Multiphoton simulator for non-Abelian anyons

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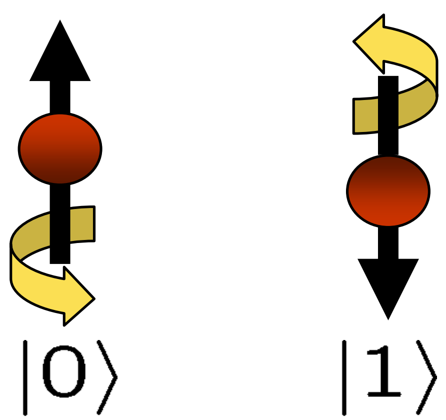
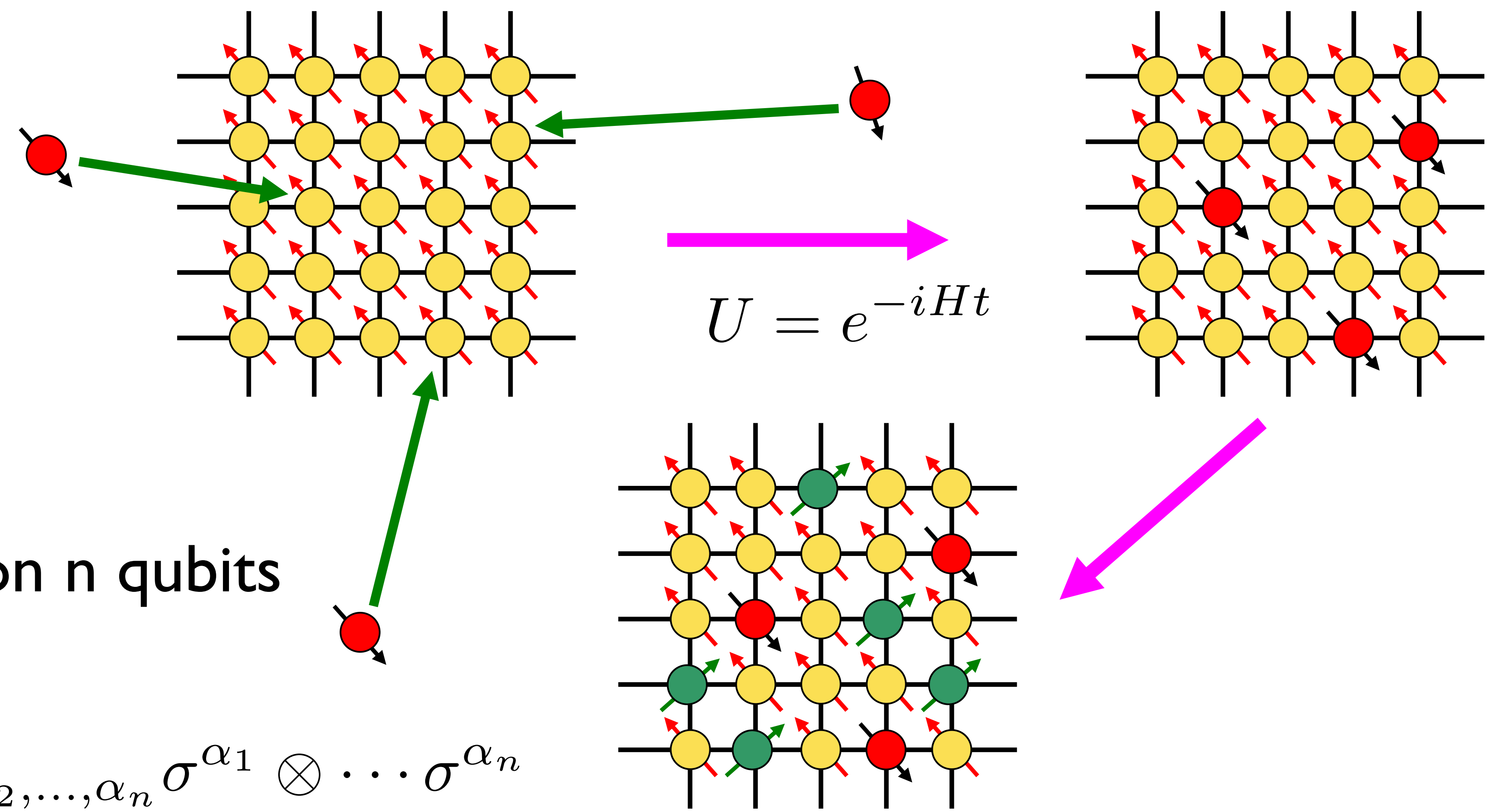
arXiv:0906.4578

- *``We should try to find out what kinds of quantum mechanical systems are mutually intersimulatable, and try to find a specific class, or character of that class which will simulate everything"* Feynman 1982

- Many physical systems have an isomorphic state space, so if we can control the state of one then that simulates another
    - one qubit=polarization space of a photon= spin space of an electron= occupation space of a single photon in two orthogonal modes, etc.



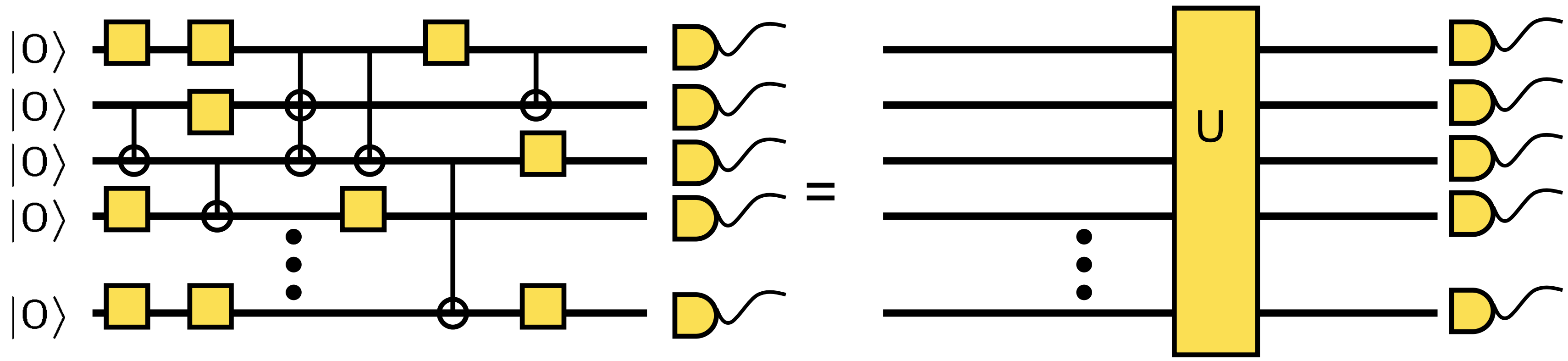


Generic Hamiltonian on n qubits

$$H = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n=0}^{3} h_{\alpha_1,\alpha_2,\ldots,\alpha_n} \sigma^{\alpha_1} \otimes \cdots \sigma^{\alpha_n}$$

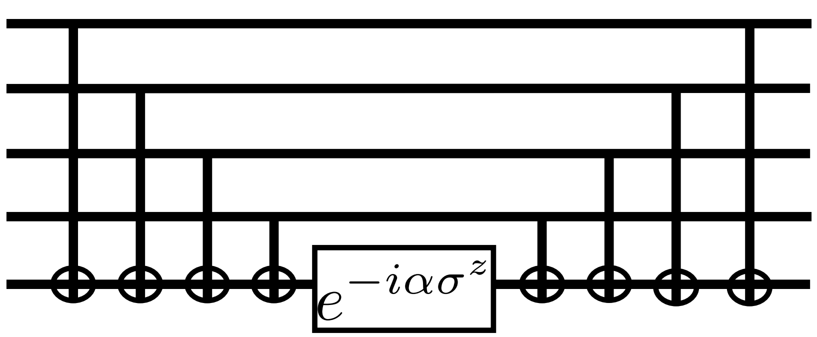
Digital Simulators

- A fault tolerant quantum computer can simulate any dynamics over finite dimensional systems



