Are large quantum systems controllable?

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New Directions in the Quantum Control Landscape

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Noise and Controllability

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27 February 2013

Coherent Control

Control of interference of matter waves.
Control knobs are external fields.

1) Controllability:
For a given scenario does a control strategy exist that will drive the initial state to a specified target.

2) Synthesis:
Constructively finding the control field that will achieve the goal.

Direct forward design using templates.
STIRAP, Two–photon, ...

3) Optimization:
Finding the optimal control field that will achieve the goal subject to constraints.

Inversion: Optimal Control Theory (OCT)
Random search, genetic algorithms ...

Controllability

Weak: $\Psi_j \rightarrow \Psi_k$

Can we make any state to state transformation within a closed Hilbert space?

Strong: U

Can we generate any Unitary transformation within a closed Hilbert space?

The control Hamiltonian:

$$H = H_0 + \sum_{j=1}^{L} u_j(t) X_j$$

X_j are control operators

$u_j(t)$ control fields

L controls $L \ll N$

N size of Hilbert space

Weak

$$H_0 \varphi_j = \varepsilon_j \varphi_j$$

$$\Psi = \sum c_j \varphi_j$$

$$\varphi_j \longrightarrow \varphi_k$$

Strong

$$\frac{dU}{dt} = H(t)\, U$$

$$U(0)=I \longrightarrow U(T)$$

Controllability

$$H = H_0 + \sum u_j(t) X_j$$

Strong

If the commutators of H_0 and X_j generate all the operators in Hilbert space then the system is completely controllable

$$[H_0, X_j] , \ldots , [[H_0,X_j],X_k] , \ldots , [[[H_0,X_j],X_k],X_l]$$

Clark and Tarn, J. Math. phys. 24 2608 (1983)
Ramakrishna and Rabitz, J. Math. phys. 54 1715 (1996)

Any Unitary transformation can be generated.

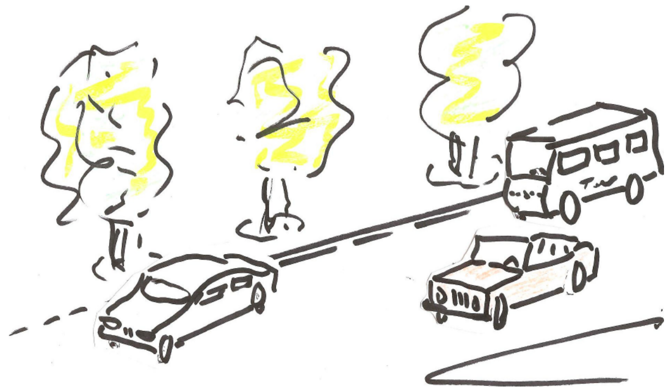
Good for quantum computers

Controllability

$$H = H_0 + \sum u_j(t) X_j$$

Parking a car:

Generating lateral motion by a series of forward–backward maneuvers



For the harmonic oscillator: $$H_0 = \ 1/2(P^2 + X^2)$$

Our control part $$H_c = u_x(t)X + u_p(t)P$$

If $\Psi(0)$ is the ground state we can generate any coherent state.

But we cannot generate a cat state!

If we change H_0 to a Morse oscillator the same controls are sufficient.

We can generate a cat state!

Controllability

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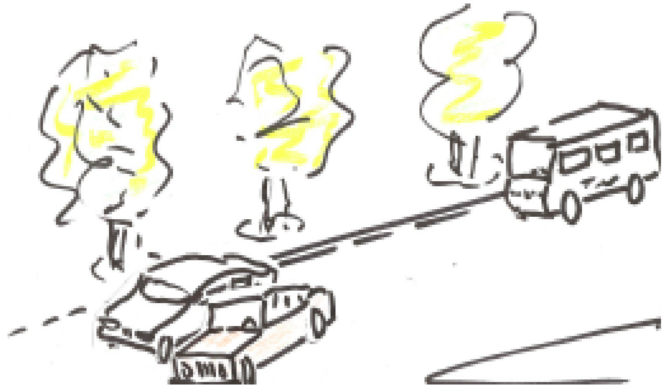

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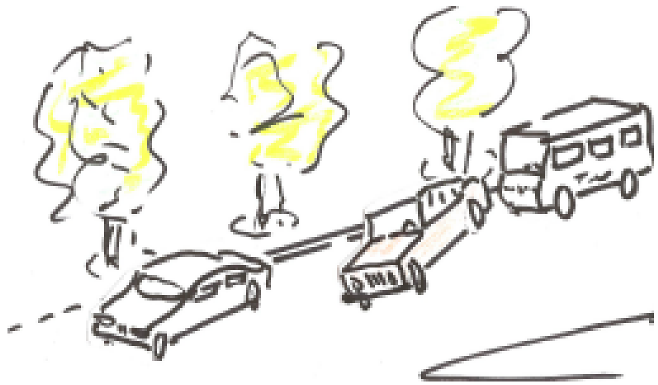

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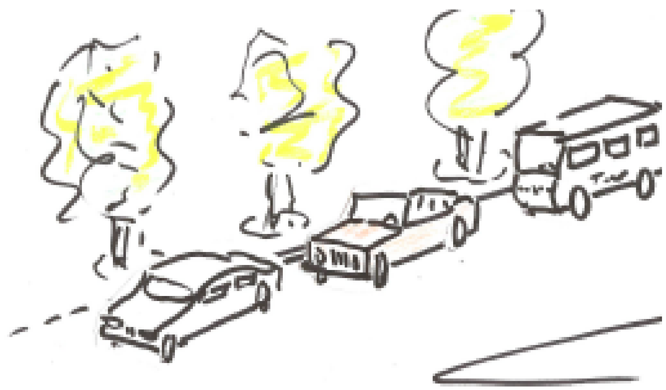

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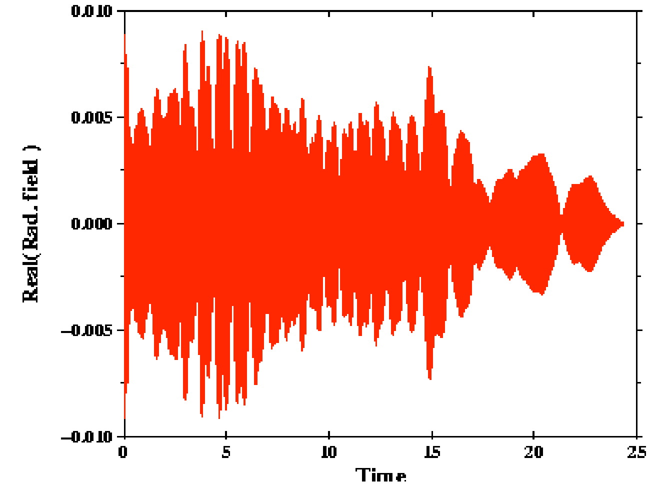
Optimal control solutions

