frac{\cos^2 \frac{\pi n}{J+1}}{J+1}\right) \\ &+ \frac{\lambda^3}{\pi^6} \sin^6 \frac{\pi n}{J+1}\left(\frac{1}{8} + \frac{\cos^2 \frac{\pi n}{J+1}}{4(J+1)^2}\left(3J+2(J+6)\cos^2\frac{\pi n}{J+1}\right)\right) \\ &+ \mathcal{O}(\lambda^4)\end{aligned}}$$

This agrees perfectly with the BMN prediction to leading order in J: $\lambda' = \frac{\lambda}{J^2}$

$$\Delta_n = J + 2\sqrt{1+\lambda' n^2} = J + 2 + \lambda' n^2 - \frac{1}{4}\lambda'^2 n^4 + \frac{1}{8}\lambda'^3 n^6 + \mathcal{O}(\lambda'^4)$$

What about $\frac{1}{J}$ corrections?

Callan, Lee, McLoughlin, Schwarz, Swanson, Wu] → They studied the leading correction to the string energies away from the plane wave limit. They find

$$\boxed{\Delta_n^{\text{string}} = J + 2 + \lambda' n^2\left(1 - \frac{2}{J}\right) + \lambda'^2 n^4\left(-\frac{1}{4} + \frac{0}{J}\right) + \lambda'^3 n^6\left(\frac{1}{8} + \frac{0}{J}\right)}$$

while the gauge result, from above, gives

$$\boxed{\Delta_n^{\text{gauge}} = J + 2 + \lambda' n^2\left(1 - \frac{2}{J}\right) + \lambda'^2 n^4\left(-\frac{1}{4} + \frac{0}{J}\right) + \lambda'^3 n^6\left(\frac{1}{8} + \frac{\frac{1}{2}}{J}\right)}$$

Disagreement at three loops!

Breakthrough II:? Fully dynamical tests of AdS/CFT

Such tests are hard even for free ($g_s = 0$) strings, that is, planar ($N = \infty$) gauge theory.
We would like to compare spectra of string and gauge states. (beyond supergravity!)

Recently, Frolov + Tseytlin found string solitons:

 (hep-th/0306143)

Folded string, "sits" at center of AdS, and rotates in two planes with angular momenta J_1, J_2.
For $J_1, J_2 \to \infty$ can find energy semi-classically:

$$\boxed{J = J_1 + J_2 \qquad E = J + \varepsilon_1 \frac{\lambda}{J} + \varepsilon_2 \frac{\lambda^2}{J^3} + \dots}$$

Without quantization, a result for arbitrary tension $\sqrt{\lambda} = \frac{R^2}{\alpha'}$!
Furthermore, this predicts perturbative operator dimensions in GT.
E.g. one-loop $\mathcal{O}(\lambda)$: Elliptic integrals

$$\boxed{\varepsilon_1 = \frac{2}{\pi^2} K(t_0)\left[E(t_0) - (1-t_0) K(t_0)\right] \; ; \quad \frac{J_2}{J} = 1 - \frac{E(t_0)}{K(t_0)}}$$

Identify gauge operators carrying charges J_1, J_2 on gauge side

$$\operatorname{Tr}\left(Z^{J_1} \phi^{J_2}\right) + \dots \quad \text{(all orderings)}$$

two complex scalars in $\mathcal{N}=4$ SYM_4

Treat these as spin chain Minahan + Zarembo (hep-th/0212208) by Bethe ansatz, in the thermodynamic limit $J \to \infty$
Beisert, Minahan, M.S. + Zarembo (hepth/0306139).

⇒ $\boxed{\text{Same result}}$!

One- and Two-Loop Success, Three-Loop Trouble II

Using integrable systems techniques we derived the scaling dimensions of FT operators up to three loops:

$$Q_2^{\text{gauge}} = 1 + \frac{1}{2\pi^2} K(q_0)\big(2E(q_0) - (2-q_0)K(q_0)\big)\lambda' +$$

$$+ \frac{1}{8\pi^4} K(q_0)^3\big(4(2-q_0)E(q_0) - (8-8q_0+q_0^2)K(q_0)\big)\lambda'^2$$

$$+ \frac{1}{4\pi^6}\frac{K(q_0)^5}{\big(E(q_0)-K(q_0)\big)\big(E(q_0)-(1-q_0)K(q_0)\big)} \times \Big[(8-8q_0+3q_0^2)E(q_0)^3 - (2-q_0)(12-12q_0+q_0^2)E(q_0)^2K(q_0) + 3(1-q_0)(8-8q_0+q_0^2)E(q_0)K(q_0)^2 - 4(1-q_0)^2(2-q_0)K(q_0)^3\Big]\lambda'^3$$

Gauge modulus: $\boxed{\alpha = \frac{J_2}{J} = \frac{1}{2} - \frac{1}{2}\frac{1}{\sqrt{1-q_0}}\frac{E(q_0)}{K(q_0)}}$ $\quad J = J_1 + J_2 = L$, $J_2 = M$

