Integrability in high-energy QCD

Gregory Korchemsky

LPT, Orsay

in collaboration with:

Andrei Belitsky (Arizona State U.),
Vladimir Braun (Regensburg U.),
Sergey Derkachov (St.Petersburg U.),
Alexander Gorsky (ITEP, Moscow),
Alexander Manashov (Regensburg U.),
Dieter Müller (Regensburg U.)

For references see review hep-th/0407232

Hidden symmetry of QCD

- Integrable models = QM systems with a *finite* number of degrees of freedom and the same number of conserved charges.
- Gauge theories in four dimensions = complex systems with *infinite* number of degrees of freedom which are not integrable *per se*.
- Integrability emerges as a hidden symmetry of *effective* Yang-Mills dynamics in two *different* limits:
  - High-energy (Regge) behaviour of scattering amplitudes in QCD

$$\mathcal{A}_{\rm BFKL}(s,t) \sim s^{E}\,, \qquad E = \frac{g^2 N_c}{8\pi^2}{\rm Re}[\psi(J) - \psi(1)]$$

  - Scale dependence of composite (Wilson) operators in QCD

$$\mathcal{O}_N(0) = \bar{q}^{\uparrow}\gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_N} q^{\uparrow} \sim \Lambda_{\rm UV}{}^{-\gamma_N}\,, \qquad \gamma_N = \frac{g^2 N_c}{8\pi^2}[\psi(N+2) - \psi(1)]$$

- “Landau paradigm”: $\psi(x) = d\ln\Gamma(x)/dx$ is not just a function ... but indication of hidden integrability (= Heisenberg spin chains)
- Integrability is not tied to QCD and is a general feature of (super) YM dynamics in four dimensions

Heisenberg spin chains

- One-dimensional chain of atoms with exchange interaction *Heisenberg'26*

$$\mathbb{H}_{s=1/2} = -\sum_{n=1}^{L} \left(\mathbf{S}_n \cdot \mathbf{S}_{n+1} - \frac{1}{4} \right)$$

- The model is completely integrable and can be solved exactly *Bethe'31*
- It can be generalized to arbitrary $SU(2)$ and $SL(2)$ spins while preserving integrability

Kulish,Reshetikhin,Sklyanin'81; Faddeev,Tarasov,Takhtajan'83

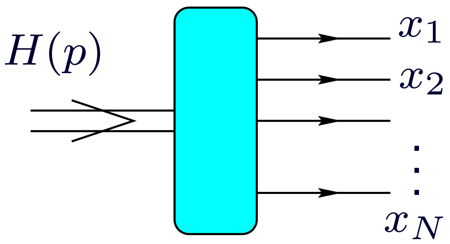
$$\mathbb{H}_s = \sum_{n=1}^{L} H(J_{n,n+1}) \,, \qquad J_{n,n+1}(J_{n,n+1}+1) = (\mathbf{S}_n + \mathbf{S}_{n+1})^2 \,.$$

  - $J_{n,n+1}$ = the sum of two neighboring spins, $\mathbf{S}_n^2 = s(s+1)$
  - $H(x) = \sum_{l=x}^{2s-1} \frac{1}{l+1} = \psi(2s+1) - \psi(x+1) =$ the Euler $\psi-$function, harmonic sum

- Integrable structures in high-energy QCD:
  - QCD in the Regge limit $\Longrightarrow SL(2,\mathbb{C})$ spin chain *Lipatov'93; Faddeev, GK'94*
  - QCD on the light-cone $\Longrightarrow SL(2,\mathbb{R})$ spin chain *Braun,Derkachov,Manashov'98; Belitsky; GK'99*
  - 'Accidental' symmetries $\Longrightarrow SU(2)$ spin chains XXX and XXZ of the dilatation operator *Ferretti,Heise,Zarembo'04; Di Vecchia, Tanzini'04*
- 'Evolution time' = $\log$(relevant energy scale) $[$'time' $= \ln s, \ln \Lambda_{\rm UV}]$