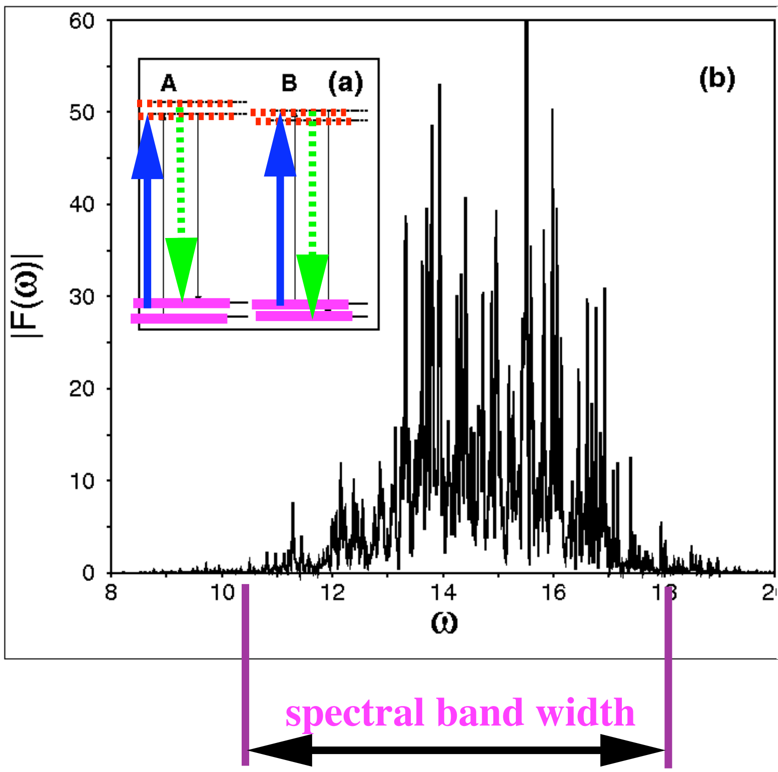
Vibrational Cooling pulse



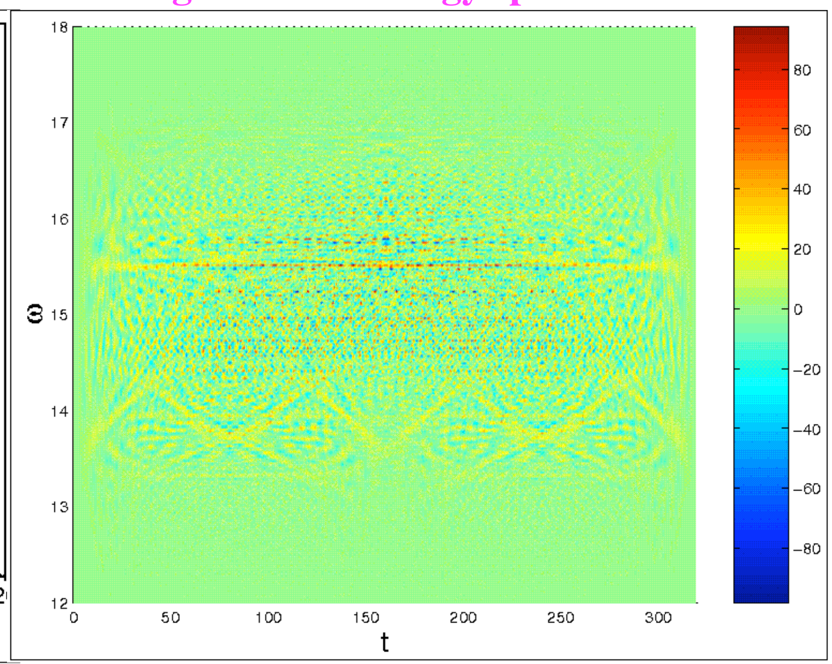
Can these pulses be executed without error?

The Fourier transform:

The optimal field $\varepsilon(t)$



Wigner time–energy spectrum



José P. Palao and Ronnie Kosloff
Quantum Computing by an Optimal Control Algorithm for Unitary Transformations
Phys. Rev. Lett. 89, 188301 (2002).

The control problem (With noise on the controls)

State to state control $|\psi_i\rangle \to |\psi_f\rangle$ at time T.
The control Hamiltonian:

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_0 + \sum_k [u_k(t) + \xi_k(t)]\hat{\mathbf{X}}_k.$$

$u_k(t)$ are control fields. $\hat{\mathbf{X}}_k$ are control operators
The unavoidable noise $\xi_k(t)$ is modelled by Gaussian noise:
$\langle \xi_k(t)\xi_l(t')\rangle = 2\Gamma_k(t)\delta_{kl}\delta(t-t')$.
where $\Gamma_k(t)$ depend on the control field $\Gamma_k(t) = f(u_k(t))$.

The equation of motion for this noisy system (Gorini-Kossakowski):

$$\frac{\partial}{\partial t}\hat{\rho} \;=\; -i\left[\hat{\mathbf{H}}_0 + \sum_k u_k(t)\hat{\mathbf{X}}_k, \hat{\rho}\right] - \sum_k \Gamma_k(t)\left[\hat{\mathbf{X}}_k, \left[\hat{\mathbf{X}}_k, \hat{\rho}\right]\right].$$

weak quantum measurement

Purity and fidelity

In the absence of noise the system is completely controllable.
Due to the noise the purity

$$\boxed{\mathscr{P} \equiv \mathtt{Tr}\left\{\hat{\rho}^2\right\}}$$

of an initially pure state $\hat{\rho} = |\psi\rangle\langle\psi|$ will decrease.
For a noisy control we define complete controllability when the purity loss during the target transformation is small, i.e., $\Delta\mathscr{P} \ll 1$.
This purity loss can be accounted by the average fidelity:

$$\boxed{\mathscr{F} = Tr\left\{\rho_f\,|\psi_f\rangle\langle\psi_f|\right\},}$$

where ψ_f is the target final state, and ρ_f is the mixed final state attained using noisy controls. For high fidelity, i.e., $1-\mathscr{F} \ll 1$,:

$$\boxed{\mathscr{F} \leq \frac{1}{2}\left(2 - \Delta\mathscr{P}\right).}$$

If purity loss is large, the state-to-state objective is lost, $\rightarrow$,
complete controllability is not true any more.

Purity loss and uncetianty

For a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, the instantaneous rate of purity loss becomes (Viola): Boxio, Viola and Ortiz,EPL 79 40003 (2007).

$$\boxed{\dot{\mathscr{P}} \equiv -\frac{d}{dt}\mathtt{Tr}\left\{\hat{\rho}^2\right\}|_{\hat{\rho}=|\psi\rangle\langle\psi|} = 4\sum_{k,u_k\neq 0}\Gamma_k(t)\Delta_{\hat{\mathbf{X}}_k}[\psi],}$$

where $\Delta_{\hat{\mathbf{X}}_k}[\psi]$ is the variance of the control operator $\hat{\mathbf{X}}_k$ in the state ψ:

$$\boxed{\Delta_{\hat{\mathbf{X}}_k}[\psi] \equiv \left\langle\psi\middle|\hat{\mathbf{X}}_k^2\middle|\psi\right\rangle - \left\langle\psi\middle|\hat{\mathbf{X}}_k\middle|\psi\right\rangle^2}.$$

The variance of a generic state scales as $\Delta_{\hat{\mathbf{X}}_k}[\psi] \sim N^2$ where N is the size of Hilbert space.

In contrast the purity loss of generalised coherent states (GCS) scale as $\Delta_{\hat{\mathbf{X}}_k}[\psi] \sim N$.

Bounds on minimum control time

Metric of change

A measure of the change $|\psi_i\rangle$ to $|\psi_f\rangle$
Using the drift Hamiltonian $\hat{\mathbf{H}}_0$ basis set $|n\rangle$,
the transformation from the initial state

$$\boxed{|\psi_i\rangle = \sum_n r_{i,n} e^{i\phi_{i,n}} |n\rangle \rightarrow, \ |\psi_f\rangle = \sum_n r_{f,n} e^{i\phi_{f,n}} |n\rangle,}$$

is characterised by the Eculidean norm $\|\Delta \mathbf{r}\|$:

$$1 \gg \|\Delta \mathbf{r}\| \geq \varepsilon > 0,$$

where $\Delta \mathbf{r} \equiv \mathbf{r}_f - \mathbf{r}_i$, $\mathbf{r}_i = (r_{i,1}, r_{i,2}, \ldots)$ and $\mathbf{r}_f = (r_{f,1}, r_{f,2}, \ldots)$. The choice of norm excludes changes to states that can be reached by free propagation generated by the drift Hamiltonian $\hat{\mathbf{H}}_0$.

For estimation of this bound an auxiliary operator $\hat{\mathbf{A}}$ is defined such that: (i) it commutes with $\hat{\mathbf{H}}_0$; (ii) its expectation value changes during the transformation. Since $\hat{\mathbf{A}}$ commutes with $\hat{\mathbf{H}}_0$ the change of its expectation value during the transformation is due to the operation of the control fields. We define

$$\hat{\mathbf{A}} = \sum_n s_n \left| n \right\rangle \left\langle n \right|,$$

where $s_n = \mathtt{sign}\left\{\Delta r_n\right\}$. The change of the expectation value of the operator $\hat{\mathbf{A}}$ $\psi_i \rightarrow \psi_f$ is given by

$$\left\langle \hat{\mathbf{A}} \right\rangle_f - \left\langle \hat{\mathbf{A}} \right\rangle_i \;=\; \sum_n \left| \Delta r_n \right| \left(r_{i,n} + r_{f,n} \right) \geq \sum_n \Delta r_n^2 = \left\| \Delta \mathbf{r} \right\|^2,$$

we obtain

$$\left\langle \hat{\mathbf{A}} \right\rangle_f - \left\langle \hat{\mathbf{A}} \right\rangle_i \geq \varepsilon^2,$$

which gives the minimal change of the expectation value of the operator $\hat{\mathbf{A}}$ during the transformation $\psi_i \to \psi_f$.
The change of the expectation value of $\hat{\mathbf{A}}$ can be estimated from the Heisenberg equations:

$$\frac{d}{dt}\hat{\mathbf{A}} = i \sum_k u_k(t) \left[\hat{\mathbf{X}}_k, \hat{\mathbf{A}} \right]$$

Let the time of the transformation be T. Then,

$$\left\langle \hat{\mathbf{A}} \right\rangle_f - \left\langle \hat{\mathbf{A}} \right\rangle_i = \int_0^T \frac{d}{dt} \left\langle \hat{\mathbf{A}} \right\rangle \, dt \leq \sum_k \int_0^T |u_k(t)| \, dt \max_{0 \leq t \leq T} \left| \left\langle \left[\hat{\mathbf{X}}_k, \hat{\mathbf{A}} \right] \right\rangle \right|$$

$$\leq 2 \sum_k \int_0^T |u_k(t)| \, dt \, |\Lambda_k| \,,$$

Bounds on minimum control time

where $\Lambda_k \sim N$ stands for the eigenvalue of $\hat{\mathbf{X}}_k$, maximal by the absolute value.

