# Non-Diffractive Scattering in $CFT_4/AdS$

Matthias Staudacher

MPI für Gravitationsphysik (AEI)
Potsdam

based on

- 0401057 Didina Serban, M.S.
  Planar $\mathcal{N}=4$ Gauge Theory and the Inozemtsev Long-Range Spin Chain
- 0403077 Gleb Arutyunov, M.S.
  Two-Loop Commuting Charges and the String/Gauge Duality
- 0405001 Niklas Beisert, Virginia Dippel, M.S.
  A Novel Long-Range Spin Chain and Planar $\mathcal{N}=4$ Super Yang-Mills
- 0406256 Gleb Arutyunov, Sergey Frolov, M.S.
  Bethe Ansatz for Quantum Strings
- In preparation M.S.
  Non-Diffractive Scattering in $CFT_4/AdS$

Plus earlier work with
Gleb Arutyunov, Niklas Beisert, Charlotte Kristjansen and Sergey Frolov, Joe Minahan, Jan Plefka, Arkady Tseytlin, Kostya Zarembo.

## Quantum Integrability

Folklore has it that a system is integrable if # degrees of freedom = # conserved charges.

This definition is only meaningful, and directly useful, in classical mechanics.

It becomes, as concerns its meaning and usefulness, problematic in classical field theory ($\rightarrow$ infinitely many d.o.f.'s) and in quantum mechanics ($\rightarrow$ conserved "charges" become operators).

It surely becomes even more problematic in quantum field theory.

A more practical, and extremely useful, definition of integrability applies to any system that supports scattering.
This means that the interactions between the true elementary excitations of the system are such that

- the total number of excitations is conserved.
- there are only two-body (as opposed to 3-body, ..) interactions
- Two-body scattering is non-diffractive, meaning that momenta (and possibly other quantum numbers) may get exchanged, but not changed in magnitude.

# One-Loop "Scattering" in Planar $\mathcal{N}=4$

su(2) bosonic sector

Consider single trace op's made from two complex scalars:

$$\boxed{\mathrm{Tr}\,\phi^M Z^{L-M} + \ldots}$$

How to solve the one-loop mixing problem?

Open up the trace, and replace by a state:

$$\mathrm{Tr}(\phi Z Z \phi \ldots \phi Z) \longrightarrow |\phi Z Z \phi \ldots \phi Z\rangle$$

1 2 3 4 … L-1 L → x

The mixing problem is solved by diagonalizing the dilatation operator:

Minahan & Zarembo

$$\boxed{H_2 = \sum_{x=1}^{L} \left(1 - P_{x,x+1}\right)}$$

Permutation Operator: permutes partons at site $x$ and $x+1$

Its eigenvalues $E_2$ give the one-loop anomalous dimensions $\Delta$

$$\boxed{H_2|\Psi\rangle = E_2|\Psi\rangle \quad \Rightarrow \quad \Delta = L + \frac{g_{YM}^2 N}{8\pi^2} E_2 + \mathcal{O}(g_{YM}^4)}$$

This Hamiltonian is integrable, and we will now explain in what sense. It may also be considered as a spin chain, but here we prefer a "parton" or "lattice gas" interpretation. So, let us think of the $\phi$'s as particles, and the $Z$'s as empty sites:

$$|\phi Z Z \phi \ldots \phi Z\rangle \longrightarrow$$

$\phi$ at $x=1$, $\phi$ at $x=4$, … $\phi$ at $x=L-1$

# Two-Body Scattering

Consider the two-body states

$$\boxed{|\Psi\rangle = \sum_{1\le x_1 < x_2 \le L} \Psi(x_1,x_2)\, |\ldots Z \underset{x=x_1}{\phi} Z \ldots Z \underset{x=x_2}{\phi} Z \ldots\rangle}$$

Write the Schrödinger equation in "position space":

$x_2 > x_1+1$: ($Z\phi Z \ldots Z\phi Z$)

$$E_2\,\Psi(x_1,x_2) = 2\Psi(x_1,x_2) - \Psi(x_1-1,x_2) - \Psi(x_1+1,x_2) + 2\Psi(x_1,x_2) - \Psi(x_1,x_2-1) - \Psi(x_1,x_2+1) \qquad \text{(I)}$$

$x_2 = x_1+1$: ($Z\phi\phi Z$)

$$E_2\,\Psi(x_1,x_2) = 2\Psi(x_1,x_2) - \Psi(x_1-1,x_2) - \Psi(x_1,x_2+1) \qquad \text{(II)}$$

This difference equation is solved by Bethe's ansatz (1931):

$$\boxed{\Psi(x_1,x_2) = e^{ip_1x_1 + ip_2x_2} + S(p_2,p_1)\, e^{ip_2x_1 + ip_1x_2}}$$

free 2-particle wavefunction; S-matrix; "free" wavefunction, with exchanged momenta.

