QCD and String Theory (KITP Conference, November 2004)

Computing Tree And One-Loop Amplitudes In Gauge Theory

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Introduction

The goal of this talk is to introduce new techniques, derived from or inspired by twistor string theory (Witten 2003), for the computation of scattering amplitudes of gluons in $\mathcal{N}=4$ gauge theory to one-loop order in perturbation theory.

Motivation

Reviews: Z.Bern TASI 92, L.Dixon TASI 95

- At tree-level, $\mathcal{N}=4$ amplitudes of gluons coincide with QCD amplitudes. This is because no fermions or scalars can propagate in the internal lines.
- At one-loop, $\mathcal{N}=4$ amplitudes of gluons are part of QCD one-loop amplitudes. To see this note that,

$$A^{\rm QCD}=g=(g+4f+3s)-4(f+s)+s=A^{\mathcal{N}=4}-4A^{\mathcal{N}=1}+A^{scalar}$$

- Why do we compute perturbative QCD amplitudes? LHC Background.

Definition of the Amplitudes

We want to compute scattering amplitudes of n gluons. Each gluon carries the following information: $g_i=\{p_i^\mu,\epsilon_i^\mu,a_i\}$.

Color Decomposition

(Berends, Giele, Mangano, Parke, Xu)

$$A_n^{\rm tree}(\{p_i^\mu,\epsilon_i^\mu,a_i\})=$$

$$\sum_\sigma {\rm Tr}(T^{a_{\sigma(1)}}\ldots T^{a_{\sigma(n)}})A^{\rm tree}(\sigma(p_1^\mu,\epsilon_1^\mu),\ldots,\sigma(p_n^\mu,\epsilon_n^\mu))$$

Color Ordered Partial Amplitudes. At one-loop the same decomposition can be done but it also includes double trace terms. We are interested in the leading-color partial amplitudes.

Spinor-Helicity Formalism

$\{p_i^\mu, \epsilon_i^\mu\} \Longrightarrow$ Large number of Lorentz invariant combinations. Highly redundant.

In four dimensions: $P_{a\dot{a}} = \sigma^\mu_{a\dot{a}} P_\mu$. Spinors of $\pm$ chirality: $P_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} + \lambda'_a \tilde{\lambda}'_{\dot{a}}$. Null vector: $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$.

Lorentz invariant inner products:

$$\langle \lambda\, \lambda' \rangle = \epsilon_{ab} \lambda^a \lambda'^b \qquad [\tilde{\lambda}\, \tilde{\lambda}'] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}^{\dot{a}} \tilde{\lambda}'^{\dot{b}} \qquad 2p \cdot q = \langle p\, q \rangle [p\, q]$$

$$(+) - \text{helicity}: \quad \epsilon^{(i)}_{a\dot{a}} = \frac{\lambda^{(i)}_a \tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}^{(i)}\, \tilde{\mu}]}$$

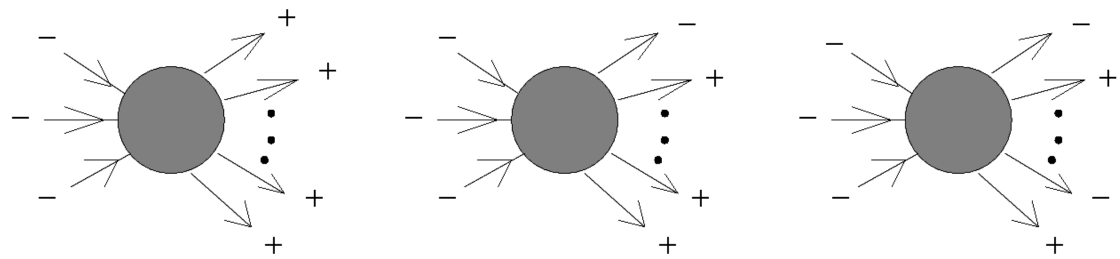
$$(-) - \text{helicity}: \quad \epsilon^{(i)}_{a\dot{a}} = \frac{\mu_a\, \tilde{\lambda}^{(i)}_{\dot{a}}}{\langle \mu\, \lambda^{(i)} \rangle}$$