Multi-particle operators in QCD on the light-cone

- Parton model: hadrons in the infinite momentum frame $\approx$ system of quasi-free partons



$$\underbrace{0 \le x_k \le 1\,, \quad \sum_k x_k = 1}_{\text{momentum fractions}}$$

- Distribution (baryon) amplitude: *Brodsky, Lepage'79*

$$\langle 0|q(z_1 n)q(z_2 n)q(z_3 n)|H(p)\rangle \stackrel{n^2=0}{=} \int_0^1 dx_1 dx_2 dx_3\, \delta(\sum x_k - 1)\, e^{-i(pn)\sum_k x_k z_k} \varphi_B(x_i,\mu^2)$$

- Nonlocal light-cone correlator = sum of plane waves
- Moments of distribution amplitudes $\Longleftrightarrow$ local operators:

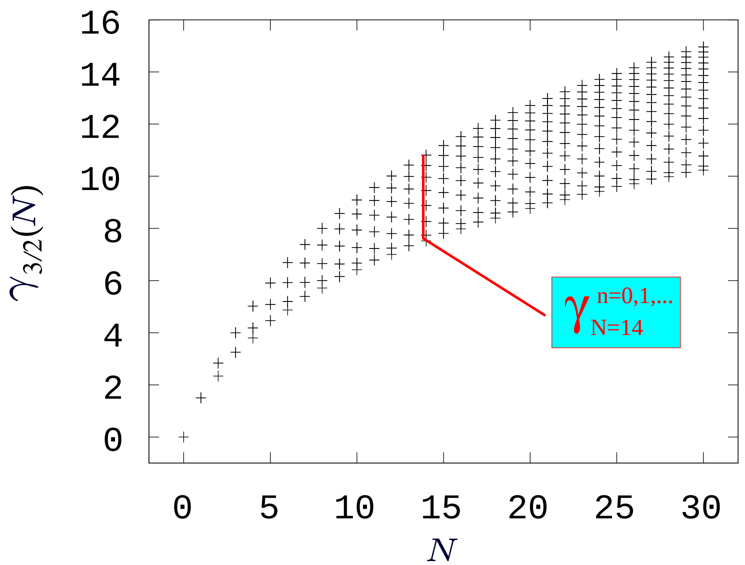
$$\varphi_B(x_i) \to \varphi_B(k_i) \quad = \quad \int \mathcal{D}x_k\, x_1^{k_1} x_2^{k_2} x_3^{k_3}\, \varphi_B(x_i,\mu^2) = \langle 0|(D_+^{k_1} q)\,(D_+^{k_2} q)\,(D_+^{k_3} q)|H(p)\rangle$$

- Scale dependence of the distribution amplitudes

$$\mu\frac{d}{d\mu}\varphi_B(k_i) = \sum_{m_j} \underbrace{V(k_i|m_j)}_{\text{mixing matrix}} \varphi_B(m_j)$$

Conformal symmetry on the light-cone

- ✔ Conventional QCD strategy: diagonalize the mixing matrix and find the anomalous dimensions
- ✔ Rich spectrum of anomalous dimensions:



Where does this structure come from?
Conformal symmetry + Integrability!

- ✔ QCD Lagrangian is invariant under the $SO(4,2)$ transformations
- ✔ $SO(4,2)$ reduces on the light-cone $x_\mu = z n_\mu$ ($n^2 = 0$) to the $SL(2)$ subgroup:

$$z \to z' = \frac{az+b}{cz+d}\,, \qquad q(z) \to q'(z) = q\left(\frac{az+b}{cz+d}\right) \cdot (cz+d)^{-2j_q}$$