Clearly (I) is solved for no matter what $S(p_2,p_1)$ iff

For the moment, $M=2$

$$\boxed{E_2 = \sum_{k=1}^{M} 4\sin^2\frac{p_k}{2}}$$

: dispersion relation

Note $2 - e^{-ip} - e^{ip} = 4\sin^2\frac{p}{2}$

For the "colliding" situation (II) $x_2 = x_1+1$ the Schrödinger equation is not automatically satisfied.

But, elementary algebra shows that (II) holds as well

iff

$$\boxed{S(p_1,p_2) = -\frac{e^{ip_1+ip_2} - 2e^{ip_1} + 1}{e^{ip_1+ip_2} - 2e^{ip_2} + 1}}$$

This is the S-matrix!

# Bethe's Equations su(2) sector

All this is true for arbitrary momenta $p_1, p_2$.

As always in QM, the spectrum gets fixed through the boundary conditions: $\Psi(x_1, x_2) = \Psi(x_2, x_1 + L)$.

This yields Bethe's equations:

$$\boxed{e^{ip_1 L} = S(p_1, p_2) \quad \text{and} \quad e^{ip_2 L} = S(p_2, p_1)}$$

Their solution often leads to complex "quasimomenta" $p_k$.

This indicates the formation of bound states.

Now we take a big leap, and immediately solve the $M$-parton problem.

If the scattering is non-diffractive, the total phasefactor aquired by a parton as it circles around should be

$$\boxed{e^{ip_k L} = \prod_{\substack{j=1 \\ j \neq k}}^{M} S(p_k, p_j) \qquad k = 1, \dots, M}$$

It "hits" all other partons exactly once!

But note: $\boxed{\sum_{k=1}^{M} p_k = 0}$ trace "cyclicity"

Morale: If it's integrable, a "back-of-the-envelope" computation will do!

Relation to the algebraic Bethe ansatz: $u_k = \frac{1}{2} \cot \frac{p_k}{2}$

$$\left( \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L = \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}$$

Bethe equations

$$\prod_{k=1}^{M} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} = 1$$ trace "cyclicity"

$$E_2 = \sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}}$$ energy (~anom. dim.)

# String Predictions for $\mathcal{N} = 4$

A Key proposal of AdS/CFT:

$$\boxed{\text{"} E = \Delta \text{"}}$$

energy of string state; dimension of conformal operator

Quantization of IIB superstrings on $AdS_5 \times S^5$ ill-understood.

Recent, dramatic progress in certain "semiclassical" limits.

These involve states with large angular momentum $J_1, J_2, J_3$ on the five-sphere $S^5$. Should correspond to operators (say $J_3 = 0$):

$$\boxed{\mathrm{Tr}\, Z^{L-M} \Phi^M} \qquad J_1 = L - M \ ; \ J_2 = M$$

Two such limits were considered:

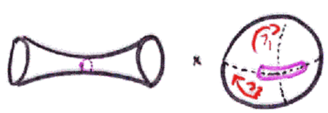
I BMN "Berenstein-Maldacena-Nastase" 0202021

$J = L - M \gg 1$ ; $M = 2, 3, \dots$ "plane-wave limit"

II FT "Frolov-Tseytlin" 0304255

$J_1 \gg 1$ ; $J_2 \gg 1$ "spinning string limit"

Semiclassical string rotations:



effective expansion parameter: $\boxed{\lambda' = \frac{\lambda}{L^2}}$

# Thermodynamic Limit and String Theory

$su(2)$ sector

1/3

Diagonalize $\mathrm{Tr}\,\phi^M Z^{L-M}$ with both $M$ and $L$ large.
Consider the "fundamental equation" = log of Bethe equation:

$$p_k L = 2\pi n_k - i \sum_{j=1}^{M}{}' \log S(p_k, p_j)$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{1}{u_k} L = 2\pi n_k + 2 \sum_{j=1}^{M}{}' \frac{1}{u_k - u_j}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

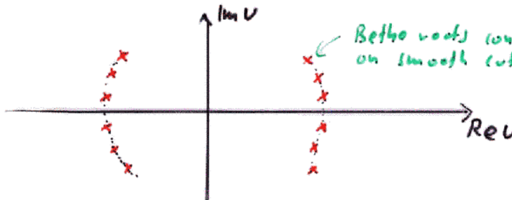
$$\boxed{\frac{1}{u} = 2\pi n_\nu + 2 \fint_C du' \rho(u') \frac{1}{u-u'}}$$

momentum, mode numbers, density of Bethe roots, scattering phase shift

Beisert, Minahan, M.S., Zarembo 0306139
see also: Sutherland (Shastry)

