Towards a c-theorem in Four Dimensions

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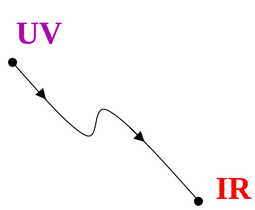
QCD and Strings Workshop, September 2004

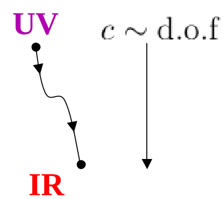
Outline

I. RG flows and the c-theorem
II. a-maximization and SCFTs
III. a-maximization Along the Entire RG Flow
IV. Open Questions and Future Work



I. RG Flows and the c-theorem

In 2d, we can define a quantity *c* which **monotonically decreases** along RG flows. (Zamolodchikov's *c*-theorem)



$$\frac{d}{dt}c(g(t)) < 0 \quad \text{with } t = -\log\mu$$

The endpoints of this flow are **CFTs**, with $\beta(g_*) = 0$

and at these points, $c(g_*) = c_{CFT}$

Recall that the central charge appears in the trace **anomaly**: $\langle T_\mu{}^\mu \rangle = \frac{-c}{12}R$

One can actually make an even stronger claim here:

Zamolodchikov argued that 2d RG flows are **gradient flows**,

$$\dot{g}^I(t) = -\beta^I(g(t)) = -G^{IJ}(g)\frac{\partial c(g)}{\partial g^J}$$

This implies $\frac{d}{dt}c(g(t)) = \frac{\partial c}{\partial g^I}\dot{g}^I = -G^{IJ}\frac{\partial c}{\partial g^I}\frac{\partial c}{\partial g^J} < 0$

positive definite

In 2d, the c-theorem confirms our intuition that d.o.f. should decrease along RG flow!

What about in four dimensions?

In 4d, we have more options! One way to see this:

$$\langle T_\mu{}^\mu \rangle = a(\text{Euler}) + c(\text{Weyl})^2$$

Cardy sez: It's ***a***! In **ALL KNOWN EXAMPLES**,
a **decreases** under RG flow
(but we know flows where *c* increases).

The **conjectured** *a*-theorem then states that

$$a_{UV} > a_{IR}$$

QuickTime™ and a TIFF (Uncompressed) decompressor are needed to see this picture.

but there is currently no general (and generally accepted) proof!

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But this a **weaker** statement than the 2d c-theorem!

Stronger Claim: One can define a quantity *a* along the entire flow; this quantity monotonically decreases and agrees with the conformal anomaly at the fixed points.

Even Stronger Claim: As in 2d, this is a gradient flow

$$\beta^I(g(t)) = G^{IJ}\frac{\partial a}{\partial g^J}$$ with positive definite metric.

Osborn *et al* investigated this latter claim and found it to be true perturbatively!

Can we say anything exact?

We can use SUSY to get exact results!

Let's do a **quick** review of 4d $\mathcal{N} = 1$ SCFTs:

- The **superconformal algebra** is

$$SU(2,2|1) \supset SO(4,2) \times U(1)_R$$

- $\Delta(\mathcal{O}) \geq \frac{3}{2} R(\mathcal{O})$, saturated for chiral primaries.

- The beta function for $\mathcal{N} = 1$ theories is given by

$$\beta_{NSVZ} = -\frac{g^3}{16\pi^2} \frac{3T(G) - \sum_i T(r_i)(1-\gamma_i)}{1 - \frac{g^2 T(G)}{8\pi^2}}$$

$\mathrm{Tr}\, T^A_{r_i} T^B_{r_i} = T(r_i)\delta^{AB}$

so $\beta_{NSVZ} \sim T(G) + \sum_i T(r_i)(R_i - 1)$

this means that $\beta_{NSVZ} = 0$ implies that the R-current is anomaly free!