Defining the average control amplitude $\bar{u}_k \equiv \frac{1}{T}\int_0^T |u_k(t)|\, dt$, the inequality:

$$\boxed{T \geq \frac{\varepsilon^2}{(2\sum_k \bar{u}_k |\Lambda_k|)} \sim \frac{\varepsilon^2}{(2N\sum_k \bar{u}_k)},}$$

which bounds the time of the transformation for given $\bar{u}_k$ (Rabitz Calarco).

Bounds on purity loss

The bounds on purity loss are obtained under assumption that the purity loss $\Delta\mathscr{P}$ during the transformation is small. In this case the evolving state can be approximated by $\rho(t) = \rho^{(0)} + \rho^{(1)} \approx \rho^{(0)} = |\psi(t)\rangle\langle\psi(t)|$. Taking the leading contribution of $\rho^{(1)}$ into account, we estimate the lower bound on the purity loss:

$$
\begin{aligned}
\Delta\mathscr{P} \;\; &\geq \;\; 4T \sum_k \overline{\Gamma_k} \min_{0\leq t\leq T} \{ \Delta_{\hat{\mathbf{X}}_k} [\psi(t)] \\
&+ \;\; \frac{1}{2} T \left\langle \psi(t) \middle| [\hat{\mathbf{X}}_k, [\hat{\mathbf{X}}_k, \rho^{(1)}(t)]] \middle| \psi(t) \right\rangle \}.
\end{aligned}
$$

where $\overline{\Gamma}_k \equiv T^{-1} \int_0^T \Gamma_k(t) dt$ is the average dephasing rate over the transformation. We further assume that during the transformation the system follows generic states so that $\Delta_{\hat{\mathbf{X}}_k} [\psi(t)] \sim (\Lambda_k)^2 \sim N^2$.

Using the above inequality,

$$\Delta\mathscr{P} \geq \frac{2\varepsilon^2 \sum_k \bar{\Gamma}_k \min_{0\leq t\leq T}\left\{\Delta_{\hat{\mathbf{X}}_k}[\psi(t)]\right\}}{\sum_k \bar{u}_k |\Lambda_k|}.$$

To estimate $\min_{0\leq t\leq T}\left\{\Delta_{\hat{\mathbf{X}}_k}[\psi(t)]\right\}$ we find the lower bound on the variance of $\hat{\mathbf{X}}_k$ in the states $|\psi\rangle = \sum_n r_n e^{i\phi_n}|n\rangle$ such that $\|\mathbf{r}-\mathbf{r}_i\| \leq \varepsilon$. The variance $\Delta_{\hat{\mathbf{X}}_k}[\psi]$ is a function of the amplitudes $\mathbf{r} = (r_1, r_2, \ldots)$ and the phases $\phi_1, \phi_2, \ldots$. The free evolution can change the phases at no cost in purity. Therefore, the minimal variance attainable for given amplitudes is sought:

$$\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r}) \equiv \min_{\phi_1,\phi_2,\ldots}\left\{\Delta_{\hat{\mathbf{X}}_k}[\psi]\right\}$$

We assume, that $\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r})$ is a smooth function of $\mathbf{r}$ for $\|\mathbf{r}-\mathbf{r}_i\| \leq \|\Delta\mathbf{r}\|$, i.e., for sufficiently small $\Delta\mathbf{r}$ and $|\delta\mathbf{r}| \leq \|\Delta\mathbf{r}\|$ we can expand $\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r}_i+\delta\mathbf{r}) \approx \tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r}_i) + \nabla\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r}_i)\cdot\delta\mathbf{r}$.

$$\left|\delta\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r})\right| \equiv \left|\nabla\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r})\cdot\delta\mathbf{r}\right| \leq \left\|\nabla\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r})\right\| \|\delta\mathbf{r}\|.$$

The minimum is obtained at $\phi_1^*(\mathbf{r}), \phi_2^*(\mathbf{r}), \ldots$. Then $\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r}) = \Delta_{\hat{\mathbf{X}}_k}[\psi^*]$ and

$$\nabla\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r}) \;\; = \;\; \nabla\Delta_{\hat{\mathbf{X}}_k}[\psi^*]$$

It should be noted that $\Delta_{\hat{\mathbf{X}}_k}[\psi^*]$ depends on r_n both through the amplitudes of ψ^* and through the phases $\phi_n^*(\mathbf{r})$, which are also functions of $\mathbf{r}$. Nonetheless, since $\phi_n^*(\mathbf{r})$ are defined as giving the minimum of $\Delta_{\hat{\mathbf{X}}_k}[\psi]$, derivatives of $\Delta_{\hat{\mathbf{X}}_k}[\psi^*]$ with respect to $\phi_n^*(\mathbf{r})$ vanish and $\phi_n^*(\mathbf{r})$ may be considered as $\mathbf{r}$-independent for the operator ∇ in the rhs.

$$\nabla\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r}) = \nabla\left\langle\psi^*\left|\hat{\mathbf{X}}_k^2\right|\psi^*\right\rangle - 2\left\langle\psi^*\left|\hat{\mathbf{X}}_k\right|\psi^*\right\rangle\nabla\left\langle\psi^*\left|\hat{\mathbf{X}}_k\right|\psi^*\right\rangle. \tag{2}$$

Using the explicit form of ψ^* and the fact that ∇ act only on the state's amplitudes we can show that the Euclidean norm of the rhs of Eq.(2) is bounded by $3\sqrt{2}(\Lambda_k)^2$. Then, Eqs. (2) and (2) imply $\left|\delta\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r})\right| \leq 3\sqrt{2}(\Lambda_k)^2\|\delta\mathbf{r}\|$. It follows that for $|\psi\rangle = \sum_n r_n e^{i\phi_n}|n\rangle$ with $\|\mathbf{r}-\mathbf{r}_i\| \leq \varepsilon \ll 1$

$$\Delta_{\hat{\mathbf{X}}_k}[\psi] \geq \tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r}_i) - 3\sqrt{2}(\Lambda_k)^2\varepsilon \tag{3}$$

From the inequalities we obtain

$$\boxed{\Delta\mathscr{P} \geq 2\varepsilon^2 \frac{\sum_k \overline{\Gamma}_k}{\sum_k \bar{u}_k} \quad \times \quad \left(\frac{\min_l \left\{ \tilde{\Delta}_{\hat{\mathbf{X}}_l}(\mathbf{r}_i) - 3\sqrt{2}(\Lambda_l)^2 \varepsilon \right\}}{\max_l \{|\Lambda_l|\}} \right)}$$

The variance $\Delta_{\hat{\mathbf{X}}_k}[\psi]$ scales as N^2 in a generic state of the system . The scaling of $\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r}_i)$ is a more subtle question, since it is the outcome of the minimization with respect to the phases in the eigenstates basis.

For an arbitrary state-to-state objective it is sufficient to show that $\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r}_i) \sim N^2$ for *some* $|\psi_i\rangle$. Let's consider a generic eigenstate $|\phi\rangle$ of $\hat{\mathbf{H}}_0$. The variance $\Delta_{\hat{\mathbf{X}}_k}[\phi]$ scales as N^2 for all k. Moreover, the variance is independent of phases. Therefore, taking $|\psi_i\rangle = |\phi\rangle$ we shall have $\tilde{\Delta}_{\hat{\mathbf{X}}_k}(\mathbf{r}_i) \sim N^2$, and, to the leading order in ε,

$$\left(\frac{\min_l \left\{ \tilde{\Delta}_{\hat{\mathbf{X}}_l}(\mathbf{r}_i) - 3\sqrt{2}(\Lambda_l)^2 \varepsilon \right\}}{\max_l \{|\Lambda_l|\}} \right) = cN^{-1},$$

where the number c is of the order of unity. We conjecture that approximation (4) holds for a generic state-to-state transformation, not necessarily from an eigensatate of the Hamiltonian. The reason is that generically the total uncertainty of a state evolving under the free evolution will remain $\sim N^2$.

Bounds on purity loss

$$\boxed{\frac{\sum_k \bar{\Gamma}_k}{\sum_k \bar{u}_k} \leq \frac{\Delta \mathscr{P}}{2c\varepsilon^2 N}.}$$

This inequality holds for $\Delta\mathscr{P}, \varepsilon \ll 1$; the number c is of the order of unity.

This result, obtained in less general form in M. Khasin and RK, PRL. 106 123002 (2011), relates the relative noise strength on the controls with the size of the system for a high-fidelity transformation.

The main result can be stated as follows. For systems and controls defined by the control Hamiltonian and for a generic state-to-state transformation such that the expectation value of the operator $\hat{\mathbf{A}}$ changes by $\varepsilon^2 \ll 1$, the purity loss associated with the noise on the controls will be small, $\Delta\mathscr{P} \ll 1$, only if the noise complies with the condition. This condition determines the upper bound on the noise strength. For a generic transformation, where the total uncertainty of the evolving state $\sim N^2$, the number c is of the order of unity. For fixed change ε^2 and purity loss $\Delta\mathscr{P}$ the upper bound on the noise strength for a generic transformation will decrease as N^{-1}. For large N the relative noise must decrease indefinitely with the size of the system in order to provide high fidelity.