Example:

$$A_5^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) = \frac{\langle 1\, 2\rangle^4}{\langle 1\, 2\rangle \langle 2\, 3\rangle \langle 3\, 4\rangle \langle 4\, 5\rangle \langle 5\, 1\rangle}$$

- Berends, Kleiss, De Causmaecker, Gastmans, Wu (1981)
- De Causmaecker, Gastmans, Troost, Wu (1982)
- Kleiss, Stirling (1985)
- Xu, Zhang, Chang (1987)
- Gunion, Kunszt (1985)

Maximal Helicity Violating or MHV Amplitudes



$$A(1^+, 2^+, 3^+, 4^+, \ldots n^+) = 0$$

$$A(1^-, 2^+, 3^+, 4^+, \ldots n^+) = 0$$

$$A(1^-, 2^+, 3^-, 4^+, \ldots n^+) = \frac{\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle \ldots \langle n-1\,n\rangle\langle n\,1\rangle}$$

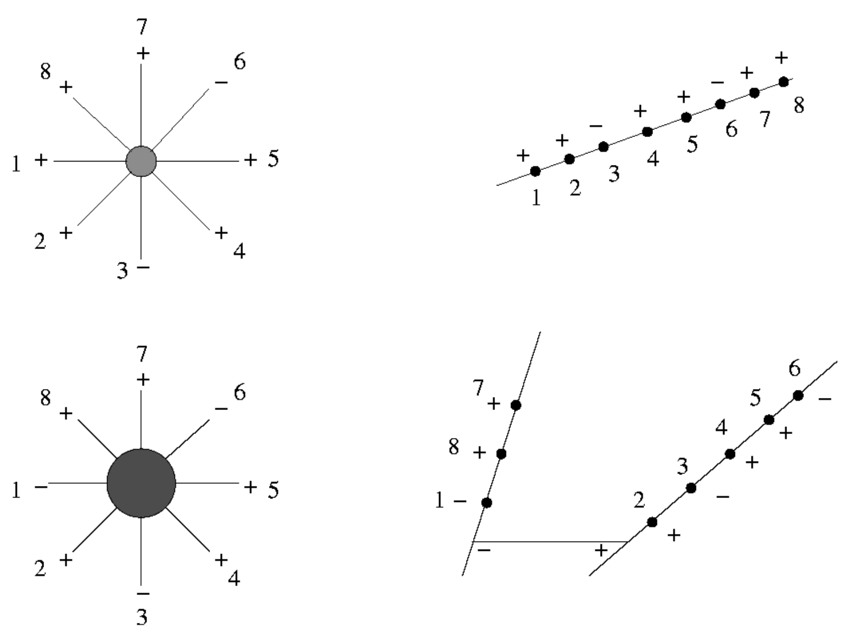
Parke-Taylor (1986); Berends-Giele (1989)

$A(1^-, 2^-, 3^-, 4^+, \ldots n^+) =$ Complicated Formula with both $\langle i\,j\rangle$ and $[i\,j]$. (Kosower 1990)

Twistor String Theory: (Witten 2003)

Twistor space (Penrose 1967): $(Z_1, Z_2, Z_3, Z_4) = (\lambda^1, \lambda^2, \mu^{\dot 1}, \mu^{\dot 2})$, with $\mu_{\dot a} = -i\partial/\partial\tilde\lambda^{\dot a}$.

Twistor string theory: Topological B-model on the Calabi-Yau supermanifold $\mathrm{CP}^{3|4}$. (More details in the discussion session!)



Conjecture: (F.C., P. Svrček, E. Witten 2004)

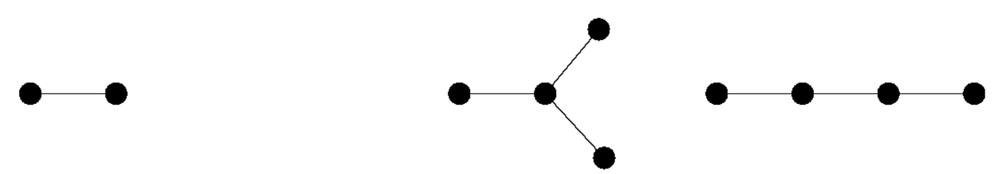
All tree-level amplitudes can be computed by sewing MHV amplitudes (continued off-shell) with Feynman propagators. These new diagrams are called MHV diagrams.

