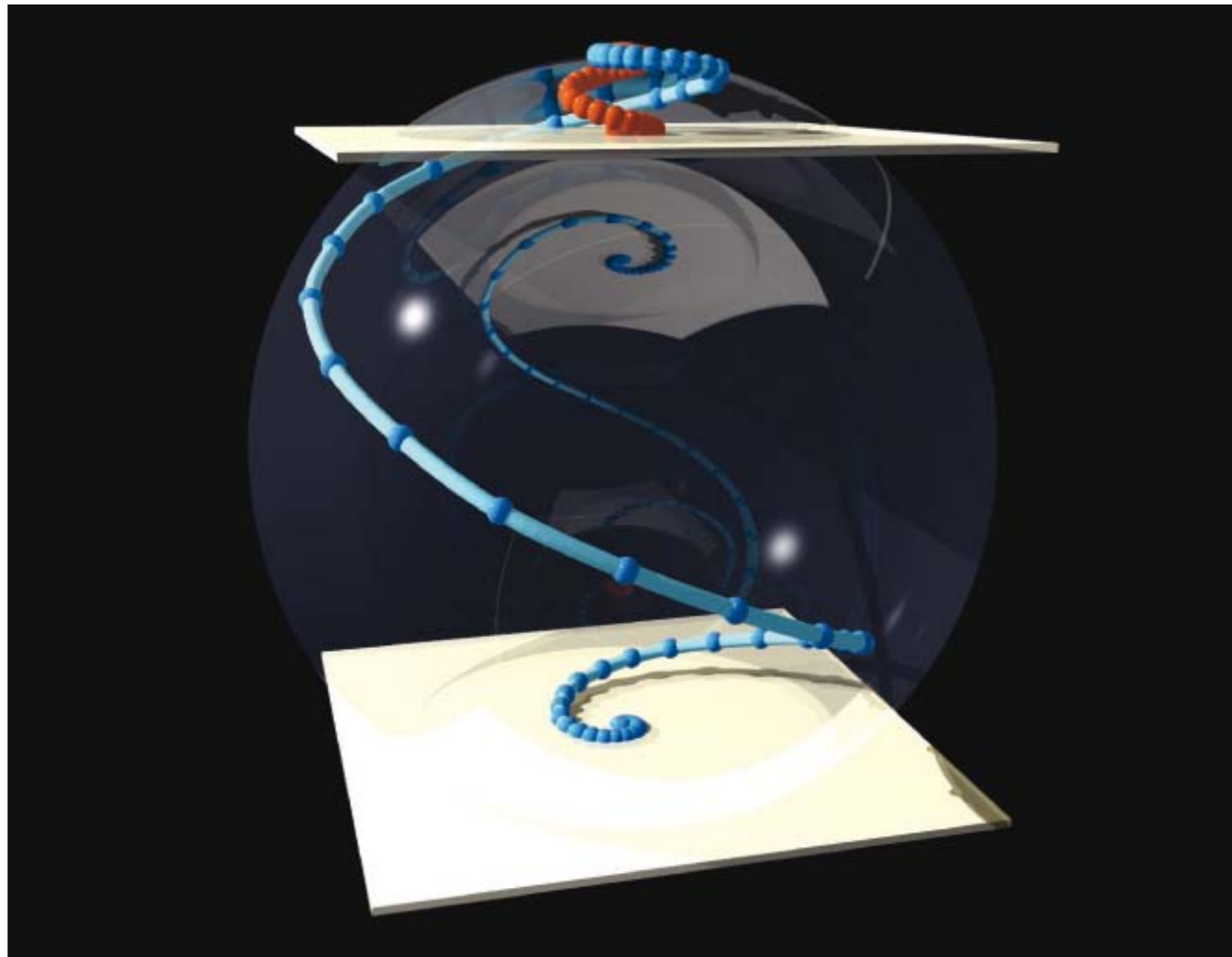
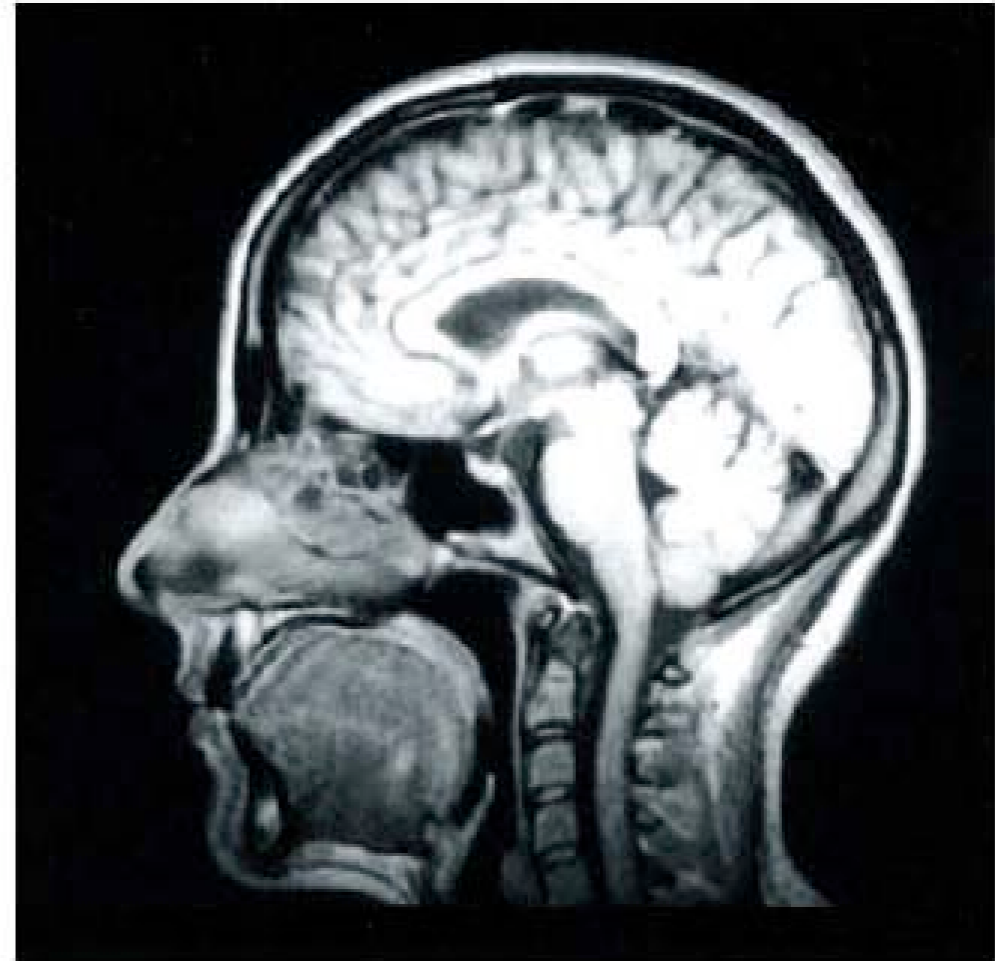
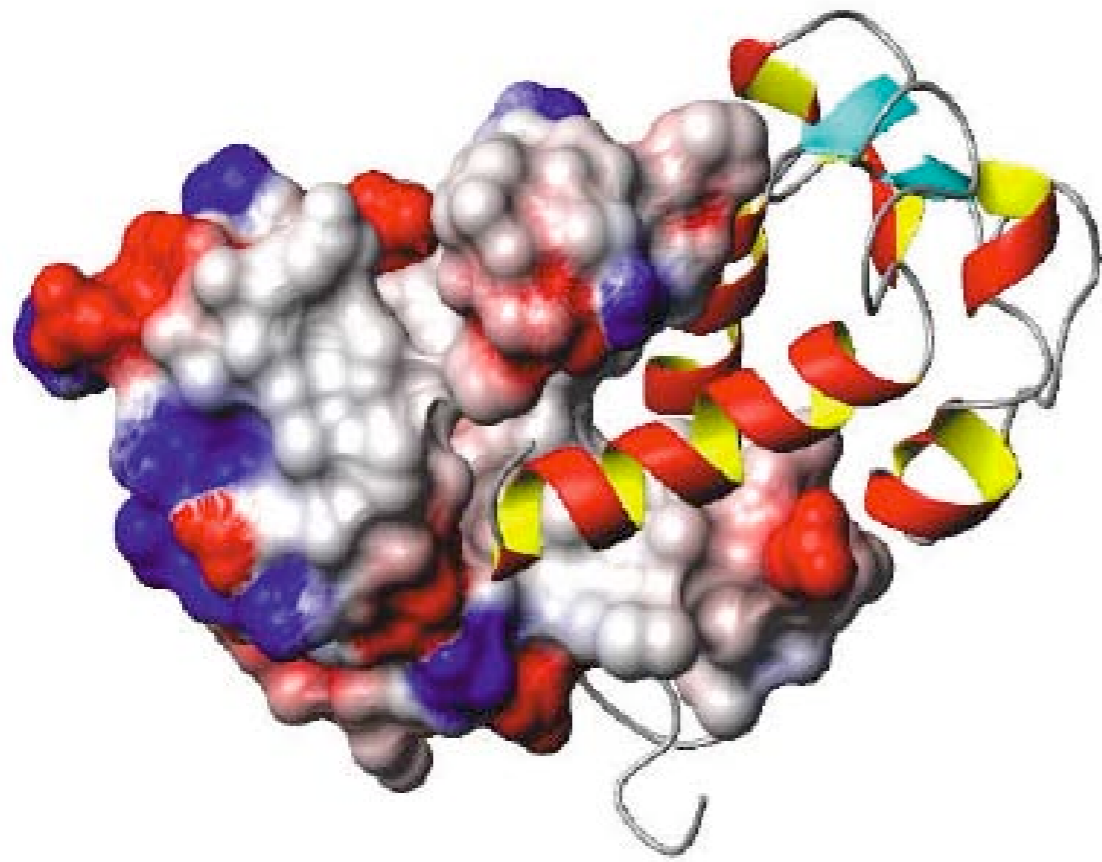


Optimal Control of Spin Dynamics

Steffen Glaser, TU München





Nobel Prizes:

1952: Edward Purcell, Felix Bloch (Physics)

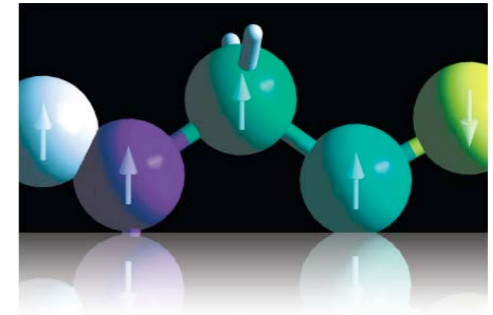
1991: Richard Ernst (Chemistry)

2002: Kurt Wüthrich (Chemistry)

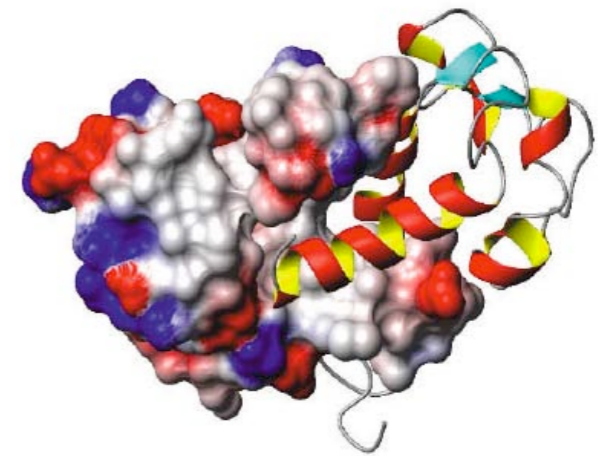
2003: Paul Lauterbur, Peter Mansfield (Medicine)

Optimal Control in Spin Systems

physical limits of spin dynamics

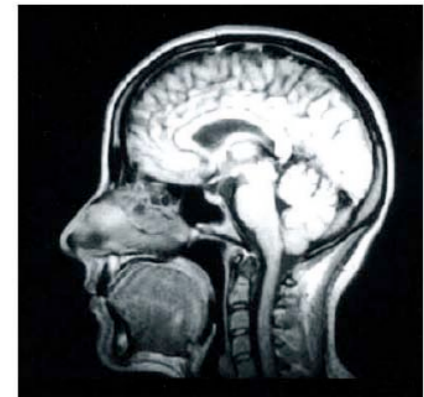


spectroscopy

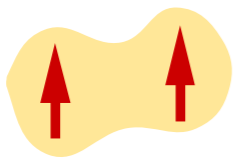


quantum computing

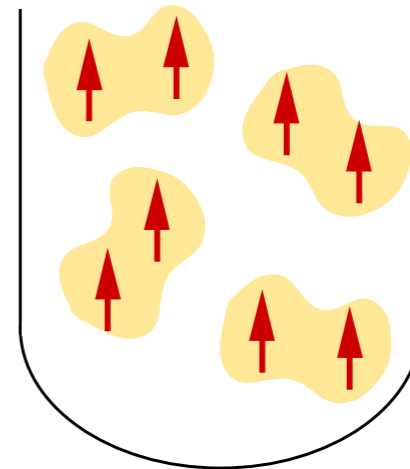
local spin manipulation and imaging







Pure state $|\Psi\rangle$



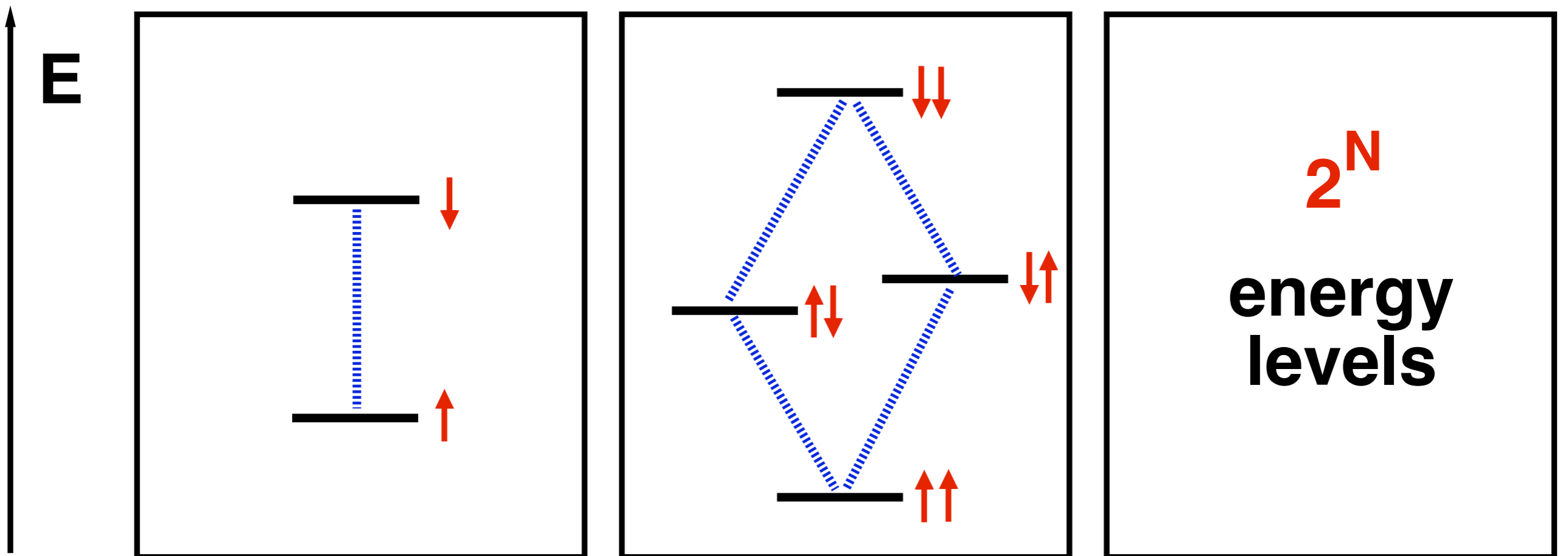
Density operator $\rho = \overline{|\Psi\rangle\langle\Psi|}$

Energy levels

1 spin 1/2

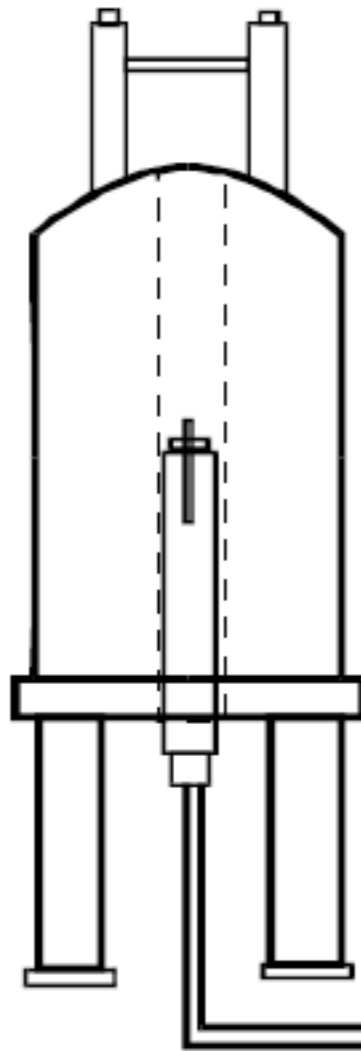
2 spins 1/2

N spins 1/2

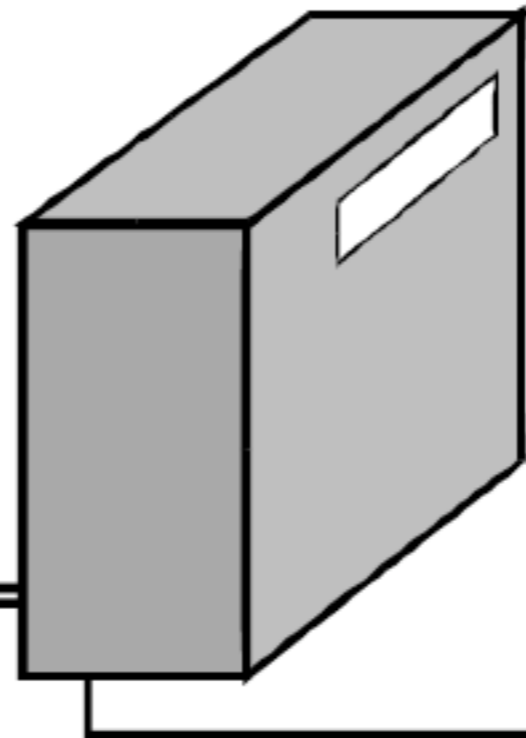


How do you measure an NMR signal?