Noise Model

Typical noise includes a static part and a dynamical part:

$$\boxed{\Gamma_k(t) = \Gamma_k + c_k u_k(t)^2.}$$

The model reflects the following properties of noise:
(i) for weak field, the dephasing rate Γ_k is independent on the amplitude of field.
(i) for large amplitude of the control field the noise $\xi_k(t)$ in becomes proportional to the amplitude, $\xi_k(t) \sim u_k(t)$.

$$\Gamma_k(t) \geq 2|u_k(t)|\sqrt{\Gamma_k c_k}.$$

The necessary condition for state to state controllability:

$$\boxed{\min_k \sqrt{\Gamma_k c_k} \leq \frac{\Delta \mathscr{P}}{2c\varepsilon^2 N},}$$

Noise and control

For (approximate) state to state controllability the purity loss should be managed to a minimum close to zero for every initial and final state. $\varphi_j \longrightarrow \varphi_k$

For a large quantum system there is a class of states for which the purity loss is unmanageable.
As a result large quantum systems are uncontrollable!

These uncontrollable states are characterized by a purity loss that scales with the size of the system.

Michael Khasin and Ronnie Kosloff
Noise and controllability: Suppression of controllability in large quantum systems
Phys. Rev. Lett. 106 123002 (2011).

Noise and control

Can controllability be maintained with noise even approximately?

$$H = H_0 + \sum_{j=1}^{L} (u_j(t) + \xi_j(t)) X_j$$

where ξ represent a delta correlated noise

A Markovian Model of noise associated with the control

$$\langle \xi_j(t) \xi_j(t') \rangle = 2\eta_j \, |u_j(t)| \, \delta(t-t')$$

Then the Master equation becomes:

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum \eta_j \, |u_j(t)| \, [X_j,[X_j,\rho]]$$

The same Master equation is obtained for a system subject to a continuous measurement of the observables associated with X_j.

Any control field has to involve noise!

Generalized coherent states $\boxed{\text{GCS}}$ (pointer states)

Looking for the states with minimum uncertainty with respect to the generators of the noise algebra:

$$\boxed{\Delta(\Psi) = \sum \langle \Delta X_j^2 \rangle = \sum \left(\langle X_j^2 \rangle - \langle X_j \rangle^2 \right)}$$

These states will be weakly invariant to the master noise equation

Example **for pure dephasing**

$$\frac{d\rho}{dt} = -i[H,\rho] + \eta\,[H,[H,\rho]]$$

The eigenstates of H are GCS

Any superposition of GCS will collapse to a mixture of GCS

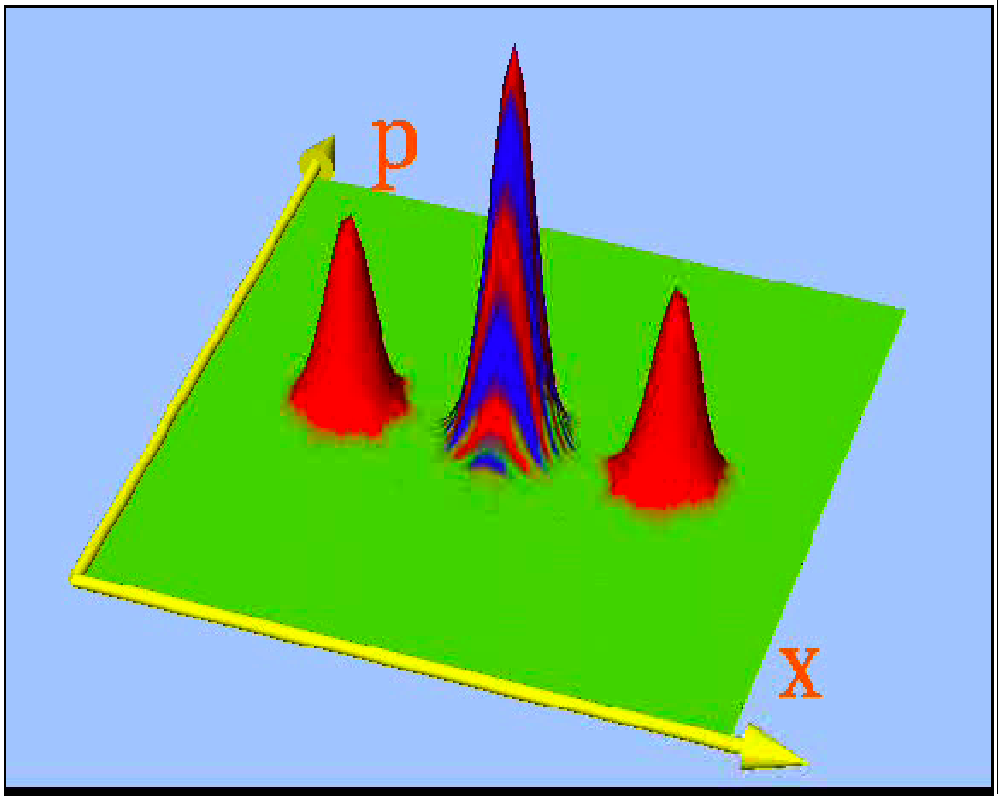
$$\boxed{\rho(0) = |\Psi\rangle\langle\Psi|} \Rightarrow \boxed{\rho(\infty) = \Sigma d_k |\varphi_k\rangle\langle\varphi_k|}$$