normalization
$$\boxed{\frac{M}{L} = \int du\, \rho(u)}$$

energy
$$\boxed{E_2 = \int du \frac{\rho(u)}{u^2}}$$




Bethe roots condense on smooth cuts

Agrees with string predictions of Frolov-Tseytlin. May be proved in generality by effective action approach of Kruczenski.
Alternative approach: Solve classical string $\sigma$-model using its classical integrability: Kazakov, Marshakov, Minahan, Zarembo 0402207

$$\boxed{\frac{M}{L} = \int dx\, \sigma(x) \left(1 - \frac{\omega^2}{x^2}\right)}$$

$$\boxed{E = \int dx \frac{\sigma(x)}{x^2}}$$

$$\omega^2 = \frac{N g^2_{YM}}{16\pi^2 L^2}$$

$$\boxed{(1 + 2\omega^2 E)\, \frac{x}{x^2 - \omega^2} = 2\pi n_\nu + 2 \fint dx' \sigma(x') \frac{1}{x - x'}}$$

- One-loop gauge-string agreement manifest ($\omega \to 0$)
- scattering interpretation obscure ...

# One-Loop "Scattering" with Fermions

$psu(1|1)$ fermionic sector

Consists of one complex scalar $Z$ and one "gaugino" $\psi$:

$$\boxed{\mathrm{Tr}\, \psi^M Z^{L-M} + \ldots}$$

Planar one-loop Hamiltonian:

$$\boxed{H_2 = \sum_{x=1}^{L} (1 - \Pi_{x,x+1})}$$

Graded permutation op (permutes partons at sites $x$ and $x+1$ with Fermi statistics.)

Well-known fact from condensed matter: XY-model !

$$\boxed{H_2 = \sum_{x=1}^{L} \left[ -\tfrac{1}{2}\left(\sigma_x^1 \sigma_{x+1}^1 + \sigma_x^2 \sigma_{x+1}^2\right) + (1 - \sigma_x^3)\right]}$$

As before, consider two-body states:

$$\boxed{|\Psi\rangle = \sum_{1 \le x_1 < x_2 \le L} \Psi(x_1, x_2)\, |\ldots Z \underset{x=x_1}{\psi} Z \ldots Z \underset{x=x_2}{\psi} Z \ldots\rangle}$$

The Schrödinger equation reads as in $su(2)$ for $x_2 > x_1 + 1$.
But the collision equation at $x_2 = x_1 + 1$ is different:

$$x_2 = x_1 + 1 \quad \ldots Z \psi\psi Z \ldots \qquad E_2 \Psi(x_1, x_2) = 4\, \Psi(x_1, x_2) - \Psi(x_1 - 1, x_2) - \Psi(x_1, x_2 + 1)$$

Making the "plane wave" Bethe ansatz works again; you find

$$\boxed{S(p_1, p_2) = -1}$$

The fermions are free !

First noticed by Callan, Heckmann, McLoughlin, Swanson 0407096
using super spin chain of Beisert, M.S. 0307042

# Bethe's Equations psu(1|1) sector

So the two-body wavefunction is just

$$\Psi(x_1,x_2)=e^{ip_1x_1+ip_2x_2}-e^{ip_2x_1+ip_1x_2}$$

Slater determinant → True also for the M-body problem!

Boundary conditions: $\Psi(x_1,x_2) = (-)\,\Psi(x_2,x_1+L)$

Fermi statistics!

This leads to the "Bethe equations"

$$\boxed{e^{ip_1L}=-S(p_1,p_2)=1 \qquad e^{ip_2L}=-S(p_2,p_1)=1}$$

Factorized scattering leads again immediately to the solution of the M-body problem:

$$\boxed{e^{ip_kL}=1 \quad k=1,\dots,M}$$

trace cyclicity: $\sum_{k=1}^{M} p_k=0$

For the Fermi-chain, these may be solved immediately:

$$\boxed{p_k=\frac{2\pi}{L}n_k}$$

So the exact one-loop anomalous dimensions of $\mathrm{Tr}\,\psi^M Z^{L-M}+\dots$ are

$$\boxed{\Delta = L+\frac{1}{2}M+\frac{g_{YM}^2N}{8\pi^2}E_2 \quad \text{with} \quad E_2=\sum_{k=1}^{M} 4\sin^2\frac{\pi n_k}{L}}$$

Because of Fermi statistics, all integers $n_k$ are distinct.

Hence there is no thermodynamic limit of the previous kind!

But there is a "near-BMN" limit.