(MHV) Rules: $A(q(-),(n-q)(+))$

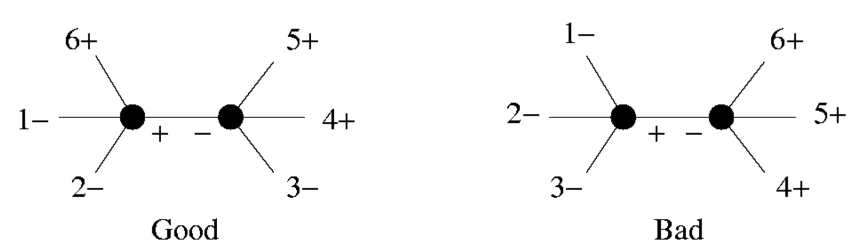
- Draw all possible “quivers” with $q-1$ nodes and $q-2$ links.

Examples:

$$(1)\;\; A_n(1^-,2^-,3^-,4^+,\ldots,n^+),\quad (2)\;\; A_n(1^-,2^-,3^-,4^-,5^-,6^+,\ldots,n^+)$$



- Attach external gluons to the nodes and assign helicities $(+)$ ——— $(-)$ to the links in such a way that each node has the helicity assignments of an MHV amplitude.



- Each graph gives a contribution equal to the product of MHV amplitudes (one for each node) times Feynman propagators (one for each link). Consider for example the “Good” graph:

$$\frac{\langle 1\,2\rangle^3}{\langle 2\,P\rangle\langle P\,6\rangle\langle 6\,1\rangle}\times\frac{1}{P^2}\times\frac{\langle P\,3\rangle^3}{\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,P\rangle}$$

What do we mean by $\langle i\ P \rangle$?

$$p_{a\dot{a}}\eta^{\dot{a}} = \lambda_a[\tilde{\lambda}, \eta] \Longrightarrow \lambda_a = \frac{p_{a\dot{a}}\eta^{\dot{a}}}{[\tilde{\lambda}, \eta]}$$

Definition:

$$\lambda_a^{(P)} = P_{a\dot{a}}\eta^{\dot{a}}$$

Does this work at all?

Yes, it does! Analytic checks for all known amplitudes in the literature. ($n \leq 7$).

Extensions to fermions, scalars (Georgiou, Khoze, Wu, Zhu, Glover).

Application to Higgs plus partons (Dixon, Glover, Khoze) .

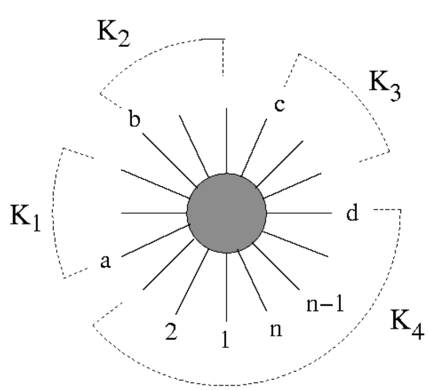
Conclusion:

All tree-level amplitudes of gluons are under control!

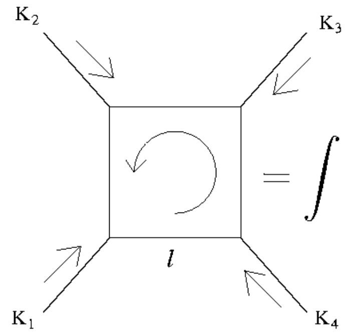
One-Loop Amplitudes in $\mathcal{N} = 4$ SYM

- Supersymmetric amplitudes of gluons are four-dimensional cut-constructible. This means that the amplitude is completely determined by its branch cuts and discontinuities. (Bern, Dixon, Dunbar, Kosower 1994)