$$|\Psi\rangle = \Sigma c_k |\varphi_k\rangle$$

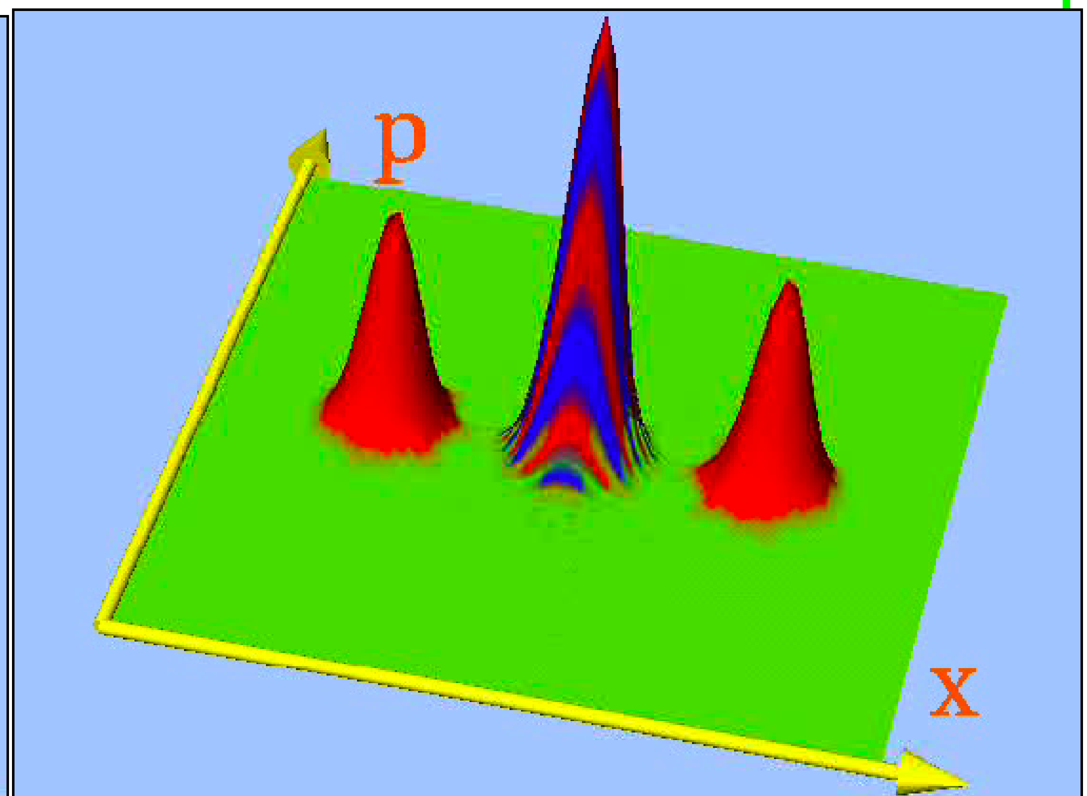
Generalized coherent states GCS

Collapse of a Cat state due to position or momentum diffusion

$$\frac{d\rho}{dt} = -i[H,\rho] + \eta\,[X,[X,\rho]] + \eta\,[P,[P,\rho]]$$



Free evolution

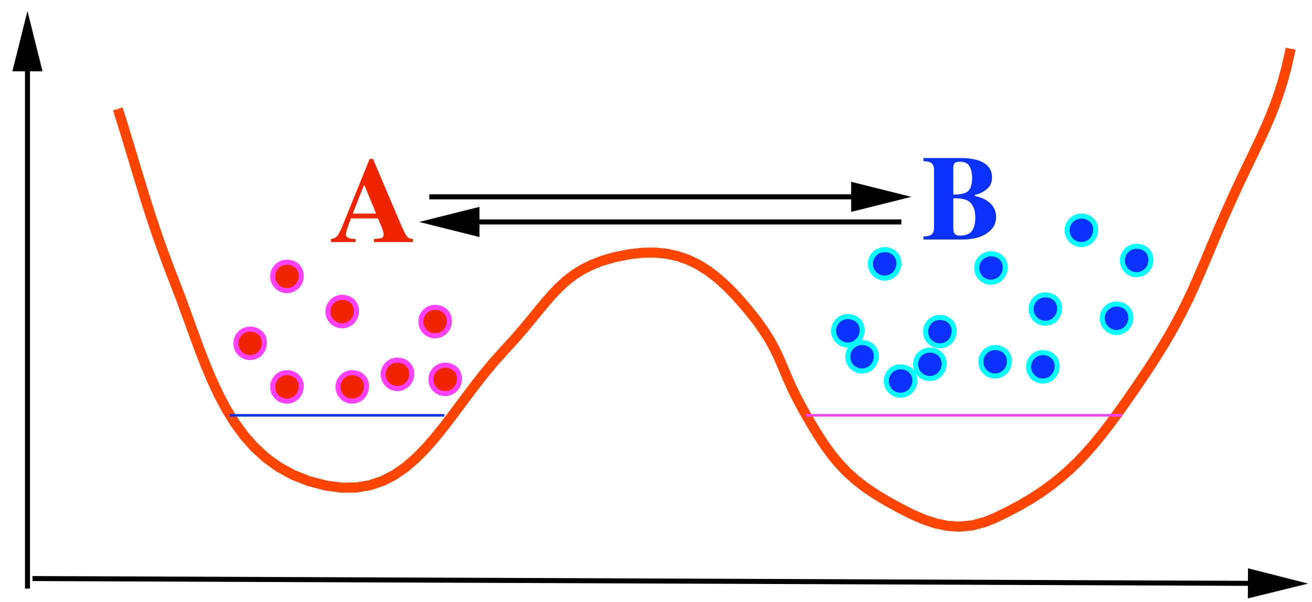


Noisy Evolution

localization in coherent states

Scaling of Optimal control soultions with system size

Example: **Tunneling Hamiltonian**



$$H = \omega_a N_a + \omega_b N_b + \Delta(a^{\dagger} b + b^{\dagger} a) + U(N_a^2 + N_b^2)$$

single particle tunneling term

inter−particle interaction

Obtaining the many body control Hamiltonian

We define

$$J_x = \frac{1}{2}\left(a^\dagger b + b^\dagger a\right)$$
$$J_y = \frac{1}{2i}\left(a^\dagger b - b^\dagger a\right)$$
$$J_z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right)$$

What is the # of states?

and the total number of particles is conserved $\boxed{N = N_a + N_b}$

Then:

$$\boxed{H = \frac{U}{N} J_z^2 - \sum \omega_\kappa(t) J_k}$$

The # of states
= size of Hilbert space
D=N+1

is the effective many body non linear control Hamiltonian

This system is completely controllable

The dynamics:

Competition between localization and dispersion.

Kahsin & Kosloff, PRA 81 043635 (2010).

$$\mathbf{H} = -\sum \omega_k(t) \mathbf{J}_k + \frac{U}{N} \mathbf{J}_z^2$$

The Heisenberg equation of motion:

$$\dot{\mathbf{X}} = i[\mathbf{H}, \mathbf{X}] - \eta \sum_{i=1}^{3} [\mathbf{J}_i, [\mathbf{J}_i, \mathbf{X}]]$$

The eigenvalue of the linear part: $\mathbf{Y}(t) = \exp((-i\,\omega - c\gamma)\tau)$

Therefore when $\eta\, c << \omega$ the dynamics of $\mathbf{J}_i$ is not affected

We have a competition between localization caused by the dissipator and dispersion on all states caused by the non linear term $\mathbf{J}_z^2$

What is the # of states?

We define

$$J_x = \frac{1}{2}\left(a^\dagger b + b^\dagger a\right)$$

$$J_y = \frac{1}{2i}\left(a^\dagger b - b^\dagger a\right)$$

$$J_z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right)$$

and the total number of particles is conserved $\boxed{N = N_a + N_b}$

Then:

$$\boxed{H = -\omega J_x + \frac{U}{N} J_z^2}$$

The # of states = size of Hilbert space

$D = N + 1$

is the effective many body non linear Hamiltonain

Generalized Coherent states (GCS) for SU(2)

Looking for the states with minimum uncertainty with respect to the operators of the algebra: $\Delta[\Psi] = \langle \Delta J_x^2 \rangle + \langle \Delta J_y^2 \rangle + \langle \Delta J_z^2 \rangle$

$$= \langle J_x^2 + J_y^2 + J_z^2 \rangle - (\langle J_x \rangle^2 + \langle J_y \rangle^2 + \langle J_z \rangle^2)$$

Generalized purity: $\boxed{P(\psi) = (\langle J_x \rangle_\psi^2 + \langle J_y \rangle_\psi^2 + \langle J_z \rangle_\psi^2)}$

Casimir $C = J_x^2 + J_y^2 + J_z^2 \quad \langle C \rangle = j(j+1)$

Maximum purity = Minimum uncertainty

All the extreme states are GCS such as:

$$|j\rangle_z \ , \ |-j\rangle_z \ , \ |j\rangle_x, \ |-j\rangle_x, \ldots$$

OCT

Experimenting with Optimal control theory

Convergence properties as a function of the Hilbert space size

Employing Krotov's method.

targets of state to state Control:

1) GCS → GCS $|j\rangle_z \rightarrow |j\rangle_x$ unrestricted.

2) GCS → GCS $|j\rangle_z \rightarrow |j\rangle_x$ guided.

3) GCS → cat state $|j\rangle_z \rightarrow (1/\sqrt{2})(|j\rangle_x + |-j\ \rangle_x$

4) cat state → GCS $(1/\sqrt{2})(|j\rangle_x + |-j\ \rangle_x\,) \rightarrow |j\rangle_z$

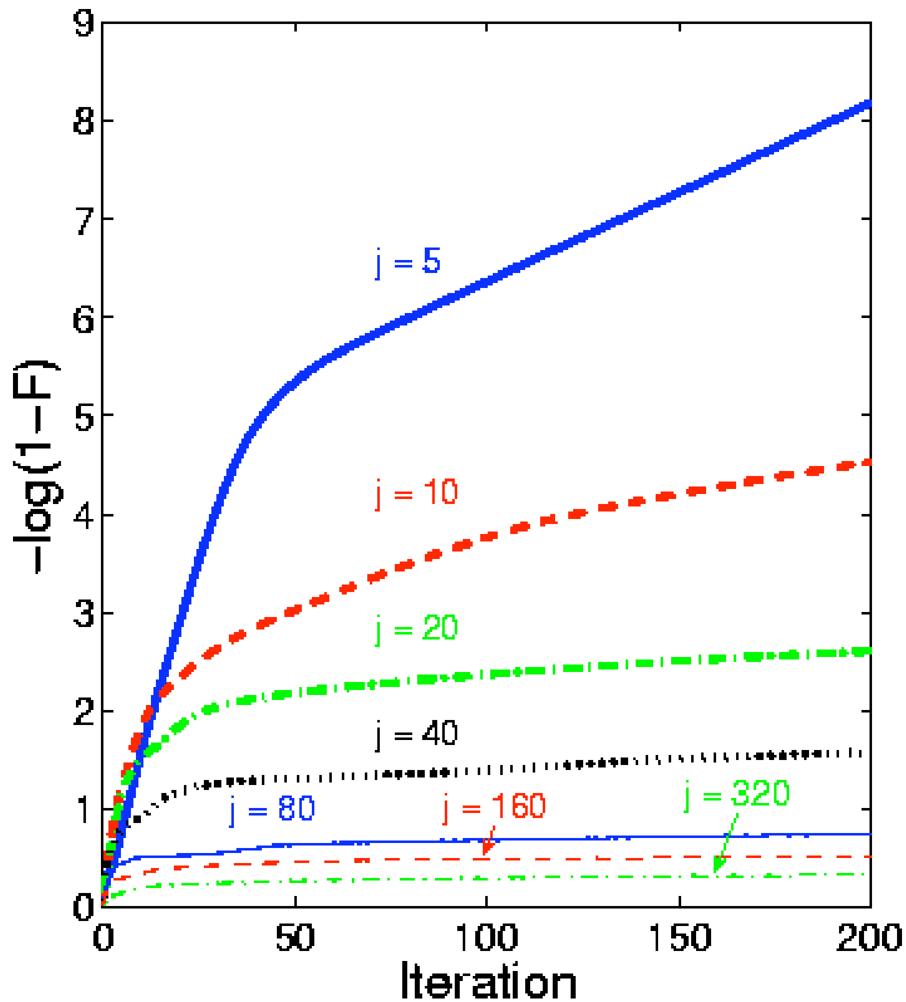
Is there a relation between the optimal field for different j ?