# One-Loop "Scattering" of Derivatives

$sl(2)\simeq su(1,1)$ sector

Final one-loop example: One scalar $Z$, one light-cone derivative $D$:

$$\boxed{\mathrm{Tr}\,D^M Z^L+\dots}$$

The Hamiltonian was obtained in Beisert 0307015:

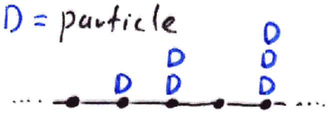
$$\boxed{H_2=\sum_{x=1}^{L}\mathcal{H}_{x,x+1}}$$

cf Beisert's talk: $\mathcal{H}_{x,x+1}=2\psi(J_{x,x+1}+1)-2\psi(1)$

Integrable spin $=-\frac{1}{2}$ $sl(2)$ spin chain Beisert, M.S. 0307042:

Lattice gas interpretation: $Z$ = hole $D$ = particle

$$|\dots Z(DZ)(D^2Z)Z(D^3Z)\dots\rangle \Longrightarrow$$



Multiple occupancy is allowed!

Two-body scattering: $\boxed{|\Psi\rangle=\sum_{1\le x_1\le x_2\le L}\Psi(x_1,x_2)\,|\dots \underset{x=x_1}{Z(DZ)Z}\dots \underset{x=x_2}{Z(DZ)Z}\rangle}$

Schrödinger equation:

$x_2>x_1$: $\boxed{Z\overset{D}{Z}Z\dots Z\overset{D}{Z}Z}$

$$E_2\,\Psi(x_1,x_2)=2\,\Psi(x_1,x_2)-\Psi(x_1-1,x_2)-\Psi(x_1+1,x_2)+2\,\Psi(x_1,x_2)-\Psi(x_1,x_2-1)-\Psi(x_1,x_2+1)$$

$x_2=x_1$: $\boxed{\dots Z\overset{D}{\overset{D}{Z}}Z\dots}$

$$E_2\,\Psi(x_1,x_2)=\frac{3}{2}\Psi(x_1,x_2)-\Psi(x_1-1,x_2)-\frac{1}{2}\Psi(x_1-1,x_2-1)+\frac{3}{2}\Psi(x_1,x_2)-\Psi(x_1,x_2+1)-\frac{1}{2}\Psi(x_1+1,x_2+1)$$

Bethe ansatz: (again)

$$\boxed{\Psi(x_1,x_2)=e^{ip_1x_1+ip_2x_2}+S(p_2,p_1)\,e^{ip_2x_1+ip_1x_2}}$$

# Bethe's Equations $sl(2)$ sector

(1) Schrödinger equation is satisfied for $x_2 > x_1$ for any $S(p_1,p_2)$ and any $p_1, p_2$ if and only if (here $M=2$):

$$\boxed{E_2 = \sum_{k=1}^{M} 4 \sin^2 \frac{p_k}{2}}$$

(2) Schrödinger equation is satisfied for $x_2 = x_1$ for any $p_1, p_2$ if and only if (elementary algebra!)

S-matrix

$$\boxed{S(p_1,p_2) = -\frac{e^{ip_1+ip_2} - 2e^{ip_2} + 1}{e^{ip_1+ip_2} - 2e^{ip_1} + 1}}$$

same as in $su(2)$ but with $p_1 \leftrightarrow p_2$ interchanged

(3) Imposition of boundary conditions yields Bethe equations:

$$\Psi(x_1,x_2) = \Psi(x_2, x_1+L) \Rightarrow e^{ip_1 L} = S(p_1,p_2)\ ;\ e^{ip_2 L} = S(p_2,p_1)$$

(4) Integrability allows to immediately go from two to an arbitrary number of particles:

$$e^{ip_k L} = \prod_{\substack{j=1\\ j\neq k}}^{M} S(p_k,p_j) \xrightarrow[u_k = \frac{1}{2}\cot\frac{p_k}{2}]{} \boxed{\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k - u_j - i}{u_k - u_j + i}}$$

- Comment: Note that we only needed a "small part" of the $sl(2)$ Hamiltonian! Important for higher loops!
- Morale (repeated): If it's integrable, a "back-of-the-envelope" computation will do ...

# Thermodynamic Limit and String Theory

$sl(2)$ sector

Diagonalize $\mathrm{Tr}\, D^M Z^L$ for both $M$ and $L$ large.