- All tensor integrals in a Feynman graph calculation of the amplitudes can be reduced to a set of scalar box integrals. (In Dim. Reg: Bern, Dixon, Kosower 1993)



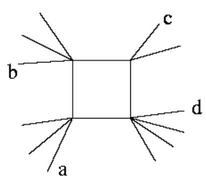
Scalar Box Integrals



$$= \int d^{4-2\epsilon}\ell \frac{1}{\ell^2(\ell - K_1)^2(\ell - K_1 - K_2)^2(\ell + K_4)^2}$$

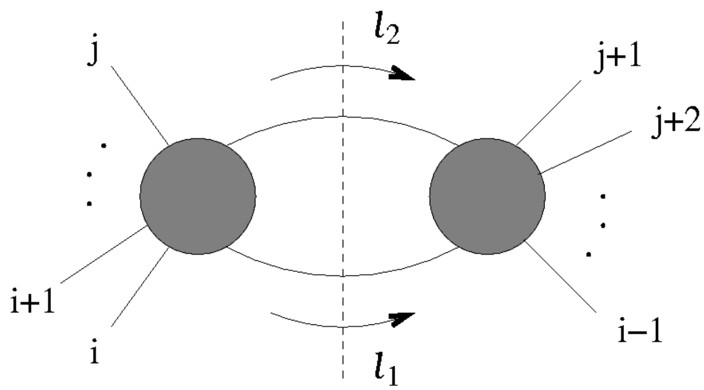
Any n-gluon one-loop amplitude can be written as: (Bern, Dixon, Kosower 1993 & with Dunbar 1994)

$$A_n^{1-\text{loop}} = \sum_{1<a<b<c<d<n} B_{abcd} \times$$



The goal of this part of the talk is to introduce an efficient way of computing the coefficients B_{abcd}, which are rational functions of $\langle i\, j\rangle$ and $[i\, j]$.

Unitarity Cuts



Cut Integral:

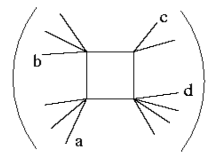
$$C_{i,\ldots,j} = \int d\mu A^{\text{tree}}(-\ell_1, i, \ldots, j, -\ell_2) A^{\text{tree}}(\ell_2, j{+}1, \ldots, i{-}1, \ell_1)$$

where

$$d\mu = d^4\ell_1 d^4\ell_2 \delta^{(4)}(\ell_1 + \ell_2 - P_L)\delta^{(+)}(\ell_1^2)\delta^{(+)}(\ell_2^2)$$

The cut can also be computed directly as the discontinuity Δ of the amplitude across the branch cut of interest.

$$C_{i,\ldots,j} = \Delta A_n^{1-\mathrm{loop}} = \sum B_{abcd} \times \Delta \left(\right)$$



Now it is clear that if a given scalar box function has a nonzero discontinuity across the cut in the P_L channel, then the cut integral has the information about its coefficient.

How can we extract it?

Direct Approach

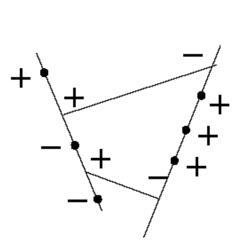
Use powerful reduction techniques to produce a function with the right discontinuity (Zvi's talk!).(Bern, Dixon, Dunbar, Kosower 1994)

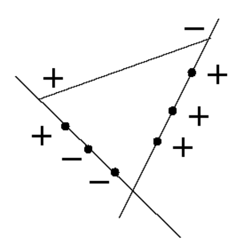
$$A_{MHV}^{1-\mathrm{loop}}(1^+, 2^+, \ldots, k^-, \ldots, l^-, \ldots, n-1^+, n^+),$$
$$A_{NMHV}^{1-\mathrm{loop}}(3(-), 3(+)).$$

What Can We Learn from Twistor String Theory?

The original twistor string theory not only contains $\mathcal{N} = 4$ SYM but it also has conformal supergravity in it (Berkovits, Witten 2004). At one-loop the two contributions mix! Nevertheless, we can ask about the localization of loop amplitudes in twistor space (F.C., P. Svrček, E. Witten 2004) .