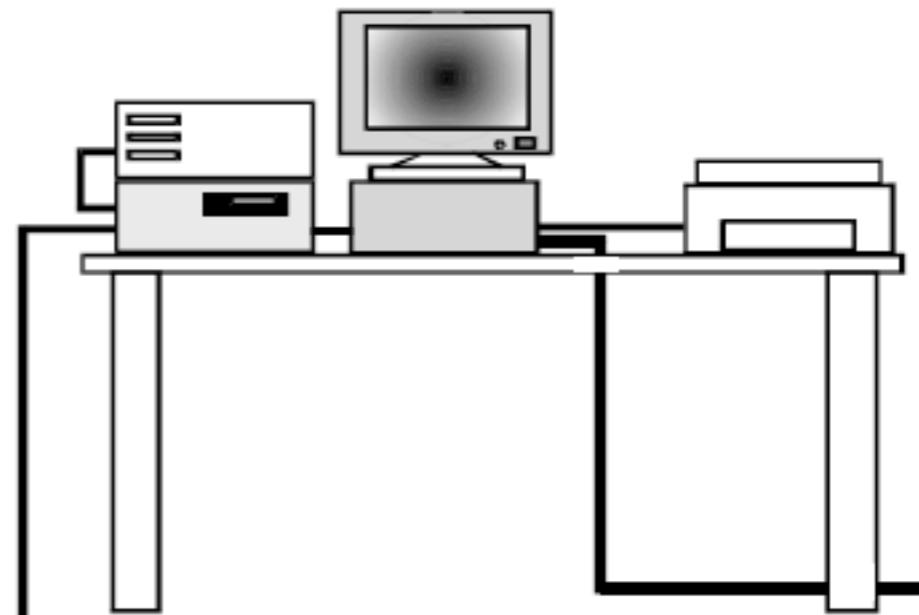
superconducting magnet incl. probe (with rf coil)



rf electronics (frequency generators, amplifiers, receiver)

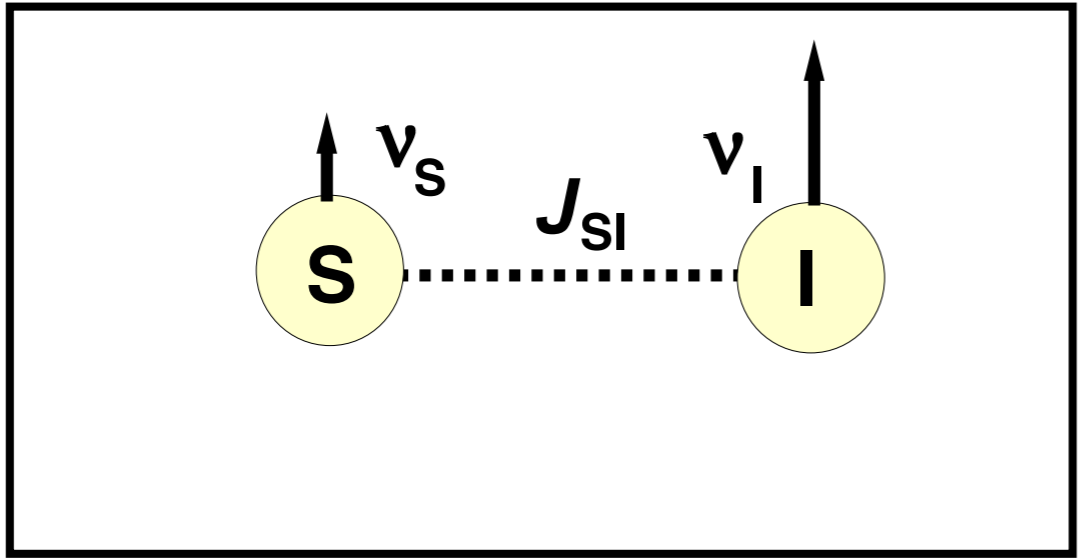


computer for spectrometer control and data processing

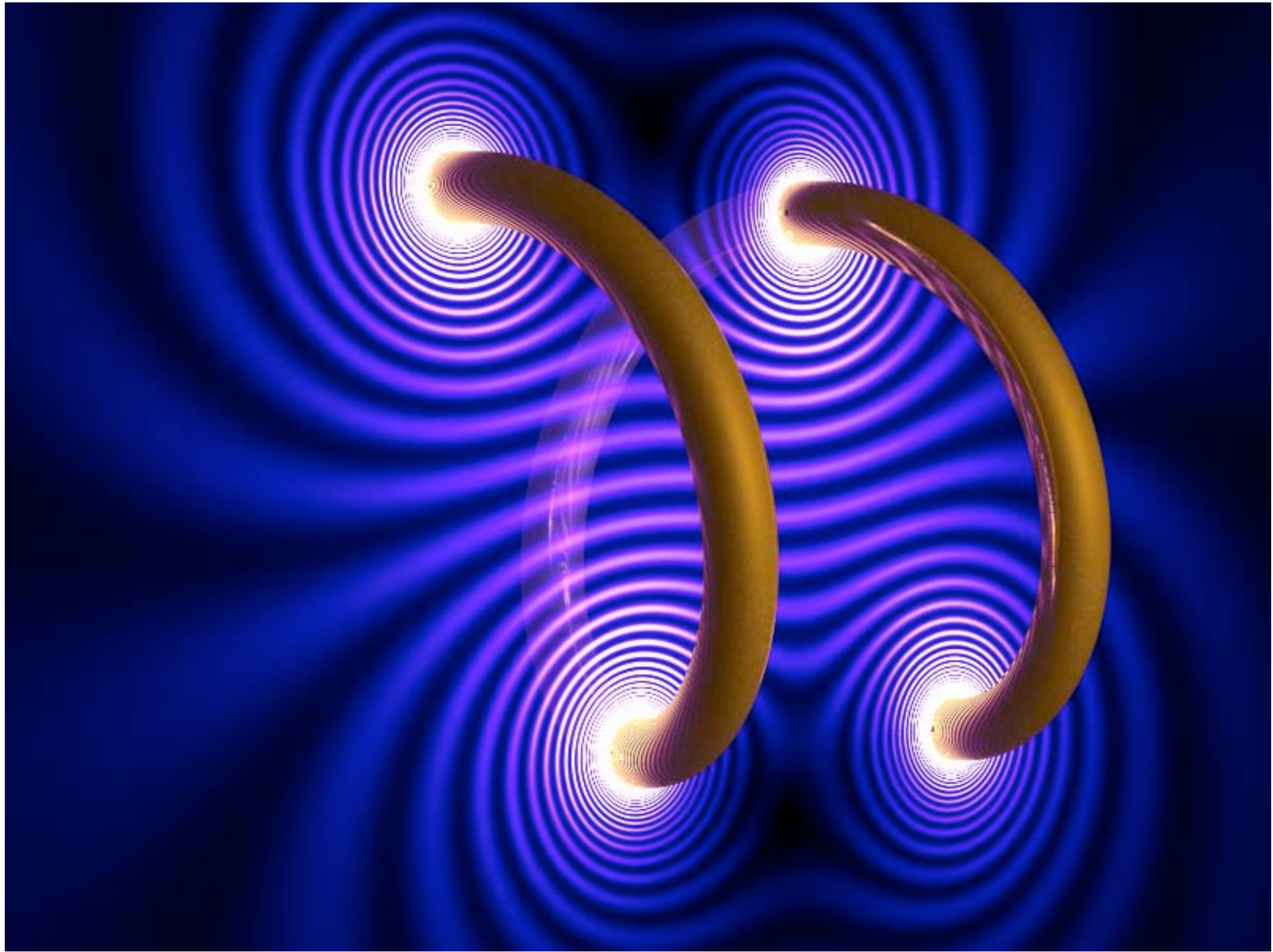


data link

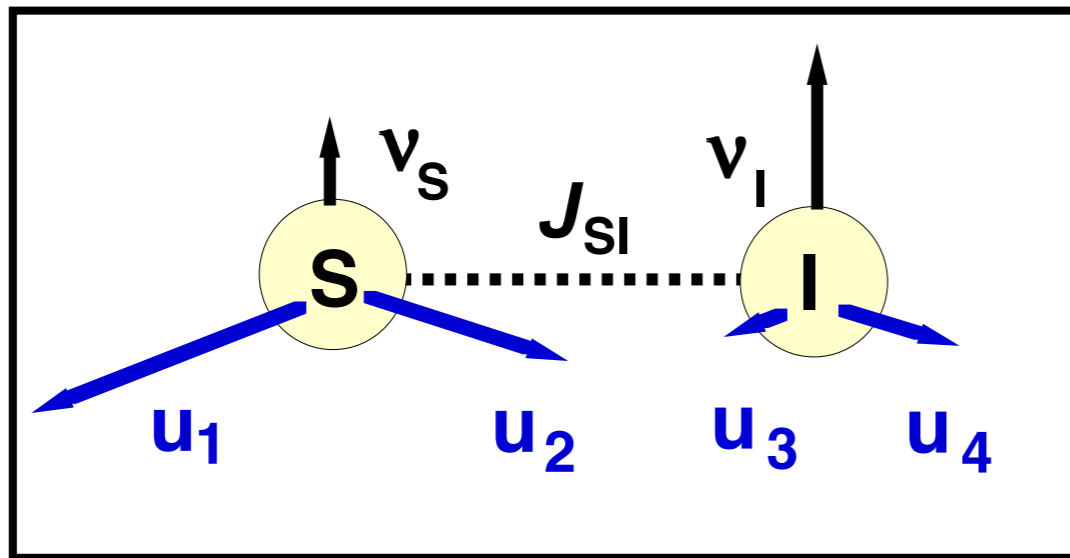




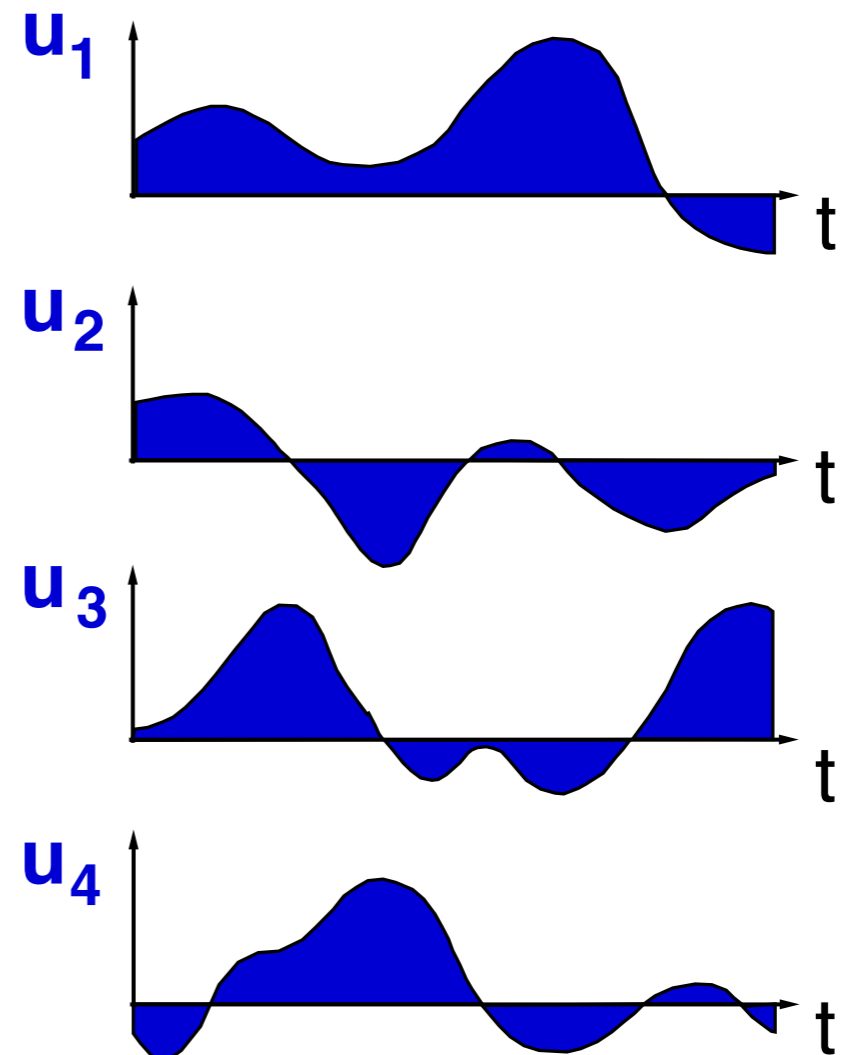
H_0



Control Parameters $u_k(t)$



$$H_0 + \sum_k u_k(t) H_k$$

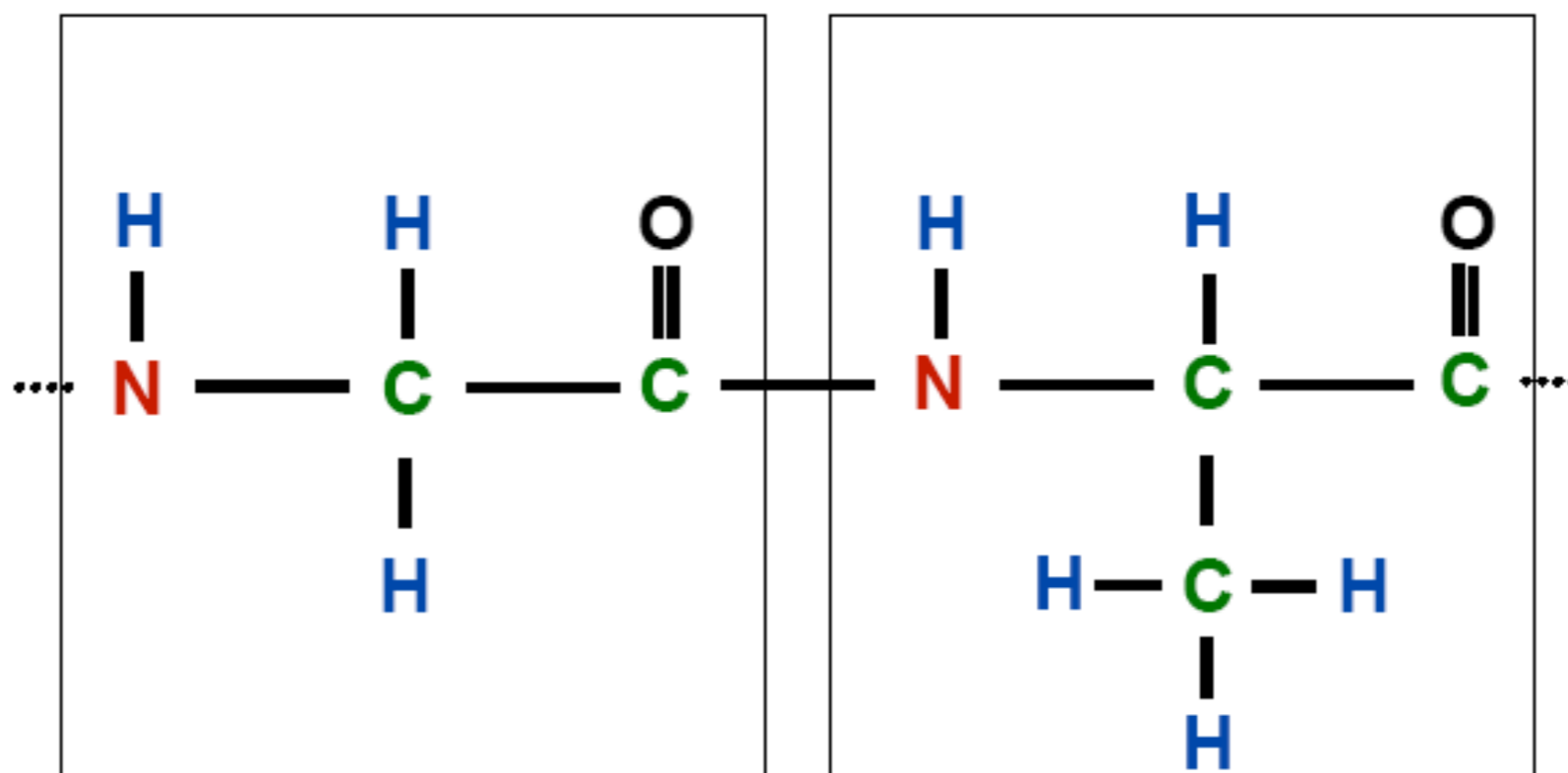
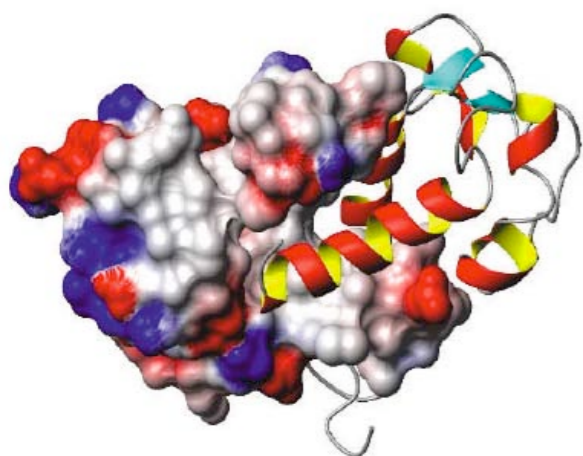


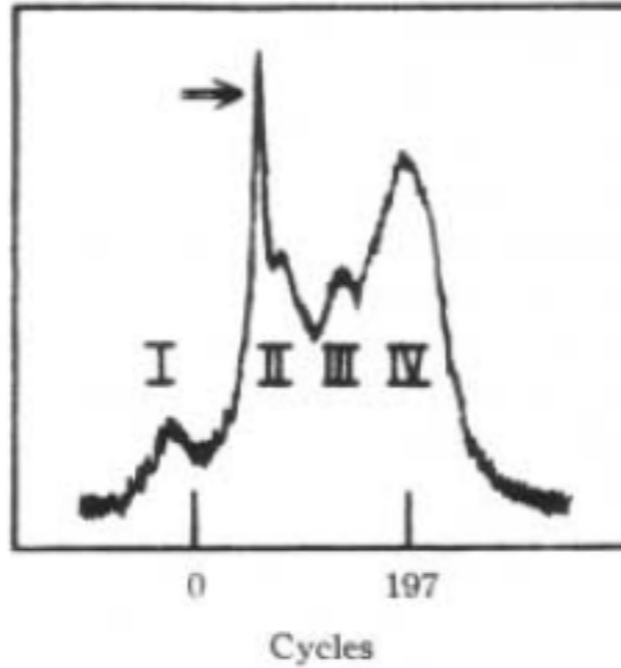
Resonance frequencies at 14 Tesla:

^1H 600 MHz

^{15}N 60 MHz

^{13}C 150 MHz

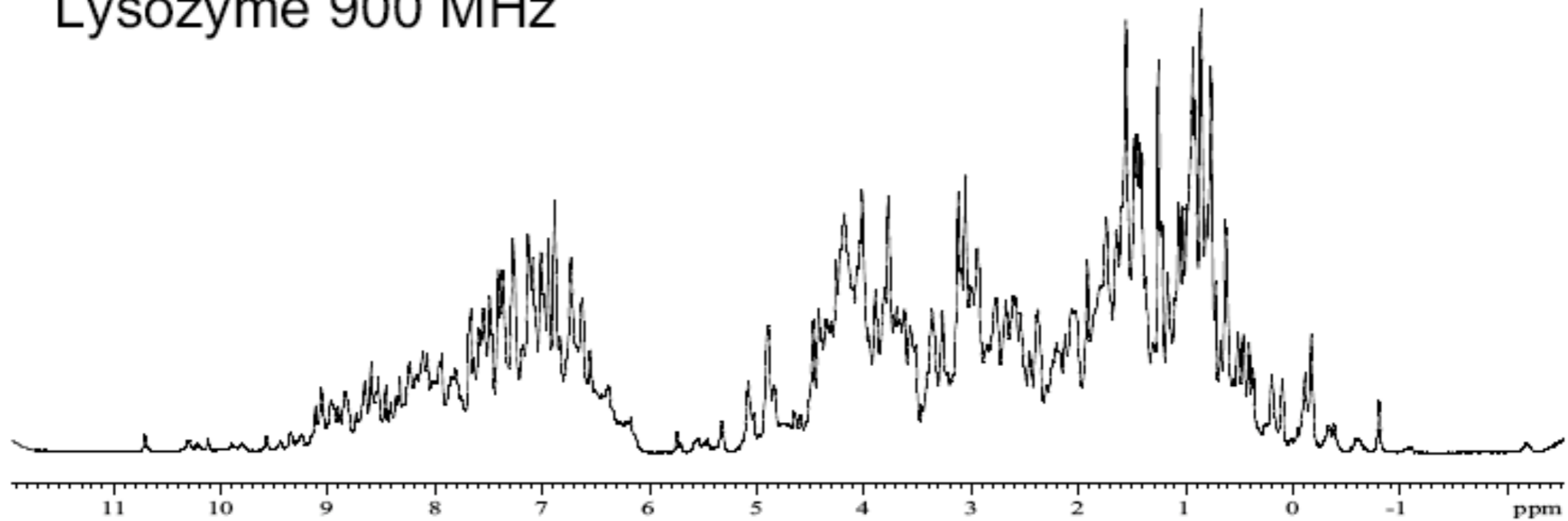




Ribonuclease
40 MHz

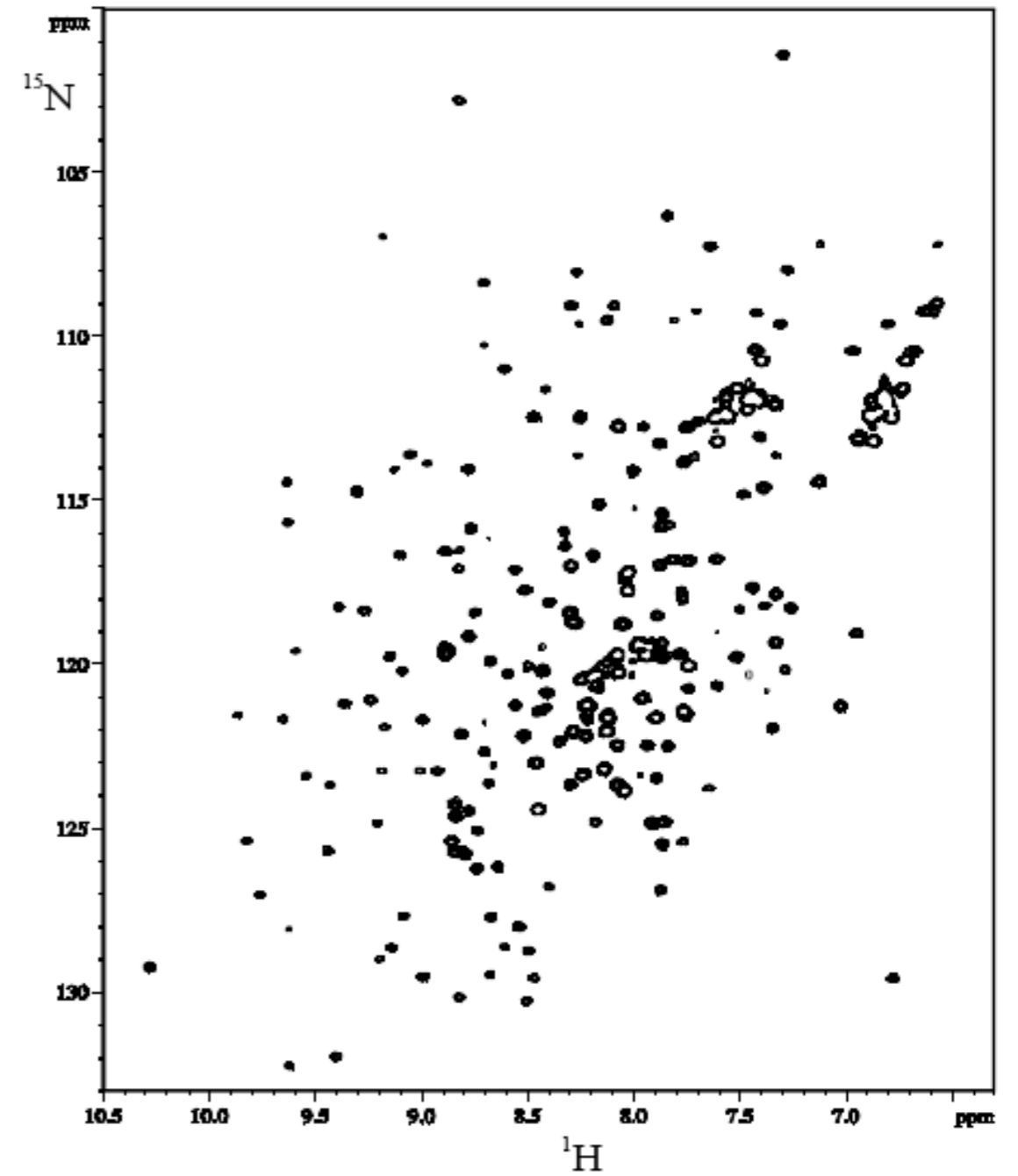
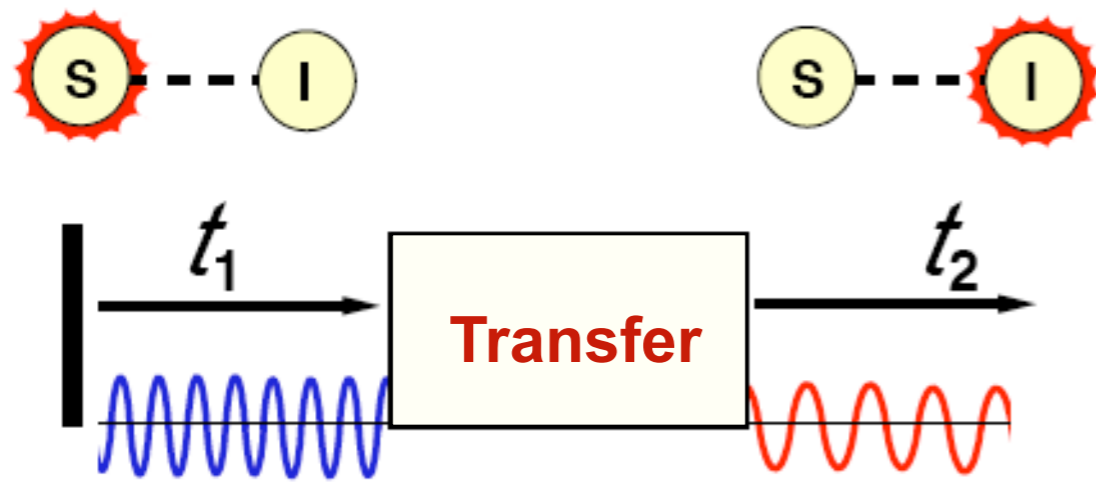
M. Saunders et al.
J.Amer.Chem.Soc. **1957**,
79, 3289