  - Problem: This may not be efficient. General unitaries on n qubits systems require 2^n gates (for qudits d^{2n})

  - Some dynamics are efficient



$$= e^{-i\alpha\sigma^{z\otimes 5}}$$

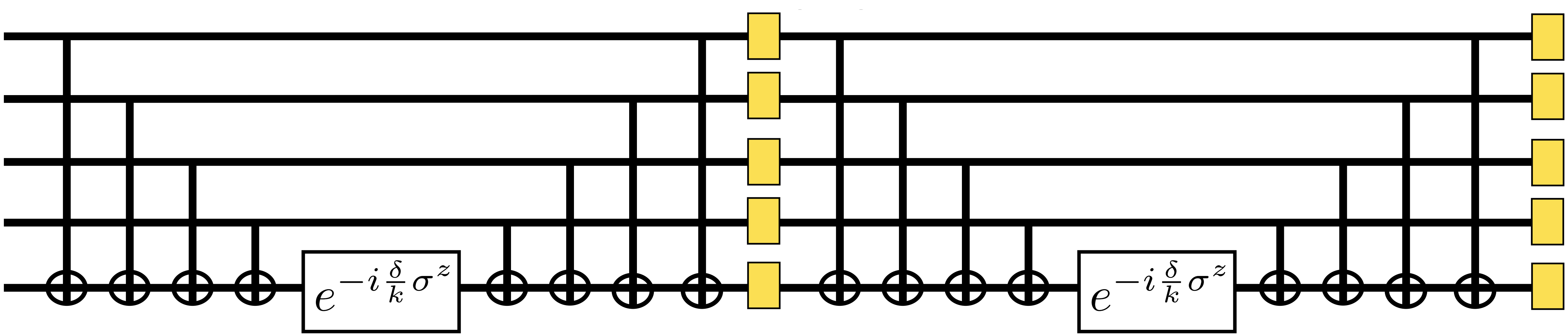
  - Is there a constructive approximate method? Yes

Stroboscopic techniques

- Control Hamiltonian $H(t) = H_{\mathrm{ent}} + \sum_m f_m(t) H_m$
  - System governed by TDSE $i\hbar\dot{U} = H(t)U(t)$
- Using properties of commutators (Trotterization) we can digitize a simulation.
  - Advantage: Digitized circuits can be made fault tolerant

$$e^{-i\Delta H_a} e^{i\Delta H_b} e^{i\Delta H_a} e^{-i\Delta H_b} = e^{-i(i[H_a,H_b])\Delta^2} + O(\Delta^3)$$

$$e^{-i\Delta(H_a+H_b)} = [e^{-i\Delta H_a/k} e^{-i\Delta H_b/k}]^k + O(\Delta^2/k)$$



...

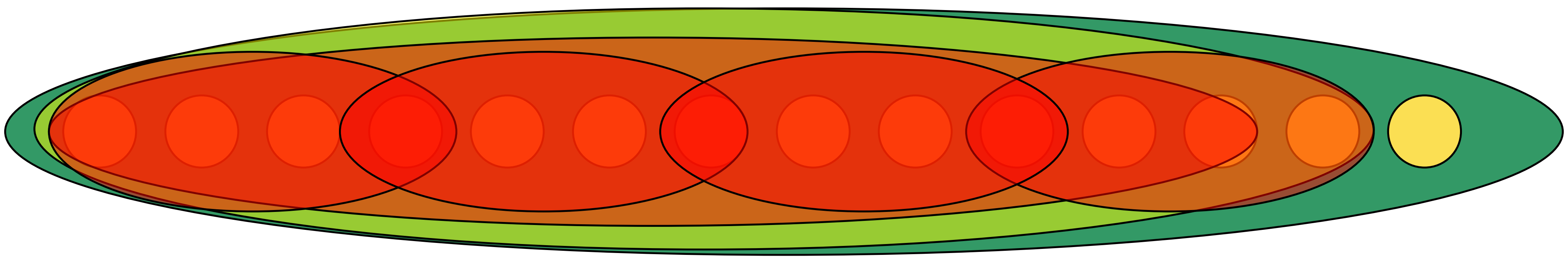
k times

$$\approx e^{-i\delta(\sigma^{z\otimes 5} + \sigma^x\sigma^z\sigma^y\sigma^z\sigma^x)}$$

M. Bremner, D. Bacon, and M.A. Nielsen,PRA 71, 052312 (2005).

k-local Hamiltonians

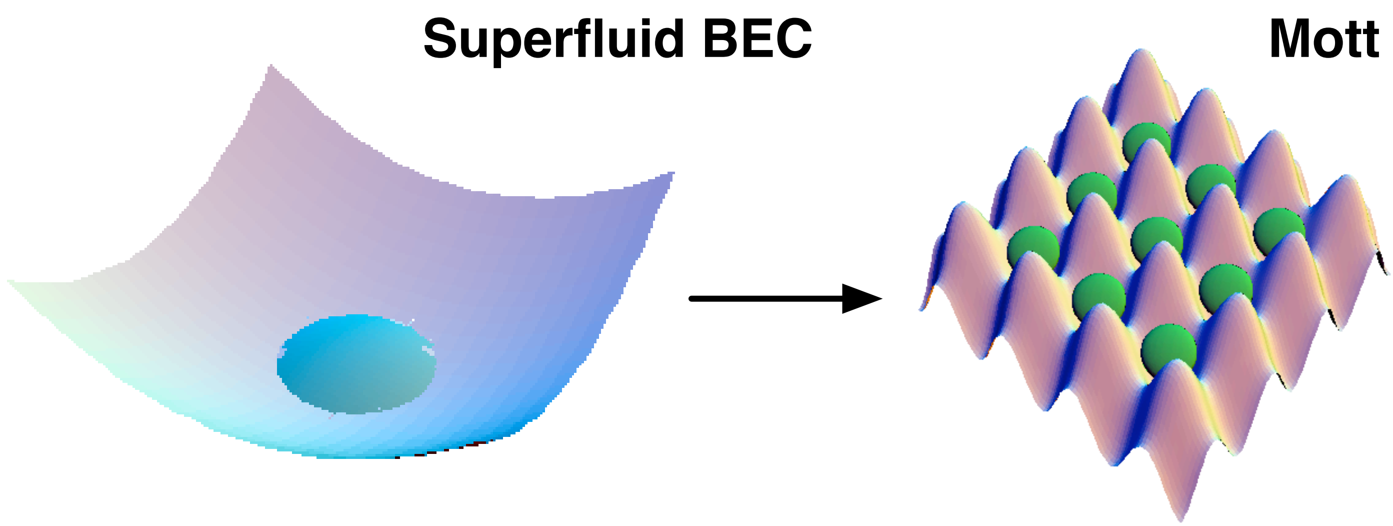
- Bounded Hamiltonians that can be written as a sum of terms involving tensor products of no more than k terms



- It is frequently the case that k is small for physically relevant Hamiltonian (often k=2)
- Complexity of stroboscopic circuit for k-local Hamiltonians is $O(n^k)$