Original expectation for 1-loop MHV amplitudes (Witten 2003):

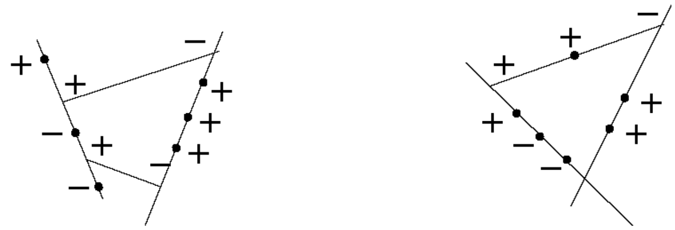




Collinear Operator: If three points Z_i, Z_j, Z_k are collinear then the vector $V_L = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K$ must vanish. For $L = \dot{a}$ we find a spinor-valued first order differential operator:

$$F_{ijk;\dot{a}} = \langle i\, j\rangle \frac{\partial}{\partial \tilde{\lambda}_k^{\dot{a}}} + \langle k\, i\rangle \frac{\partial}{\partial \tilde{\lambda}_j^{\dot{a}}} + \langle j\, k\rangle \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{a}}}$$

But, we found a surprise!:



However, Brandhuber, Spence, Travaglini (hep-th/0407214) recomputed 1-loop MHV amplitudes from MHV diagrams!

How is this be possible?

The reason for the apparent discrepancy is a holomorphic anomaly in the action of the operator on the amplitude. (F.C., P. Svrček, E. Witten 2004)

What is the holomorphic anomaly?

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z-b} = 2\pi\delta(z-b)$$

This means that

$$\frac{\partial}{\partial \bar{\lambda}_i} \frac{1}{\langle i\, m\rangle} \neq 0 \qquad \Rightarrow \qquad F_{ijk} \frac{1}{\langle i\, m\rangle} \neq 0$$

Consider for example:

$$F_{ijk} C_{i,\ldots,j} = \int d\mu F_{ijk} A^{\rm tree}_{\rm MHV}(-\ell_1, i, \ldots, j, -\ell_2) A^{\rm tree}(\ell_2, j{+}1, \ldots, i{-}1, \ell_1)$$

This is the case $m = \ell_1$. The delta function is not zero and it localizes the integral to produce a rational function!

New Technique For All Next-to-MHV One-Loop Amplitudes

(F.C. October 2004)

Consider a first order differential operator $\mathcal{O}$ in the spinor variables such that when $\mathcal{O}$ acts on the cut integral

$$C_{i,\ldots,j} = \int d\mu A^{\rm tree}_{MHV}(-\ell_1, i, \ldots, j, -\ell_2) A^{\rm tree}(\ell_2, j{+}1, \ldots, i{-}1, \ell_1)$$

it produces a rational function. $\mathcal{O}C_{i,\ldots,j} = R$.

Consider the action of $\mathcal{O}$ on the discontinuity of the amplitude in the kinematical regime of interest

$$\mathcal{O}C_{i,\ldots,j} = \sum \left(\mathcal{O}(B_{abcd}) \times \Delta(\Box_{abcd}) + B_{abcd} \times \mathcal{O}\Delta(\Box_{abcd})\right)$$

Observation: $\Delta(\Box_{abcd}) \sim {\rm Log}\,(G_{abcd})$, where G_{abcd} is a rational function (“unique”). This implies that

$$\mathcal{O}(B_{abcd}) = 0 \qquad \Rightarrow \qquad F_{ijk}(B_{abcd}) = 0$$

The coefficients B_{abcd} have some interesting twistor space structure!

This implies that $\mathcal{O}$ acting on this produces a rational function with a pole of the form

$$\mathcal{O}\Delta(\Box_{abdc}) \sim \frac{\mathcal{O}(G_{abcd})}{G_{abcd}}$$

Signature Poles and The Computation of The Coefficients

Let us denote by G_k the signature pole of each scalar box integral. Then the rational function $\mathcal{O}C_{i,\ldots,j}$ can be written in two ways

$$\frac{P}{Q\prod_k G_k} = \sum_k B_k \times \frac{\mathcal{O}(G_k)}{G_k}$$

where P is a polynomial in the numerator. Generically $\mathcal{O}P \neq 0$. Q is a polynomial defined as all factors in the denominator such that $\mathcal{O}Q = 0$.