Lysozyme 900 MHz



frequency dispersion: 10 kHz

Two-dimensional NMR

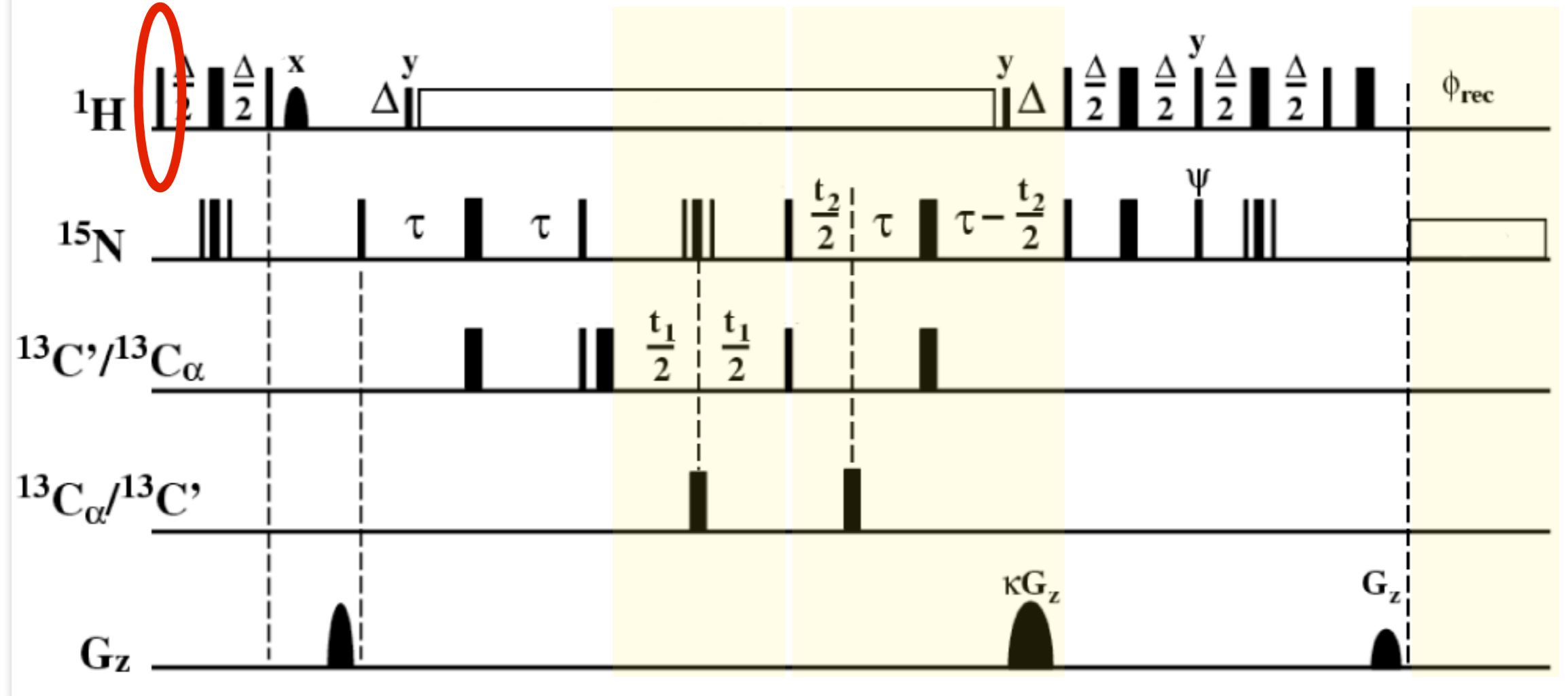


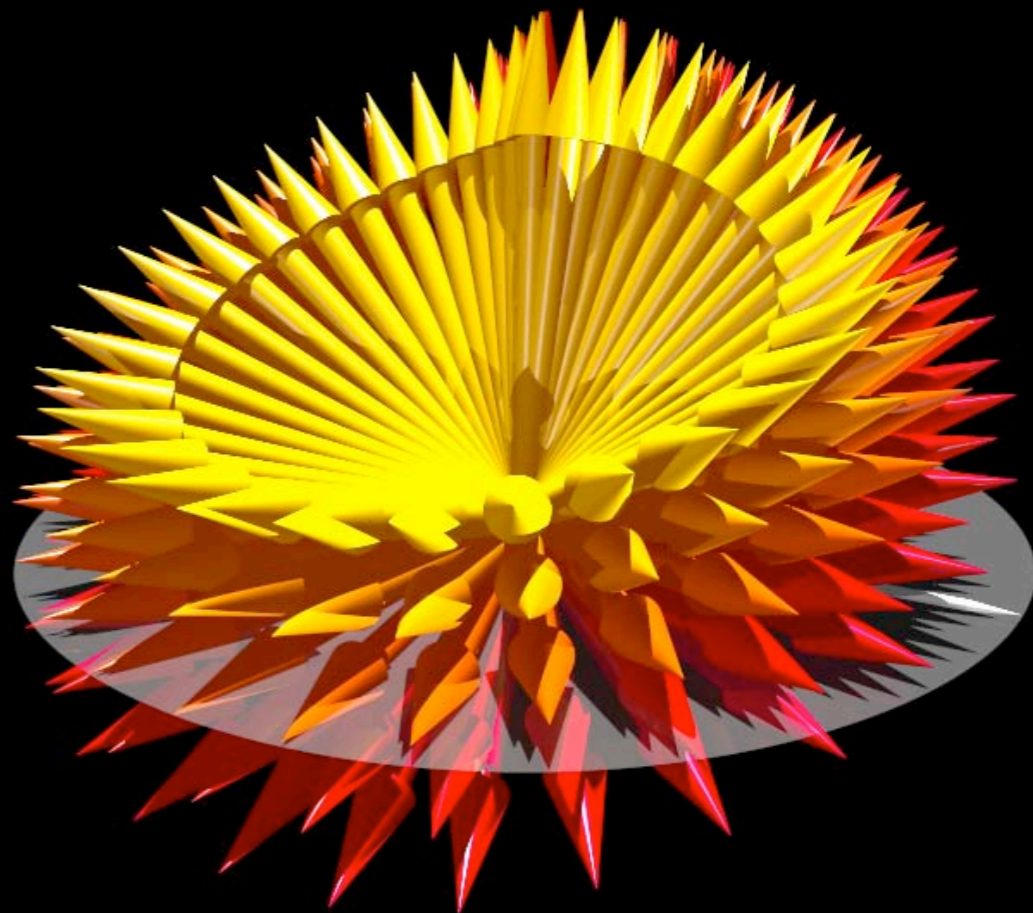
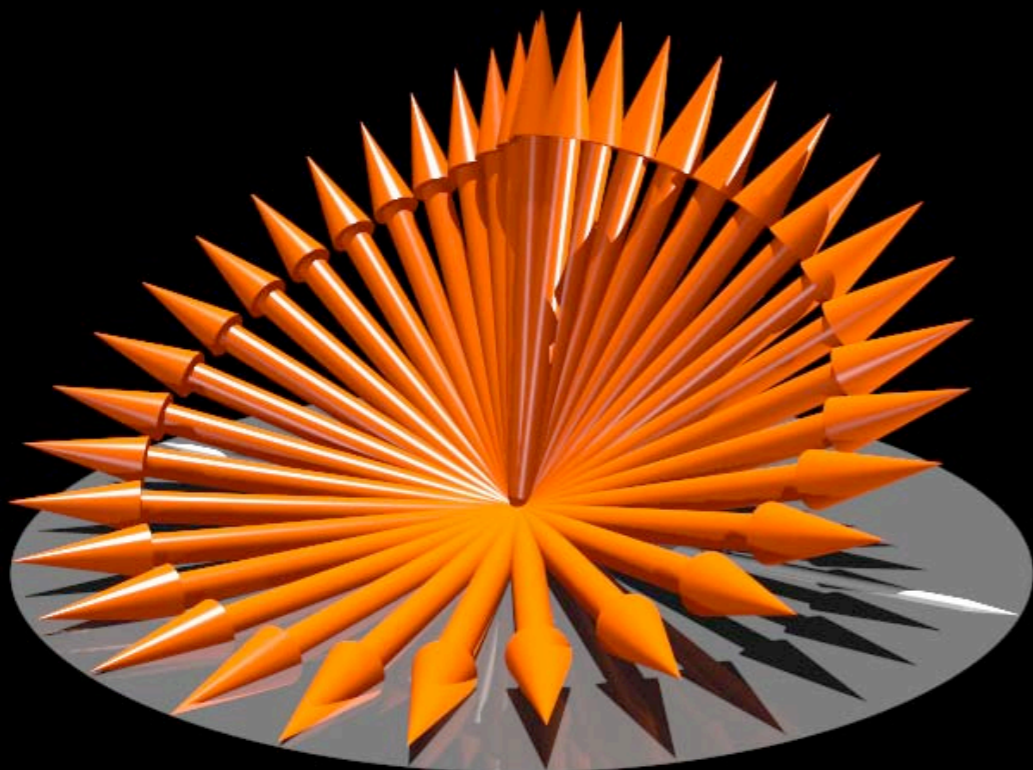
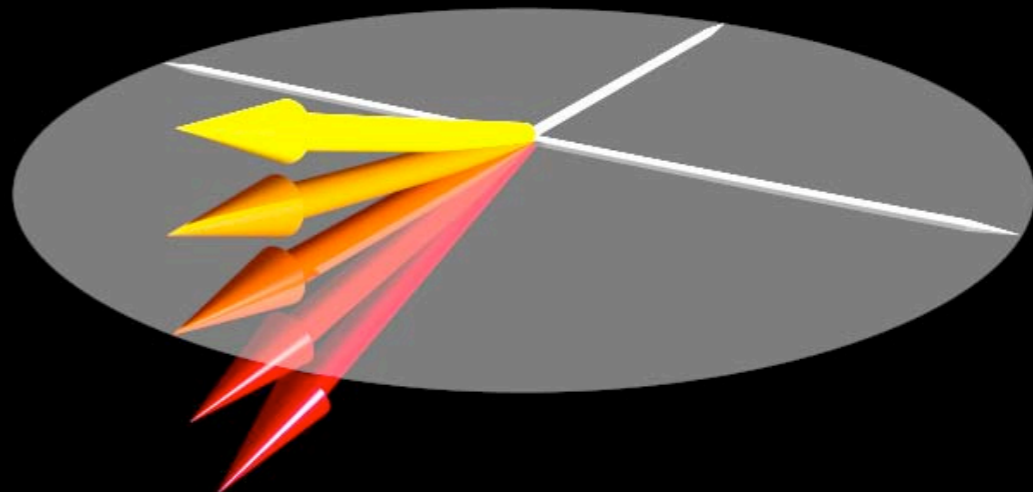
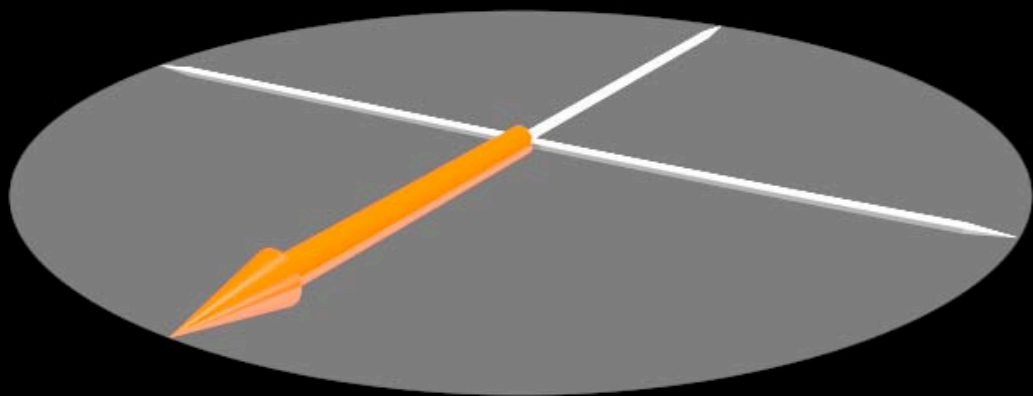
t1