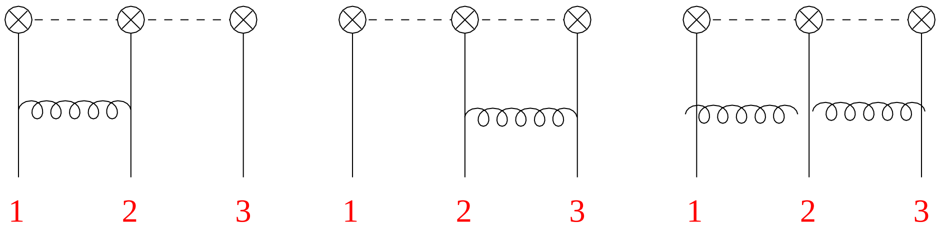
- ◆ The $SL(2)$ 'spin' generators: $L_- = -\frac{d}{dz}\,, \quad L_+ = \left(z^2\frac{d}{dz} + 2zj_q\right)\,, \quad L_0 = \left(z\frac{d}{dz} + j_q\right)$
- ◆ $j_q = 1$ is the *conformal spin* of the quark field

Integrability on the light-cone

- Callan-Symanzik equation (helicity$-\frac{3}{2}$ baryon operator $B \equiv q^{\uparrow}(z_1 n) q^{\uparrow}(z_2 n) q^{\uparrow}(z_3 n)$)

$$\mu \frac{d}{d\mu} B(z_1, z_2, z_3) = [\mathbb{H} \cdot B](z_1, z_2, z_3),$$

- One-loop dilatation operator:

$\mathbb{H}_{1-\mathrm{loop}} =$ 

- Two-particle structure: $\mathbb{H} = \frac{g_s^2 N_c}{8\pi^2} [\mathcal{H}_{12} + \mathcal{H}_{23} + \mathcal{H}_{13}]$

$$\mathcal{H}_{12} B(z_1, z_2, z_3) = \int_0^1 \frac{d\alpha}{\alpha}(1-\alpha)\Big[2B(z_1, z_2, z_3) - B(z_1 - \alpha z_{12}, z_2, z_3) - B(z_1, z_2 + \alpha z_{12}, z_3)\Big]$$

  - quark fields are displaced along the light-cone
  - one-loop dilatation operator is conformal invariant

$$[\mathbb{H}, \vec{L}_1 + \vec{L}_2 + \vec{L}_3] = 0 \Longleftrightarrow \mathcal{H}_{jk} = H(J_{jk})$$

Integrability on the light-cone (II)

- QCD anomalous dimensions are eigenvalues of the dilatation operator

$$\mathbb{H}\,\Psi_N = \gamma_N\,\Psi_N$$

- $SL(2)$ invariant form of the dilatation operator

$$\mathbb{H} = \frac{g_s^2 N_c}{8\pi^2}\left[\mathcal{H}_{12} + \mathcal{H}_{23} + \mathcal{H}_{13}\right], \qquad \mathcal{H}_{jk} = 2\Big[\psi(J_{jk}) - \psi(1)\Big]$$

Two-particle conformal spin $\mathbf{J}_{jk}^2 = J_{jk}(J_{jk}-1) \equiv (\vec{L}_j + \vec{L}_k)^2$

- 1-loop dilatation operator $\equiv$ Hamiltonian of the $SL(2,\mathbb{R})$ Heisenberg spin chain

  Braun, Derkachov, Manashov'98

  - Number of sites = number of quark operators
  - Spin operators = Generators of the $SL(2,\mathbb{R})$ 'collinear' group

- The spectrum of anomalous dimensions can be found exactly using the Bethe Ansatz *GK'95*

$$\gamma_N = -i\frac{Q'(ij_q)}{Q(ij_q)} + i\frac{Q'(-ij_q)}{Q(-ij_q)}, \qquad Q(u) = \prod_k (u - \lambda_k)$$

Baxter equation

$$(u+ij_q)^3 Q(u+ij_q) + (u-ij_q)^3 Q(u-ij_q) = (2u^3 + q_2 u + q_3) Q(u)$$

q_2, q_3- conserved charges; $j_q = 1$ conformal spin of quark

Integrable "zoo" in multi-color QCD

- ✔ Interaction between partons with the *aligned* helicities (quarks $q^{\uparrow}$, gluons $G^{\uparrow}$) is integrable

  1-loop dilatation operator $\mathbb{H} =$ Hamiltonian of a noncompact $SL(2,\mathbb{R})$ Heisenberg magnet:

  Braun, Derkachov, Manashov; Belitsky; GK'99

  - Three-quark $[q^{\uparrow}q^{\uparrow}q^{\uparrow}] \Longrightarrow$ *closed* spin $j_q = 1$ chain
  - Multi-gluon $[G^{\uparrow}G^{\uparrow}...G^{\uparrow}] \Longrightarrow$ *closed* spin $j_g = 3/2$ chain
  - Antiquark-Glue-Quark $[\bar{q}\, G^{\uparrow}...G^{\uparrow} q] \Longrightarrow$ *open inhomogeneous* spin chain

- ✔ Integrability is broken in the 'mixed' helicity sectors
  - Symmetry breaking terms generate a mass gap in the spectrum of $\gamma(N)$
  - ... but they do not affect large N asymptotics

$$\gamma(N) \sim \lambda \ln N + N^0 \times (\text{nonintegrable terms})$$

- ✔ Does the $SL(2,\mathbb{R})$ integrability hold beyond one-loop in which case the conformal symmetry *is broken*? Yes, it does! Integrability is *not* tied to conformal symmetry *Belitsky, GK, Müller'04*

- ✔ For large N, in (super-)Yang-Mills theories $\gamma(N)$ has universal asymptotics in all sectors to all loops *Belitsky, Gorsky, GK'03*

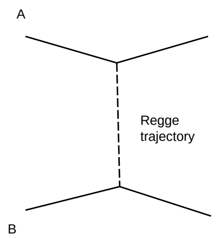
$$2 \cdot \Gamma_{\text{cusp}}(\lambda) \ln N \le \gamma(N) \le L \cdot \Gamma_{\text{cusp}}(\lambda) \ln N$$

$\Gamma_{\text{cusp}}(\lambda) =$cusp anomaly of Wilson loop; $L =$ # of 'good' fields

Scattering in QCD at high-energy (Regge asymptotics)

- Regge phenomena in strong interactions (since 60's):

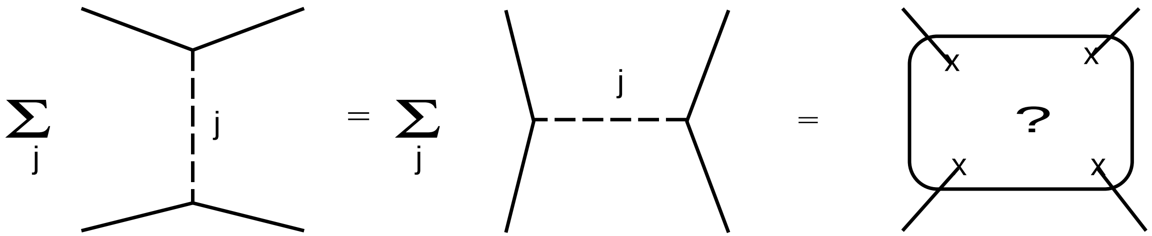
$$\sigma_{\rm AB}(s) =$$



$$= \sum_j \beta_A^j(t)\beta_B^j(t)\, s^{\alpha_j(t)-1}$$

Scattering amplitudes grow at high energy s as a power $\sim s^{\alpha_j(t)}$

- Dual model:



Regge trajectories + duality condition = Hadronic string (?)

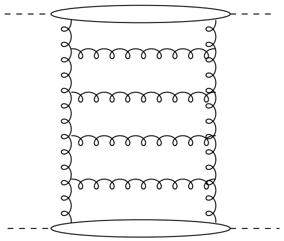
- High-energy asymptotics in QCD: interaction induces large corrections which need to be resummed to all order of perturbation theory *Balitsky-Fadin-Kuraev-Lipatov '78*:

$$\sigma_{\rm AB}(s) = \sum_{n=0,1,\dots} w_n \left(g_s^2 \ln s\right)^n \sim s^{\alpha_{\mathbb{P}}-1}$$

BFKL Pomeron + Unitarity

- Leading contribution: BFKL Pomeron ($\lambda = g_s^2 N_c/(4\pi^2)$)