Convergence of OCT as a function of system size j

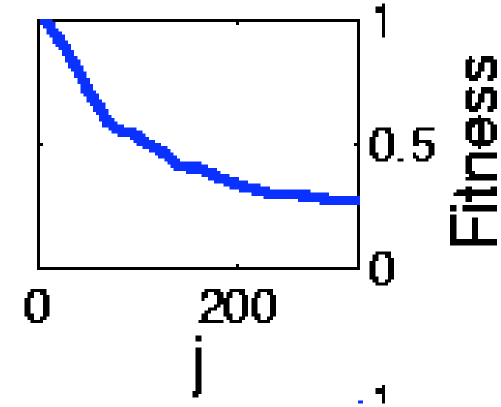
OCT

1) GCS → GCS $|j\rangle_z \rightarrow |j\rangle_x$ **unrestricted.**

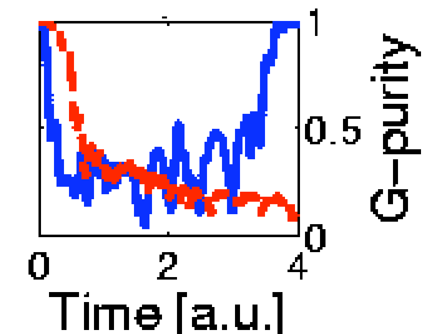
Expectation of the target $F = \langle P \rangle$, $P = |\psi(T)\rangle\langle\psi(T)|$



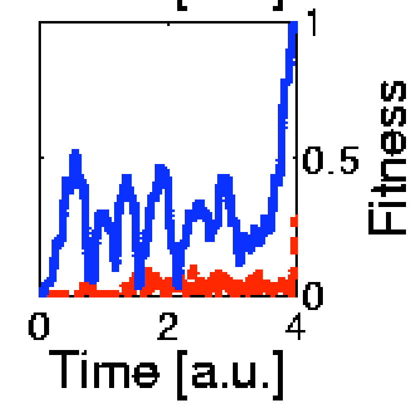
200 iterations



j=5



j=320

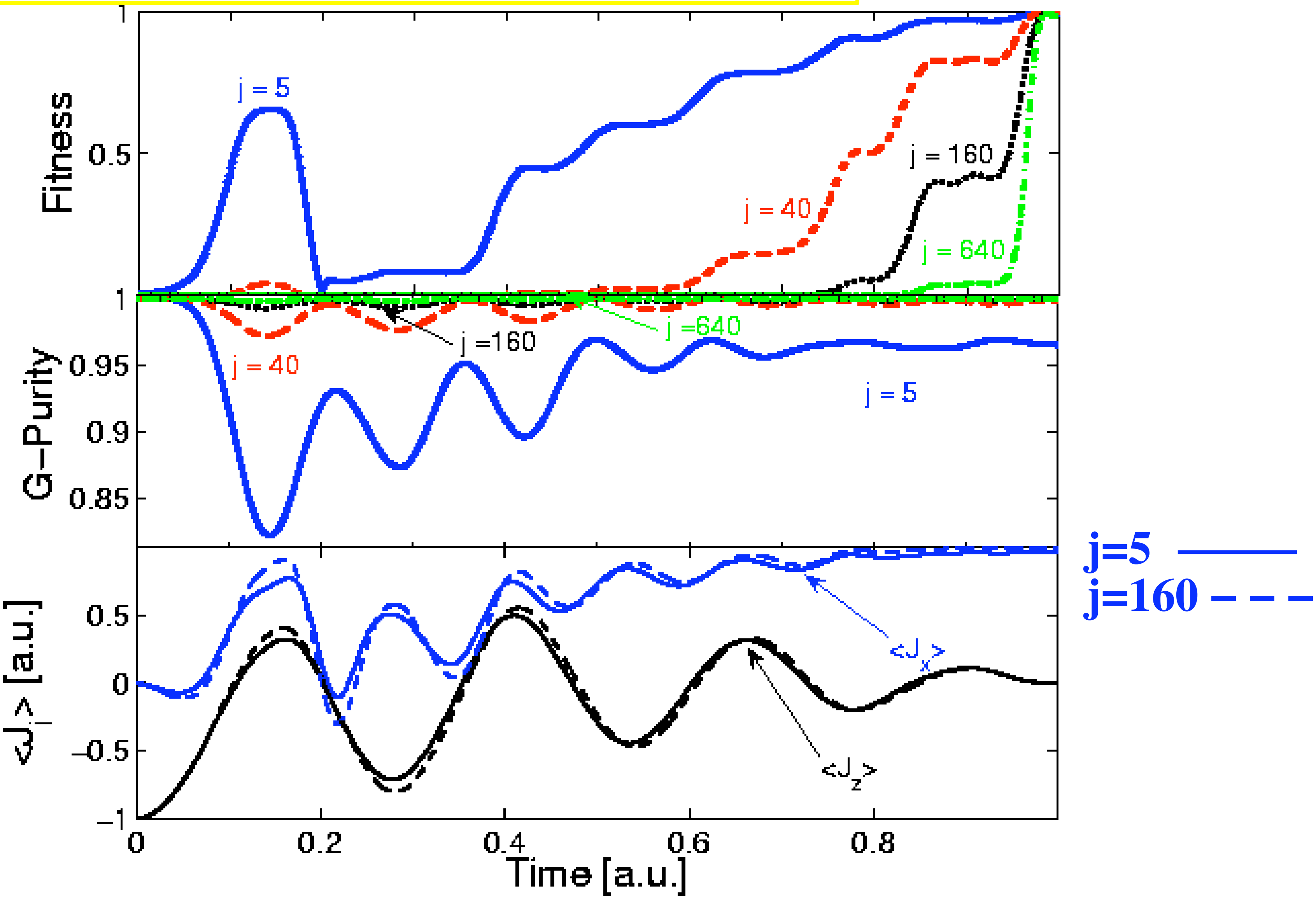


Optimal Control Theory

OCT

2) GCS → GCS $|j\rangle_z \rightarrow |j\rangle_x$ guided.

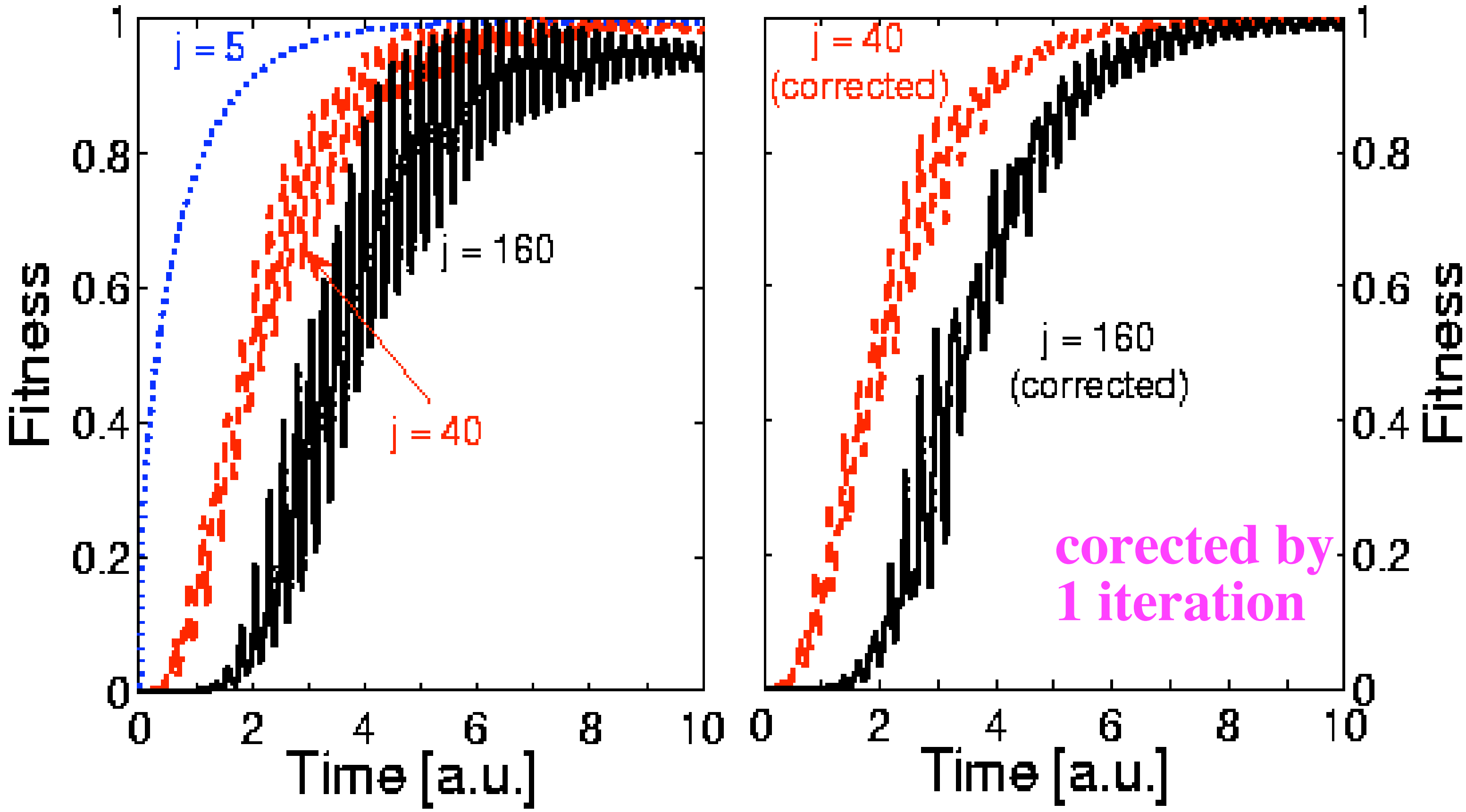
Pilot field for j=5



Local Control Theory

2) GCS → GCS $|j\rangle_z \rightarrow |j\rangle_x$ guided.