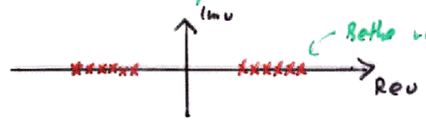
Very similar to $su(2)$, but the sign of the phase shift has flipped:

$$\boxed{\frac{1}{u} = 2\pi n_\nu - 2 \int_c du' \rho(u') \frac{1}{u-u'}}$$

momentum, mode numbers, Bethe root density, scattering phase shift

Beisert, Frolov, M.S., Tseytlin 0308117

Solutions tend to be "analytic continuations" of the $su(2)$ case:




Corresponds on the string side to semiclassical strings with one large angular momentum $J = L$ on $S^5$, and one large spin $S = M$ in $AdS_5$. (Frolov, Tseytlin)

Again, one-loop gauge-string agreement may be shown in generality, either by Kruczenski's effective action approach, or by "solving" the integrable classical $\sigma$-model Kazakov, Zarembo 0410105:

$$\boxed{\frac{x}{x^2-\omega^2} - 2\omega^2 P \frac{1}{x^2-\omega^2} = 2\pi n_\nu - 2 \int_c dx' \sigma(x') \frac{1}{x-x'}} \qquad \omega^2 = \frac{N g_{YM}^2}{16\pi^2 L^2}$$

$$\boxed{\underbrace{\frac{M}{L} = \int dx\, \sigma(x) \left(1 - \frac{\omega^2}{x^2}\right)}_{\text{normalization}} \qquad \underbrace{P = \int dx \frac{\sigma(x)}{x} = 2\pi m}_{\text{momentum}} \qquad \underbrace{E = \int dx \frac{\sigma(x)}{x^2}}_{\text{energy}}}$$

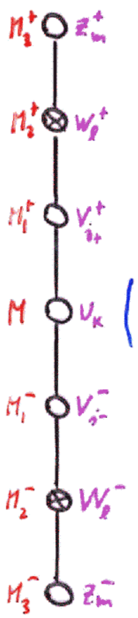
- One loop gauge-string agreement manifest ($\omega \to 0$)
- Scattering interpretation obscure ...

# The Integrable Super Spin Chain psu(2,2|4)

Beisert, M.S. 0307042

We showed that the complete $\mathcal{N}=4$ one-loop dilatation operator (derived in Beisert 0307015) is integrable.

We also proposed **Bethe equations**:




$$1=\prod_{m'=1}^{M_3^+}{}' \frac{z_m^+-z_{m'}^+-i}{z_m^+-z_{m'}^++i}\prod_{\ell=1}^{M_2^+}\frac{z_m^+-w_\ell^++\frac{i}{2}}{z_m^+-w_\ell^+-\frac{i}{2}}$$

$$1=\prod_{m=1}^{M_3^+}\frac{w_\ell^+-z_m^++\frac{i}{2}}{w_\ell^+-z_m^+-\frac{i}{2}}\prod_{j_+=1}^{M_1^+}\frac{w_\ell^+-v_{j_+}^+-\frac{i}{2}}{w_\ell^+-v_{j_+}^++\frac{i}{2}}$$

$$1=\prod_{k=1}^{M}\frac{v_{j_+}^+-u_k-\frac{i}{2}}{v_{j_+}^+-u_k+\frac{i}{2}}\prod_{j_+'=1}^{M_1^+}{}'\frac{v_{j_+}^+-v_{j_+'}^++i}{v_{j_+}^+-v_{j_+'}^+-i}\prod_{\ell=1}^{M_2^+}\frac{v_{j_+}^+-w_\ell^+-\frac{i}{2}}{v_{j_+}^+-w_\ell^++\frac{i}{2}}$$

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L=\prod_{j_-=1}^{M_1^-}\frac{u_k-v_{j_-}^--\frac{i}{2}}{u_k-v_{j_-}^-+\frac{i}{2}}\prod_{k'=1}^{M}\frac{u_k-u_{k'}+i}{u_k-u_{k'}-i}\prod_{j_+=1}^{M_1^+}\frac{u_k-v_{j_+}^+-\frac{i}{2}}{u_k-v_{j_+}^++\frac{i}{2}}$$

$$1=\prod_{k=1}^{M}\frac{v_{j_-}^--u_k-\frac{i}{2}}{v_{j_-}^--u_k+\frac{i}{2}}\prod_{j_-'=1}^{M_1^-}{}'\frac{v_{j_-}^--v_{j_-'}^-+i}{v_{j_-}^--v_{j_-'}^--i}\prod_{\ell=1}^{M_2^-}\frac{v_{j_-}^--w_\ell^--\frac{i}{2}}{v_{j_-}^--w_\ell^-+\frac{i}{2}}$$

$$1=\prod_{m=1}^{M_3^-}\frac{w_\ell^--z_m^-+\frac{i}{2}}{w_\ell^--z_m^--\frac{i}{2}}\prod_{j_-=1}^{M_1^-}\frac{w_\ell^--v_{j_-}^--\frac{i}{2}}{w_\ell^--v_{j_-}^-+\frac{i}{2}}$$

$$1=\prod_{m'=1}^{M_3^-}{}'\frac{z_m^--z_{m'}^--i}{z_m^--z_{m'}^-+i}\prod_{\ell=1}^{M_2^-}\frac{z_m^--w_\ell^-+\frac{i}{2}}{z_m^--w_\ell^--\frac{i}{2}}$$

Their solution gives anomalous dimension $\Delta=\Delta_0+g^2\sum_{k=1}^{M}4\sin^2\frac{p_k}{2}$ of **any** single trace op in $\mathcal{N}=4$ at one loop!