We're not done yet…

- We can write other beta functions in terms of R-charges:

$$\beta(h) = (\Delta(W) - 3)h = \frac{3}{2} h(R(W) - 2)$$

superpotential coupling

- There is a supermultiplet with the R-current, stress tensor, and SUSY current in it.
- There's a unitarity bound, $\Delta(\mathcal{O}) \geq 1$
- $a = 3\,\mathrm{Tr}\, R^3 - \mathrm{Tr}\, R$ (Freedman *et al*)

can compute via **'t Hooft anomaly matching**!

The point:

The R-symmetry is a very useful tool – provided that we can find it!

II. a-maximization and SCFTs

In some cases, it is easy to find the anomaly-free R symmetry:

For SU(N_c) SQCD with N_f flavors, $R(Q) = R(\widetilde{Q}) = 1 - \frac{N_c}{N_f}$

But in general the R-charges are not uniquely determined by anomaly freedom!

For SQCD with an extra adjoint matter field *X*,

$$N_c + N_c(R(X) - 1) + N_f(R(Q) - 1) = 0$$

means there's a **one-parameter family of possible R-charges!**

Which of these is the $U(1)_R \subset SU(2,2|1)$?

The answer: (K. Intriligator, BW)

real parameters

Define $R_{trial} = R_0 + \sum_I s_I F_I$

any anomaly-free R-symmetry

all other U(1) symmetries

Finding the $U(1)_R \subset SU(2,2|1)$ means finding the appropriate values of s_I

To do this, just find the local max of the function

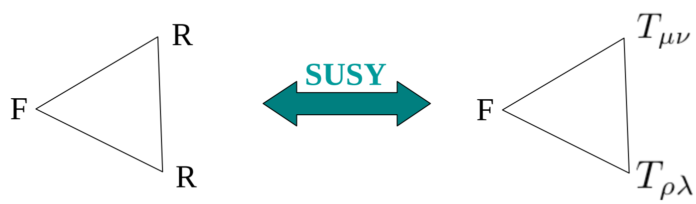
$$a_{trial}(s_I) = 3\,\mathrm{Tr}\, R_{trial}^3 - \mathrm{Tr}\, R_{trial}$$

Why does this work?

Maximizing $a_{trial}(s_I) = 3\,\mathrm{Tr}\,R^3_{trial} - \mathrm{Tr}\,R_{trial}$

is equivalent to the two conditions

1) $9\mathrm{Tr}R^2F_I = \mathrm{Tr}F_I$ **(SUSY)**



2) $\mathrm{Tr}RF_IF_J < 0$ **(SUSY and CFT)**

$$\langle RF_IF_J\rangle \xleftrightarrow{\text{SUSY}} \langle T_{\mu\nu}F_IF_J\rangle \sim \langle F_IF_J\rangle < 0$$

So we now know the R-charges in ANY 4d SCFT!

This means we also know the exact dimensions of lots of operators, all for the **low low price** of maximizing a cubic function.

some restrictions apply

Also, at the maximum, *a* is the central charge of the SCFT:

$$a = 3\,\mathrm{Tr}\,R^3 - \mathrm{Tr}\,R$$

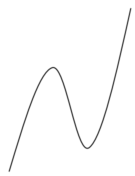
(there's a similar expression for *c*, too)

This means that SUSY gauge theories are excellent testing grounds for the *a*-theorem.

Quick quick example: Consider a free chiral superfield Φ

$$a = 3(r-1)^3 - (r-1)$$

This has extrema at *r=2/3* and *r=4/3*.



max! **min!**

And it's basically just as easy for interacting theories!

Note one other thing: Since we're maximizing a cubic function, R-charges, chiral primary operator dimensions, and central charges must be quadratical irrationals.

$$\frac{n+\sqrt{m}}{p}$$

These cannot depend on any continuous moduli.

a-maximization almost proves the *a*-theorem!

Since relevant deformations generally break the flavor symmetries,

$$\mathcal{F}_{IR} \subset \mathcal{F}_{UV}$$

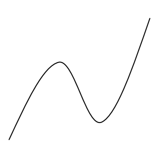
Maximizing over a subset then implies that $a_{IR} < a_{UV}$

But this is wrong!

Loopholes:

1) Accidental symmetries.

2) Only a local max.



