1

FROM SPIN CHAINS TO SIGMA MODELS:

THE LANDAU-LIFSHITZ EQUATION

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See Also M. Kruczenski
0311203
M. Kruczenski, A. Ryzhov, A. Tseytlin
0403120
R. Hernandez, E. Lopez
0403139
M. Kruczenski, A. Tseytlin
0406189

2

WITHIN THE AdS/CFT CORRESP.
LARGE CHARGE SECTORS
HAVE RECENTLY RECEIVED ATTENTION

- CHARGES IN CARTAN OF
  $SO(2,4) \times SO(6)$ $(E, S_1, S_2; J_1, J_2, J_3)$

FOR EXAMPLE

- BMN SECTOR: LARGE E, J_1
- GKP CONSIDERED LARGE E, S_1

3

ON STRING TH. SIDE ONE LOOKS AT CLASSICAL SPINNING STRINGS WITH LARGE SPINS & SMALL QUANTUM CORRECTIONS

THEIR ENERGIES, $E(J)$, ARE COMPARED WITH ANOMALOUS DIMENSIONS OF g. inv. OPERATORS IN THE GAUGE THEORY

4

IN THE LARGE N LIMIT ($g_s \to 0$)

- THE GAUGE THEORY HAS AN EXPANSION IN λ AND $\frac{1}{J}$

- THE STRING THEORY HAS AN EXPANSION IN $\sqrt{\lambda}$ AND $\frac{1}{J}$

HOWEVER...

Frolov, Tseytlin

REGULAR FT SOLUTIONS

SOLUTIONS OF CLASSICAL STRING SIGMA MODEL WITH REGULAR EXPANSION IN λ WITH QUANTUM CORRECTIONS SUPPRESSED BY $\frac{1}{J}$

$$E(J) = J\left[1 + \sum_{k=1}^{\infty}\left(\frac{\lambda}{J^2}\right)^k\left(c_k + \sum_{n=1}^{\infty}\frac{d_{nk}}{J^n}\right)\right]$$

ON GAUGE THEORY SIDE, IN LARGE CHARGE SECTORS, THE λ, $\frac{1}{J}$ EXPANSION ALSO REARRANGES ITSELF INTO A λ/J^2, $\frac{1}{J}$ EXPANSION SO:

COMPARISON OF THE TWO EXPANSIONS IS POSSIBLE

BUT $\longrightarrow$ ORDER OF LIMITS

GAUGE THEORY CALCULATIONS ARE DONE FOR SMALL λ

STRING THEORY CALCULATIONS ARE DONE FOR LARGE λ

WE FOCUS (INITIALLY) ON ANALYTIC STRING SOLUTIONS WITH LARGE

$(E; J_1, J_2) \longleftrightarrow$ SU(2) SECTOR

$(E; S_1; J_3) \longleftrightarrow$ SL(2) SECTOR

$(E; J_1, J_2, J_3) \longleftrightarrow$ SU(3) SECTOR

THESE CORRESPOND TO GAUGE THEORY OPERATORS

$\mathrm{Tr}(X^{J_1} Z^{J_2}) + \ldots$

$\mathrm{Tr}(D_{1+i2}^{S_1} Z^{J_3}) + \ldots$

$\mathrm{Tr}(X^{J_1} Y^{J_2} Z^{J_3}) + \ldots$

WHERE X, Y, Z ARE CHIRAL SCALARS $X = \phi_1 + i\phi_2$ ETC

COMPARISONS BETWEEN ENERGIES OF STRING SOLUTIONS AND ANOMALOUS DIMENSIONS OF OPERATORS HAVE BEEN CARRIED OUT IN THESE SECTORS

PRECISE AGREEMENT EXISTS AT 1 LOOP AND 2 LOOPS

BUT:

AT 3 LOOPS IN THE SU(2) SECTOR THE ANSWERS DISAGREE!

THE HOPE IS TO RESOLVE THIS WITHOUT HAVING TO RESUM THE ENTIRE THEORY

MICROSCOPIC EQUIVALENCE?