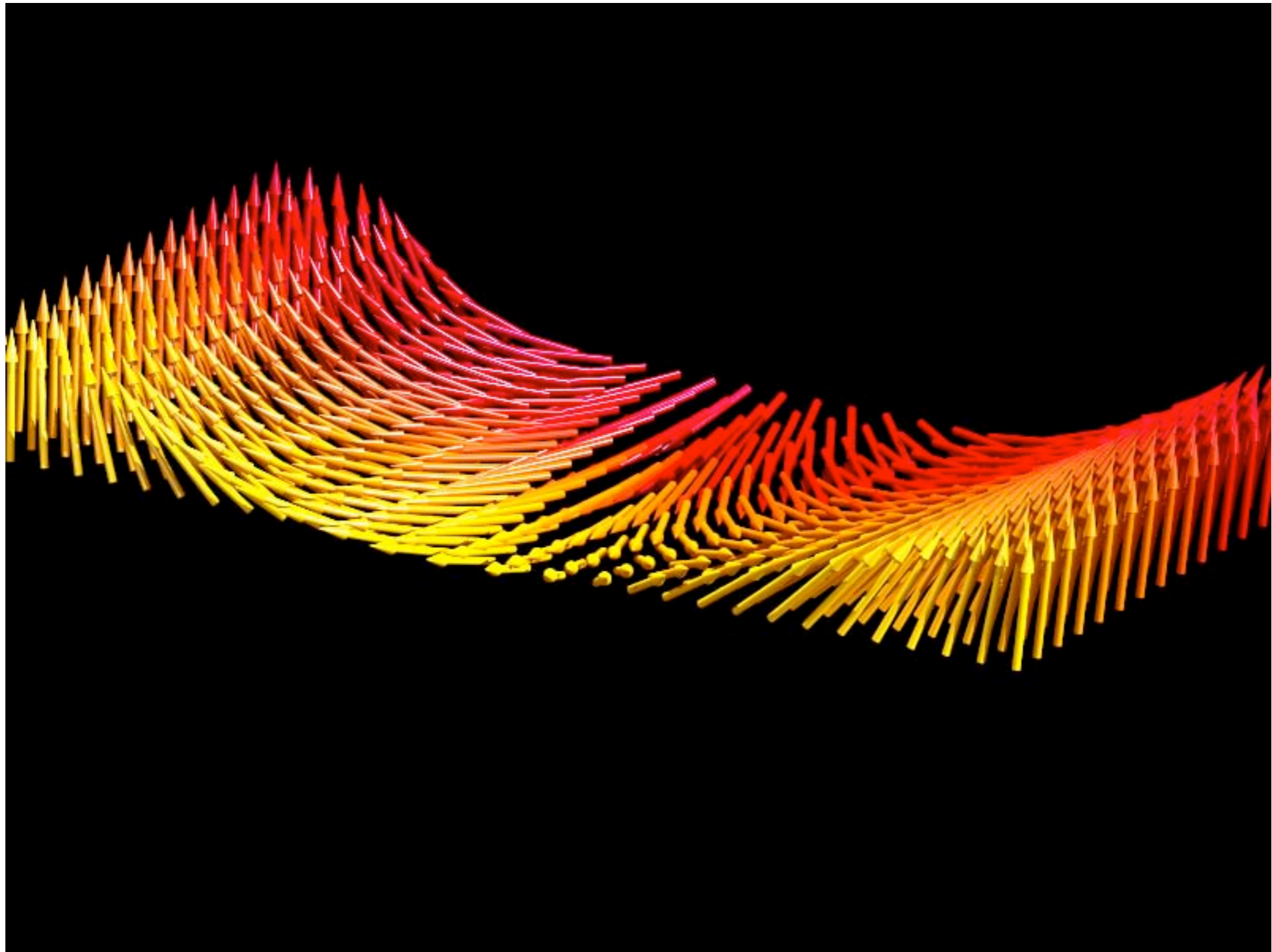
t2

t3

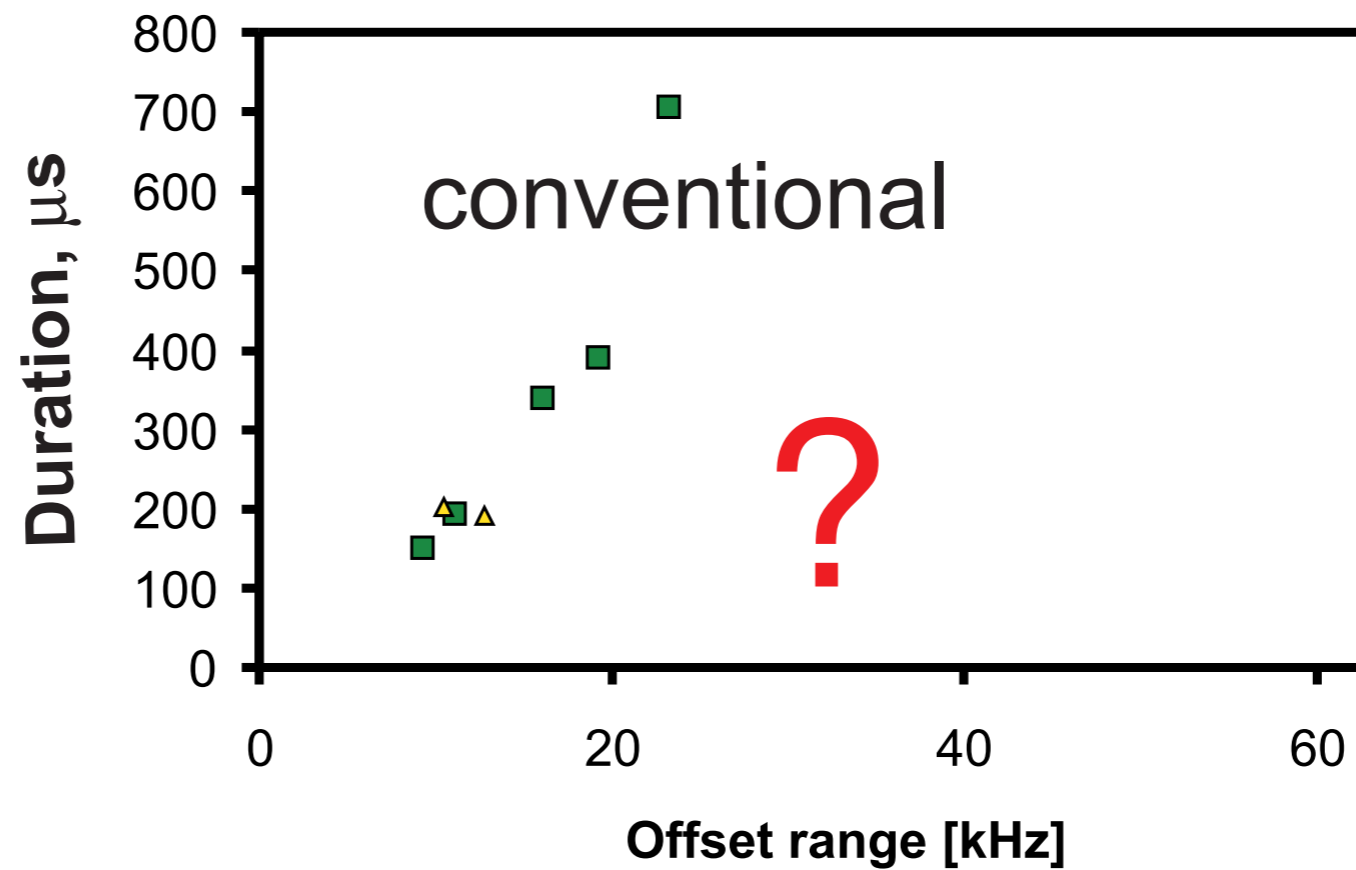
3D HNCO / HNCA







Performance of conventional composite pulses for broadband (robust) excitation



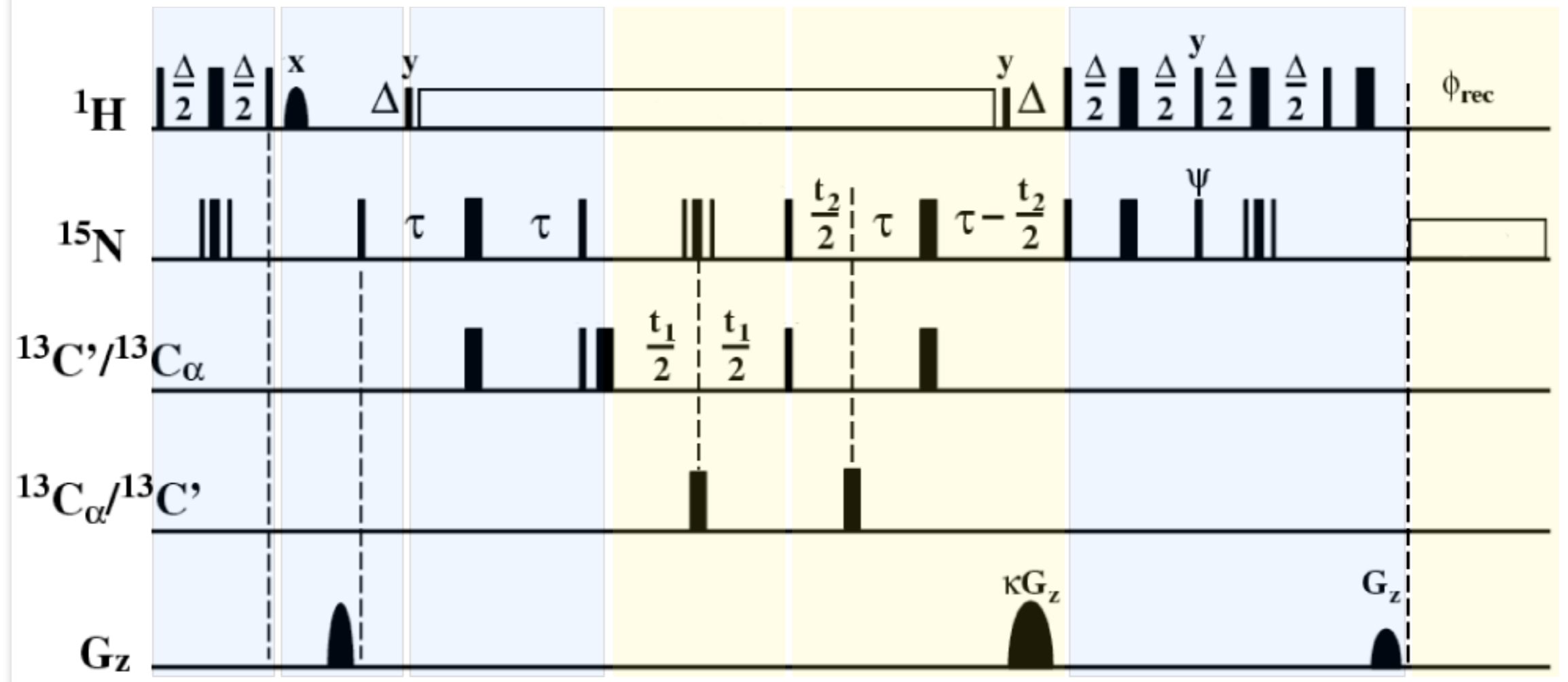
(excitation efficiency: 98%, max. rf amplitude: 10 kHz, no rf inhomogeneity)

t1

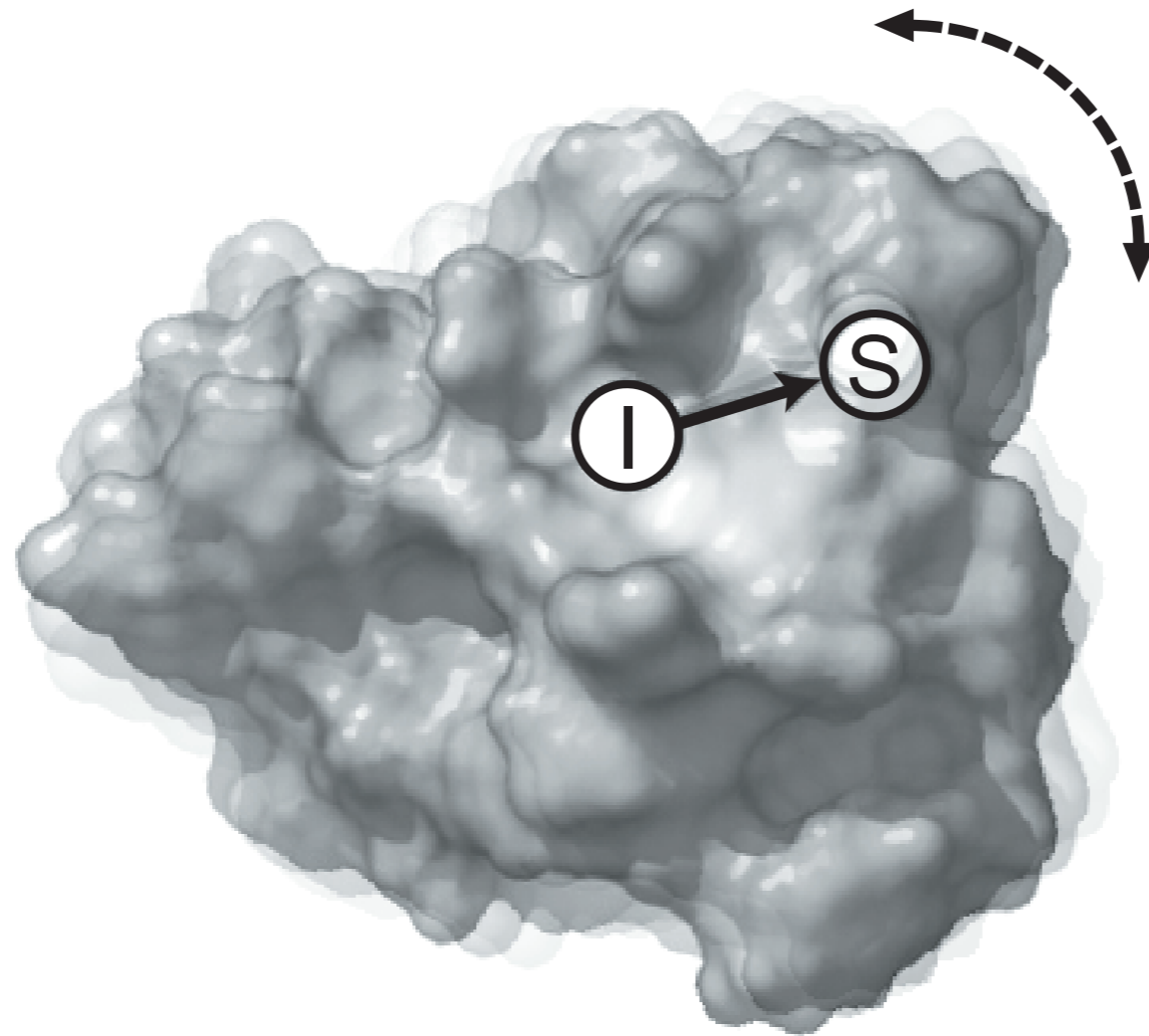
t2

t3

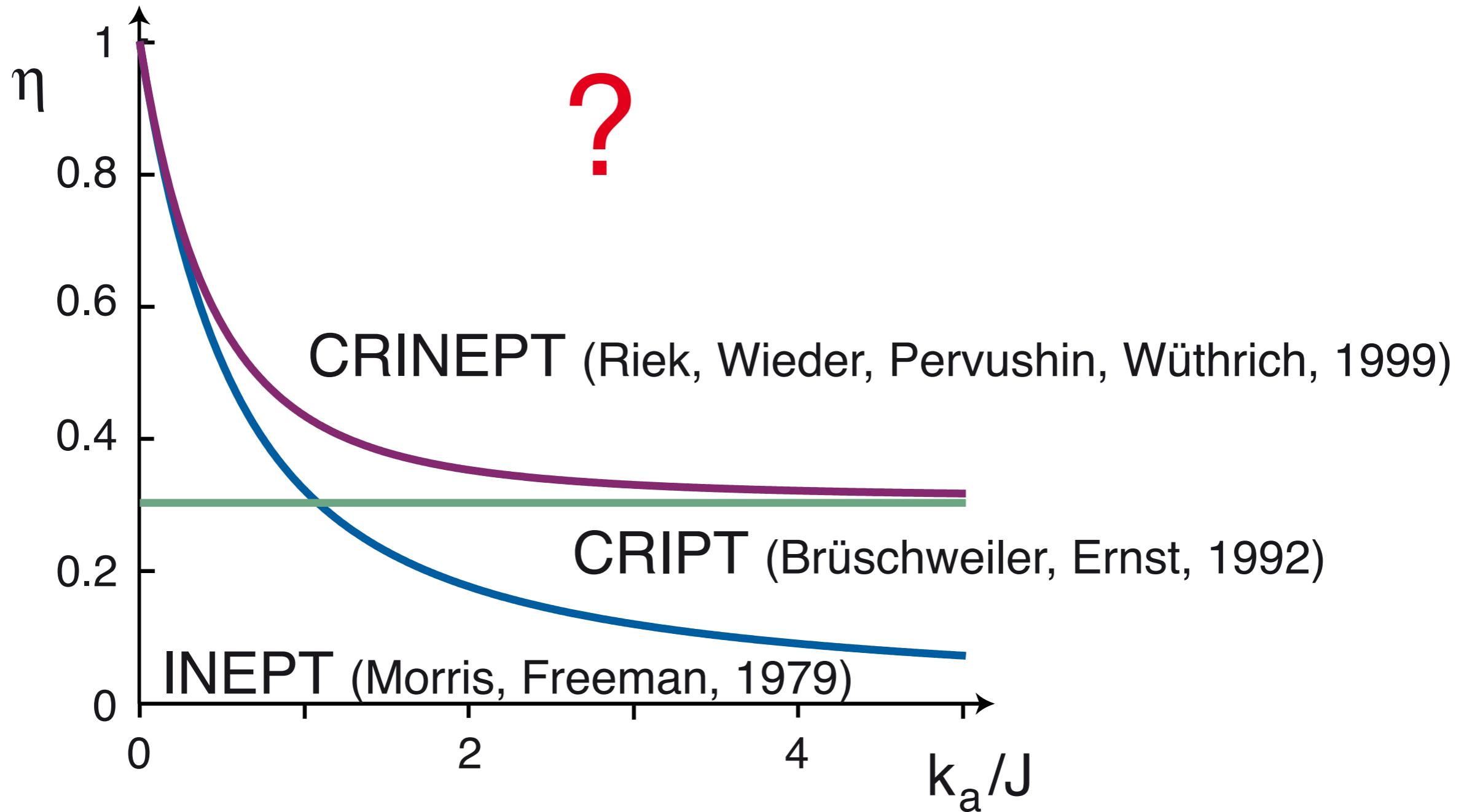
3D HNCO / HNCA



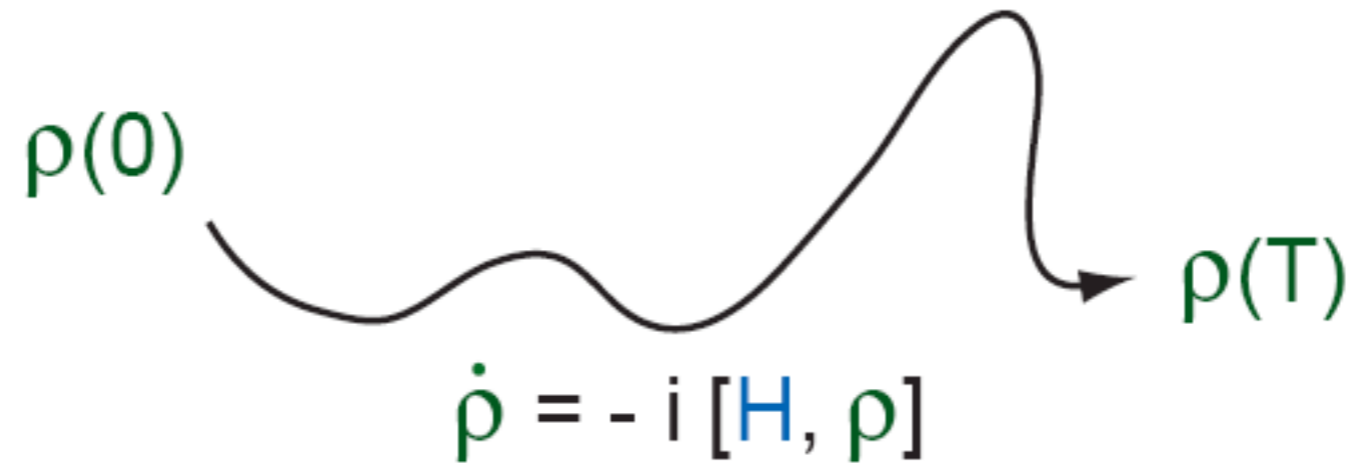
Relaxation rates k increase with molecular weight



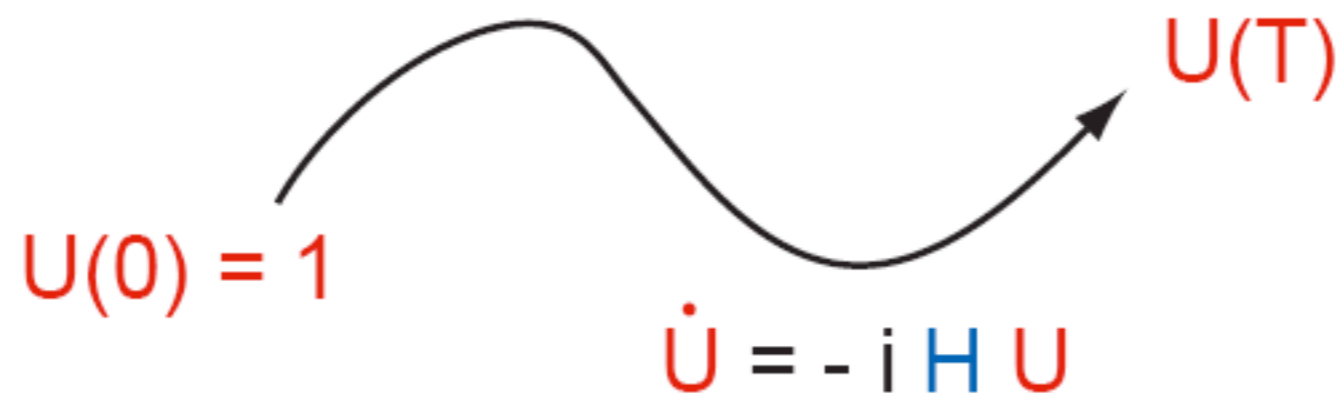
Transfer Efficiency $I_x \rightarrow 2I_z S_y$



Transformation of the Density Operator



Generation of Unitary Operators



NMR Quantum Computing

INPUT

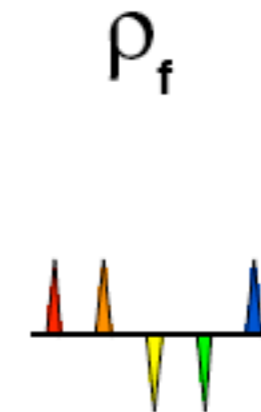
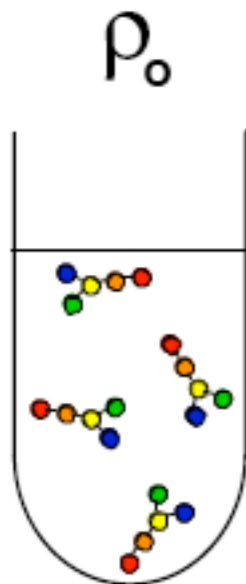
CALCULATION

OUTPUT

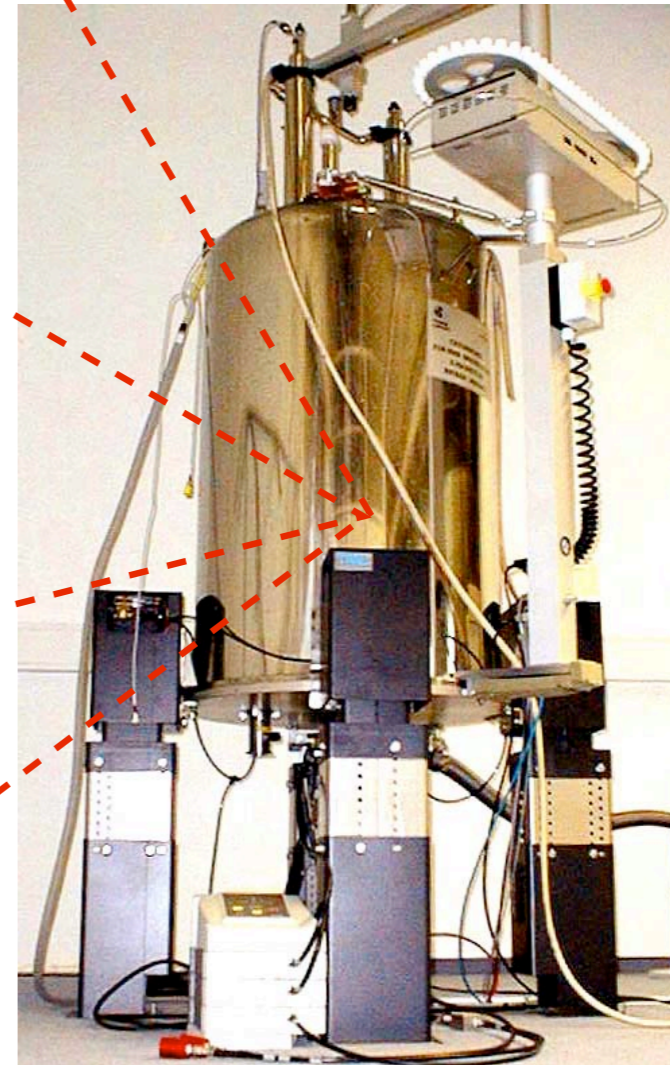
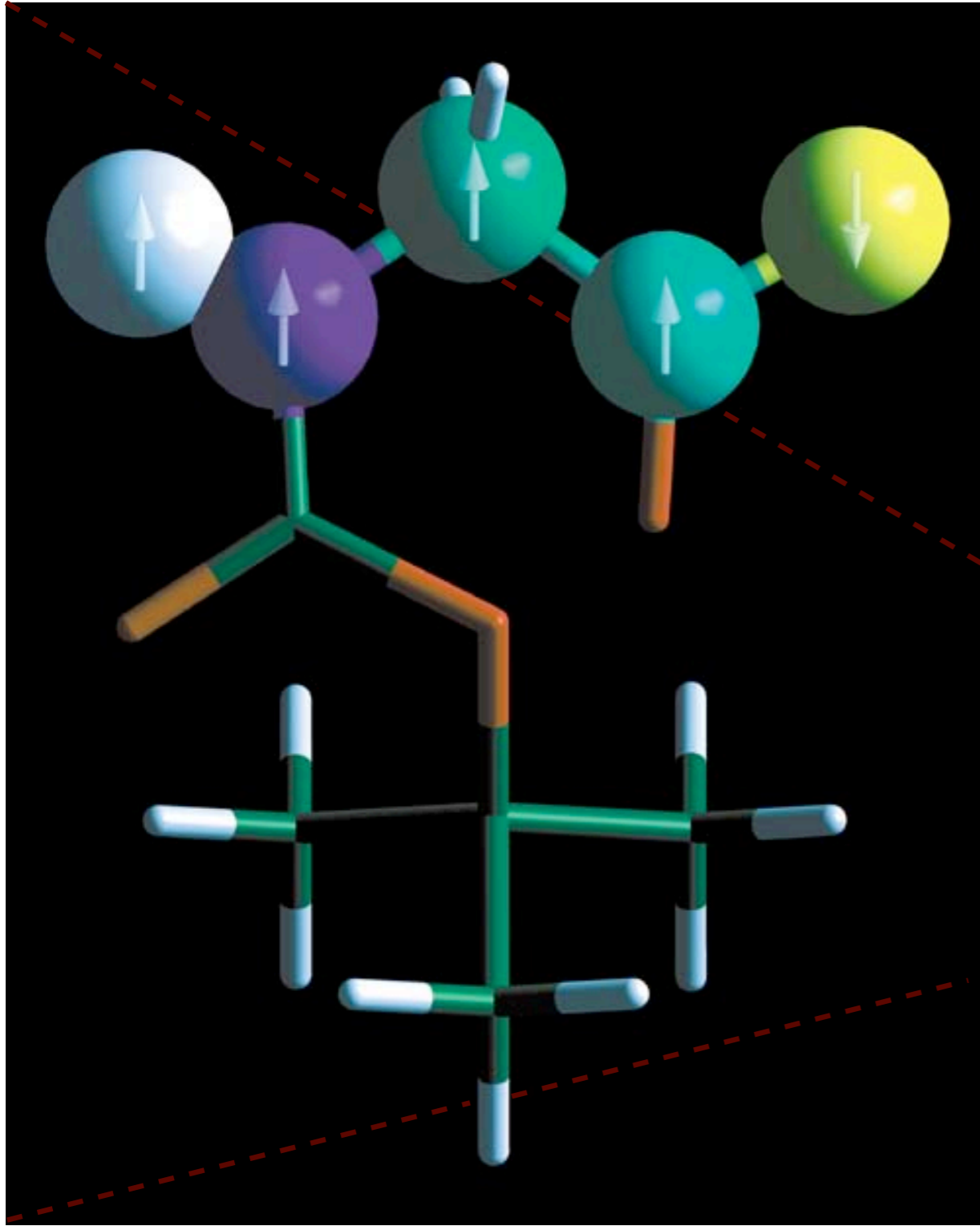
prepare
initial state