In principle possible to derive them using the "pedestrian" approach above: "**nested Bethe ansatz**" C.N. Yang 1967.

Possible to derive psu(1|1) and sl(2) equations directly from the above (M.S., to appear)

# Higher Loop Integrability

2/3

- Higher loop integrability was proposed in Beisert, Kristjansen, M.S., 0303060. Three-loop dilatation operator in su(2) derived, assuming integrability. Three-loop prediction for Konishi (now confirmed).
- Three-loop dilatation operator in su(2|3) (obviously includes su(2)) rigorously derived in Beisert 0310252. Spectral degeneracies consistent with integrability observed.
- Three-loop su(2) dilatation operator embedded into integrable long-range Inozemtsev spin chain in Serban, M.S. 0401057. Three-loop factorized scattering established.
- "Asymptotic" 5-loop su(2) dilatation operator proposed in Beisert, Dippel, M.S. 0405001, along with a conjecture for an all-loop (asymptotic) Bethe ansatz.

This ansatz reads

$$\boxed{e^{ip_kL}=\prod_{\substack{j=1\\ j\neq k}}^{M}S(p_k,p_j)\quad\text{with}\quad S(p_k,p_j)=\frac{\varphi(p_k)-\varphi(p_j)+i}{\varphi(p_k)-\varphi(p_j)-i}\quad\text{where}\quad\varphi(p)=\frac{1}{2}\cot\frac{p}{2}\sqrt{1+8g^2\sin^2\frac{p}{2}}}$$

The proposed anomalous dimension $\Delta$ is

$$\boxed{\Delta=L+g^2E(g)\quad\text{with}\quad E(g)=\sum_{k=1}^{M}\frac{1}{g^2}\left(\sqrt{1+8g^2\sin^2\frac{p}{2}}-1\right)}$$

It is supposed to yield correct results to L-1 loops!

## Three-Loop Bethe Ansatz for the Fermionic Sector

M.S., to appear

Does higher-loop non-diffractive scattering persist beyond su(2)?

Next simplest case: $psu(1|1) \subset su(2|3) \subset psu(2,2|4)$

su(2|3) 3-loop Hamiltonian known in "algorithmic form" Beisert 0310252

Can extract corrections to the XY model:

$$H_4 = \sum_{x=1}^{L} \Big[ 2(\sigma_x^3 - 1) - \tfrac{1}{4}(\sigma_x^3 \sigma_{x+1}^3 - 1) + \tfrac{9}{8}(\sigma_x^1 \sigma_{x+1}^1 + \sigma_x^2 \sigma_{x+1}^2) - \tfrac{1}{16}(\sigma_x^1 \sigma_{x+1}^1 + \sigma_x^2 \sigma_{x+1}^2)\sigma_{x+2}^3 - \tfrac{1}{16}\sigma_x^3(\sigma_{x+1}^1 \sigma_{x+2}^1 + \sigma_{x+1}^2 \sigma_{x+2}^2) - \tfrac{1}{8}\sigma_x^1(1+\sigma_{x+1}^3)\sigma_{x+2}^1 - \tfrac{1}{8}\sigma_x^2(1+\sigma_{x+1}^3)\sigma_{x+2}^2 \Big]$$

$H_6$ is even messier ...

Naive Bethe ansatz does not work anymore!

But, we still expect it to be true asymptotically:

$$\boxed{\Psi(x_1,x_2) \sim e^{ip_1x_1 + ip_2x_2} + S(p_2,p_1)\, e^{ip_2x_1 + ip_1x_2} \qquad \text{if} \quad x_1 \ll x_2}$$

2-body case

Indeed, consistent with BMN scaling, a plane wave $e^{ip_1x_1 + \ldots + ip_Mx_M}$ solves the Schrödinger equation with $g^2 = \frac{g_{YM}^2 N}{8\pi^2}$ iff

$$\boxed{E = \sum_{k=1}^{M} \Big[ 4 \sin^2 \tfrac{p_k}{2} - 8 g^2 \sin^4 \tfrac{p_k}{2} + 32 g^4 \sin^6 \tfrac{p_k}{2} + \mathcal{O}(g^8) \Big]}$$