AT 1- & 2- LOOP LEVEL IT WAS SHOWN THAT WHOLE INTEGRABLE STRUCTURES MATCH!

Arutyunov
Staudacher

THIS GAVE STRONG EVIDENCE THAT THE MICROSCOPIC SYSTEMS HAD TO BE THE SAME (IN SOME LIMITS)

Q: CAN WE MAKE PRECISE THE

$$E = \Delta - J$$

RELATION NOT BY COMPARING EIGENVALUES BUT BY MATCHING THE HAMILTONIANS/LAGRANGIANS?

ANOMALOUS DIMENSIONS FROM SPIN CHAINS 1

MINAHAN
ZAREMBO

THINK OF $Tr(X^{J_1} Z^{J_2})$ AS A PERIODIC CHAIN WITH $J_1 + J_2$ LENGTH

AT EACH SITE
$$|\downarrow\rangle \longleftrightarrow Z$$
$$|\uparrow\rangle \longleftrightarrow X$$

GENERALISING THIS TO FULL N=4 SYM IS POSSIBLE WITH ∞ DIML REPS AT INDIVIDUAL SITES

BEISERT
KRISTJANSEN
STAUDACHER

Eg: $SL(2)$ $\quad Tr(D_{1+i2}^{S} Z^{J})$

IS DESCRIBED BY PERIODIC CHAIN OF LENGTH J

AT EACH SITE

$$\frac{1}{n!}(D_{1+i2})^n Z \longleftrightarrow (a^+)^n |0\rangle$$

ANOMALOUS DIMENSIONS FROM SPIN CHAINS 2

FINDING Δ IN PERT. THEORY HAS BEEN MAPPED TO FINDING THE EIGENVALUES OF A SPIN CHAIN HAMILTONIAN

AT 1 LOOP $H = \sum_{\ell=1}^{J} H_{\ell,\ell+1}$

SECTOR	OP	$H_{\ell,\ell+1}$
SU(2)	$Tr(X^{J_1} Z^{J_2}) + \ldots$	$1 - P_{\ell,\ell+1}$ $= \frac{1}{2} - \frac{1}{2}\sigma_\ell^i \sigma_{\ell+1}^i$
SU(3)	$Tr(X^{J_1} Y^{J_2} Z^{J_3}) + \ldots$	$1 - P_{\ell,\ell+1}$ $= \frac{2}{3} - \frac{1}{2}\lambda_\ell^r \lambda_{\ell+1}^r$
SO(6)	$Tr(\phi_1^{J_1} \ldots \phi_6^{J_6}) + \ldots$	$\frac{1}{2}K_{\ell\ell+1} + 1 - P_{\ell\ell+1}$ $= \frac{1}{2}M_\ell^{ij} M_{\ell+1}^{ij} + \frac{9}{8} - \frac{1}{16}(M_\ell^{ij} M_{\ell+1}^{ij})^2$

ANOMALOUS DIMENSIONS FROM SPIN CHAINS 3

IN THE SL(2) SECTOR WE ALSO HAVE

$$H = \sum_{\ell=1}^{J} H_{\ell,\ell+1}$$

FOR 1 LOOP DILATATION OPERATOR WHERE NOW

$$H_{\ell\,\ell+1}\,(a_\ell^+)^k (a_{\ell+1}^+)^{n-k}\,|0,0\rangle = \sum_{p=0}^{n} c_p(k,n)(a_\ell^+)^p (a_{\ell+1}^+)^{n-p}\,|0,0\rangle$$

WITH

$$c_p(k,n) = \begin{cases} \frac{-1}{|k-p|} & k \neq p \\ h(k) + h(n-k) & k = p \end{cases}$$