apply
quantum-algorithm

detect
NMR spectrum



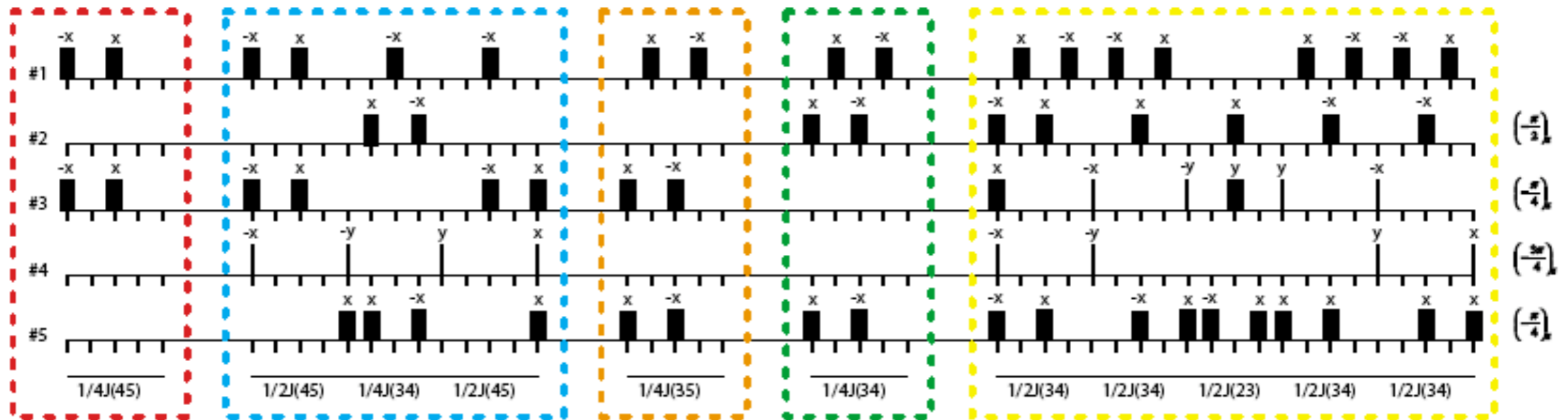
Cory, Fahmy, Havel (1996)
Gershenfeld, Chuang (1997)



Implemented Test Functions for Thermal Deutsch Jozsa Algorithm

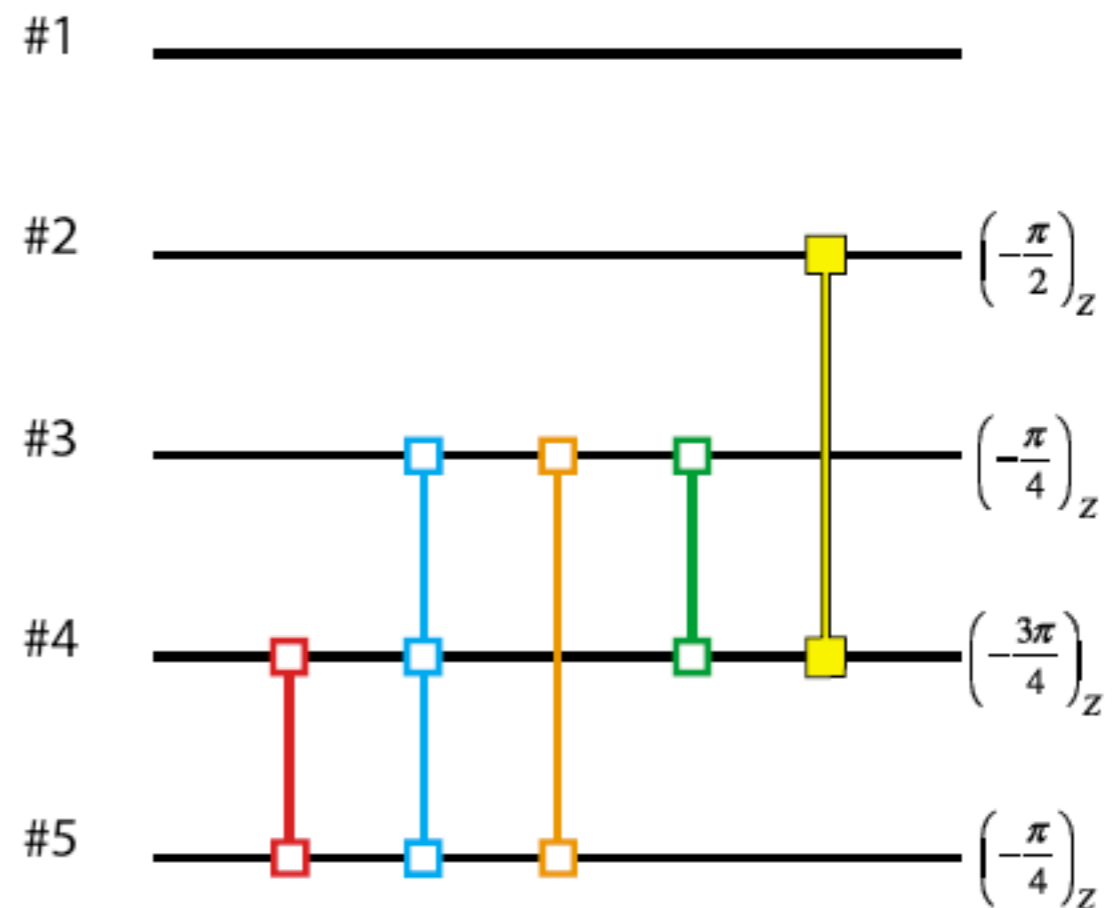
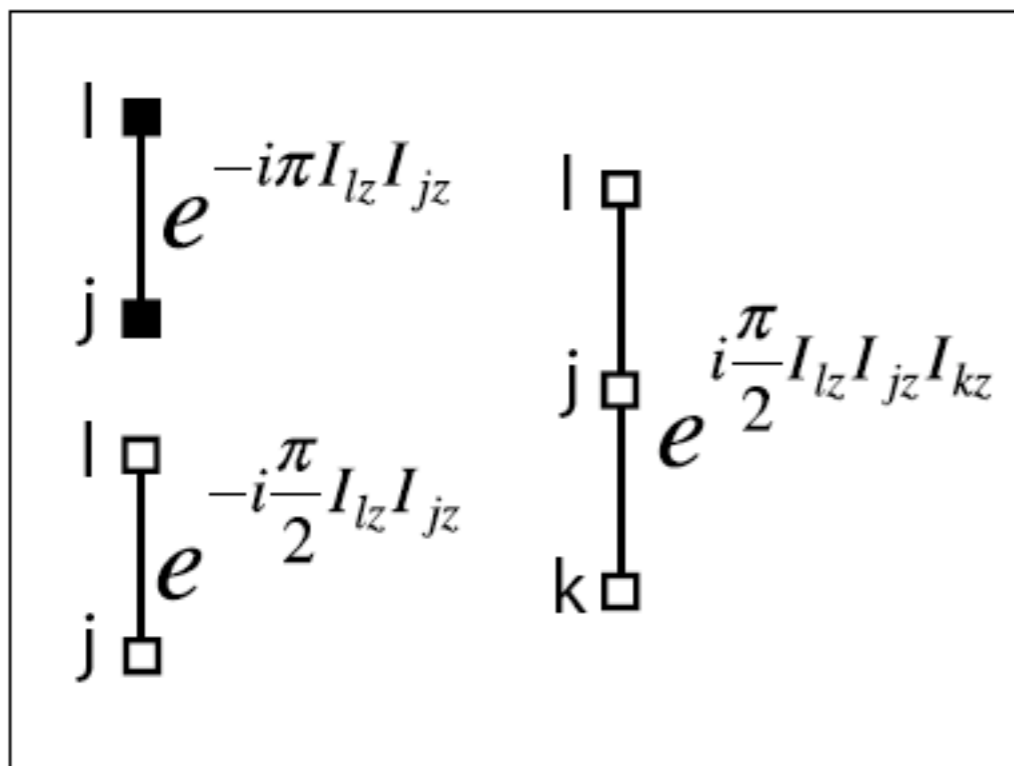
constant function: $f(\vec{x}) = 0$
 $f'(\vec{x}) = f(\vec{x}) \cdot \overline{x_4} = 0$

balanced function: $f(\vec{x}) = x_2 \oplus x_3 \cdot x_5$
 $f'(\vec{x}) = f(\vec{x}) \cdot \overline{x_4} = x_2 \cdot \overline{x_4} \oplus x_3 \cdot \overline{x_4} \cdot x_5$

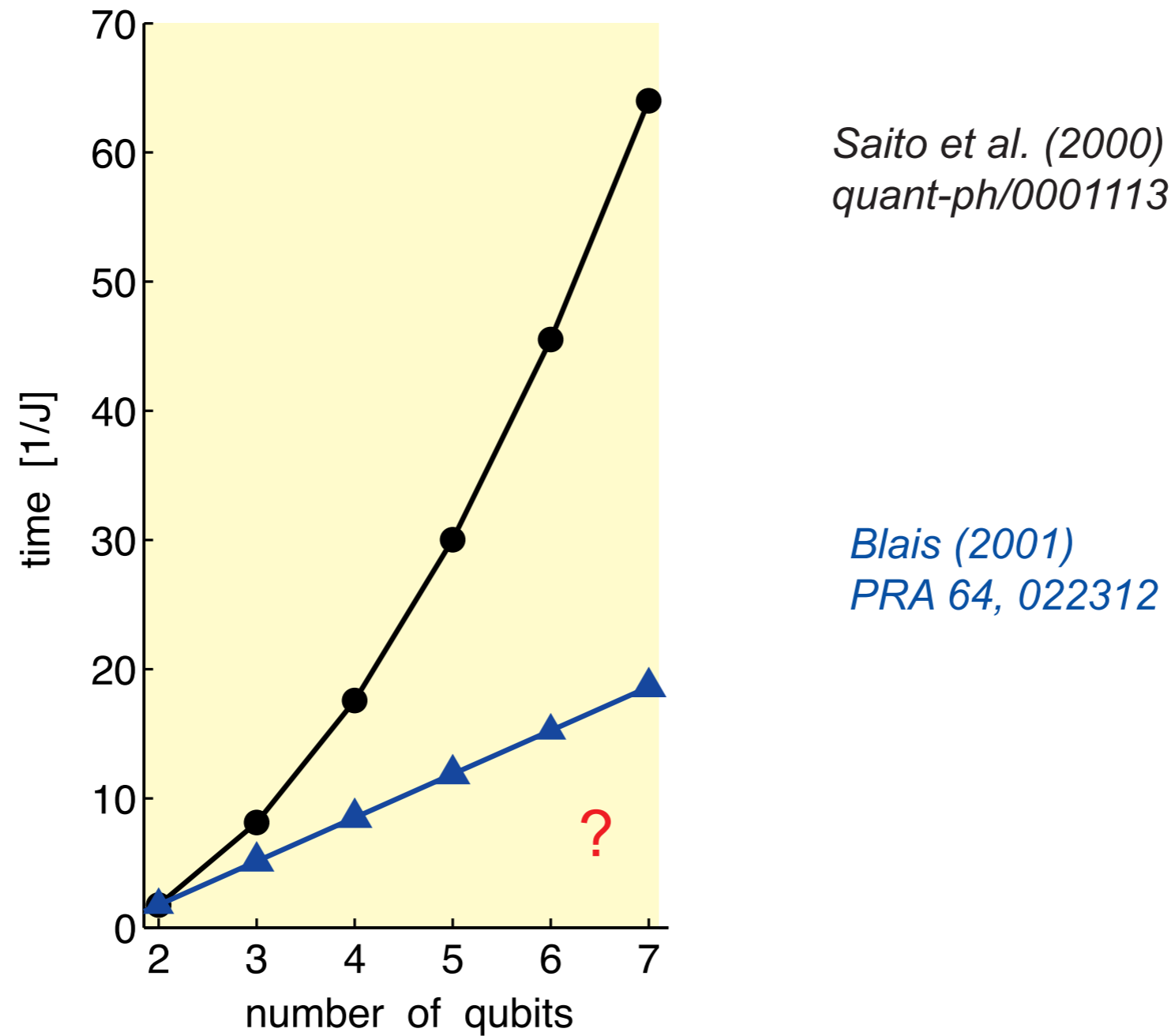


Scheme for the implementation of the function

$$f'(\vec{x}) = f(\vec{x}) \cdot \overline{x_4} = x_2 \cdot \overline{x_4} \oplus x_3 \cdot \overline{x_4} \cdot x_5$$



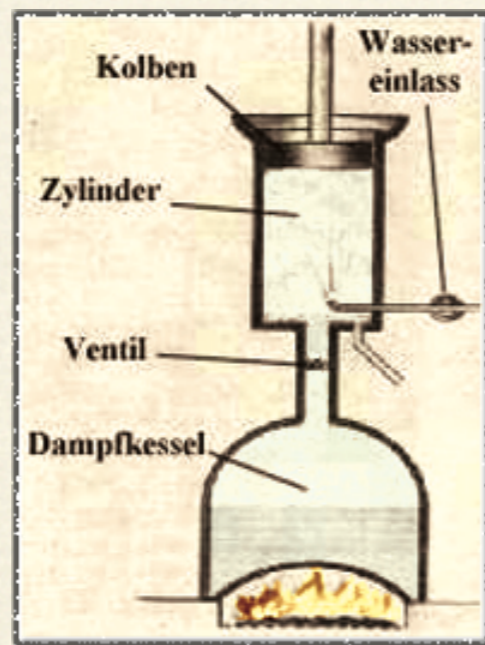
Time-optimal implementation of the quantum Fourier transform ?



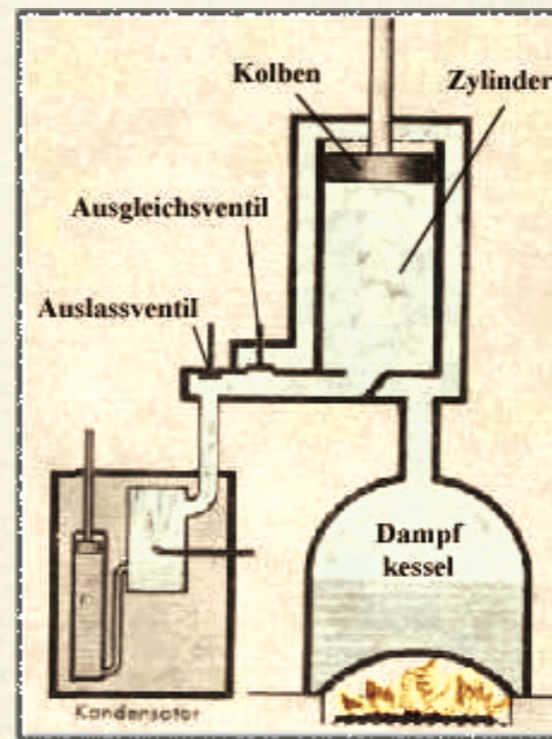
Steam Engine



1697
D. Papin



1712
T. Newcomen



1765
J. Watt



Steam Engine



„The theory of its operation is rudimentary and attempts to improve its performance are still made in an almost haphazard way.“

1824

RÉFLEXIONS
SUR LA
PUISSANCE MOTRICE
DU FEU

ET

SUR LES MACHINES
PROPRES A DÉVELOPPER CETTE PUISSANCE.

PAR S. CARNOT,

ANCIEN ÉLÈVE DE L'ÉCOLE POLYTECHNIQUE.

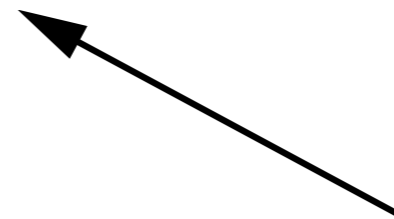
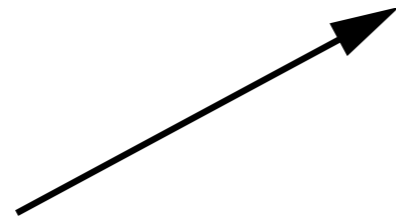


A PARIS,
CHEZ BACHELIER, LIBRAIRE,
QUAI DES AUGUSTINS, N^o. 55.



1824.