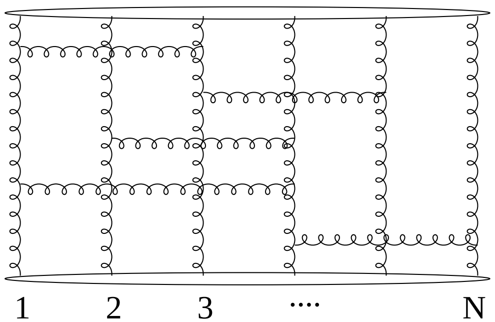
$$\sigma_{\rm LO} = \sum_{\rm rungs}$$



$$\sim \lambda^2 \frac{\exp(4\ln 2 \cdot \lambda \ln s)}{\sqrt{\lambda \ln s}} \sim \underbrace{s^{4\ln 2 \cdot \lambda}}_{\text{violates unitarity}}$$

- BFKL Pomeron + Unitarity $\Rightarrow$ generalized ladder diagrams

$$\sum_{N=2,3,..}$$



- Multi-Regge kinematics: $\int d^4k = \int dk_+ dk_- \int d^2k_\perp$

– strong ordering in the longitudinal momenta $y = \ln \frac{k_+}{k_-}$

$$y_1 \gg y_2 \gg y_3 \gg ... = \text{“evolution time” in the t} - \text{channel}$$

– “random walk” in the transverse momenta

$$k_{1,\perp} \sim k_{2,\perp} \sim k_{3,\perp} \sim ...$$

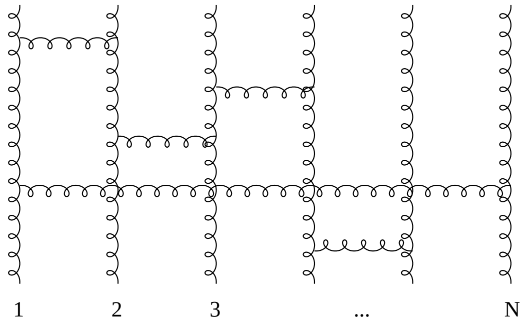
✌ Elastic pair-wise interaction of $N = 2, 3, ...$ particles “living” on the two-dimensional $k_\perp-$plane and propagating in the “time” $y = \ln s$.

✌ Nontrivial QCD dynamics occurs on the two-dimensional transverse space

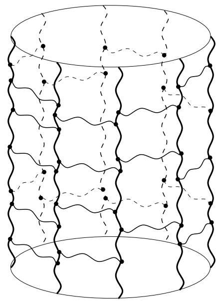
Color-singlet compound gluonic states

- The effective QCD Hamiltonian $\mathcal{H}_N$ has remarkable properties in the multi-color limit:

$\mathcal{A}(s,t) =$



=



$\sim \exp\left(y\mathcal{H}_N\right) = s^{\mathcal{H}_N}$

$\hookrightarrow$ Elastic scattering of N reggeized gluons

✔ The Bartels-Kwiecinski-Praszalowicz equation $\equiv$ 2-dim Schrödinger equation

$$\mathcal{H}_N \Psi\underbrace{(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)}_{2-\text{dim coordinates}} = E_N \Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)$$

✔ $\Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N) =$ colour-singlet compound states built from N reggeized gluons

✔ High-energy asymptotics of the scattering amplitudes is governed by the contribution of these states

$$\mathcal{A}(s,t) \sim -is \sum_{\substack{N-\text{gluon}\\ \text{states}}} (i\lambda)^N \underbrace{s^{\lambda E_N}}_{\text{Regge behaviour}} \beta_N(t)$$

✔ Intercept = maximal energy E_N

Integrability

✔ Interaction occurs only between nearest neighbours

$$\mathcal{H}_N = \sum_{k=1}^{N} \underbrace{H(\vec{z}_k, \vec{z}_{k+1})}_{\text{BFKL kernel}} + \mathcal{O}(1/N_c^2) = N-\text{body QM system with periodic boundary conditions}$$