$h(k) = \sum_{j=1}^{k} \frac{1}{j}$ IS THE k^{th} HARMONIC #

FROM SPIN CHAIN TO SIGMA MODEL

THE BETHE ANSATZ GIVES EIGENVALUES OF INTEGRABLE SYSTEMS

INSTEAD, WE OBTAIN A 2D LAGRANGIAN AS A CONTINUUM LIMIT OF THE COHERENT STATE EXPECTATION VALUE OF SPIN CHAIN LAGRANGIAN

Kruczenski

COHERENT STATES

GIVEN A LIE GROUP G, A SUB-GROUP H, AND CORRESP. LIE ALGEBRA IN REP Λ

- WE FIND VACUUM $|0\rangle$ ST

$$\langle 0|0\rangle = 1$$

$$\Lambda(h)|0\rangle = e^{i\phi(h)}|0\rangle \quad h\in H$$

THIS VACUUM IS H INVARIANT

NB: WE REQUIRE H TO BE MAXIMAL

- THE COHERENT STATE IS

$$\exp\left\{\sum_{i\in G/H} w_i E_{-i} - w_i^* E_i\right\}|0\rangle$$

WITH THE w_i COORDINATES ON G/H

COHERENT STATES
SU(2) EXAMPLE

CONSIDER $SU(2)/U(1) \sim S^2$

IN $s=\frac{1}{2}$ REPRESENTATION.

THE VACUUM IS $|\frac{1}{2},\frac{1}{2}\rangle$ $\quad S_z|\frac{1}{2}\frac{1}{2}\rangle = \frac{1}{2}|\frac{1}{2}\frac{1}{2}\rangle$, $S^2|\frac{1}{2}\frac{1}{2}\rangle = \frac{3}{4}|\frac{1}{2}\frac{1}{2}\rangle$

THE COHERENT STATE IS

$$|n\rangle = e^{i(n_x S_x - n_y S_y)}|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

WHICH SATISFIES $\langle n|n\rangle = 1$

$$\langle n|S_i|n\rangle = \frac{1}{2}n_i$$

WHERE
$$n_1 = \frac{n_y \sin\Delta}{\Delta} \qquad \Delta = \sqrt{n_x^2 + n_y^2}$$
$$n_2 = -\frac{n_x \sin\Delta}{\Delta}$$
$$n_3 = \cos\Delta$$

$\sum_{i=1}^{3} n_i n_i = 1$ SO $n_i \in S^2$

SIGMA MODEL FROM
SPIN CHAIN SU(3) SECTOR 1

THE $SU(3)/SU(2)\times U(1)$ COHERENT STATE IS

$$|N\rangle = e^{i(a\lambda^4 + b\lambda^5 + c\lambda^6 + d\lambda^7)}|0\rangle$$

$\lambda_3|0\rangle = 0$

$\lambda^8|0\rangle = \frac{-2}{\sqrt{3}}|0\rangle$

AND $\langle N|\lambda^r|N\rangle = \frac{2}{3}a_r(a,b,c,d)$

DEFINE $N = \sum_{r=1}^{8} a_r \lambda^r$

THIS CAN BE WRITTEN AS

$$N_{ij} = 3U_i^* U_j - \delta_{ij} \qquad \Sigma|U_i|^2 = 1$$

THE U_i ARE COORDS ON

$$CP^2 \cong SU(3)/SU(2)\times U(1)$$

SIGMA MODEL FROM SPIN CHAIN SU(3) SECTOR 2

BS + A Tseytlin

WITH $|N\rangle = \prod_{\ell=1}^{J} |N\rangle_\ell$

THE COHERENT STATE EXPECTATION VALUE OF SU(3) HAMILTONIAN IS

$$\langle N| D_{SU(3)} |N\rangle = \frac{\lambda}{288\pi^2} \sum_{\ell=1}^{J} \left[Tr(N_\ell \lambda^r) - Tr(N_{\ell+1} \lambda^r) \right]^2$$

TAKING THE CONTINUUM LIMIT

$J \to \infty$, λ/J^2 FIXED

$$N_{\ell+1} - N_\ell = \frac{2\pi}{J} \partial_1 N + O\left(\frac{1}{J^2}\right)$$

AND $t_{OLD} \to \frac{J^2}{\lambda} t_{NEW}$

WE GET

$$I = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} \mathcal{L}_{WZ} - \frac{1}{36} Tr(\partial_1 N \partial_1 N)$$