Optimal Control of Spin Systems



Optimal Control Theory

Spin Physics

liquid
state
NMR

solid
state
NMR

point to
point
transform

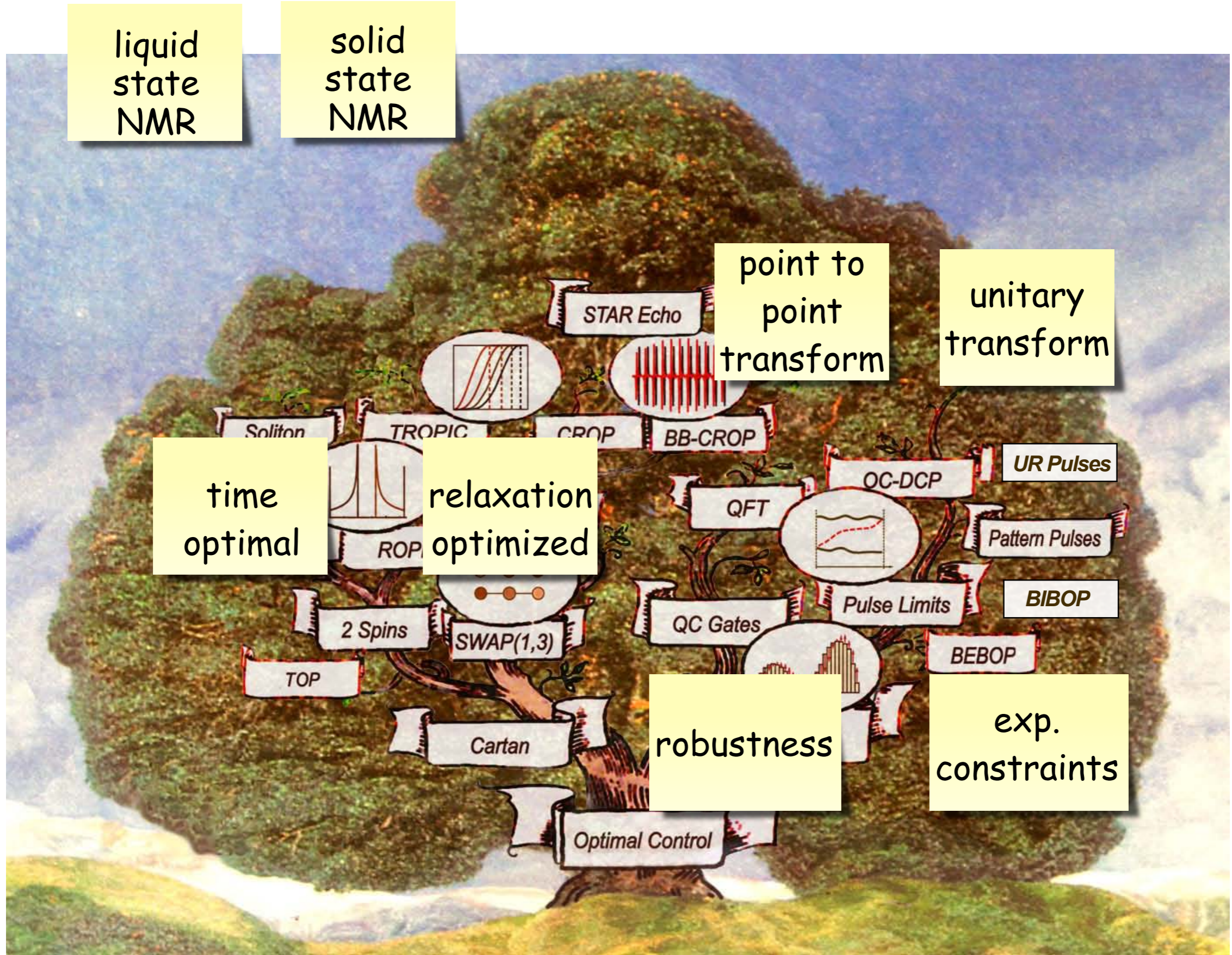
unitary
transform

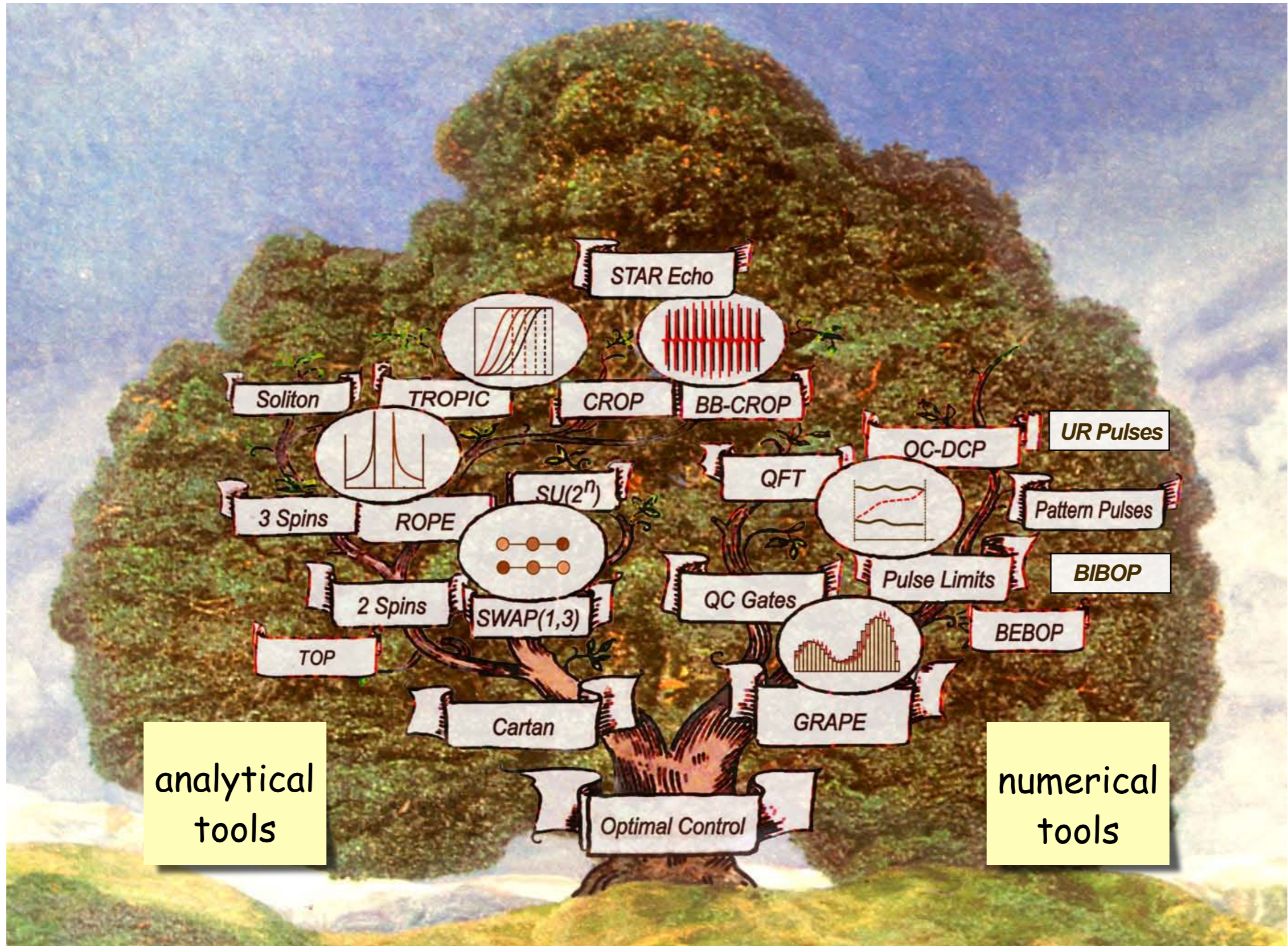
time
optimal

relaxation
optimized

robustness

exp.
constraints





Optimal control in NMR: band-selective excitation and inversion

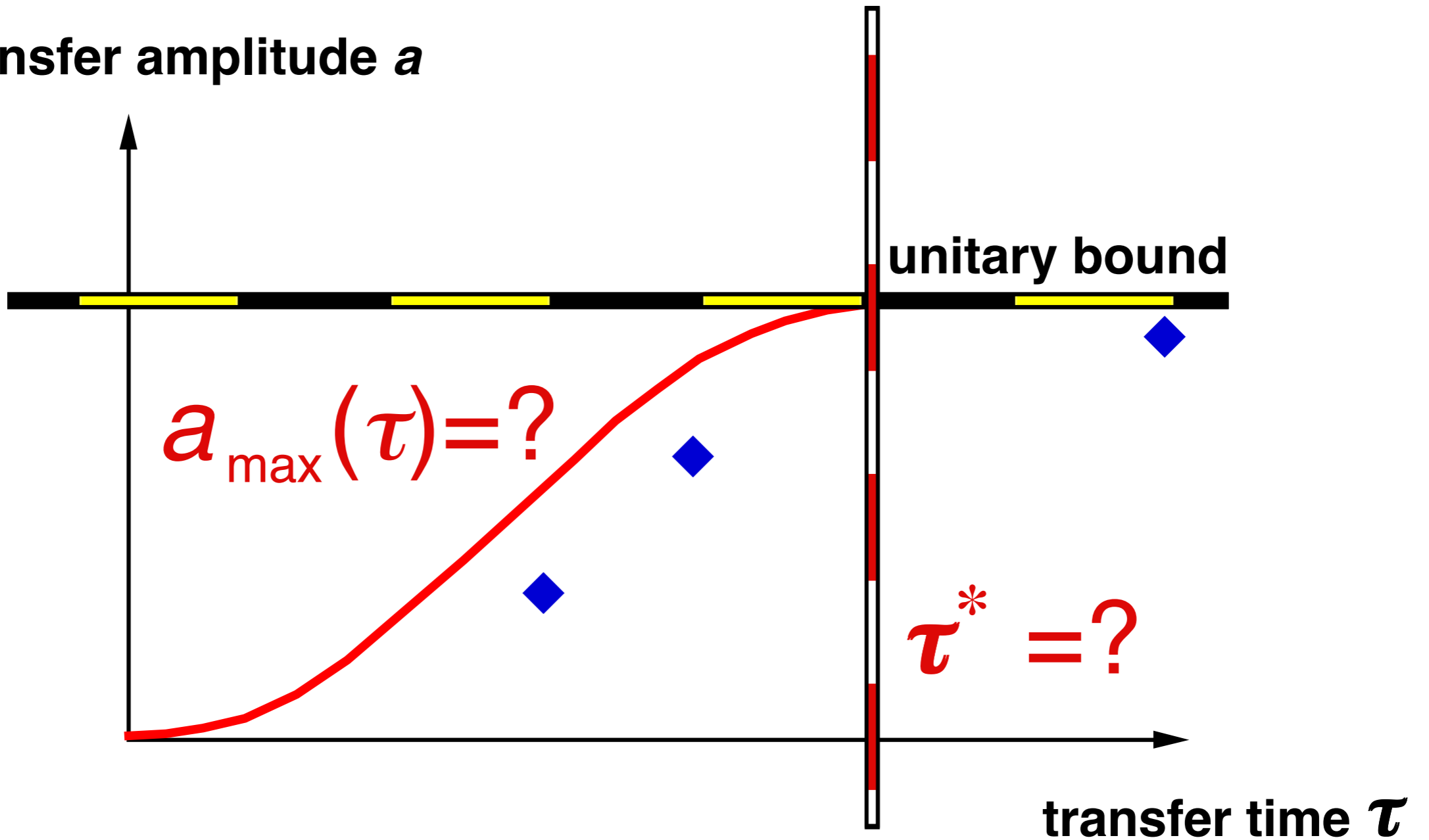
S. Conolly, D. Nishimura, A. Macovski, Optimal control solutions to the magnetic resonance selective excitation problem, *IEEE Trans. Med. Imaging* MI-5 (1986) 106–115.

J. Mao, T.H. Mareci, K.N. Scott, E.R. Andrew, Selective inversion radiofrequency pulses by optimal control, *J. Magn. Reson.* 70 (1986) 310–318.

D. Rosenfeld, Y. Zur, Design of adiabatic selective pulses using optimal control theory, *Magn. Reson. Med.* 36 (1996) 401–409.

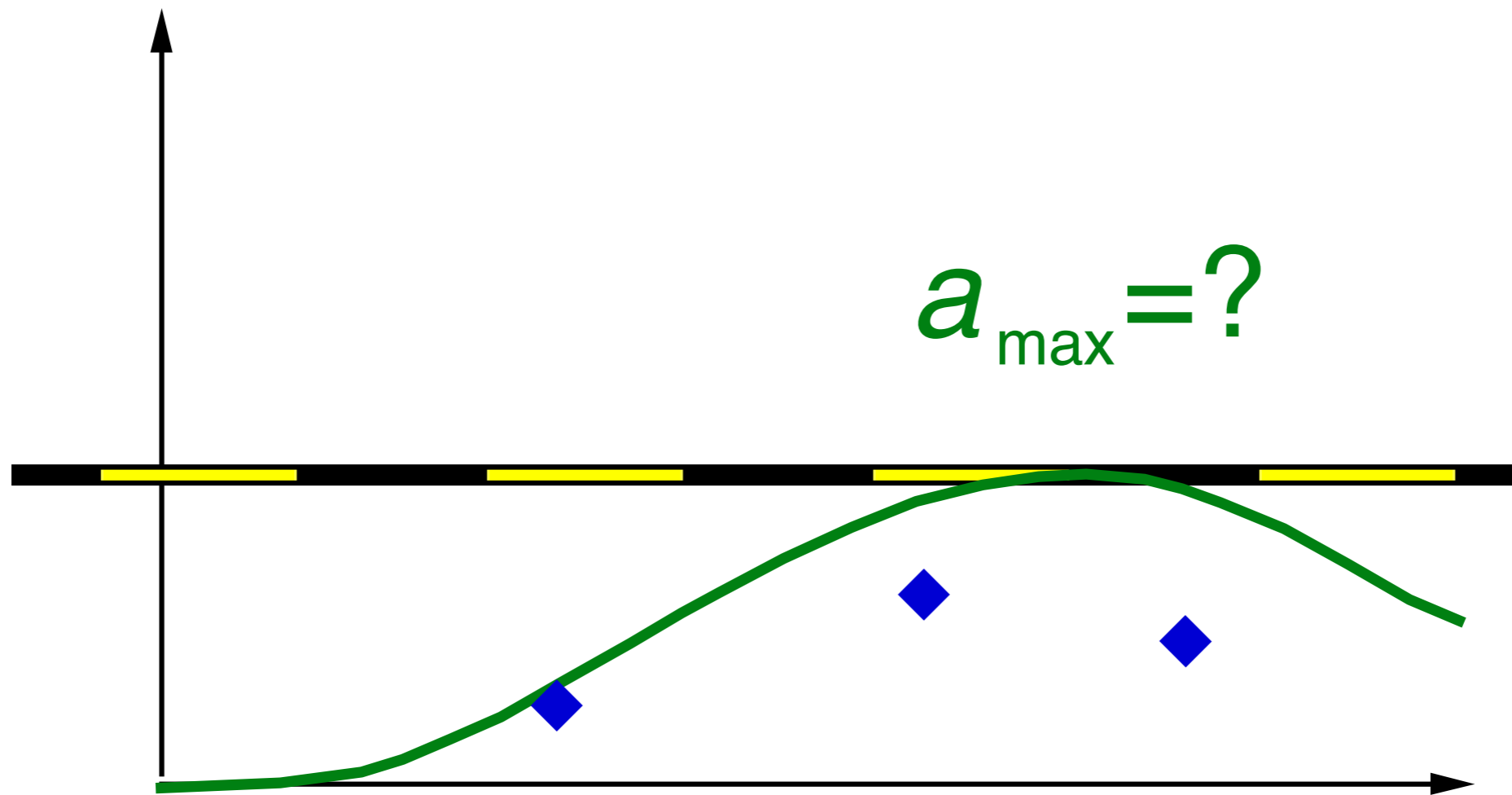
Unitary Quantum Evolution (no Relaxation)

transfer amplitude a



Quantum Evolution in Presence of Relaxation

transfer amplitude a



transfer time τ

Time-Optimal Control of Two-Spin Systems

Strong-Pulse Limit: $H_{rf} \gg H_c$ (2 time scales)

Cartan Decomposition

Characterization of ALL unitary operators
that can be created in time T

Derivation of - time-optimal transfer function (TOP curve)

- minimum time for maximum transfer
- pulse sequence

Khaneja, Brockett, Glaser (2001)

Khaneja, Kramer, Glaser (2005)

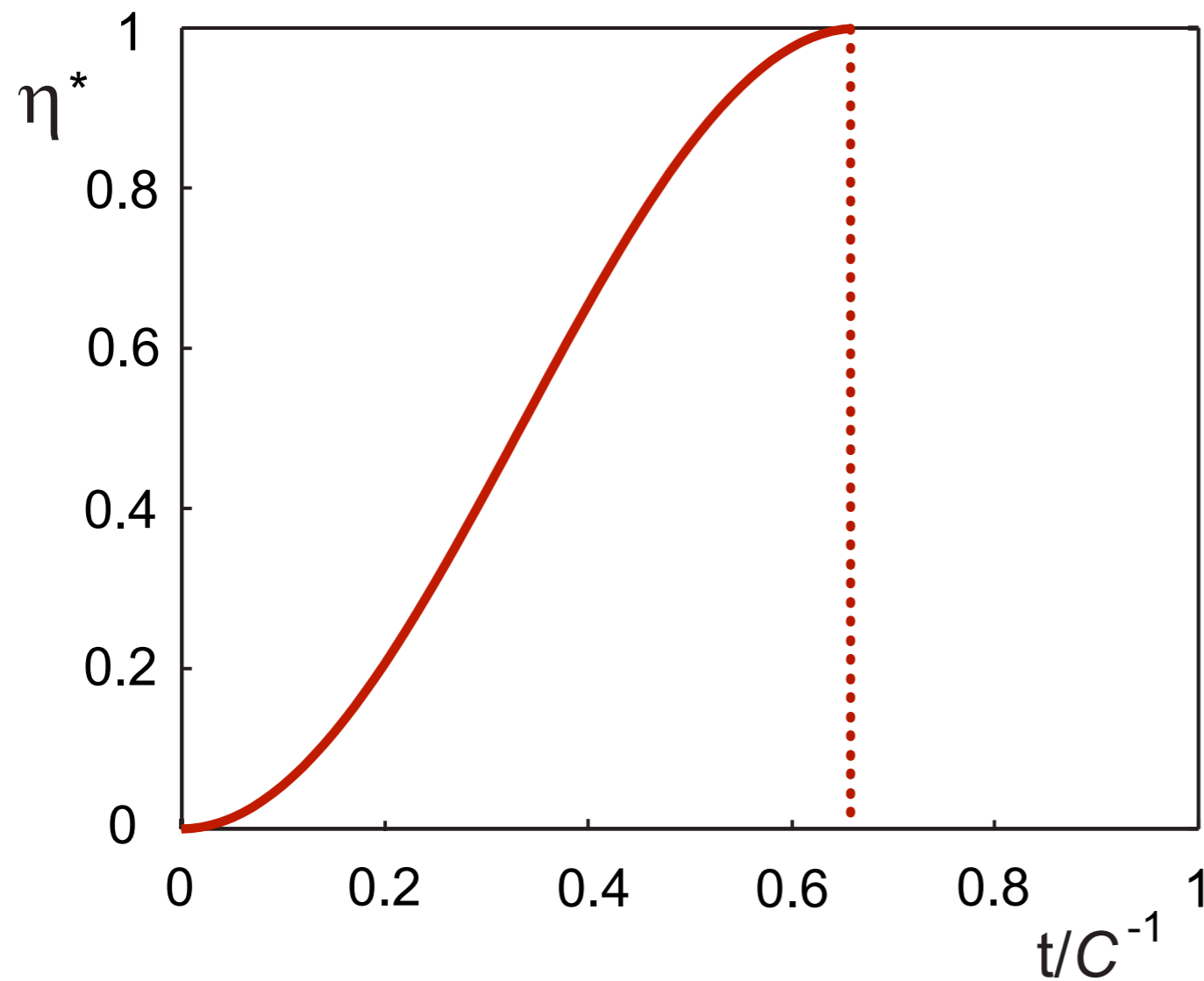
Maximum transfer efficiency $\eta^*(t)$ and minimum time t_{\min} for complete transfer

Transfer	$\eta^*(t)$	t_{\min}^{-1}
$I_x \rightarrow S_x$	$\sin^2\left(\frac{\pi}{2}C(\mu_3 + \mu_2)t\right)$	$C(\mu_3 + \mu_2)$
$I^- \rightarrow S^-$	$\sin(\pi Ca) \sin(\pi Cb)$	$\frac{2}{3}C(\mu_3 + \mu_2 + \mu_1)$
$I_x \rightarrow 2I_z S_x$	$\sin(\pi C \mu_3 t)$	$2C \mu_3 $
$I^- \rightarrow 2I_z S^-$	$\max_x \sin\left(\frac{\pi}{2}C\{ \mu_3 + \mu_2 - \mu_1 + x\}t\right) \cos(\pi Ctx)$	$C(\mu_3 + \mu_2 - \mu_1)$
$I_x S_\beta \rightarrow I_\beta S_x$	$\sin\left(\frac{\pi}{2}C(\mu_3 + \mu_2)t\right)$	$C(\mu_3 + \mu_2)$
$I^- S_\beta \rightarrow I_\beta S^-$	$\sin\left(\frac{\pi}{2}C(\mu_3 + \mu_2)t\right)$	$C(\mu_3 + \mu_2)$

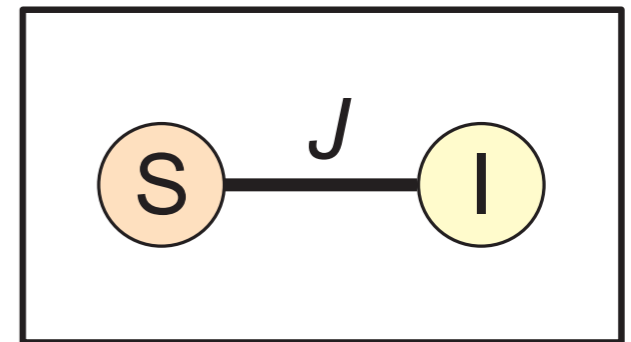
Note: $I^- = I_x - iI_y$ and $I_\beta = \frac{1}{2} - I_z$. For the transfer $I^- \rightarrow S^-$, the optimal values of a and b are completely characterized by the two conditions $a + 2b = (|\mu_3| + |\mu_2| + |\mu_1|) t$ and $\tan(\pi Ca) = 2 \tan(\pi Cb)$.