Analogue simulators

- Engineer an “always on” interaction that mimics a physics you want to simulate
- Ex: Superfluid to Mott insulator phase transition

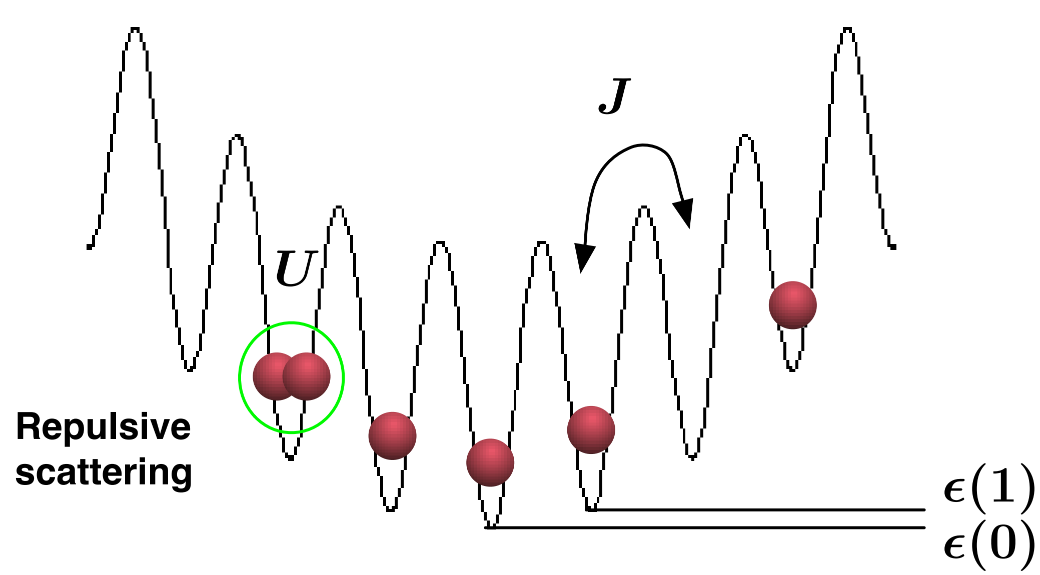


Theory: Jaksch *et al.* PRL 81, 3108 (1998)

Exp: M. Greiner *et al.* Nature 415, 39 (2003)

- Analogue simulation of Bose-Hubbard interaction

$$H_{BH} = -\sum_{<j,k>} J(a_j^\dagger a_k + a_k^\dagger a_j) + \sum_j \frac{U}{2} n_j(n_j - 1) + \epsilon(j) n_j$$



Pros/Cons

- Analogue simulators:
  - Pros: A more direct method, much simpler control
  - Cons: Not fault tolerant, architectural limitations, physical limitations on locality and strength of interactions in the simulating system

- Digital simulators:
  - Pros: Universality, can be made fault tolerant, no architectural constaints in principle
  - Cons: Complicated control pulse sequences, evidence* that fault tolerant simulation has complexity that scales like $1/\epsilon$ for a tolerated error ϵ

***K. Brown, R.J. Clark, I.L. Chuang, PRL 97, 050504 (2006).**

A quantum simulation algorithm

- Problem: Compute the energy gap ΔE between the ground and first excited state of a Hamiltonian H

L.A. Wu, M.S. Byrd, and D.A. Lidar, PRL 89, 057904 (2002).

- Algorithm:
  - Map the Hilbert space of the system to be simulated to n qubits
  - Prepare QC in the ground state of a local Hamiltonian H_0
    - e.g. $|\psi(0)\rangle = |+_x\rangle^{\otimes n} \quad H_0 = -\sum_j \sigma_j^x$
  - Evolve, using stroboscopic circuits, according to the following adiabatic Hamiltonian

$$H(s) = (1-s)H_0 + sH \qquad s = t/T \quad T \gg \frac{||\frac{\partial H(s)}{\partial s}||}{(\Delta E)^2}$$

  - Failure to remain adiabatic results in a final state which has some small admixture of ground and excited states of H $\quad |\psi\rangle_i = c_g|\lambda_g\rangle + c_e|\lambda_e\rangle$
  - Evolve by simulated H for time t_i
  - Measure some operator that couples ground and excited states $\langle O(t_i)\rangle$
  - Repeat for a polynomial number of times steps (polynomial number of steps enough to resolve ΔE). Compute Fourier transform of $\{\langle O(t_i)\rangle\}$

Caveats

- To estimate of the gap with error ϵ # of digits precision needed is $\log(1/\epsilon)$
  - A good quantum algorithm would have a complexity $O(poly \log(1/\epsilon))$
- Error in trotter approximation of evolution due to a sum of Hamiltonians scales like $O(t^2/k)$. Higher order Trotter expansion has error $O(t^{m+1}/k^m)$ using gates $O(2^m)$
- Total time to implement algorithm is independent of k. *If control gates are perfect*, then precision improves with larger k.
- If not, then each subcircuit must be replaced with a fault tolerant version.
- But time to implement $e^{-iHt/k}$ is independent of k (for standard control Hamiltonians)! Hence for a fixed error the total time for the simulation is $O(1/\epsilon)$.

K. Brown, R.J. Clark, I.L. Chuang, PRL 97, 050504 (2006).

- Workarounds:
  - Find better gate libraries (maybe global pulses with correlated errors) combined with noiseless subsystems
  - Hard wire the Hamiltonian into a system so no Trotterization is needed!

What to simulate?

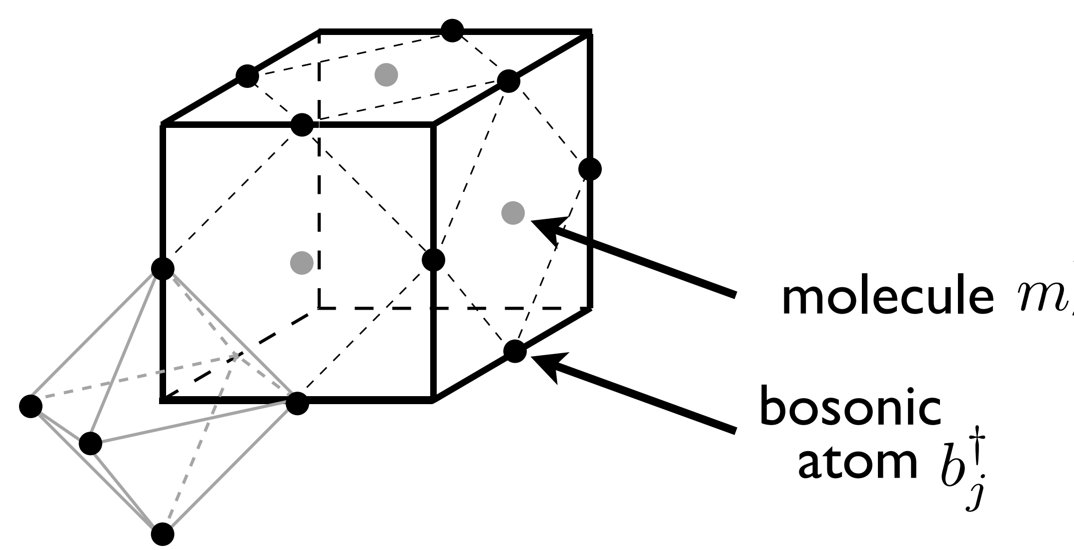
- Hubbard models
  - 2D Fermi-Hubbard Model

$$H = -\sum_{\langle \mathbf{r},\mathbf{r}'\rangle,\sigma} t_{\mathbf{r},\mathbf{r}'} \left(c^\dagger_{\mathbf{r},\sigma} c_{\mathbf{r}',\sigma} + H.c. \right) + U \sum_{\mathbf{r}} n_{\mathbf{r},\uparrow} n_{\mathbf{r},\downarrow} \qquad \{c_{\mathbf{r},\sigma}, c^\dagger_{\mathbf{r}',\sigma'}\} = \delta_{\mathbf{r},\mathbf{r}'}\delta_{\sigma,\sigma'}$$