WHERE $\mathcal{L}_{WZ}$ IS THE CONTINUUM LIMIT OF $\langle N| \frac{d}{dt} |N\rangle$

EXPLICITLY

$$\mathcal{L}_{WZ} = \frac{i}{18} \int_0^1 dz \, Tr\left(N[\partial_z N, \partial_0 N]\right)$$

THE EQUATION OF MOTION IS

$$\partial_0 N = -\frac{i}{6} [N, \partial_1^2 N]$$

IE THE MATRIX LANDAU LIFSHITZ EQ

IN TERMS OF u_i $\quad N_{ij} = u_i^* u_j - 1_{ij}$

$$\mathcal{L}_{SU(3)} = \int_0^{2\pi} \frac{d\sigma}{2\pi}\left[-i\, u_i^* \partial_0 u_i - \tfrac{1}{2} |D_1 u_i|^2\right]$$

WHERE $D_1 u_i = (\partial_1 + u_j^* \partial_1 u_j)\, u_i$

TO RESTRICT TO SU(2) SECTOR SET $u_3 = 0$ AND DEFINE

$$n^i = (u_1^*, u_2^*)\, \sigma^i \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad n^i n^i = 1$$

THE n^i EQ. OF MOTION IS

$$\partial_0 n^i = \tfrac{1}{2} \varepsilon^{ijk} n^j \partial_1^2 n^k$$

WHICH IS THE ORDINARY LANDAU LIFSHITZ EQ

SIGMA MODEL FROM SPIN CHAIN SL(2) SECTOR

THE SL(2)/U(1) COHERENT STATE IS (BS + A Tseytlin)

$$|\ell\rangle = \sqrt{1-|\zeta|^2} \sum_{k=0}^{\infty} \zeta^k a^{+k} |0\rangle$$

THE CONTINUUM LIMIT OF

$$\langle \ell | \frac{d}{dt} | \ell \rangle + \langle \ell | D_{SL(2)} | \ell \rangle$$

IS

$$-\frac{1}{2}\int_0^1 dz \left(\varepsilon^{ijk} \ell_i \partial_z \ell_j \partial_0 \ell_k\right) - \frac{1}{8} \eta^{ij} \partial_1 \ell_i \partial_1 \ell_j$$

WITH $\eta^{ij} \ell_i \ell_j = -1$

IE "MINKOWSKI" LL EQUATION

STRING σ MODEL TO LANDAU-LIFSHITZ EQ.

START WITH BOSONIC σ MODEL ON

$$ds^2 = dY_i^* dY^i + dX_i^* dX_i \qquad |X_i|^2 = 1$$

$$\eta^{ij} Y_i Y_j = -1$$

AND DO HOPF FIBRATION

$$Y_i = e^{iy} V_i \qquad X_i = e^{i\alpha} u_i$$

WE CONSIDER TWO CASES

- $V_1 = 1 \quad V_2 = V_3 = 0$ mostly S^5
- $u_3 = 1 \quad u_1 = u_2 = 0$ mostly AdS_5

IN THE FORMER CASE WE GO TO CONFORMAL GAUGE & SET

$$y = t = \kappa\tau$$

THIS TURNS OUT TO BE CONSISTENT

THE RESULTING ACTION DEPENDS ON u_i AND α

IN THE $\kappa \to \infty$ LIMIT, THE CONFORMAL GAUGE CONSTRAINTS CAN BE USED TO ELIMINATE α TO GET

$$I = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[-i\, u_i^* \partial_t u_i - \tfrac{1}{2} |D_\sigma u_i|^2 \right]$$

WHICH IS PRECISELY THE SU(3) ACTION OBTAINED ON SPIN CHAIN SIDE

A SIMILAR PROCEDURE REPRODUCES THE SL(2) SECTOR ACTION

WE HAVE SHOWN THAT IN THE "CHIRAL" SECTORS, AT 1-LOOP, THE STRING σ MODEL IS EQUIVALENT TO THE σ MODEL OBTAINED AS A COHERENT STATE CONTINUUM LIMIT OF SPIN CHAIN

Q: • WHAT ABOUT HIGHER LOOPS

• WHAT ABOUT NON-CHIRAL SECTORS

2 LOOP SU(2) SECTOR

Kruczenski, Ryzhov, Tseytlin

THE CLASSICAL CONTINUUM LIMIT OF 2 LOOP DILATATION OP IS

$$\langle N| D_2 + D_4 |N\rangle \longrightarrow \int \frac{\lambda'}{8}(\partial_1 n)^2 - \frac{\lambda'^2}{32}(\partial_1^2 n)^2$$

THIS DOES NOT MATCH STRING TH.