After a Gauß-Landen transformation, this agrees perfectly with the string prediction at one (Beisert, Minahan, M.S., Zarembo) and two (Serban, M.S.) loops:

$$t_0 = -\frac{\big(1-\sqrt{1-q_0}\big)^2}{4\sqrt{1-q_0}}$$

However, starting from three loops a discrepancy appears (Serban, M.S.):

$$\boxed{Q_2^{\text{string}} - Q_2^{\text{gauge}} = \frac{q_0^2}{16\pi^6} K(q_0)^5\big[2E(q_0) - (2-q_0)K(q_0)\big]\lambda'^3 + \mathcal{O}(\lambda'^4)}$$

Bootstrapping the Classical Transfer Matrix

In 0310182, Arutyunov + M.S., it was shown that not only the energies, but all commuting charges agree when comparing folded (and circular) strings + gauge theory. This may be proven in general: 0402207, Kazakov, Marshakov, Minahan, Zarembo: These authors derived integral consistency equations on the transfer matrix of the classical σ-model:

$$\boxed{\begin{gathered} 2⨍_{\mathcal{C}} dx'\,\sigma(x')\,\frac{1}{x-x'} = \Delta\,\frac{x}{x^2-\omega^2} + 2\pi n_\nu \\ \omega^2 = \frac{\lambda'}{16\pi^2} = \frac{\lambda}{16\pi^2 L^2} \\ \int_{\mathcal{C}} dx\,\sigma(x)\Big(1-\frac{\omega^2}{x^2}\Big) = \alpha = \frac{J_2}{L} \end{gathered}}$$

The charges, including the energy Q_2 and the momentum Q_1 are then given by

$$\boxed{Q_r = \int_{\mathcal{C}} dx\,\frac{\sigma(x)}{x^r} \qquad Q_1 = 2\pi m \qquad \Delta = 1 + 2\omega^2 Q_2}$$

Comparing to thermodynamic, one-loop gauge theory, we see that the structures match. What about two and three loops? We have the dilatation operator, and found evidence that it is 2- and 3-loop integrable. But what about the Bethe ansatz?

$\mathcal{N}=4$ and the Inozemtsev Long-Range Spin Chain

[Serban, M.S., 0401057]

Since we found evidence that the dil.op. is integrable beyond one loop, a Bethe ansatz should exist, given what we said earlier. Unfortunately not too much is known about the classification of int. long-range spin chains. The only thing already on the market seems to be the one of Inozemtsev. It is very closely related to the integrable Calogero-Moser multiparticle systems. The Hamiltonian reads

$$H_{Ino} = \sum_{k,n} \wp_{L,\frac{\pi}{\kappa}}(n)\,(1-\vec{\sigma}_k\cdot\vec{\sigma}_{k+n})$$

↑ Weierstrass elliptic function

Qualitatively, this "smells right": There are two coupling constants $L, \frac{\pi}{\kappa}$, related to the two periods of $\wp$. Now, if we take $L\to\infty$, we get

$$H'_{Ino} = \sum_{k,n} \frac{1}{\mathrm{sh}^2(\kappa n)}\,(1-\vec{\sigma}_k\cdot\vec{\sigma}_{k+n})$$

Then, by defining $e^{-2\kappa} = \frac{1}{2}g^2 - \frac{3}{4}g^4 \pm \ldots$ we can "emulate" the two-loop dilatation operator.

Inozemtsev, continued ...

Now, for the hyperbolic Hamiltonian H'_{Ino}, Inozemtsev managed to find the all-orders "asymptotic" phase shift, and thus the S-matrix. It reads

$$S(p_1,p_2) = \frac{\varphi(p_1)-\varphi(p_2)+i}{\varphi(p_1)-\varphi(p_2)-i}$$

with

$$\varphi(p) = \frac{1}{2}\cot\frac{p}{2} + \frac{1}{2}\sum_{n>0}\left[\cot\left(\frac{p}{2}-i\kappa n\right)+\cot\left(\frac{p}{2}+i\kappa n\right)\right]$$

To two loops, this is

$$\varphi(p) = \frac{1}{2}\cot\frac{p}{2}\left(1+4g^2\sin^2\frac{p}{2}+\ldots\right)$$

So we have a two-loop Bethe ansatz!

What about three-loops? There the trouble is that we have four-spin terms $\sigma\sigma\sigma\sigma$ in the dilatation operator! However, we showed that we can add a higher charge to H_{Ino}, so as to still emulate the dilatation operator.

So we also have a three-loop Bethe ansatz!

Using this ansatz, we can produce the above three-loop discrepancy, so it is really there!

A Novel Long Range Spin Chain

[Beisert, Dippel, M.S., 0405001]

Apparently an alternative spin chain without breakdown of scaling exists. Hamiltonian complicated.