A. Auerbach, 1994

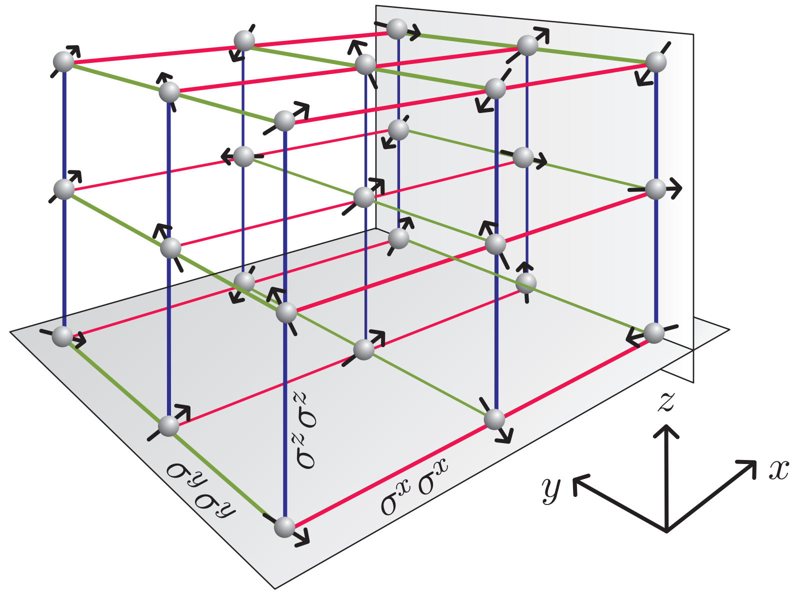
    - Observables: Energy gap, correlation functions
    - Proposed model for high Tc superconductivity. A q. simulation could falsify it
  - 2D and 3D lattice gauge theory
    - Ring exchange model: Emergent U(1) gauge theory

$$H_{\mathrm{RE}} = K \sum_{\square} (b_1^\dagger b_2 b_3^\dagger b_4 + b_1 b_2^\dagger b_3 b_4^\dagger - n_1 n_3 - n_2 n_4)$$



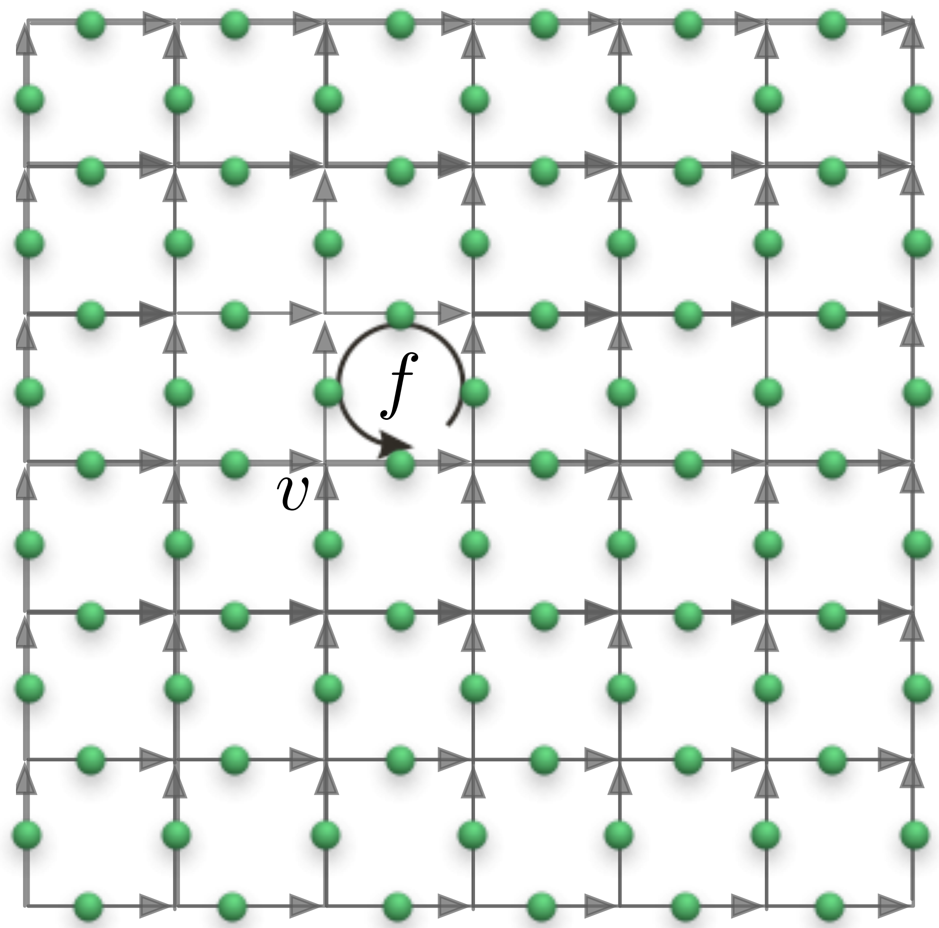
H.P. Buchler, et al. PRL 95, 040402 (2005)

- Spin lattice models



  - Study quantum phase transitions (simulate spin liquids, valence bond solids, etc)
  - Resource states for measurement based quantum computation, e.g. AKLT model
  - Emergent physics and topological order
    - Loop gas models, string net models, discrete gauge models (this talk)
    - General idea: prepare a highly entangled state of a spin network that is the vacuum state of a physical theory.

A spin lattice model* of discrete gauge theory

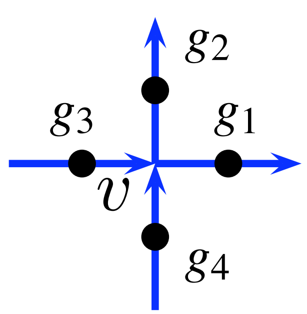
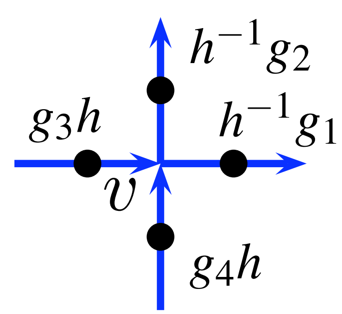


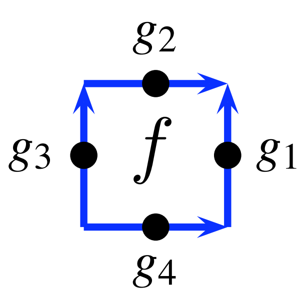
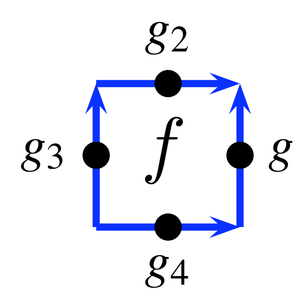
- Pick a finite group G
  - spins on each edge: local basis $\{|g\rangle ; g \in G\}$
- Local gauge transformation

$$T_g(v) = \prod_{e_j \in [v,*]} L_g(e_j) \prod_{e_j \in [*,v]} R_{g^{-1}}(e_j)$$