TOP (time-optimal pulse) curves for dipolar coupling

$$(\mu_1, \mu_2, \mu_3) = (-1/2, -1/2, 1)$$

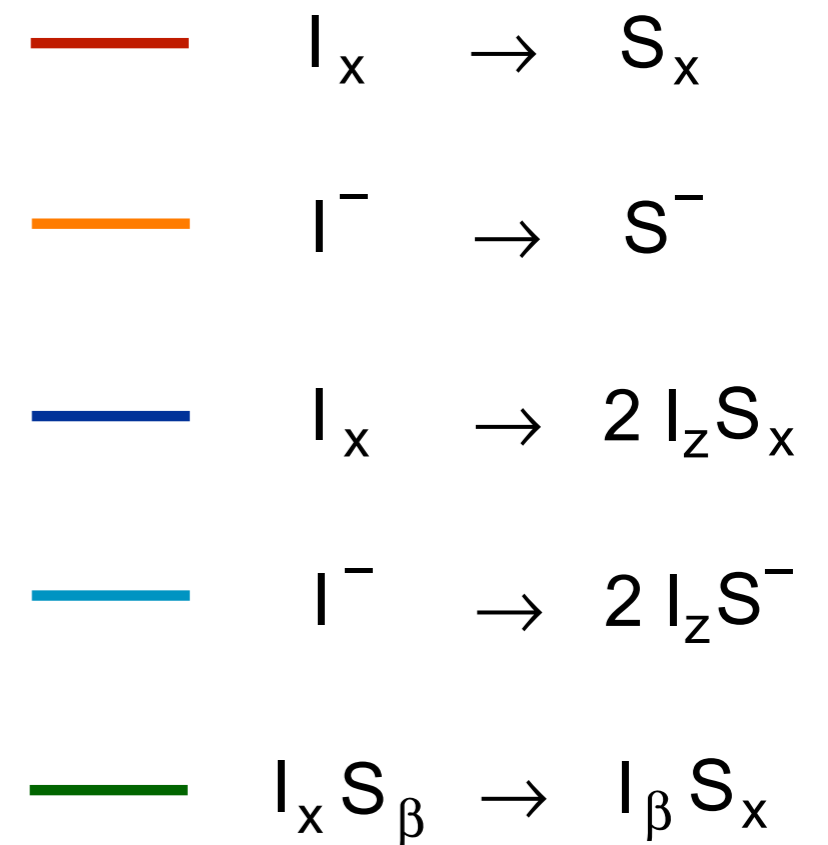
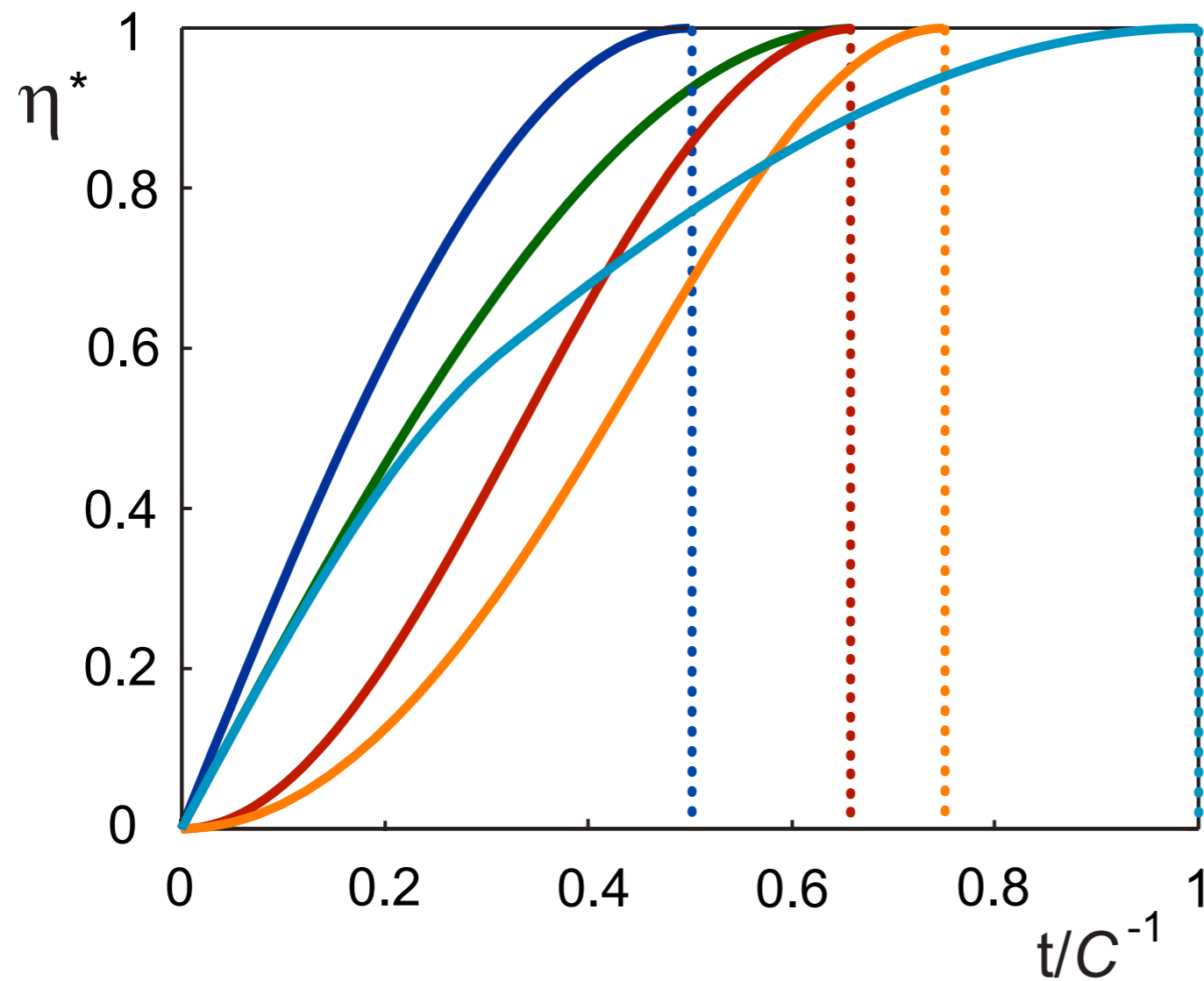


— $I_x \rightarrow S_x$



TOP (time-optimal pulse) curves for dipolar coupling

$$(\mu_1, \mu_2, \mu_3) = (-1/2, -1/2, 1)$$

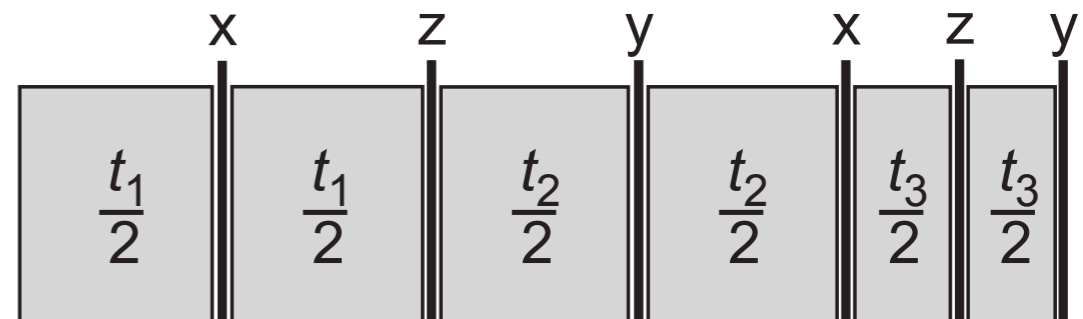


Optimal
sequence
of
effective
Hamiltonians

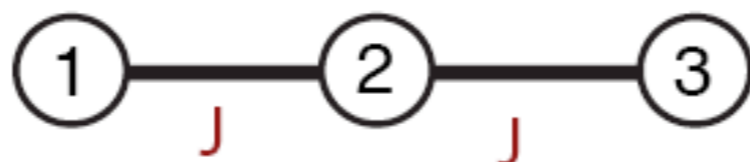
Pulse sequences

$$I^- \rightarrow S^-$$

$ \mu_1\rangle$	$ \mu_1\rangle$	$ \mu_3\rangle$	$ \mu_2\rangle$	$ \mu_2\rangle$	$ \mu_3\rangle$
$ \mu_2\rangle$	$ \mu_3\rangle$	$ \mu_1\rangle$	$ \mu_1\rangle$	$ \mu_3\rangle$	$ \mu_2\rangle$
$ \mu_3\rangle$	$ \mu_2\rangle$	$ \mu_2\rangle$	$ \mu_3\rangle$	$ \mu_1\rangle$	$ \mu_1\rangle$



Time-Optimal Simulation of Trilinear Coupling Terms



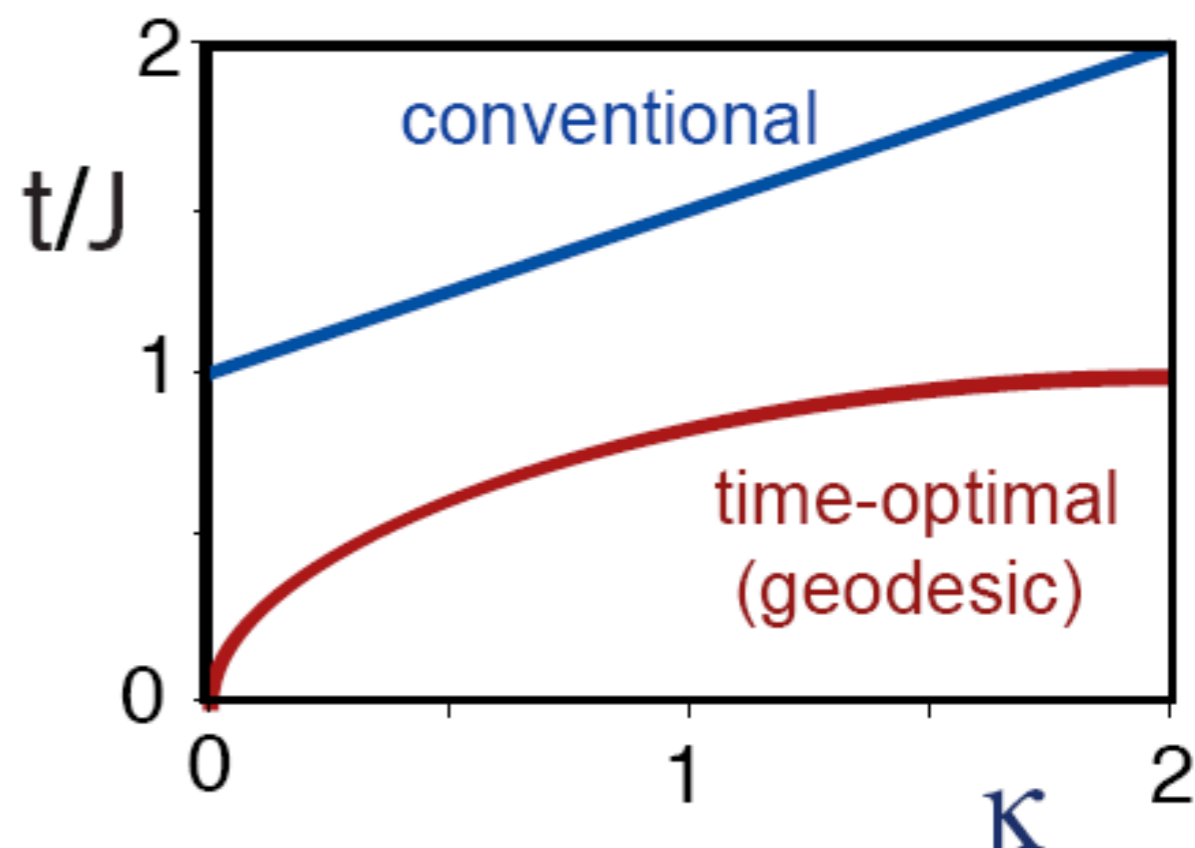
given:

$$H = 2 \pi J (I_{1z} I_{2z} + I_{2z} I_{3z})$$

desired:

$$U = \exp\{-i \kappa 2 \pi I_{1z} I_{2z} I_{3z}\}$$

$$H_{\text{eff}} = 2 \pi J_{\text{eff}} (I_{1z} I_{2z} I_{3z})$$

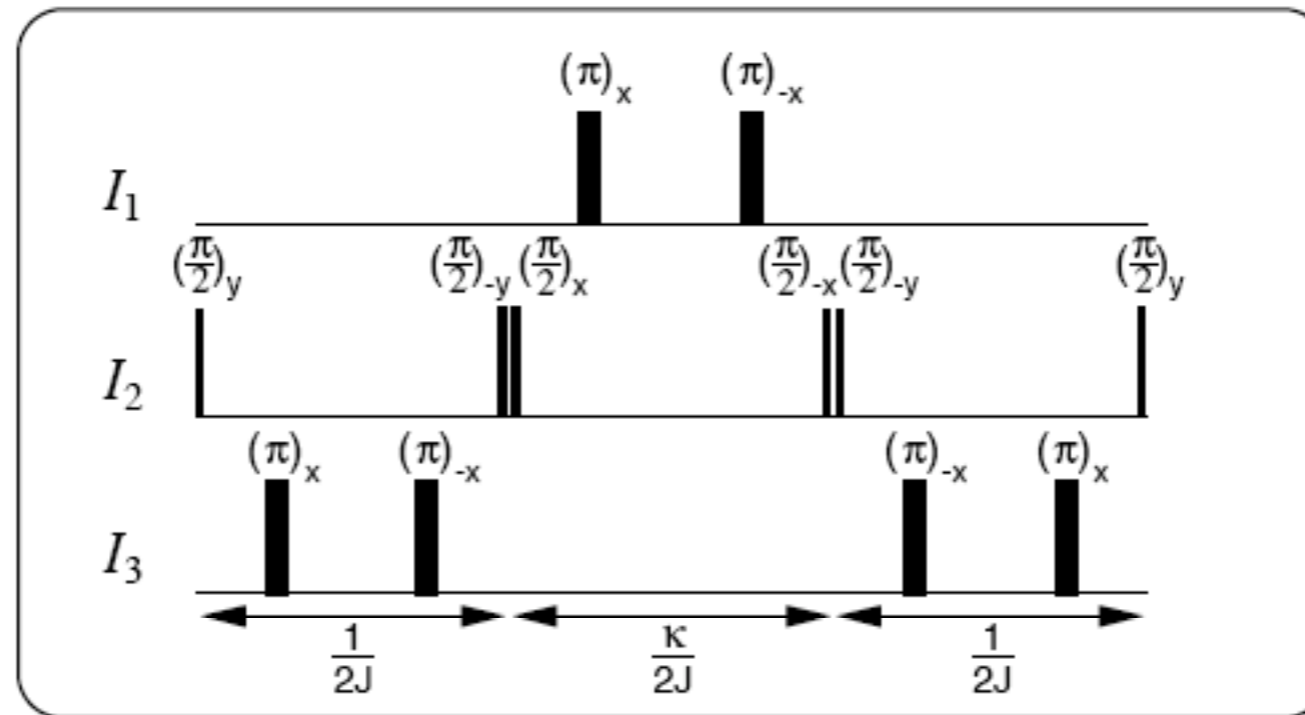


Tseng, Somaroo, Sharf, Knill, Laflamme, Havel, Cory, Phys. Rev. A 61, 012302 (2000)

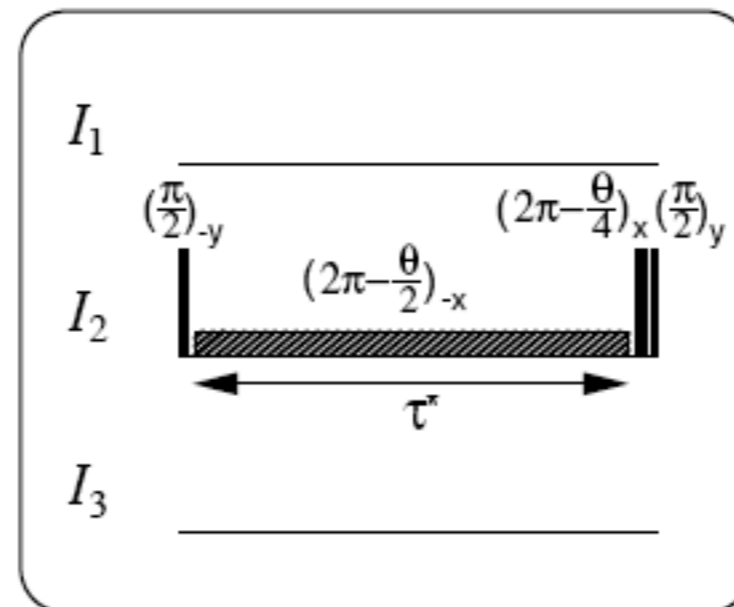
Khaneja, Glaser, Brockett, Phys. Rev. A 65, 032301 (2002)

Pulse Sequences ("zzz")

conventional

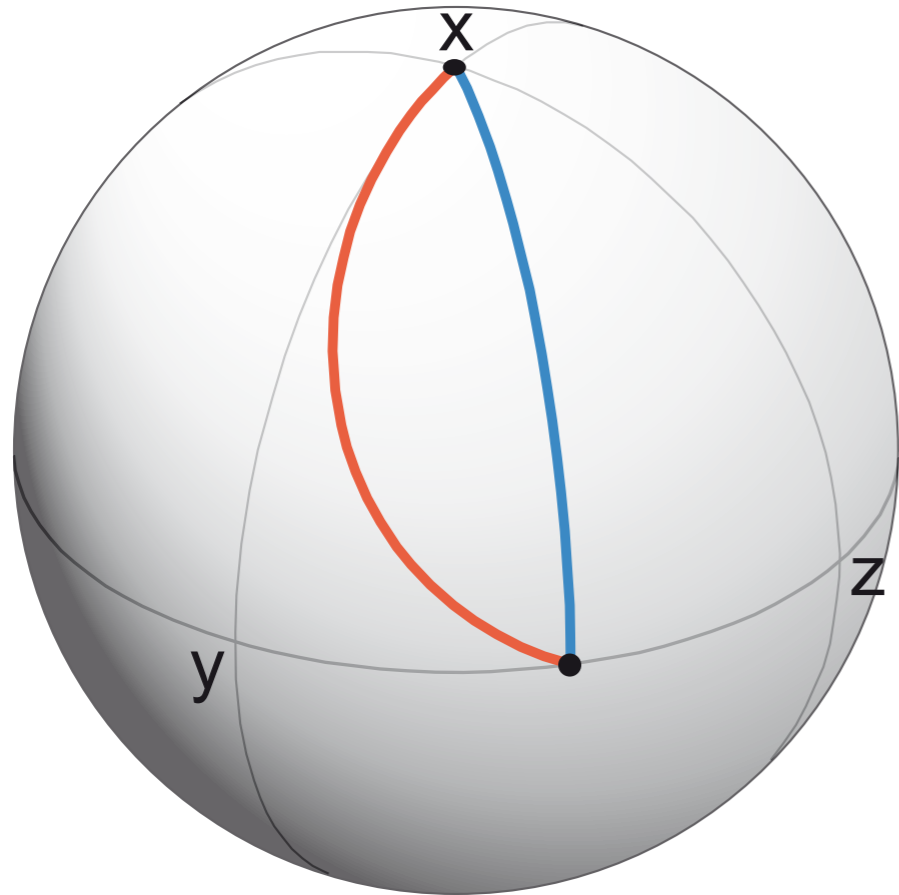


geodesic



Khaneja, Glaser, Brockett, Phys. Rev. A 65, 032301 (2002)

Geodesics on a sphere



Euklidian metric

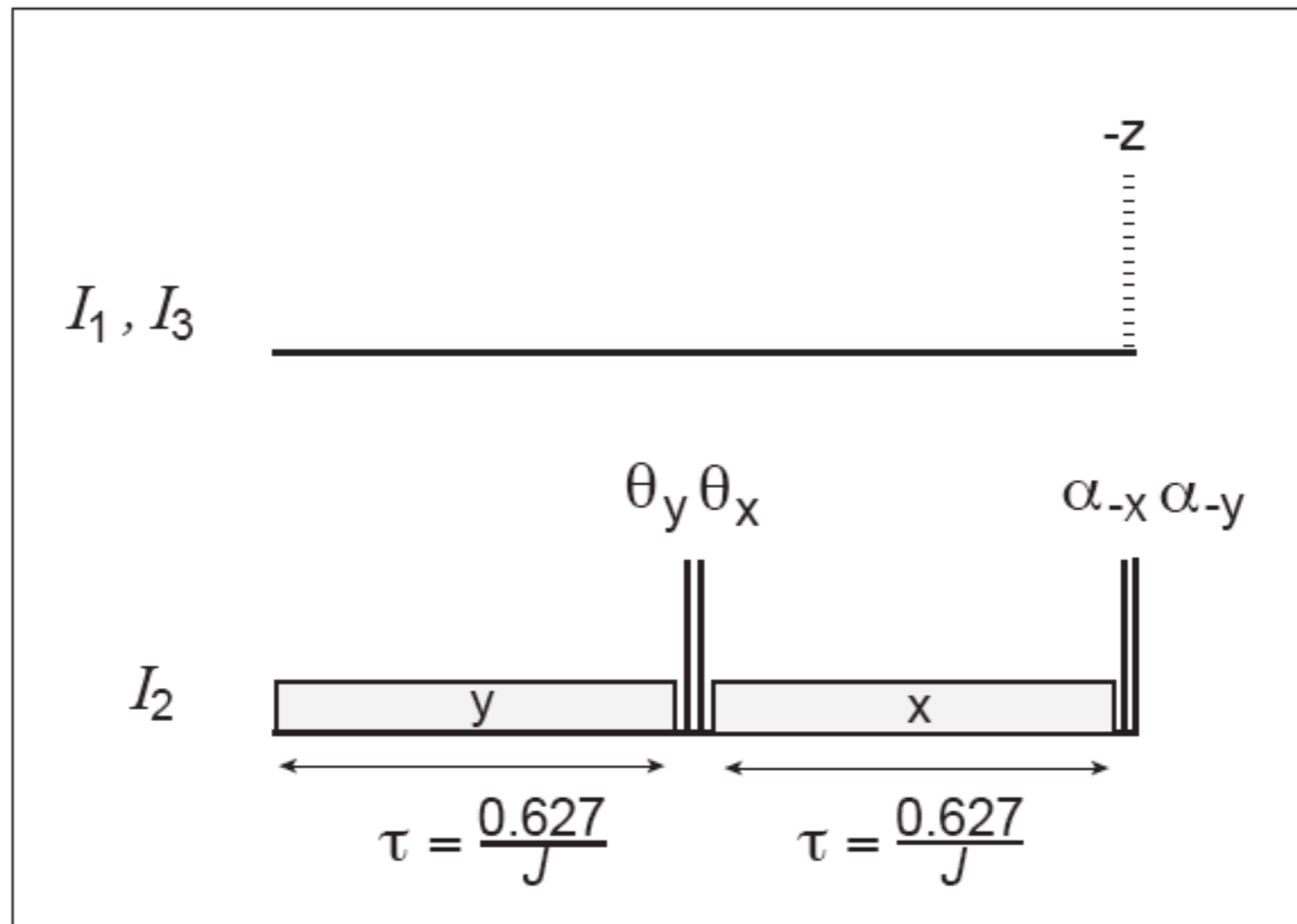
$$(dx)^2 + (dy)^2 + (dz)^2$$

“quantum gate design metric”

$$\frac{(dx)^2 + (dz)^2}{y^2}$$

Khaneja et al., Phys. Rev. A 75, 012322 (2007).