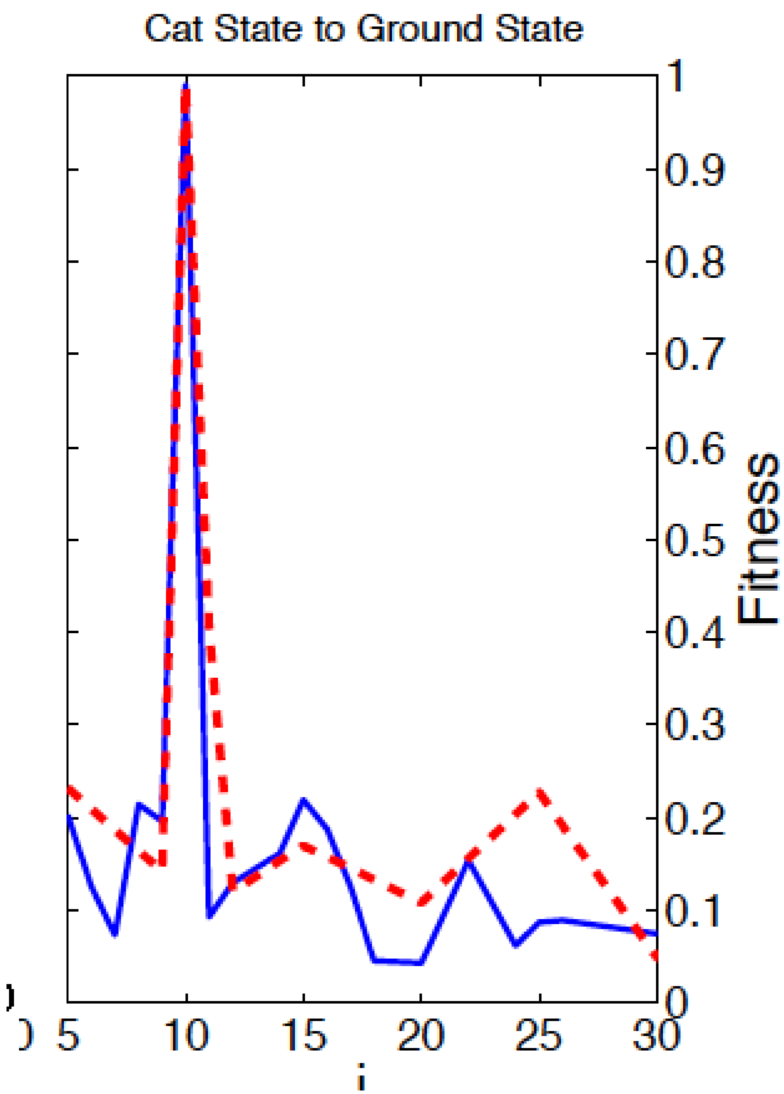
Pilot field for j=5



corected by 1 iteration

3) GCS → cat state $|j\rangle_z \rightarrow (1/\sqrt{2})(|j\rangle_x + |-j\rangle_x)$ **OCT**

4) cat state → GCS $(1/\sqrt{2})(|j\rangle_x + |-j\rangle_x) \rightarrow |j\rangle_z$



Pilot field for j=10

Extreme sensativity to increasing j

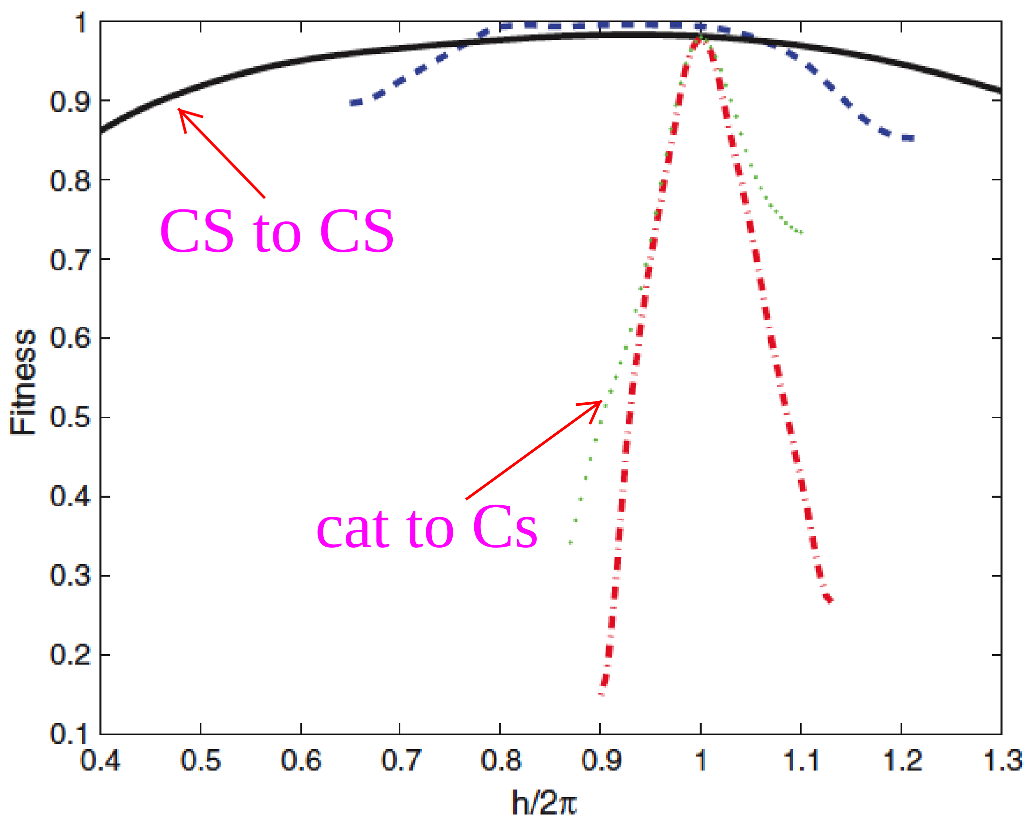
Each optimal field is completely different

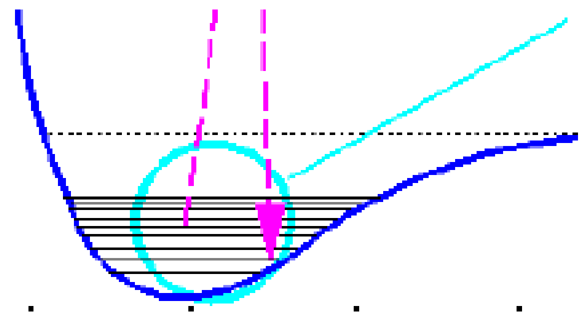
Mores oscillator $H = H_0 + f(t) X$

3) CS → cat state $|\alpha\rangle \rightarrow (1/\sqrt{2})(|\alpha\rangle.\ |-\alpha\rangle)$

4) cat state → CS $(1/\sqrt{2})(|\alpha\rangle.\ |-\alpha\rangle) \rightarrow |\alpha\rangle$

Fitness as a function of h Plank's constant





Shimson Kallush, and Ronnie Kosloff,
Scaling the robustness of the solutions for quantum controllable problems,
Phys. Rev. A 83 063412 (2011).