- Want ground states local gauge invariant

$$H_{\mathrm{TO}} = -\sum_v A(v) - \sum_f B(f)$$

$A(v)\,\Big|$  $\Big\rangle = \frac{1}{|G|}\sum_{h \in G}\Big|$  $\Big\rangle$

$B(f)\,\Big|$  $\Big\rangle = \delta(g_3 g_2 g_1^{-1} g_4^{-1}, e)\,\Big|$  $\Big\rangle$

$$[A(v), A(v')] = [A(v), B(f)] = [B(f), B(f')] = 0$$

Realizes the quantum double $D(G)$

*A. Yu. Kitaev, Annals of Physics **303**, 2 (2003)

- Gauge transformations $T_g(v) = \prod_{e_j \in [v,*]} L_g(e_j) \prod_{e_j \in [*,v]} R_{g^{-1}}(e_j)$

- Simplest non-Abelian group $S_3 = \{e, \underbrace{c_+, c_-}_{\text{cyclic perms}}, \underbrace{t_0, t_1, t_2}_{\text{transpositions}}\}$

- Regular rep:

$$L_e = \left(\begin{array}{c|ccc|cc} 1 & & & & & \\ \hline & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ \hline & & & & 1 & \\ & & & & & 1 \end{array}\right), \quad L_{t_0} = \left(\begin{array}{c|ccc|cc} & 1 & & & & \\ \hline 1 & & & & & \\ & & & & 1 & \\ & & & & & 1 \\ \hline & & 1 & & & \\ & & & 1 & & \end{array}\right)$$

$$L_{t_1} = \left(\begin{array}{c|ccc|cc} & & 1 & & & \\ \hline & & & & & 1 \\ 1 & & & & & \\ & & & & 1 & \\ \hline & & & 1 & & \\ & 1 & & & & \end{array}\right), \quad L_{t_2} = \left(\begin{array}{c|ccc|cc} & & & 1 & & \\ \hline & & & & 1 & \\ & & & & & 1 \\ 1 & & & & & \\ \hline & 1 & & & & \\ & & 1 & & & \end{array}\right)$$

$$L_{c_+} = \left(\begin{array}{c|ccc|cc} & & & & & 1 \\ \hline & & 1 & & & \\ & & & 1 & & \\ & 1 & & & & \\ \hline 1 & & & & & \\ & & & & 1 & \end{array}\right), \quad L_{c_-} = \left(\begin{array}{c|ccc|cc} & & & & 1 & \\ \hline & & & 1 & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & & & & 1 \\ 1 & & & & & \end{array}\right)$$

Left multiplication

$$R_e = \left(\begin{array}{c|ccc|cc} 1 & & & & & \\ \hline & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ \hline & & & & 1 & \\ & & & & & 1 \end{array}\right), \quad R_{t_0} = \left(\begin{array}{c|ccc|cc} & 1 & & & & \\ \hline 1 & & & & & \\ & & & & & 1 \\ & & & & 1 & \\ \hline & & & 1 & & \\ & & 1 & & & \end{array}\right)$$

$$R_{t_1} = \left(\begin{array}{c|ccc|cc} & & 1 & & & \\ \hline & & & & 1 & \\ 1 & & & & & \\ & & & & & 1 \\ \hline & 1 & & & & \\ & & & 1 & & \end{array}\right), \quad R_{t_2} = \left(\begin{array}{c|ccc|cc} & & & 1 & & \\ \hline & & & & & 1 \\ & & & & 1 & \\ 1 & & & & & \\ \hline & & 1 & & & \\ & 1 & & & & \end{array}\right)$$

$$R_{c_+} = \left(\begin{array}{c|ccc|cc} & & & & & 1 \\ \hline & & & 1 & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline 1 & & & & & \\ & & & & 1 & \end{array}\right), \quad R_{c_-} = \left(\begin{array}{c|ccc|cc} & & & & 1 & \\ \hline & & 1 & & & \\ & & & 1 & & \\ & 1 & & & & \\ \hline & & & & & 1 \\ 1 & & & & & \end{array}\right)$$

Right multiplication

- Simplification using semi-direct product structure of group $S_3 \cong \mathbb{Z}_3 \rtimes_\phi \mathbb{Z}_2$

$$L_e = \mathbf{1}_3 \otimes \mathbf{1}_2, \quad L_{t_0} = F(1,2) \otimes \sigma^x, \quad L_{t_1} = F(0,2) \otimes \sigma^x, \quad L_{t_2} = F(0,1) \otimes \sigma^x,$$

$$L_{c_+} = X^{-1} \otimes \mathbf{1}_2, \quad L_{c_-} = X \otimes \mathbf{1}_2, \quad R_e = \mathbf{1}_3 \otimes \mathbf{1}_2, \quad R_{t_0} = \mathbf{1}_3 \otimes \sigma^x,$$

$$R_{t_1} = X^{-1} \otimes \sigma^- + X \otimes \sigma^+, \quad R_{t_2} = X^{-1} \otimes \sigma^+ + X \otimes \sigma^-,$$

$$R_{c_+} = X \otimes |0\rangle\langle 0| + X^{-1} \otimes |1\rangle\langle 1|, \quad R_{c_-} = X^{-1} \otimes |0\rangle\langle 0| + X \otimes |1\rangle\langle 1|,$$

$$F(i,j) = (|i\rangle\langle j| + |j\rangle\langle i|) \oplus 1$$

- Suggests a qutrit/qubit encoding of spins

- Efficient quantum circuit exists for preparing vacuum state of model + manipulation of anyonic excitations*
  - works with or without a background Hamiltonian present

*M. Aguado, GKB, F. Verstraete, J.I. Cirac, Phys. Rev. Lett. **101**, 260501 (2008), GKB, M. Aguado, J.I. Cirac, New J. Phys. **11** 053009 (2009).

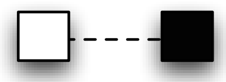
Particle spectrum of $D(S_3)$

- Labels $\Pi^{[\alpha]}_{R(N_{[\alpha]})}$ ← conjugacy class-----> magnetic charge; ← irrep of centralizer of conjugacy class------> electric charge

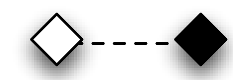
particle type		quantum dimension $= \|[\alpha]\|\|R\|$
Vacuum	$\Pi^{[e]}_{R_1^+}$	1
Pure magnetic charge	$\Pi^{[c]}_{\beta_0}$ $\Pi^{[t]}_{\gamma_0}$	(2,3)
Pure electric charge	$\Pi^{[e]}_{R_1^-}$ $\Pi^{[e]}_{R_2}$	(1,2)
Dyonic combination	$\Pi^{[c]}_{\beta_1}$ $\Pi^{[c]}_{\beta_2}$ $\Pi^{[t]}_{\gamma_1}$	(2,2,3)

- Particles with quantum dimension >1 are non-Abelian anyons

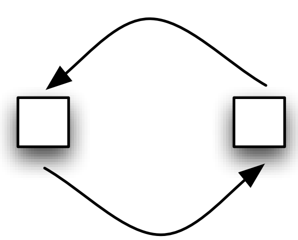
Braid relations

- All excitations come in particle/anti-particle pairs
  - Magnetic flux pair $|a, a^{-1}\rangle$ 
  - Electric charge pairs transform under the irrep R and conjugate R*.

$$|M^R\rangle = \frac{1}{\sqrt{|R|}} \sum M^R_{\mu,\nu} |\mu\rangle_R \otimes |\nu\rangle_{R^*} \qquad \sum_{\mu,\nu} |M^R_{\mu,\nu}|^2 = |R|$$