✔ The system possesses a “hidden” set of the integrals of motion *Lipatov; Faddeev, GK’94*

$$[q_k, \mathcal{H}_N] = [q_k, q_n] = 0\,, \qquad (k, n = 2, 3, \ldots, N)$$

Their number is large enough $(= N)$ for the Schrödinger equation to be *completely integrable !*

✔ The system of N reggeized gluons $\equiv$ completely integrable quantum-mechanical model

- ✗ $\mathcal{H}_N \overset{N_c \to \infty}{=\!=} SL(2, \mathbb{C})$ Heisenberg spin chain
- ✗ Number of sites = number of reggeized gluons
- ✗ Spin operators = generators of (infinite-dimensional) irreps of the $SL(2, \mathbb{C})$ group

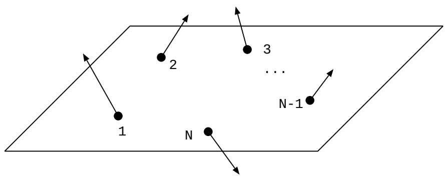
✔ The Schrödinger equation can be solved exactly by the Quantum Inverse Scattering Method

- ✌ $(-E_N) =$ The ground state energy of noncompact Heisenberg spin magnet

$$E_2 = 4 \ln 2 \quad \text{[BFKL'78]}\,, \qquad E_3 = \begin{cases} -.24717 & \text{[Janik, Wosiek'97]} \\ 0 & \text{[Bartels, Lipatov, Vacca'00]} \end{cases}, \qquad E_{N>3} = ?$$

The $SL(2,\mathbb{C})$ Heisenberg spin chain

- Quantum mechanical system of N interacting particles (reggeized gluons) on a two-dimensional $(x,y)-$plane (= transverse gluonic degrees of freedom)
- The position of the particles on the plane is defined by complex (anti)holomoprhic coordinates



$$z_k = x_k + iy_k\,, \qquad \bar{z}_k = x_k - iy_k\,, \qquad (k=1,...,N)$$

- Each particle carries the spin $S_\alpha^{(k)}$ (and similar for the antiholomorphic $\bar{S}_0$, $\bar{S}_-$ and $\bar{S}_+$)

$$S_0^{(k)} = z_k\partial_{z_k}\,, \quad S_- = -\partial_{z_k}\,, \quad S_+^{(k)} = z_k^2\partial_{z_k}\,,$$

- The spin operators associated with each particle are the $SL(2,\mathbb{C})$ generators of the small conformal transformations

$$z \longrightarrow (az+b)/(cz+d)\,, \qquad \bar{z} \longrightarrow (\bar{a}\bar{z}+\bar{b})/(\bar{c}\bar{z}+\bar{d})$$

- The effective QCD Hamiltonian, $\mathcal{H}_N$, describes the interaction between noncompact $SL(2,\mathbb{C})$ spins attached to N particles

$$\mathcal{H}_N \overset{N_c\to\infty}{=} \sum_{k=1}^{N} H(J_{k,k+1}) + H(\bar{J}_{k,k+1})$$

$J_{k,k+1}$ and $\bar{J}_{k,k+1}$ define the sum of two spins.

The $SL(2,\mathbb{C})$ Heisenberg spin chain (II)

✔ The $R-$matrix for *infinite-dim.* irreps of the $SL(2,\mathbb{C})$ (principal series) *Derkachov, GK, Manashov'01*

$$R_{12}(u,\bar{u}) = \frac{\Gamma(i\bar{u})\Gamma(1+i\bar{u})}{\Gamma(-iu)\Gamma(1-iu)} \times \frac{\Gamma(1-\bar{J}_{12}-i\bar{u})\,\Gamma(\bar{J}_{12}-i\bar{u})}{\Gamma(1-J_{12}+iu)\,\Gamma(J_{12}+iu)}\,.$$

➥ The two-particle Hamiltonian of the $SL(2,\mathbb{C})$ magnet = effective Hamiltonian in multi-color QCD!