TO GET MATCHING NEED TO THINK OF CTN LIMIT AS DESCRIBING EFFECTIVE ACTION FOR LOW ENERGY MODES WITH MOMENTA $\sim 1/J$

IE HAVE TO "INTEGRATE OUT" HIGH MOMENTUM MODES

THEN MATCHING IS OBTAINED

σ MODEL FOR NON CHIRAL SO(6) SPIN CHAIN

USING $SO(6)/SO(2)\times SO(4)$ COHERENT STATE WE GET 1-LOOP CONTINUUM HAMILTONIAN
BS+ ABe, thin

$$H = J\int \frac{d\sigma}{2\pi}\, \frac{\lambda'}{8}\left[Tr(\partial_1 m)^2 + \frac{1}{4} Tr(m\,\partial_1 m)^2 \right]$$

WITH $m^{ij} \in G_{2,6} \sim SO(6)/SO(2)\times SO(4)$

IN FACT

$$m^{ij} = v^i v^{j*} - v^j v^{i*} \qquad i = 1\ldots 6$$

WITH $v^i v^{i*} = 1 \quad v^i v^i = 0$

SO

$$L = -i\, v^{i*}\partial_t v^i - \frac{1}{2}|\partial_1 v_i|^2 + \frac{1}{2}|v^{i*}\partial_1 v^i|^2$$

THIS ACTION HAS BEEN ALSO RECENTLY OBTAINED AS A LIMIT OF STRING σ MODEL (SEE MARTIN'S TALK)
KRUCZENSKI, TSETLIN

SO(6) SECTOR: PICKING THE WRONG VACUUM

ABOVE WE CONSIDERED

$$SO(6)/SO(4)\times SO(2) \sim G_{2,6}$$

IE, OUR VACUUM WAS BPS

$$Tr(Z^J)$$

WE MIGHT HAVE BEEN TEMPTED TO INSTEAD CONSIDER

$$SO(6)/SO(5) \sim S^5$$

THIS CORRESPONDS TO PICKING

$$Tr(\phi_6^{\,J})$$

AS THE NON BPS VACUUM

THE CONTINUUM LIMIT OF
SPIN CHAIN HAMILTONIAN
GIVES

$$H = \frac{\lambda}{(4\pi)^2} J \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[1 + \frac{(2\pi)^2}{J^2} (\partial_1 v^i)^2 \right]$$

WITH $v^i \in S^5$

THE RED TERM DOES NOT
HAVE A GOOD

$\lambda \to \infty, J \to \infty, \lambda/J^2 \sim cst$

LIMIT !

EXECUTIVE SUMMARY

- WE OBTAIN A σMODEL AS A COHERENT STATE CONTINUUM LIMIT OF THE SPIN CHAIN HAMILTONIAN IN THE $SU(3)$, $SL(2)$, $SO(6)$ SECTORS AT 1 LOOP
- THIS MATCHES TO SUITABLE LIMIT OF STRING σMODEL DEMONSTRATING 1 LOOP EQUIVALENCE BETWEEN THE TWO THEORIES AT MICROSCOPIC LEVEL

29

OPEN QUESTIONS

- OTHER SECTORS
- EXTEND 2 LOOP
- 3 LOOP PROBLEM
- THIS APPROACH CIRCUMVENTS INTEGRABILITY
  COULD WE USE IT IN MORE GENERAL SETTINGS ?
- FOR WHAT TYPES OF GAUGE THEORIES IS THE BMN LIMIT WELL DEFINED ?
- HOLOGRAPHY ?