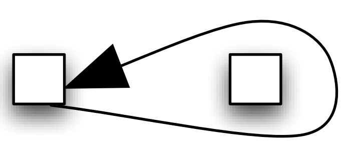


- Interchanging two fluxes



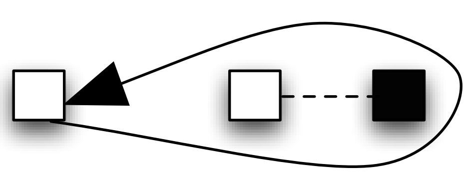
$$\mathcal{R}\,|a\rangle|b\rangle = \sigma|a\rangle|aba^{-1}\rangle = |aba^{-1}\rangle|a\rangle$$

- Braiding two fluxes



$$\mathcal{R}^2|a\rangle|b\rangle = |abab^{-1}a^{-1}\rangle|abbb^{-1}a^{-1}\rangle = |(ab)a(ab)^{-1}\rangle|aba^{-1}\rangle$$

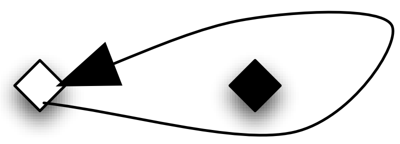
- Braiding one flux around flux pair



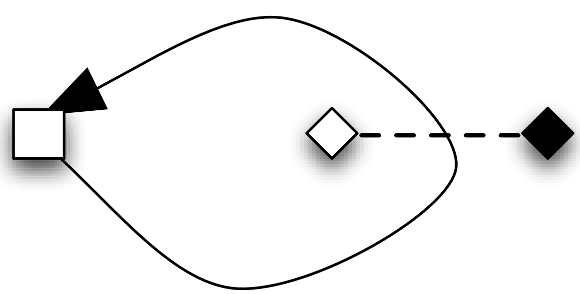
$$\mathcal{R}^2_{1,2} \otimes \mathcal{R}^2_{1,3}|b\rangle|a, a^{-1}\rangle = |b\rangle|bab^{-1}, ba^{-1}b^{-1}\rangle$$

  - action is trivial if pair prepared in *chargeless* state $|0_{[\ell]}\rangle = \frac{1}{\sqrt{|[\ell]|}} \sum_{\ell \in [\ell]} |\ell, \ell^{-1}\rangle$

- Electric charges moving past each other have no effect

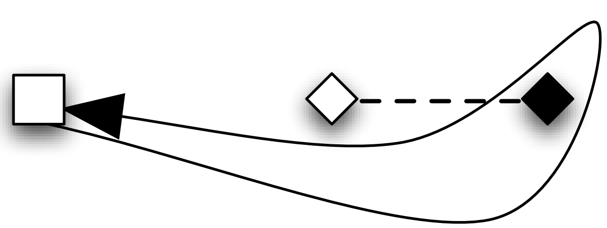


- Braiding a flux around one charge in a pair



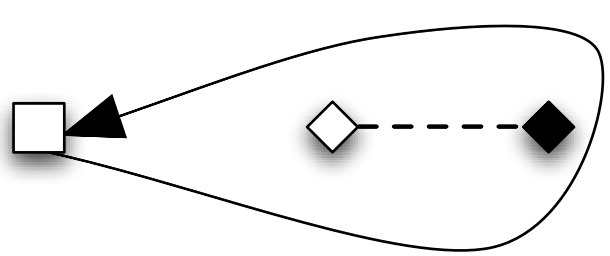
$$\mathcal{R}_{1,2}^{2}|h\rangle|M^R\rangle = |h\rangle|R(h)M^R\rangle$$

- Braiding a flux around the anti-charge in a pair



$$\mathcal{R}_{1,3}^{2}|h\rangle|M^R\rangle = |h\rangle|M^R R(h^{-1})\rangle$$

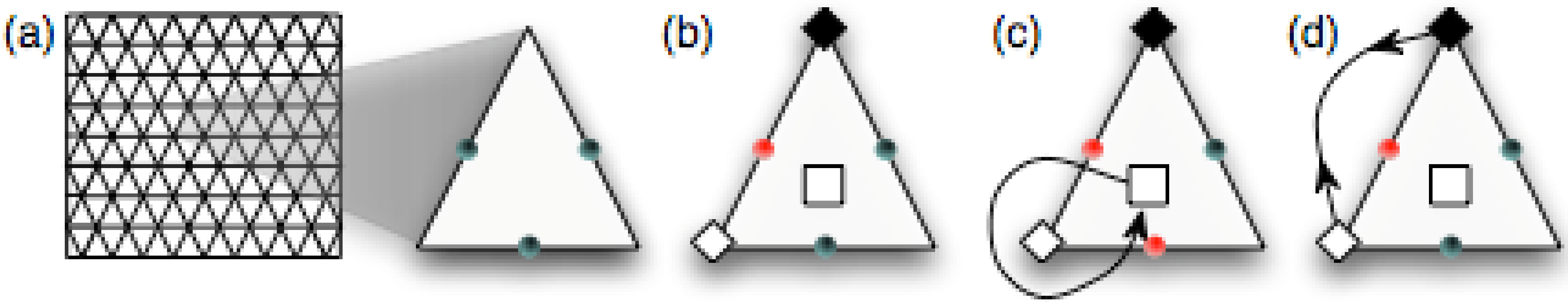
- Braiding a flux around the pair acts like conjugation



$$\mathcal{R}_{1,2}^{2}\otimes\mathcal{R}_{1,3}^{2}|h\rangle|M^R\rangle = |h\rangle|R(h)M^R R(h^{-1})\rangle$$

- For each irrep R, a unique *fluxless* state invariant under conjugation $|\mathbf{1}_{|R|}\rangle$

Sketch of proposed interferometry



(a) A spin lattice model of $D(S_3)$, we simulate a single plaquette

(b) Acting on one spin (red) produces a anyonic electric charge pair (diamonds) and also can produce a flux (square)

(c) Braiding the flux around one charge by acting on (red) spins

(d) Fusion of the electric charge pair. Incomplete fusion to vacuum is signature of non-Abelian anyonic statistics

Experimental Simulation of Abelian anyons

- Algorithmic simulation of $D(\mathbb{Z}_2)$ (toric code) with entangled photons

J.K. Pachos, W. Wieczorek, C. Schmid, N. Kiesel, R. Pohlner, H. Weinfurter, arXiv:0710.0895 New J Phys (in press);
Chao-Yang Lu, Wei-Bo Gao, Otfried Gühne, Xiao-Qi Zhou, Zeng-Bing Chen, Jian-Wei Pan, PRL **102**, 030502 (2009)