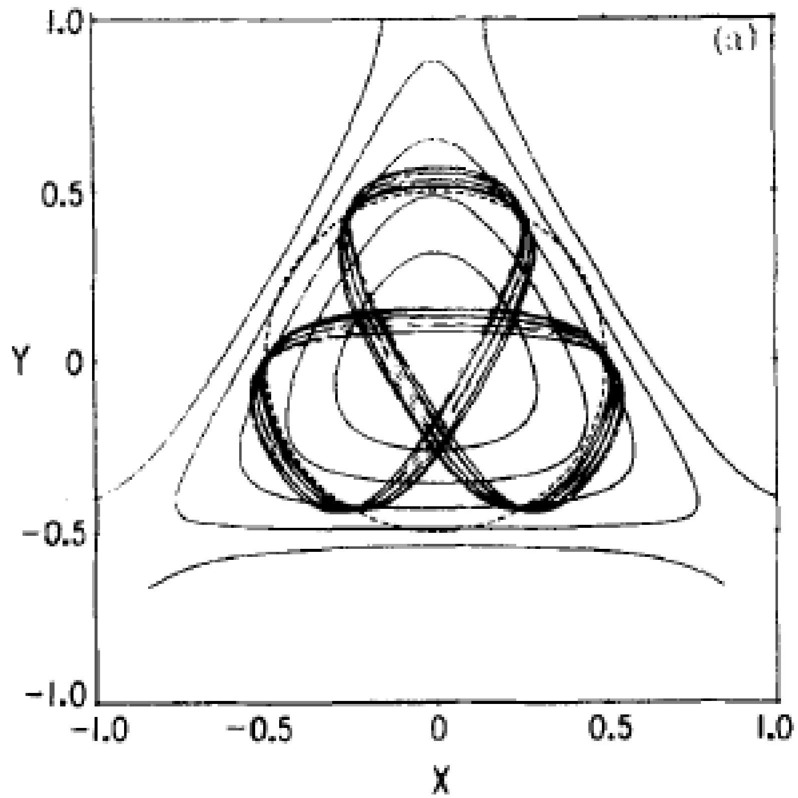
Shimshon Kallush and Ronnie Kosloff
Mutual influence of locality and chaotic dynamics on quantum controllability
Phys. Rev. A 85, 013420 (2012).

Henon Heils

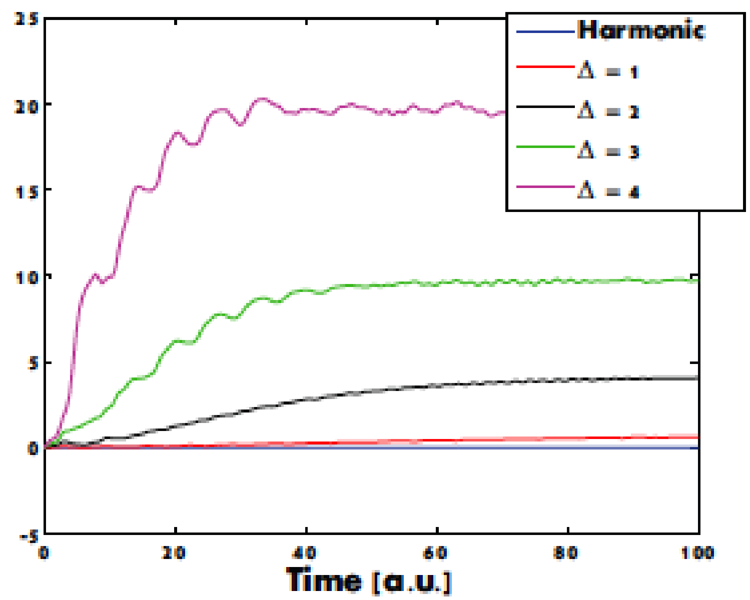
$H = H_0 + f(t)O$ $\quad H = \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + x^2 y - \frac{1}{3} y^3$

3) GCS → cat state $\quad |\alpha\rangle \rightarrow (1/\sqrt{2})(|\alpha\rangle.\ |-\alpha\rangle)$

4) cat state →GCS $\quad (1/\sqrt{2})(|\alpha\rangle.\ |-\alpha\rangle) \rightarrow |\alpha\rangle$



$\{\hat{\mathrm{A}}_{loc}\} \equiv \{\hat{\mathrm{P}}_x,\ \hat{\mathrm{P}}_y,\ \hat{\mathrm{X}},\ \hat{\mathrm{Y}}\}$ control algebra



The non linearity that generates classical chaos also generates control

Discussion Conclusion

$$\mathbf{H} = \mathbf{H}_0 + \sum_{j=1}^{L} \mathbf{u}_j(t)\mathbf{X}_j$$

1) Control Hamiltonian assumes a fixed number of controls L and an increasing size of Hilbert space N.

2) This is different than the quantum computing model where the number of controls increases logarithmicly with the size of Hilbert space $L \propto \log N$.

3) Some Markovian noise on the controls is unavoidable.

4) State to state control is lost due to loss of purity $P = tr\{\rho^2\}$.

as a result a pure state cannot be transformed to a pure state.

5) The purity loss is proportional to the uncertainty with respect to the control operators X.

Discussion Conclusion

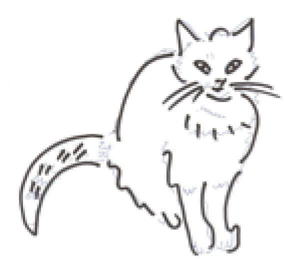
$$\dot{\rho} = -i[\mathbf{H},\rho] - \eta \sum_{i=1}^{3} [\, \mathbf{J}_i, [\mathbf{J}_i,\rho]]$$

6) Generalized coherent states GCS have minimum uncertainty eventually only they will survive.

7) Superpositions of GCS are sensative to noise: cat states are hard to maintain.

8) Using optimal control theory (without noise) state to state GCS to GCS are scalable to large size.

9) The control fields that produce cat states are not scalable.



Thank you

Generalized Coherent states (GCS)

A GCS is transformed to another GCS by a global time dependent unitary operator

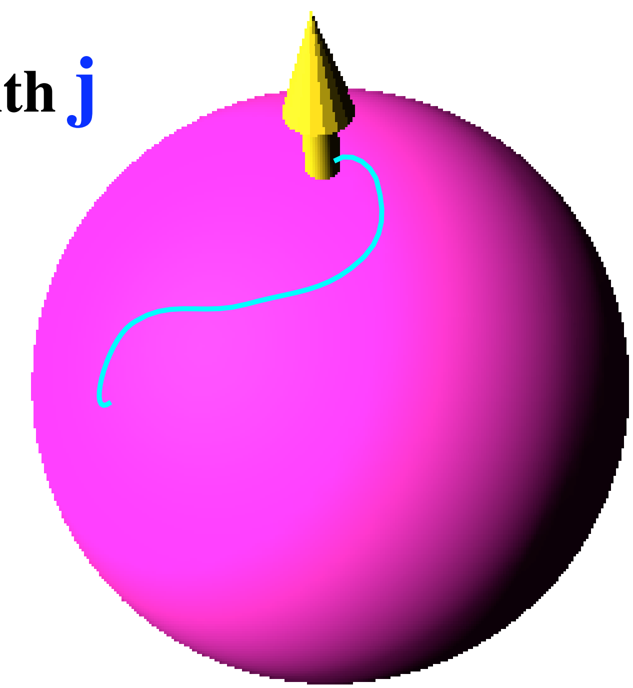
$$\mathbf{U(t)} = \exp\left(-i\,(\alpha(t)\mathbf{J}_x + \beta(t)\mathbf{J}_y + \gamma(t)\mathbf{J}_z)\right)$$

The purity is invariant to a unitary transformation U (rotation) generated by the group $\mathbf{U} = \exp\left(-i\,(\alpha\mathbf{J}_x + \beta\mathbf{J}_y + \chi\mathbf{J}_z)\right)$

$$P(\psi) = P(U\psi)$$

The uncertainty $\Delta\,[\Psi]$ **of a GCS scales linearly with** $\mathbf{j}$

The uncertainty $\Delta\,[\Psi]$ **of a cat state scales as** $\mathbf{j}^2$



The global stable solution of the Stochastic Schrodinger equation is a GCS

Khasin &Kosloff, JPA 41 (2008) 365203

$$\mathbf{X} = i\omega[\mathbf{J}_x, \mathbf{X}] - \gamma \sum [\,\mathbf{J}_i, [\mathbf{J}_i, \mathbf{X}]]$$

M. Khasin[1] and R. Kosloff[2] PRL **106**, 123002 (2011)

Time scale of Control

$$H = H_0 + \sum_{j=1}^{L} u_j(t) X_j \qquad H_0 \varphi_j = \varepsilon_j \varphi_j$$

state to state transformation

$$\Psi_i = \sum r_j^i e^{-\phi_j}\,\varphi_j \quad \Longrightarrow \quad \Psi_f = \sum r_j^f e^{-\phi_j}\,\varphi_j$$

We consider changes Δr such that $1 \gg \|\Delta r\| \geq \varepsilon > 0$.

where $\|\Delta r\|$ is the Eucledian norm between the vectors r_i and r_f

free propagation does not change this norm

We get:

$$\varepsilon^2 \leq 2\,\Sigma T \bar{u}_k |\Lambda_k|$$

Timescale of control

$$T \geq \frac{2\varepsilon^2}{\Sigma u_k |\Lambda_k|}$$

$T\bar{u}_k$ **is the action of control k**

$|\Lambda_k|$ **is the maximum eigenvalue of X_k.**

We now estimate the purity loss at the same time scale.