A recent proposal of Kutasov helps with the second, but let's talk about the first right now…

Accidental Symmetries

In general, you ***never know*** when you'll see
an accidental symmetry in the IR.

But there's a special kind we know about! Sometimes operators like

$$M = Q\widetilde{Q}$$

appear to **violate** the **unitarity bound** $\Delta(M) \geq 1$

For example, this happens in SQCD: Since $R(Q) = 1 - \frac{N_c}{N_f}$

when $N_f < \frac{3}{2}N_c$ we find $R(M) < \frac{2}{3}$

**But for this number of flavors, the theory is in
a free magnetic phase!**

Quick review of Seiberg Duality in SQCD:

$$SU(N_c) \Longleftrightarrow SU(N_f - N_c)$$

with N_f flavors — with N_f flavors and singlets

Notice that there are asymptotic freedom bounds for both theories:

$$3T(G) - \sum_i T(r_i) > 0$$

$$N_f < 3N_c \quad \text{and} \quad 3(N_f - N_c) > N_f$$

free magnetic | $\frac{3}{2}N_c$ | conformal window | $3N_c$ | free electric $\rightarrow N_f$

Seiberg says:

When a gauge invariant operator appears to violate the unitarity bound, an accidental symmetry rescues it by dragging its R-charge back up to 2/3!

That is, the operator becomes **free**.

Kutasov, Parnachev, and Sahakyan pointed out that this nontrivially affects *a*-maximization.

If an operator *M* hits the unitarity bound, we should maximize

$$a' = a^0 - 3\,[R(M) - 1]^3 + [R(M) - 1] + \frac{2}{9}$$

There are lots of examples where this is crucial! Let's do one.

SU(N_c) SQCD with N_f flavors, an adjoint matter field *X*, and $W_{tree} = 0$

R-charges are

$$R(Q) = R(\widetilde{Q}) \equiv y, \;\; R(X) = \frac{1-y}{x}, \;\; x \equiv \frac{N_c}{N_f}$$

So maximize $a(x,y)$ with respect to *y*.

In doing this, generalized mesons gradually hit *R=2/3*!

$$Q\widetilde{Q} \quad QX\widetilde{Q} \quad QX^2\widetilde{Q}$$

Each time this happens, we must correct *a*:

$$a^1(x,y) = a^0(x,y) + N_f^2\left[\frac{2}{9} - 3(2y-1)^3 + (2y-1)\right]$$

We generalized this to this case of SQCD with two adjoints *X* and *Y*, with different superpotentials.

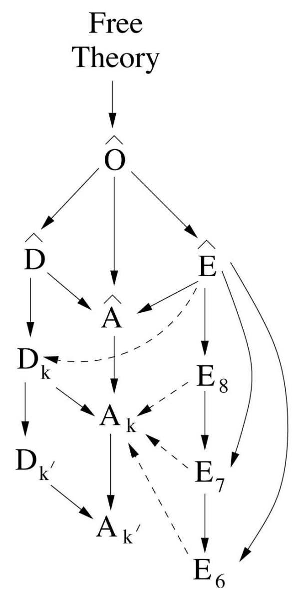
(K. Intriligator, BW)

Can use operator dimensions to figure out exactly which superpotentials are relevant, and when. Surprisingly, there is an ADE classification!

$$W_{\widehat{O}} = 0 \qquad W_{\widehat{A}} = TrY^2$$

$$W_{\widehat{D}} = TrXY^2 \quad W_{\widehat{E}} = TrY^3$$

$$W_{E_8} = Tr(X^3 + Y^5)$$ etc.



Lots of new SCFTs, and all flows compatible with the *a*-theorem!

Example: $\hat{D} \rightarrow D_{k+2}$ **(Intriligator, BW)**

$$W_{\hat{D}} = \mathrm{Tr}\, XY^2$$

$$\downarrow$$

$$W_{D_{k+2}} = \mathrm{Tr}\,(XY^2 + X^{k+1})$$

Although X^{k+1} looks irrelevant, it can get a large negative anomalous dimension!