Mathematical Problem:

Given P, Q, G_k's find B_k's such that $\mathcal{O}B_k = 0$ and the equation above is satisfied.

C_{123} cut of $A_n(1^-,2^-,3^-,4^+,\ldots,n^+)$

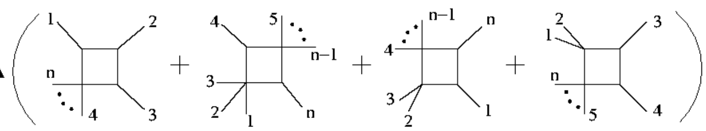
Can one compute some coefficients for all n in next-to-MHV amplitudes?

Yes!

$$C_{123} = \sum_{k=1}^{(n-2)(n-3)/2} B_i \times \Delta(\text{Box Functions})$$

Answer: Most coefficients B_i are zero. Only four are nonzero.

$C_{123} = \mathcal{B}_n \times \Delta$



where

$$\mathcal{B}_n = \frac{((p_1+p_2+p_3)^2)^3}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\ldots\langle n-1\,n\rangle\langle 4|1+2+3|1]\langle n|1+2+3|3]}$$

with

$$\langle 4|i|1] = \langle 4\,i\rangle[i\,1].$$

Systematic Procedure:

(R.Britto, F.C., B. Feng, October 2004)

We want something that allows us to "split" the signature poles.

$$R = \frac{P}{QG_1G_2\prod' G_k} = \frac{P}{Q\prod' G_k}\left(\frac{1}{G_1G_2} \times \frac{\mathcal{O}(G_1)G_2 - \mathcal{O}(G_2)G_1}{\mathcal{O}(G_1)G_2 - \mathcal{O}(G_2)G_1}\right)$$

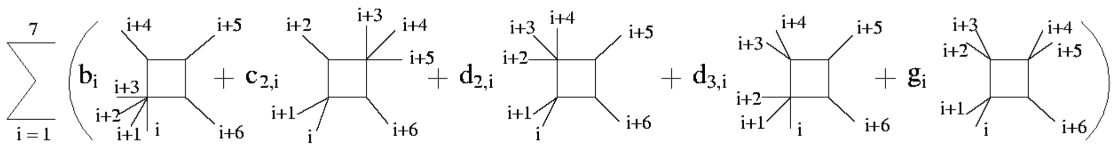
$$R = \frac{P}{Q(\mathcal{O}(G_1)G_2 - \mathcal{O}(G_2)G_1)\prod' G_k}\left(\frac{\mathcal{O}(G_1)}{G_1} - \frac{\mathcal{O}(G_2)}{G_2}\right)$$

It turns out that all G_k's satisfy $\mathcal{O}(\mathcal{O}(G_k)) = 0$! Therefore,

$$\mathcal{O}(\mathcal{O}(G_1)G_2 - \mathcal{O}(G_2)G_1) = 0.$$

Application: $A_7(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$

$$A_7(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+) =$$



We were able to obtain all 35 coefficients rather quickly!

Luckily, we did not have to wait for long to compare our coefficients with a computation based on the direct unitarity cut method. It turns out that Z. Bern, V. Del Duca, L. Dixon, D. Kosower had independently computed all seven-gluon NMHV amplitudes and their results became available shortly after ours. Their motivation was to study the twistor space structure of those amplitudes (but that is the subject of another talk!).

Conclusions

- One-loop amplitudes in $\mathcal{N} = 4$ SYM have interesting twistor space structures.
- A direct application of differential operators to test these structures fails because of a holomorphic anomaly.
- Not only the amplitudes but also the unitarity cuts are affected by the anomaly.
- The anomaly gives rise to a new and efficient technique for computing amplitudes at 1-loop. (Next-to-MHV, i.e., $3(-)$).
- As a byproduct of the new technique, we learned that the coefficients B_{abcd} are annihilated by some collinear operators F_{ijk}.