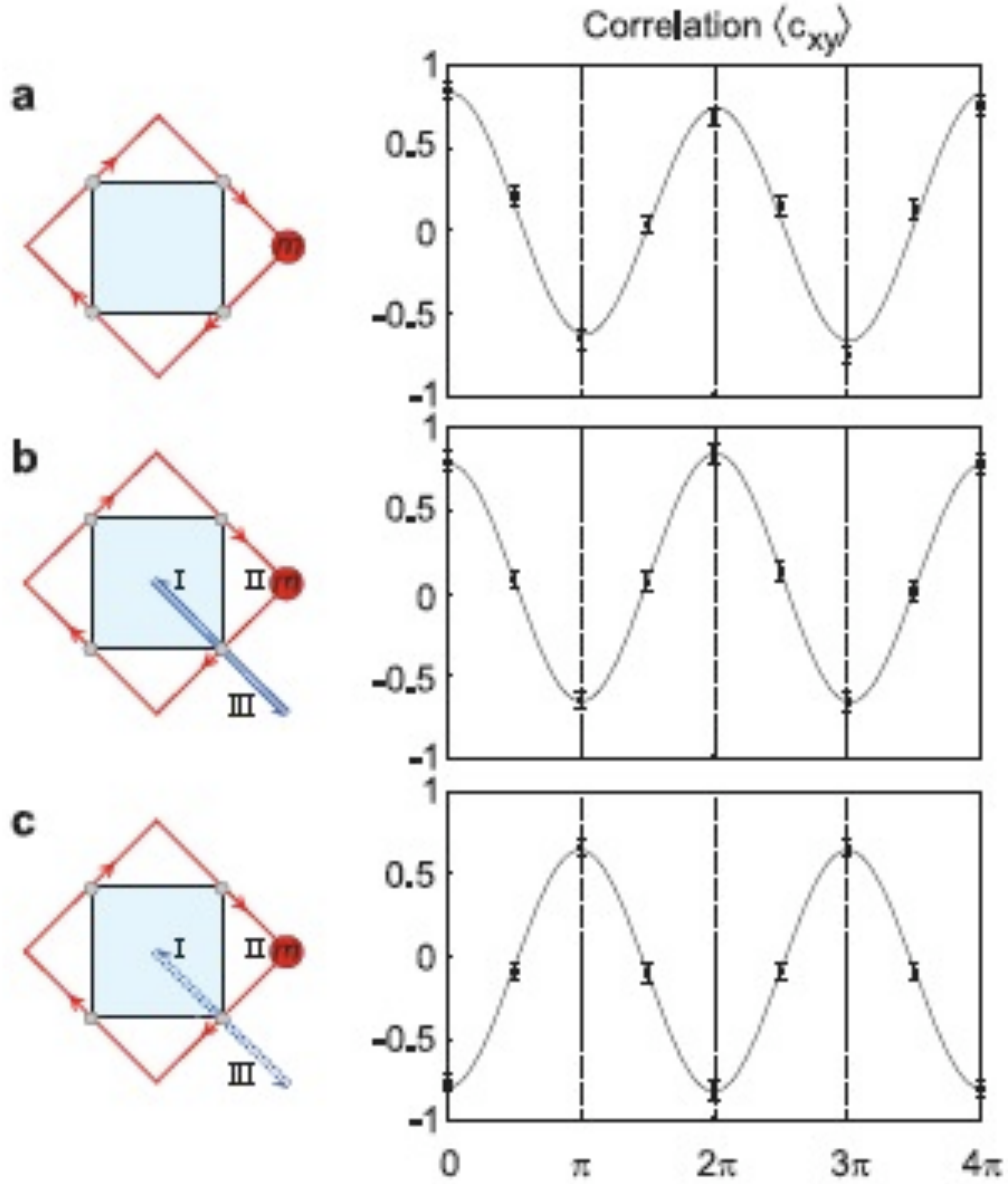
Ground state of

$$|\xi\rangle = \prod_s \frac{1}{\sqrt{2}}(\mathbb{1} + \sigma^x_{s,1}\sigma^x_{s,2}\sigma^x_{s,3}\sigma^x_{s,4})|00...0\rangle$$

Background Hamiltonian is zero!

Prepare GS on one plaquette

$$|\xi\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$$



Create flux pair braid around plaquette, annihilate

$$\sigma^x_1\sigma^x_2\sigma^x_3\sigma^x_4|\xi\rangle = |\xi\rangle$$

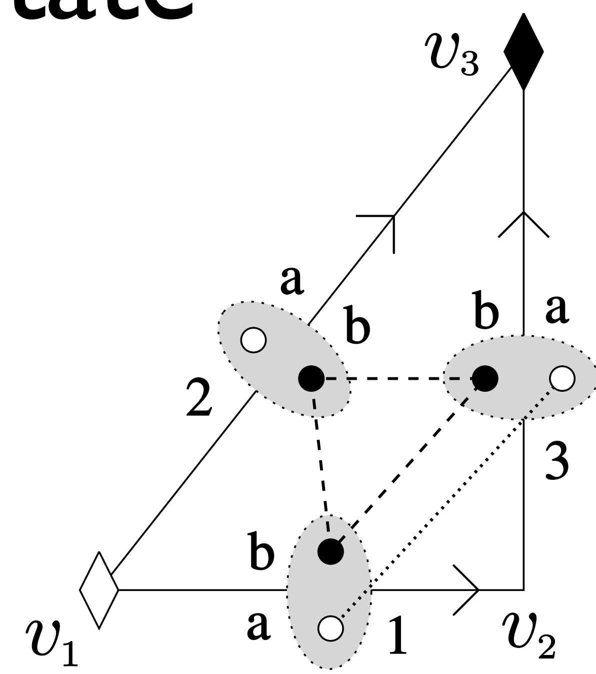
Create charge pair, braid flux pair around charge, annihilate

$$\sigma^z_3[\sigma^x_1\sigma^x_2\sigma^x_3\sigma^x_4]\sigma^z_3|\xi\rangle = -[\sigma^x_1\sigma^x_2\sigma^x_3\sigma^x_4]|\xi\rangle = -|\xi\rangle$$

Measure interference of two processes

$$e^{-i\frac{\pi}{4}\sigma^z_1}[\sigma^x_1\sigma^x_2\sigma^x_3\sigma^x_4]e^{i\frac{\pi}{4}\sigma^z_1}|\xi\rangle = (|0000> -|1111>)/\sqrt{2}$$

Building the initial state

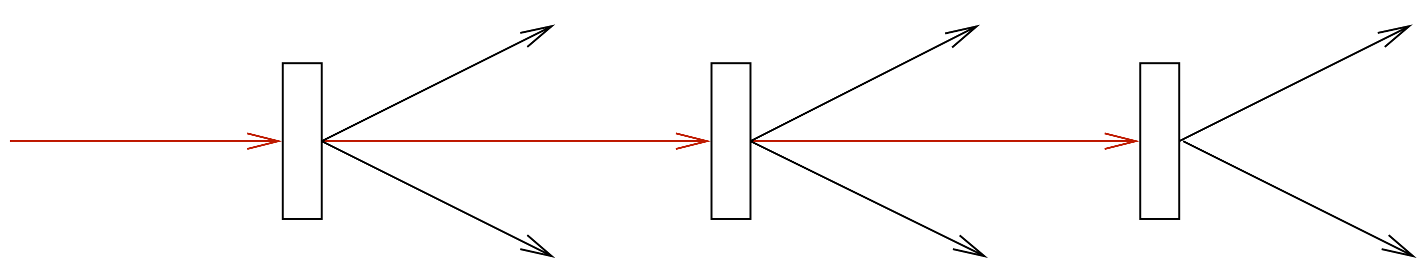


- Resources needed:
  - 3 type-I SPDC crystals
  - 6 photons, 15 modes
  - 14 beam splitters + 11 phase shifters

- Creating 2 qutrit entanglement
  - type-I SPDC: strong pulse in, entangled photon in same polarization out