In fact, one can show that for **any** *k*, there is a range $x \equiv \frac{N_c}{N_f} > x^{min}_{D_{k+2}}$ where the deformation is relevant.

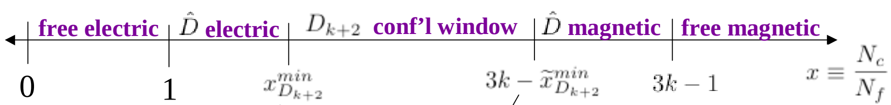
For large *k*, $x^{min}_{D_{k+2}} = \frac{9}{8}k$

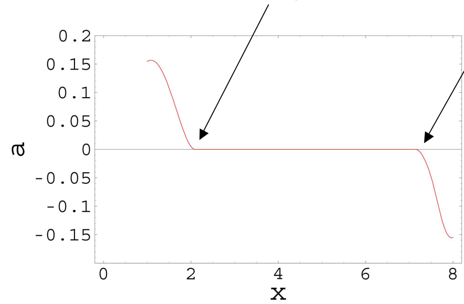
Turns out we can say more – we know a dual theory!

For $W_{D_{k+2}} = \mathrm{Tr}\left(XY^2 + X^{k+1}\right)$ (Brodie)

$$SU(N_c) \xleftrightarrow{\text{dual}} SU(3kN_f - N_c)$$

We can now find the phases of these theories!





This is a plot of $a_{elec} - a_{mag}$ which shows that there are **accidental symmetries**!

III. a-maximization along the entire RG flow

The weak form of the *a*-theorem, $a_{IR} < a_{UV}$ has survived many different tests! Can we extend it?

Kutasov says:

Use Lagrange multipliers!

$$a = 2|G| + \sum_i |r_i| \left[3(R_i - 1)^3 - (R_i - 1)\right]$$
$$-\lambda \left(T(G) + \sum_i T(r_i)(R_i - 1) \right)$$

This Lagrange multiplier enforces anomaly freedom in the IR.

Now, just **extremize** with respect to each R-charge!

$$R_i = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda T(r_i)}{|r_i|}}$$

We can now interpret these as the R-charges along the flow

$$\underset{\text{UV}}{\lambda = 0} \longrightarrow \underset{\text{IR}}{\lambda = \lambda_*}$$

Plugging back into *a* gives

$$a(\lambda) = 2|G| - \lambda T(G) + \frac{2}{9}\sum_i |r_i|(1 + \frac{\lambda T(r_i)}{|r_i|})^{3/2}$$

which interpolates between the two central charges.

This proposal effectively takes care of loophole #2.

$$\frac{da}{d\lambda} = \sum_i \cancelto{0}{\frac{\partial a}{\partial R_i}} \frac{\partial R_i}{\partial \lambda} + \frac{\partial a}{\partial \lambda}$$

and

$$\frac{\partial a}{\partial \lambda} = -\left[T(G) + \sum_i T(r_i)(R_i - 1)\right] < 0$$

until the IR fixed point, where the beta function vanishes.

So *a* is monotonically decreasing!

An Aside:

These Lagrange multipliers have an interesting interpretation.

Compare anomalous dimensions: (Kutasov)

Expand $R_i = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda T(r_i)}{|r_i|}} \approx \frac{2}{3} - \frac{\lambda T(r_i)}{6|r_i|}$

So, since $R_i = \frac{2}{3}\left(1 + \frac{\gamma_i}{2}\right)$, we find $\gamma_i = -\frac{\lambda T(r_i)}{2|r_i|}$

But we know from gauge theory that $\gamma_i = -\frac{\alpha |G| T(r_i)}{\pi |r_i|}$

We thus conclude that $\lambda = \frac{2\alpha}{\pi}|G| + \cdots$

The Lagrange mutliplier acts like *g*² in some scheme!