Bethe ansatz (conjecture): $g^2 = \frac{\lambda}{8\pi^2}$

$$\boxed{e^{i p_j L} = \prod_{k=1}^{M} \frac{\varphi_j - \varphi_k + i}{\varphi_j - \varphi_k - i}} \qquad \boxed{\varphi(p) = \frac{1}{2} \cot\frac{p}{2} \sqrt{1 + 8 g^2 \sin^2\frac{p}{2}}}$$

$$\boxed{Q_2 = \sum_{k=1}^{M} \frac{1}{g^2} \left(\sqrt{1 + 8 g^2 \sin^2\frac{p_k}{2}} - 1 \right)}$$

The Bethe ansatz is asymptotic:
Wrappings are excluded! BMN limit is manifest!
Thermodynamic limit:

$$\boxed{2 \fint_C d\varphi' \rho(\varphi') \frac{1}{\varphi - \varphi'} = \frac{1}{\sqrt{\varphi^2 - \frac{\lambda'}{4\pi^2}}} + 2\pi n_\nu}$$

Bethe equation for the string σ-model (Kazakov, Marshakov, Minahan, Zarembo)

$$\boxed{2 \fint_C dx' \sigma(x') \frac{1}{x - x'} = \Delta \frac{x}{x^2 - \left(\frac{\lambda'}{16\pi^2}\right)^2} + 2\pi n_\nu \qquad \Delta = 1 + \frac{\lambda'}{8\pi^2} Q_2}$$

One can show that under the map

$$\boxed{\varphi = x + \frac{\lambda}{16\pi^2} \frac{1}{x}} \nearrow Q_r = \int_C dx \frac{\sigma(x)}{x^r}$$

Structural Matching of Gauge + String Theory

The answers are different, unfortunately.
But let us next show that they are closely related:
$\sigma(x)$ in KMMZ is not a density:

$$\int_C dx\, \sigma(x) \left(1 - \frac{\omega^2}{x^2}\right) = \alpha \qquad \omega^2 = \frac{\lambda'}{16\pi^2}$$

Let us turn it into one:

$$d\varphi := \left(1 - \frac{\omega^2}{x^2}\right) dx$$

$$\Rightarrow \qquad \varphi = x + \frac{\omega^2}{x} \qquad x = \frac{\varphi}{2} + \frac{1}{2}\sqrt{\varphi^2 - 4\omega^2}$$

Now compare the commuting charges in the novel spin chain to classical string theory:

$$Q_r^{gauge} = \int d\varphi\, \rho(\varphi) \frac{1}{\sqrt{\varphi^2 - 4\omega^2}} \frac{1}{\left(\frac{1}{2}\varphi + \sqrt{\varphi^2 - 4\omega^2}\right)^{r-1}}$$

$$\Longleftrightarrow \quad Q_r^{string} = \int dx\, \sigma(x) \frac{1}{x^r}$$

Formally, these two expressions agree under above map.
⇒ The local dispersion laws agree!!!
It is the S-matrix which is "dressed" in string theory:

$$2 \fint_C d\varphi' \rho(\varphi') \left[\frac{1}{\varphi - \varphi'} + \vartheta(\lambda'^2) \right] = \frac{1}{\sqrt{\varphi^2 - 4\omega^2}} + 2\pi n_\nu$$

additional scattering!

Bethe Ansatz for Quantum Strings

[Arutyunov, Frolov, M.S., 0406256]

In fact, we may rewrite the "dressing factor" in a very suggestive form:

$$\underbrace{S(p,p')}_{\text{full S-matrix}} = \underbrace{S_0(p,p')}_{\text{"bare" S-matrix}} \cdot \underbrace{\exp\Big[2i\sum_{r=1}^{\infty}\omega^{2r+4}\big(q_{r+2}\,q'_{r+3}-q_{r+3}\,q'_{r+2}\big)\Big]}_{\text{"dressing factor"}}$$

We may now rediscretize the Bethe equations:

$$e^{ip_k L} = \prod_{\substack{j=1\\ j\neq k}}^{M} \underbrace{\frac{\varphi(p_k)-\varphi(p_j)+i}{\varphi(p_k)-\varphi(p_j)-i}}_{\text{bare S-matrix}} \cdot \underbrace{e^{2i\sum_{r=1}^{\infty}\left(\frac{g^2}{2}\right)^{r+2}\left[q_{r+2}(p_k)\,q_{r+3}(p_j)-q_{r+3}(p_k)\,q_{r+2}(p_j)\right]}}_{\text{"dressing factor"}}$$

- consistent with the "strong coupling" explanation of the disagreement
- reproduces near-BMN string results; for two, three [Callan, McLoughlin, Swanson, 0405153], many [McLoughlin/Swanson 0407240]
- most importantly, reproduces the generic AdS prediction

$$\boxed{\Delta \simeq 2\sqrt{\pi}\,\lambda^{\frac{1}{4}}} \qquad \underline{\lambda\to\infty} \quad \text{(still large } L\text{)}$$

cf Gubser, Klebanov, Polyakov, 9802109