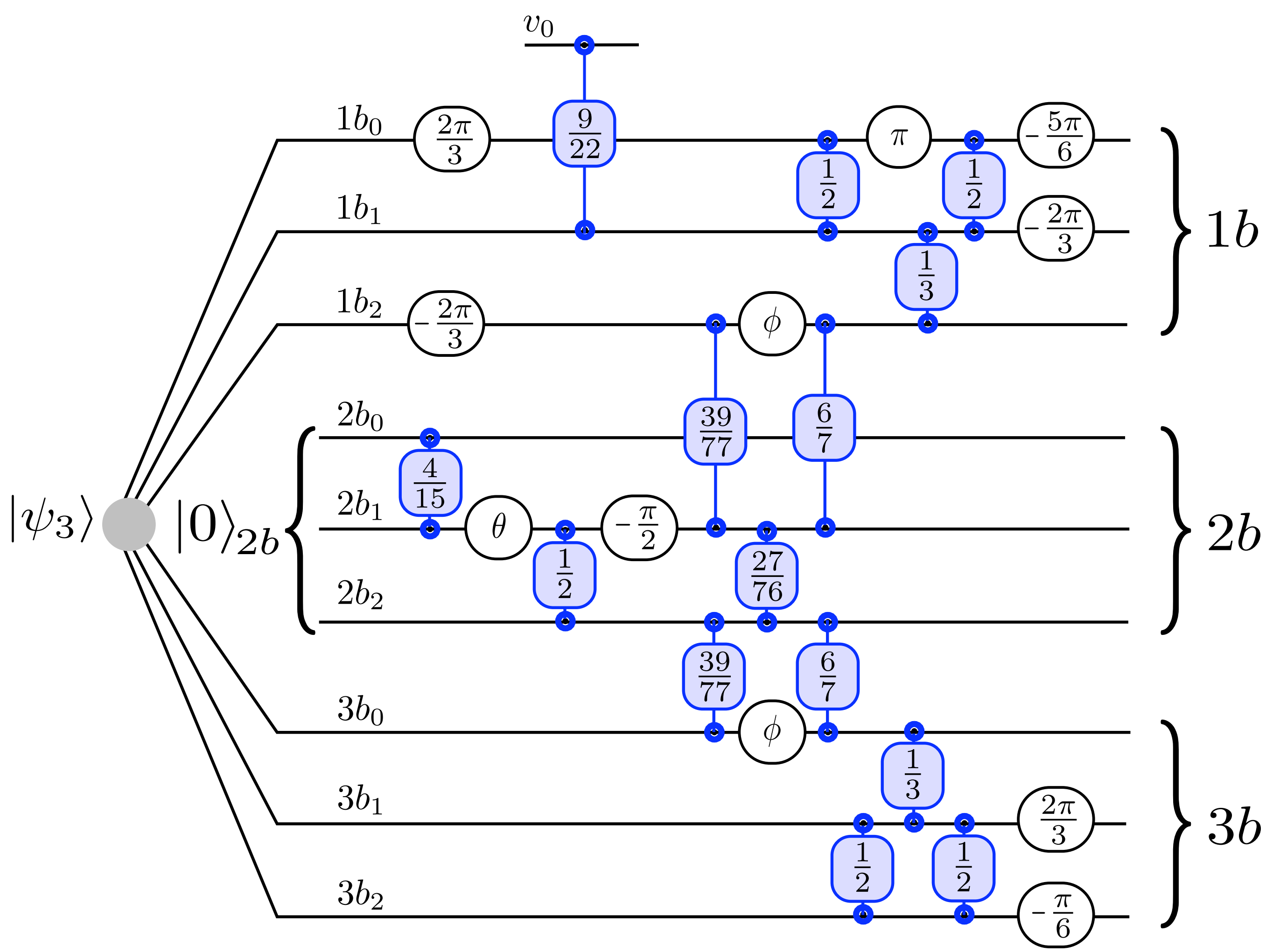
$$|\mathrm{SPDC}\rangle = \sqrt{1-\lambda^2}\sum_{n=0}^{\infty}\lambda^n|nn\rangle$$

  - three crystals $$|\mathrm{SPDC}\rangle^{\otimes 3} = (1-\lambda^2)^{3/2}\sum_{n_1,n_2,n_3=0}^{\infty}\lambda^{n_1+n_2+n_3}|n_1n_1,n_2n_2,n_3n_3\rangle$$



  - probability for one photon per triple above and below is $\lambda^2(1-\lambda^2)^3$

- **Input** $|\psi_3\rangle_{1b,3b} = (|0\rangle_{1b}|0\rangle_{3b} + |1\rangle_{1b}|1\rangle_{3b} + |2\rangle_{1b}|2\rangle_{3b})/\sqrt{3}$



- Post-select on one photon per triple mode 1,2,3. Success probability 9/55

- **Output** $\{2|0\rangle_{2b}\big(|0\rangle_{1b}|0\rangle_{3b} + |1\rangle_{1b}|1\rangle_{3b} + |2\rangle_{1b}|2\rangle_{3b}\big) - |2\rangle_{2b}\big(|0\rangle_{1b}|1\rangle_{3b} + |1\rangle_{1b}|2\rangle_{3b} + |2\rangle_{1b}|0\rangle_{3b}\big)$
$-|1\rangle_{2b}\big(|0\rangle_{1b}|2\rangle_{3b} + |1\rangle_{1b}|0\rangle_{3b} + |2\rangle_{1b}|1\rangle_{3b}\big)\}/(3\sqrt{2}),$

Measuring fusion data (a simple method)

- Braid flux around one charge at vertex v $T_h(v)|\mathbf{1}_{R_2};(v,v')\rangle = |R_2(h);(v,v')\rangle$
  - Measure projector onto irrep for fluxless charge pair

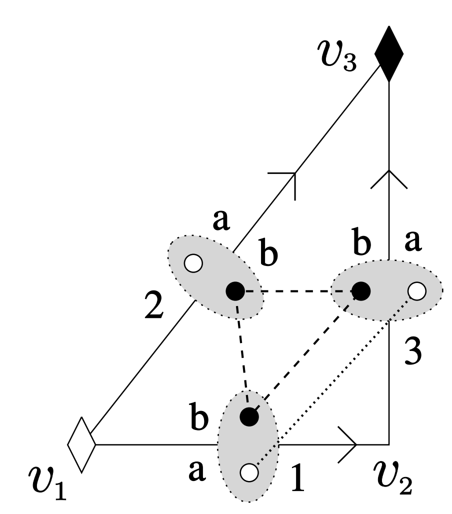
$$\langle R(h)|W_{R'}|R(h)\rangle = \sum_{a,b,d,e=1}^{|R|}\sum_{c=1}^{|R'|} Q^{[RR'^*R^*]}_{acd,bce} R^*_{ab}(h) R_{de}(h)$$

- Projectors onto vacuum fusion channel for three irreps

$$Q^{[R^{(1)}R^{(2)}R^{(3)}]}_{ace,bcf} = \frac{1}{|G|}\sum_g R^{(1)}_{ab}(g) R^{(2)}_{cd}(g) R^{(3)}_{ef}(g)$$

- Just local (non-entangling) operations + measurement on photons in 2a and 2b
  - signature of non-Abelian statistics

$$\langle R_2(h);(v,v')|W_{R_2}(e)|R_2(h);(v,v')\rangle = \begin{cases} 1 & h=e \\ -\frac{1}{2} & h=c_\pm \\ 0 & h=t_j\ \forall j \end{cases}$$



Summary

- Good quantum simulation requires good quantum control
- Can simulate emergent physics
  - Build highly entangled spin networks corresponding to vacuum states of models with exotic excitations
  - Can manipulate these excitations and measure their properties
- Photonic spin networks look promising in the near term for demonstrating prototype models
  - Integrated photonics* with waveguides etched into glass is a good platform

*A. Politi, M. J. Cryan, J. G. Rarity, S. Yu, and J. L. O'Brien, Science **320**, 646 (2008).

- Larger simulations become inefficient due to bad scaling of probability to create many spin entangled states. There are work arounds but other systems such as Josephson junctions or trapped atoms/molecules in optical lattices may be better