$$H_{12} = -i\frac{d}{du}\ln R_{12}(u,u)\Big|_{u=0} = H(J_{12}) + H(\bar{J}_{12})$$

$$H(J_{12}) = \psi(J_{12}) + \psi(1-J_{12}) - 2\psi(1)\,, \quad J_{12} = \frac{1+n}{2} + i\nu$$

➥ Integrals of motion

$$q_L = \sum_{1\le j_1<\ldots<j_L\le N} i^L (z_{j_1}-z_{j_2})(z_{j_1}-z_{j_2})\ldots(z_{j_{L-1}}-z_{j_1})\partial_{z_{j_1}}\ldots\partial_{z_{j_L}}$$

✔ Can the spectrum of the QCD Hamiltonian $\mathcal{H}_N$ be found exactly?

- The $SL(2,\mathbb{C})$ magnet does not contain a pseudovacuum state (= highest weight)
- ... the "conventional" Bethe Ansatz is *not* applicable
- The way out: Baxter $\mathbb{Q}-$operator + Sklyanin's SoV approach

Derkachov, GK, Manashov'01
de Vega, Lipatov'01

Exact solution

- Solve the Baxter equation

$$(u+i)^N Q(u+i) + (u-i)^N Q(u-i) = (2u^N + q_2 u + \ldots + q_N) Q(u)$$

The same(!) as for the $SL(2,\mathbb{R})$ except that $Q(u)$ is *meromorphic* in u

- Calculate the energy from $Q(u)$

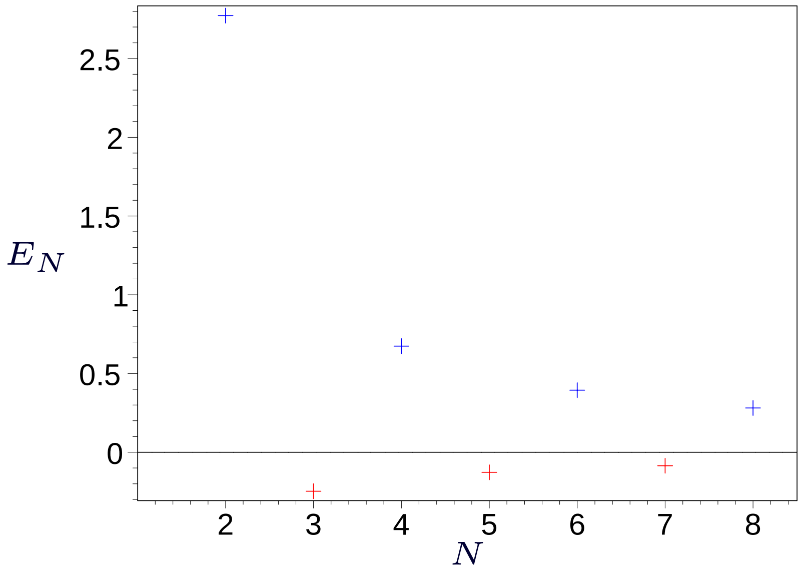
$$E_N = E_N(q_2, q_3, ..., q_N) = \text{a complicated function of the integrals of motion}$$

- The eigenvalues of the integrals of motion $q_2, ..., q_N$ are quantized
- Quantum numbers, $q_2, ..., q_N$, and the energy, E_N, of the $N-$reggeon states in multi-color QCD: *Derkachov, GK, Kotanski, Manashov' 01*

N	q_2	iq_3	q_4	iq_5	q_6	E_N
2	.25					2.77259
3	.25	.20526				-.24717
4	.25	0	.15359			.67416
5	.25	.26768	.03945	.06024		-.12751
6	.25	0	.28182	0	.07049	.39458

Spectrum

- The compound states of $N-$reggeized gluons in the Pomeron sector:

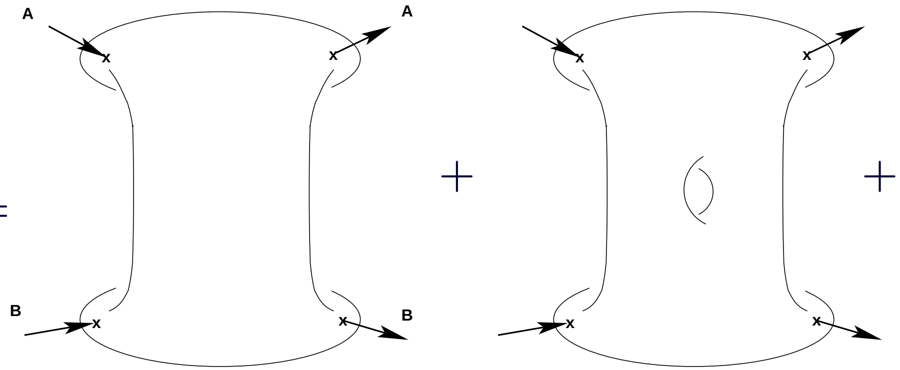


- Thermodynamical limit $E_N \to 0$ as $N \to \infty$ need to be understood...
- $\alpha_{\mathbb{P}} - 1 \sim E_N \to 0$

‘QCD String’ interpretation

- Stringy bootstrap *Veneziano ’74*

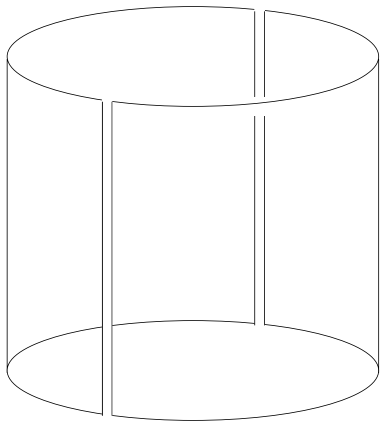
$\mathcal{A}(s,t) =$



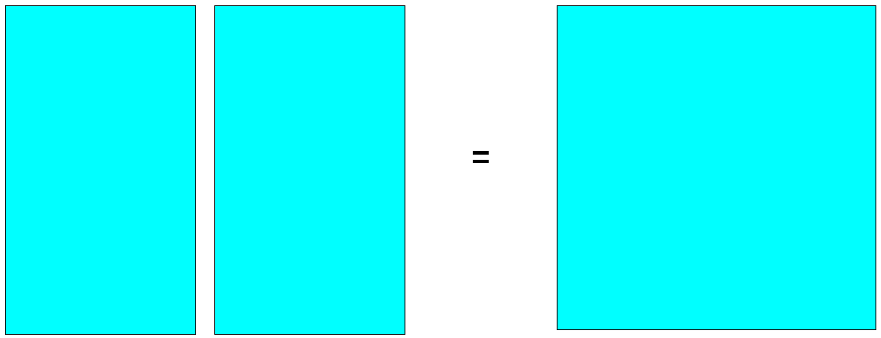
$+ \ \mathcal{O}(1/N_c^4)$

- Planar limit

Cylinder = two sheets glued together



Sheet + Sheet = Sheet



$$1/s \times (s^{\alpha_{\mathbb{R}}})^2 = s^{2\alpha_{\mathbb{R}}-1} \equiv s^{\alpha_{\mathbb{P}}}; \qquad s^{2\alpha_{\mathbb{R}}-1} = s^{\alpha_{\mathbb{R}}} \Longrightarrow \alpha_{\mathbb{R}} = 1 \Longrightarrow \alpha_{\mathbb{P}} = 1$$

Conclusions

- High-energy QCD possesses a hidden symmetry in two *different* limits:
  - The Callan-Symanzik equation on the light-cone ($\equiv$ dilatation operator) is completely integrable (large conformal spins, all loops)
  - The Schrödinger equation for partial waves of the scattering amplitudes is completely integrable in the Regge limit
- In both cases one encounters the *same* integrable structure $\equiv$ XXX Heisenberg noncompact spin chain
- The symmetry is powerful enough to calculate exactly the spectrum of anomalous dimensions/ intercepts of the compound states in multi-color QCD using the Bethe Ansatz
- *Where does integrability of high-energy QCD come from?*