3-loop lattice dispersion relation

## Three-Loop S-matrix for the Fermionic Sector

M.S., to appear

Beyond one loop, the fermions are not* free anymore,

*also noticed by Callan/Heckman/McLoughlin/Swanson 0407096

but we still expect $S(p_1,p_2)$ to be a pure phase:

$$\boxed{S(p_1,p_2) = -e^{i\theta(p_1,p_2)}}$$

Key idea: Modify fine structure of wavefunction in the vicinity of the collision area. Leave asymptotic wavefunction intact. Schrödinger equation becomes consistent, and phase may be extracted:

$$\boxed{\begin{aligned}\theta(p_1,p_2) = {} & 4 g^2 \sin\tfrac{p_1}{2} \sin\tfrac{p_1-p_2}{2} \sin\tfrac{p_2}{2} + \\ & + g^4 \sin\tfrac{p_1}{2}\Big(\sin\tfrac{p_1-3p_2}{2} - 7\sin\tfrac{p_1-p_2}{2} + \\ & \qquad + \sin\tfrac{3p_1-3p_2}{2} + \sin\tfrac{3p_1-p_2}{2}\Big)\sin\tfrac{p_2}{2} + \\ & + \mathcal{O}(g^6)\end{aligned}}$$

Complicated, but simpler than Hamiltonian!

Bethe equations:

$$\boxed{e^{ip_kL} = \prod_{\substack{j=1\\ j\neq k}}^{M} e^{i\theta(p_k,p_j)} \qquad \sum_{k=1}^{M} p_k = 0}$$

One might be tempted to term this technique "perturbative" Bethe ansatz.

# Exact Three-Loop Spectrum for $\mathrm{Tr}\, \varphi^M Z^{L-M}$

Taking a logarithm, we find the "fundamental equation"

$$\boxed{p_k L = 2\pi n_k + \sum_{j=1}^{M} \theta(p_k, p_j)} \qquad \sum_{k=1}^{M} n_k = 0$$

After linear iteration to $\mathcal{O}(g^4)$ we find:

$$E_2 = 4 \sum_{k=1}^{M} \sin^2\left(\frac{\pi n_k}{L}\right), \tag{1}$$

$$E_4 = -8 \sum_{k=1}^{M} \sin^4\left(\frac{\pi n_k}{L}\right) + \tag{2}$$
$$+ \frac{16}{L} \sum_{k,j=1}^{M} \cos\left(\frac{\pi n_k}{L}\right) \sin^2\left(\frac{\pi n_k}{L}\right) \sin\left(\frac{\pi n_j}{L}\right) \sin\left(\frac{\pi(n_k - n_j)}{L}\right),$$

$$E_6 = 32 \sum_{k=1}^{M} \sin^6\left(\frac{\pi n_k}{L}\right) + \tag{3}$$
$$- \frac{16}{L} \sum_{k,j=1}^{M} \cos\left(\frac{\pi n_k}{L}\right) \sin^2\left(\frac{\pi n_k}{L}\right) \sin\left(\frac{\pi n_j}{L}\right) \sin\left(\frac{\pi(n_k - n_j)}{L}\right) \times$$
$$\times \left(5 \sin^2\left(\frac{\pi n_k}{L}\right) + \sin^2\left(\frac{\pi n_j}{L}\right) + \sin^2\left(\frac{\pi(n_k - n_j)}{L}\right)\right) +$$
$$+ \frac{16}{L^2} \sum_{k,j,m=1}^{M} \cos\left(\frac{\pi n_k}{L}\right) \sin\left(\frac{\pi n_k}{L}\right) \sin\left(\frac{\pi n_j}{L}\right) \sin\left(\frac{\pi n_m}{L}\right) \times$$
$$\times \sin\left(\frac{\pi(n_j - n_m)}{L}\right) \left(\cos\left(\frac{2\pi n_j}{L}\right) - \cos\left(\frac{2\pi(n_k - n_j)}{L}\right)\right)$$
$$+ \frac{2}{L^2} \sum_{k,j,m=1}^{M} \sin\left(\frac{\pi n_k}{L}\right) \sin\left(\frac{\pi n_m}{L}\right) \sin\left(\frac{\pi(n_k - n_m)}{L}\right) \times$$
$$\times \left(\sin\left(\frac{2\pi n_j}{L}\right) + \sin\left(\frac{2\pi(n_j - n_k)}{L}\right) + \sin\left(\frac{2\pi(n_j + n_k)}{L}\right) - \right.$$
$$\left. - 3 \sin\left(\frac{2\pi(n_j - 2n_k)}{L}\right) - 3 \sin\left(\frac{4\pi n_k}{L}\right)\right),$$

$$\Delta = L + \tfrac{1}{2} M + g^2 E_2 + g^4 E_4 + g^6 E_6 + \mathcal{O}(g^8), \quad \text{with} \quad g^2 = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2}. \tag{4}$$

Result checked for a large number of states against direct diagonalization. The fact that an explicit spectrum may be found is one of the amazing consequences of integrability!