We can go further, too: (Barnes, Intriligator, BW, Wright; Kutasov, Schwimmer)

Expand $\gamma_i = 3R_i - 2 = 1 - \sqrt{1 + \frac{\lambda C(r_i)}{|G|}}$ to get

$$\gamma_i = \sum_{p=1}^{\infty} \frac{(2p-3)!!}{p!}\left(\frac{-\lambda}{2|G|}\right)^p C(r_i)^p$$

but in general, we expect $\frac{\lambda}{2|G|} = \frac{g^2}{4\pi^2} + \sum_{q=2}^{\infty} A_q g^{2q}$

scheme dependent

This means we can extract the scheme-independent part of γ_i

$$\gamma_i^{(p)} = \frac{(2p-3)!!}{p!}\left(-\frac{C(r_i)}{4\pi^2}\right)^p g^{2p}$$

This matches **precisely** with computations by Jack, Jones, North.
However, the scheme-dependent terms do not match.

To match up the **scheme-dependent** stuff too, we must take into account the wavefunction renormalization.

(Barnes, Intriligator, BW, Wright; Kutasov and Schwimmer)

$$\alpha \rightarrow \widetilde{\alpha}(\alpha) \qquad \Phi_i \rightarrow \sqrt{Z_i(\alpha)}\Phi_i$$

$$\widetilde{\gamma}_i(\widetilde{\alpha}) = \gamma_i(\alpha) - \beta(\alpha)\frac{d}{d\alpha}\log Z_i(\alpha)$$

So it looks like the Lagrange multipliers really work, and tell us something about physics. Not just formal!

Now, back to the a-theorem!

We can extend the Lagrange multipliers to other cases as well:

1. **Superpotential deformations**
2. **Accidental symmetries from unitarity violations**
3. **Higgsing?**

still working on it!

In these cases, we can use Lagrange multipliers to prove the a-theorem, up to the caveat of additional accidental symmetries.

(Kutasov; Barnes, Intriligator, BW, Wright)

Let's do the case of superpotential deformations:
As before, just write

$$a(R_i, \lambda_J) = 3TrR^3 - TrR + \sum_J \lambda_J \widehat{\beta}_J(R)$$

with $\widehat{\beta}_W(R) = R(W) - 2$

$$\widehat{\beta}_g(R) = -(T(G) + \sum_i T(r_i)(R_i - 1))$$

Notice that this automatically implies gradient flow, just as in 2d!

$$\frac{\partial a}{\partial \lambda_J} = \widehat{\beta}_J$$

For example, consider the superpotential case:

$$\widehat{\beta}_W = R(W) - 2$$

If $R(W) < 2$, the superpotential is **relevant**

this drives λ to increase **and *a* to decrease!**

We can use similar techniques to prove the a-theorem for accidental symmetries that come from unitarity violations, and hopefully for other cases as well.

There's another computation we can compare with as well: Osborn *et al* computed the metric in coupling space.

$$\frac{\partial a}{\partial g} = G_{gg}\beta_{NSVZ}$$

$$\frac{\partial a}{\partial h} = G_{hh}\beta_W \quad \text{(e.g. Yukawa)}$$

To leading order, there are **no** cross terms.

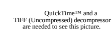
Computation with Lagrange multipliers: $G_{gg} = \frac{16|G|}{3g^2} \qquad G_{hh} = \frac{4}{72\pi^2}$

These are EXACTLY the leading order terms obtained perturbatively by Osborn *et al*.

This is evidence for the STRONGEST version of the a-theorem, that the RG flow is gradient flow.

To really check this, we'd need to go beyond leading order. There seems to be a mismatch, although there are many scheme-dependence subtleties to worry about.

ALL IN ALL, THE A-THEOREM HAS SURVIVED EVERY KNOWN TEST AND THE PROOF IS GRADUALLY FALLING TO A FULL FRONTAL ASSAULT



IV. Conclusions, Open Questions, and Future Work

_a_-maximization gives us access to lots of new information. New results for previously mysterious SCFTs!

The _a_-theorem is "almost proved," and we're closing in.

R-charges etc. all quadratic irrational numbers!

String theory reason for ADE?

Relation between Lagrange multipliers and couplings? Which scheme?

Accidental symmetries? Can we say anything general?

Pick a theory – can use a-maximization to study it!