# Higher Loop Thermodynamics and String Theory

The "Bethe" equations obtained from the classical $\sigma$-model do not "look like" scattering equations. E.g. take the normalization condition:

$$\frac{M}{L} = \int dx\, \sigma(x) \left(1 - \frac{\omega^2}{x^2}\right)$$

A density of excitations $\rho$ should obey $\frac{M}{L} = \int \rho$ !

This suggests the change of variables $x \to \varphi$

$$d\varphi = \left(1 - \frac{\omega^2}{x^2}\right) dx \quad \Rightarrow \quad \boxed{\varphi = x + \frac{\omega^2}{x}}$$

Beisert, Dippel, M.S. 0405001

Rewrite the string equations with $x \to \varphi$: $(su(2))$

$$\Rightarrow \quad \frac{1}{\sqrt{\varphi^2 - 4\omega^2}} = 2\pi n_\nu + 2 \int_C d\varphi' \rho_{\text{string}}(\varphi') \left[\frac{1}{\varphi - \varphi'} + \underbrace{\theta'(\varphi, \varphi')}_{\text{additional scattering of } \mathcal{O}(g_{YM}^4)}\right]$$

Now compare to long range gauge Bethe ansatz:

$$\underset{\text{momentum}}{\frac{1}{\sqrt{\varphi^2 - 4\omega^2}}} = 2\pi \underset{\text{mode numbers}}{n_\nu} + 2 \int_C d\varphi' \rho_{\text{gauge}}(\varphi') \frac{1}{\varphi - \varphi'}$$

$\hookrightarrow$ phase shift $= \theta(\varphi, \varphi')$

This suggests an explanation for the infamous three-loop discrepancies: The S-matrix is coupling constant dependent and changes when you go from weak (gauge) to strong (string) coupling!

Further evidence: Local dispersion laws agree for all commuting charges in gauge and string theory.

This also explains why strict BMN works (there is no scattering) !

# Bethe Ansätze for Quantum Strings

One may rediscretize the "thermodynamic" Bethe equations on the string side Arutyunov, Frolov, M.S. 0406256:

$$\boxed{e^{i p_k L} = \prod_{j=1}^{M}{}' \frac{\varphi(p_k)-\varphi(p_j)+i}{\varphi(p_k)-\varphi(p_j)-i} \times e^{2i\sum_{r=1}^{\infty}\left(\frac{g^2}{2}\right)^{r+1}\left[q_{r+1}(p_k)\,q_{r+2}(p_j)-q_{r+2}(p_k)\,q_{r+1}(p_j)\right]}}$$

where $q_r(p)$ are the local dispersion laws for the $r$-th charge.

- reproduces near-BMN string results Callan, McLoughlin, Swanson 0405153; McLoughlin, Swanson 0407240
- reproduces the generic string prediction Gubser, Polyakov, Klebanov 9802109

$$\boxed{\Delta \simeq 2\sqrt{n}\,\lambda^{\frac{1}{4}}}$$

We can furthermore test these ideas on $sl(2)$ M.S. to appear

Here one finds

$$\boxed{e^{i p_k L} = \prod_{j=1}^{M}{}' \frac{\varphi(p_k)-\varphi(p_j)-i}{\varphi(p_k)-\varphi(p_j)+i}\; e^{-2i\sum_{r=0}^{\infty}\left(\frac{g^2}{2}\right)^{r+1}\left[q_{r+1}(p_k)\,q_{r+2}(p_j)-q_{r+2}(p_k)\,q_{r+1}(p_j)\right]}}$$

↑ intriguing...

- reproduces near-BMN and $\Delta \simeq 2\sqrt{n}\,\lambda^{\frac{1}{4}}$

Finally, for $psu(1|1)$, even though no thermodynamic string equations exist, we can "extract" the S-matrix from McLoughlin, Swanson 0407240; where $q_2(p) = \frac{1}{g^2}\left(\sqrt{1+8g^2\sin^2\frac{p}{2}}-1\right)$

$$\boxed{e^{i p_k L} = \prod_{j=1}^{M}{}' \, e^{i\frac{g^2}{2}\left[q_2(p_k)\,p_j - p_k\,q_2(p_j)\right]};}$$

M.S. to appear

- Also reproduces $\Delta \simeq 2\sqrt{n}\,\lambda^{\frac{1}{4}}$ !

# Question in Lieu of Conclusion

> Can we "bootstrap" the S-matrix* of the CFT/AdS system?

* I am talking about the "internal" S-matrix, as explained in this talk!

The S-matrix appears to be much simpler than

- the dilatation operator of $\mathcal{N}=4$ SYM
- the quantum Hamiltonian of strings on $AdS_5 \times S^5$

And yet, assuming integrability holds, it encodes the entire spectrum!

## Philosophical Question

Is there a physical reason for the integrability observed in gauge and string theory?