Pulse sequence for creating $U_{13} = \exp\{-i \pi I_{1z} I_{3z}\}$



$\theta = 180^\circ - \alpha = 31.4^\circ$, weak pulse amplitude: $0.52 J$

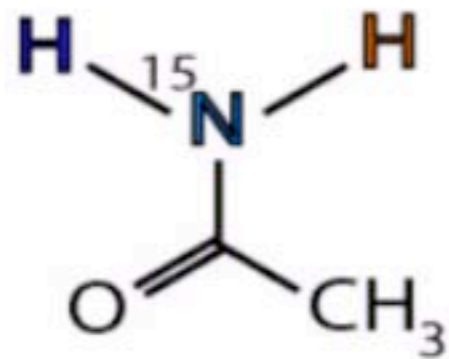
TABLE I. Duration τ_C of various implementations of the CNOT(1,3) gate.

Pulse sequence	τ_C (units of J^{-1})	<u>Relative duration (%)</u>
Sequence 1 (C1)	3.5	100
Sequence 2 (C2)	2.5	71.4
Sequence 3 (C3)	2.0	57.1
Sequence 4 (C4)	1.866	53.3
Sequence 5 (C5)	1.253	38.8

(C1, C2) D. Collins, K. W. Kim, W. C. Holton, H. Sierzputowska-Gracz, and E. O. Stejskal, Phys. Rev. A **62**, 022304 (2000).

(C3, C4, C5) *Khaneja et al., Phys. Rev. A 75, 012322 (2007)*

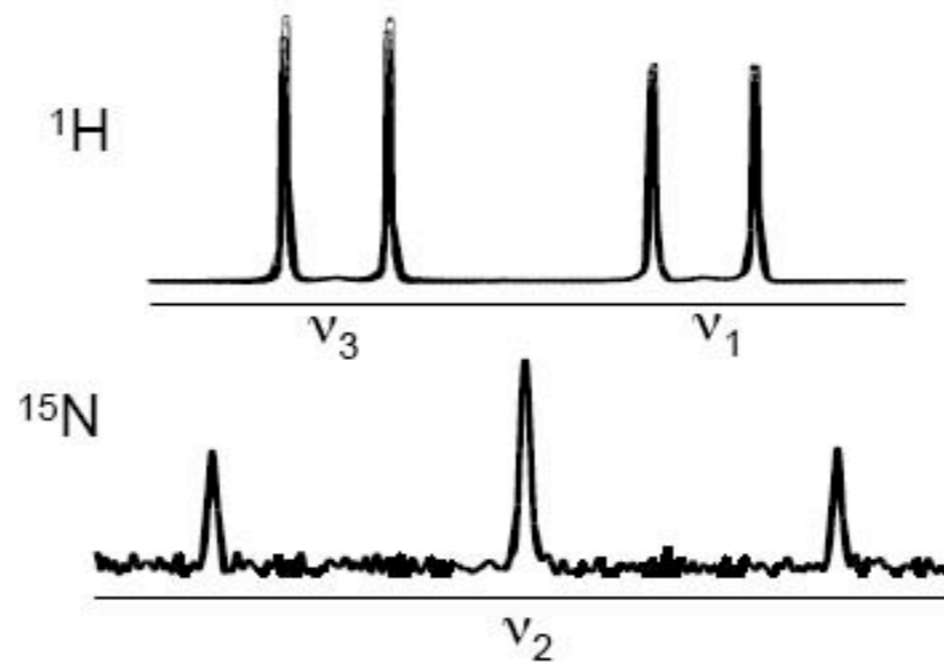
Experimental model system



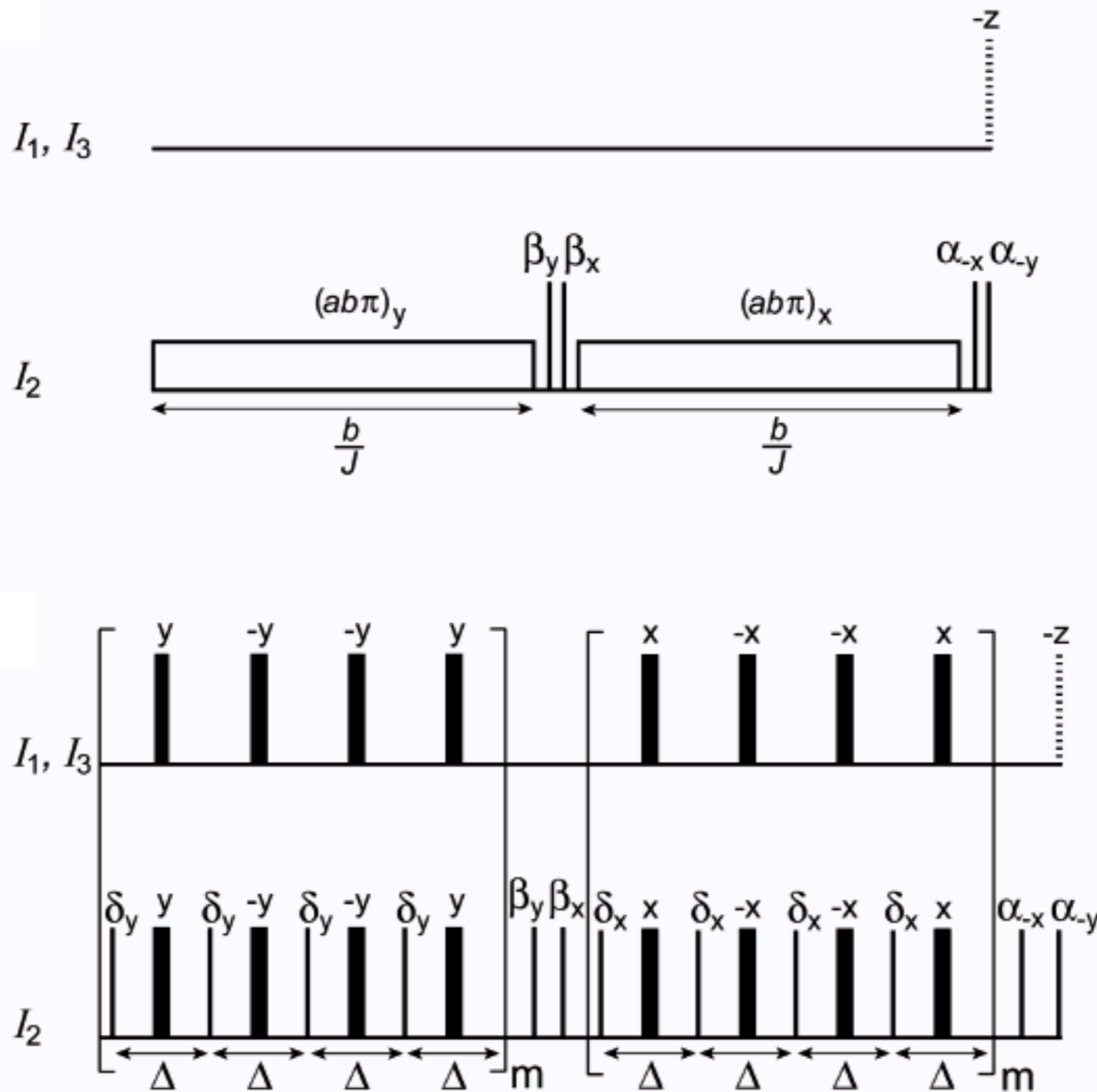
¹⁵N - acetamide

Solvent: DMSO-d₆
Temp.: 295 K
Bruker 500 Avance Spectrometer

$$J_{12} = -87.3 \text{ Hz} \approx J_{23} = -88.8 \text{ Hz} \gg J_{13} = 2.9 \text{ Hz}$$
$$\Delta\nu_{13} = 310 \text{ Hz}$$



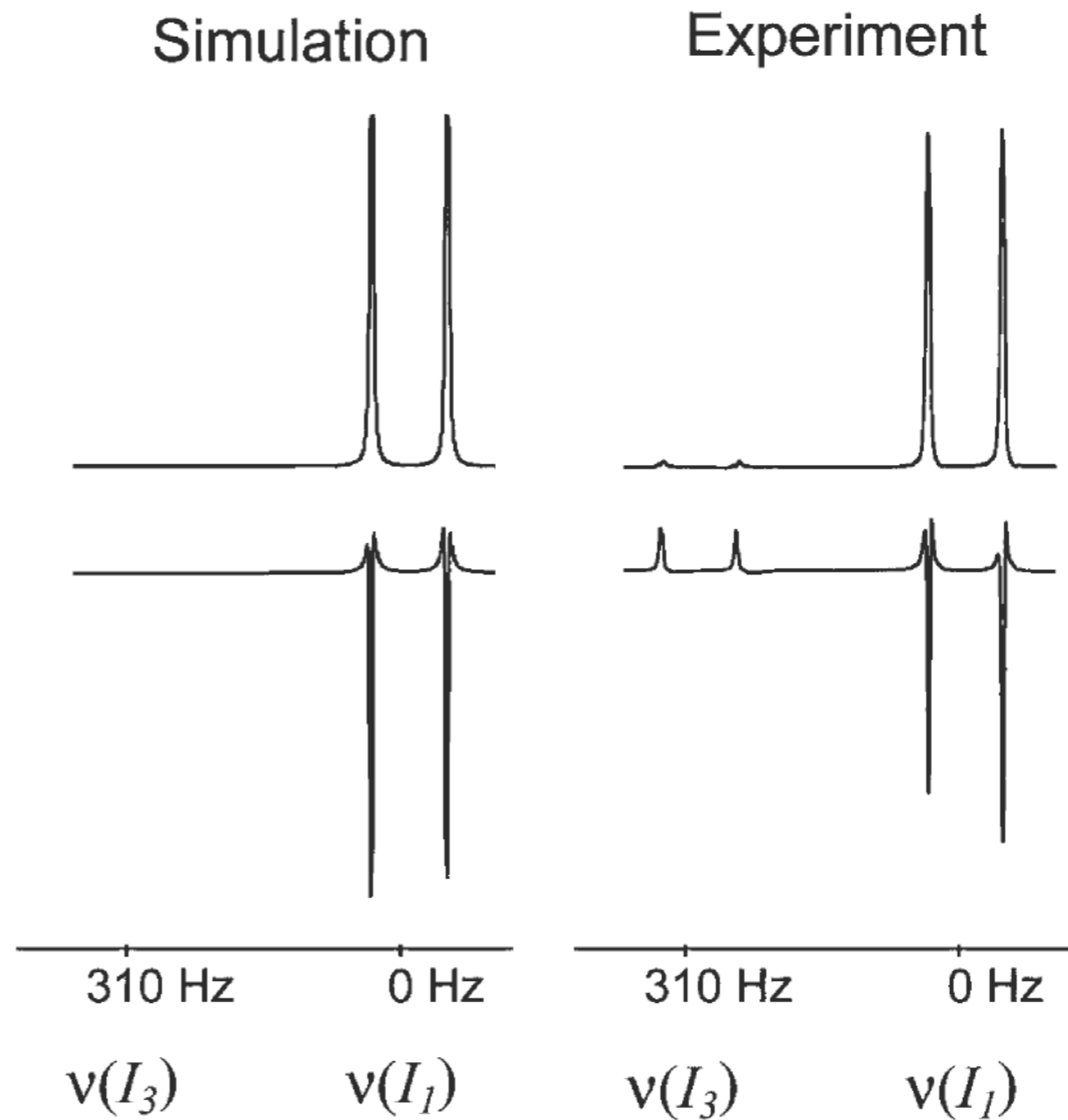
Experimental Demonstration U_{13}



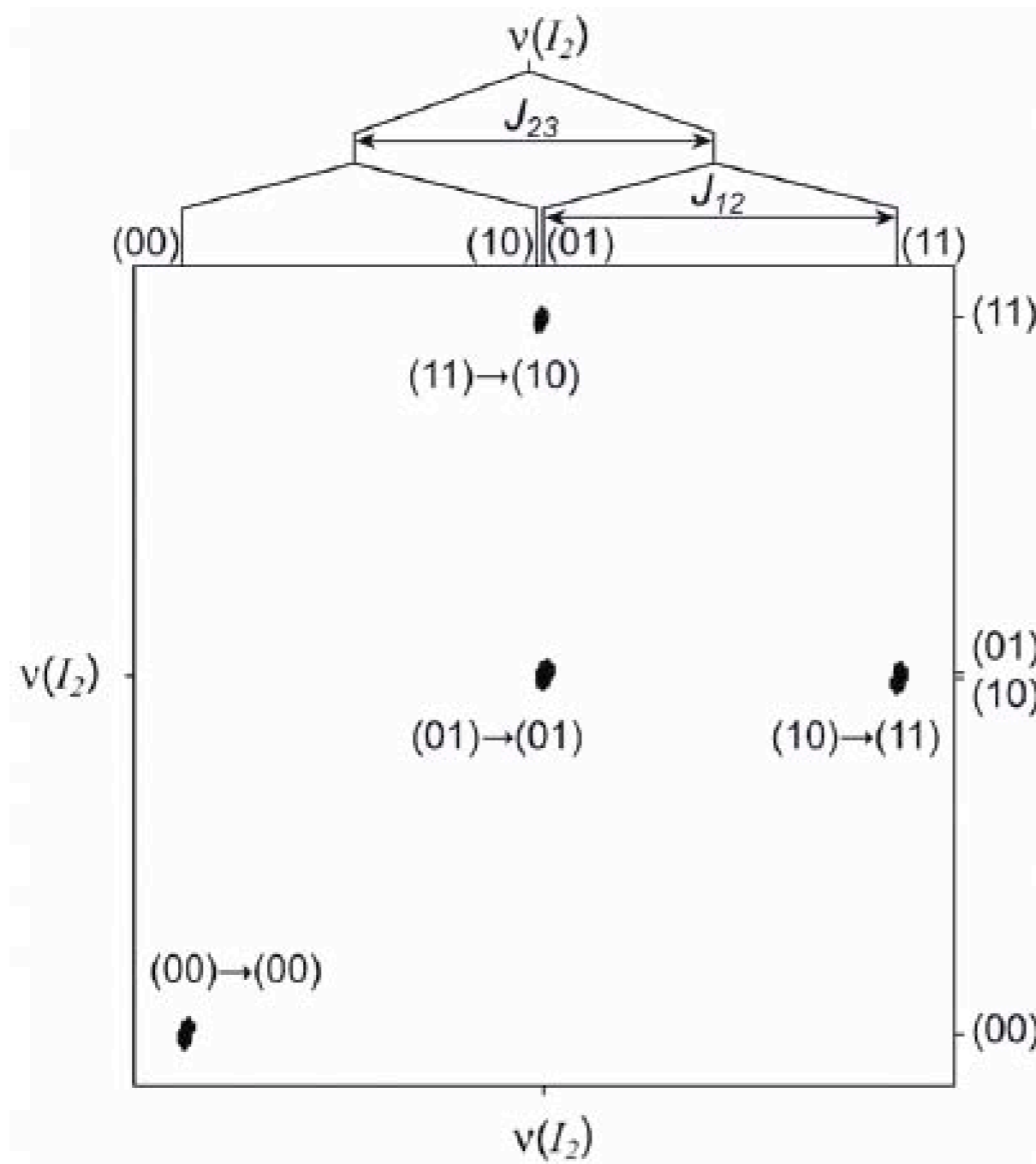
$$\mathcal{U}_{13} = \exp\left\{-i\frac{\pi}{2}2I_{1z}I_{3z}\right\}$$

$$\rho_A = I_{1x}$$

$$\rho_B = 2I_{1y}I_{3z}$$

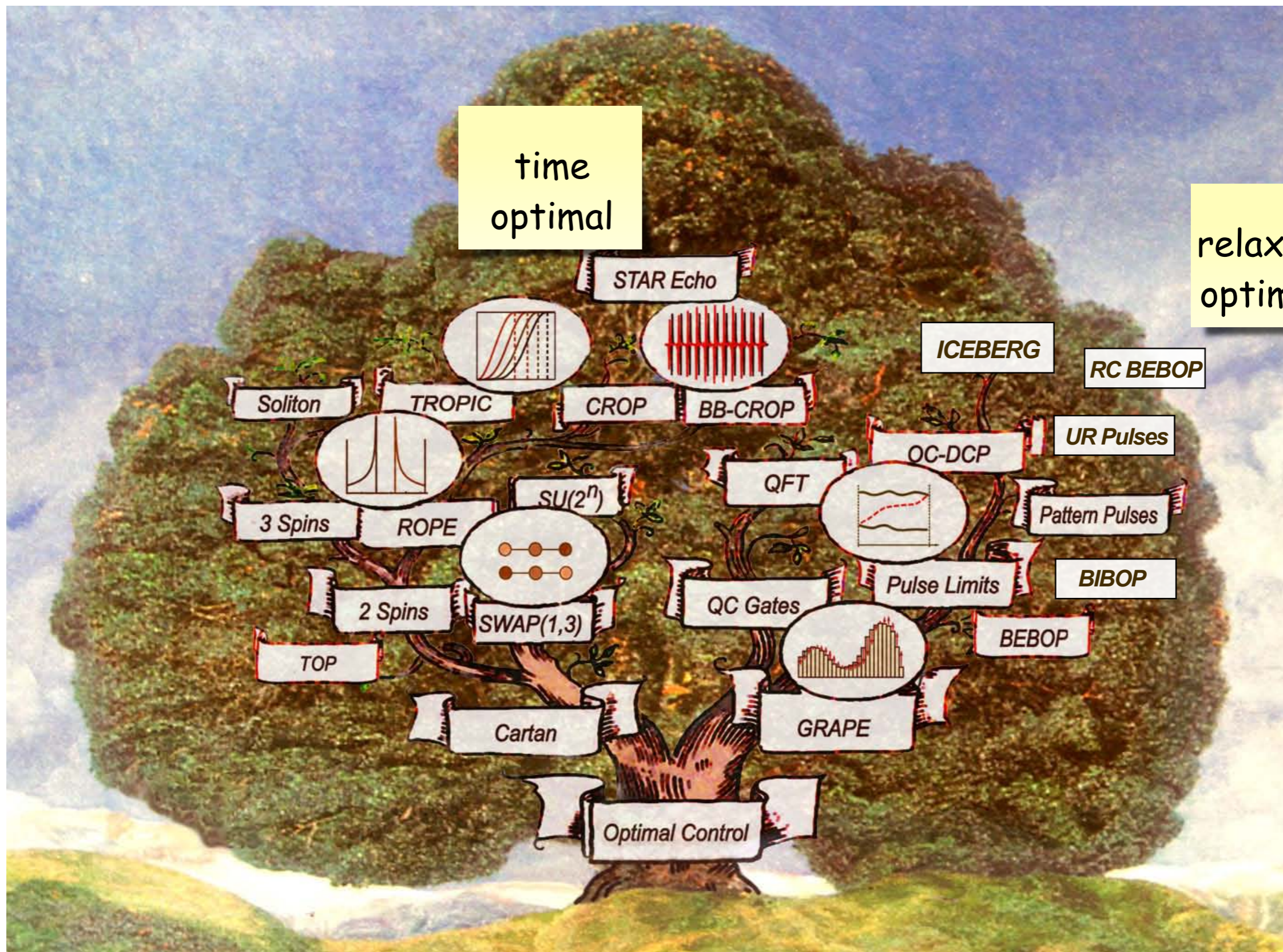


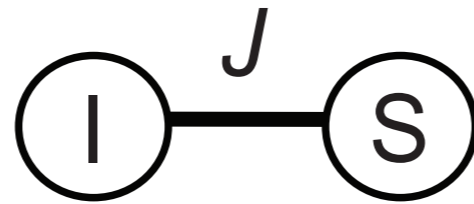
Experimental demonstration of CNOT(1,3)



time optimal

relaxation optimized





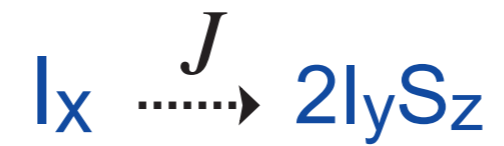
Dipol-Dipol Relaxation in the Spin-Diffusion Limit

$$\dot{\rho} = \pi J [-i 2I_z S_z, \rho] + \pi k [2I_z S_z, [2I_z S_z, \rho]]$$

$$\xi = k/J$$

$$I_x \xrightarrow{?} 2I_y S_z$$

Conventional transfer (INEPT)

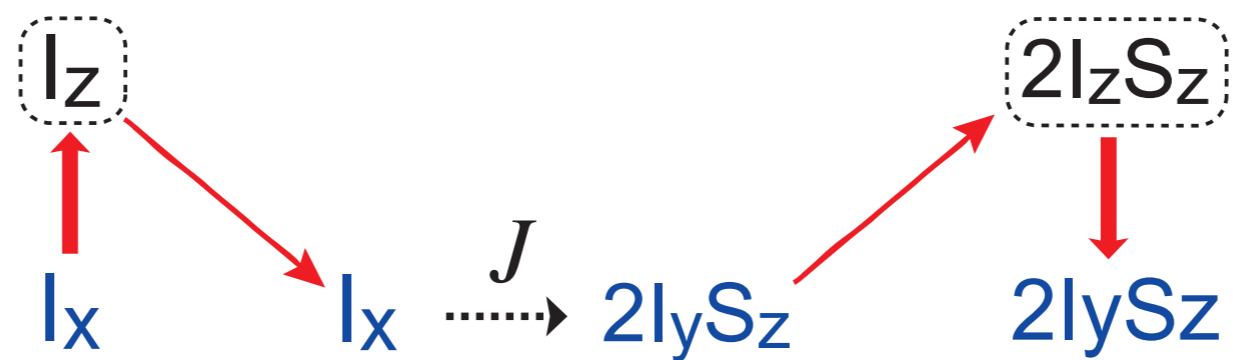


Morris, Freeman, J. Am. Chem. Soc. 101,760 (1979)

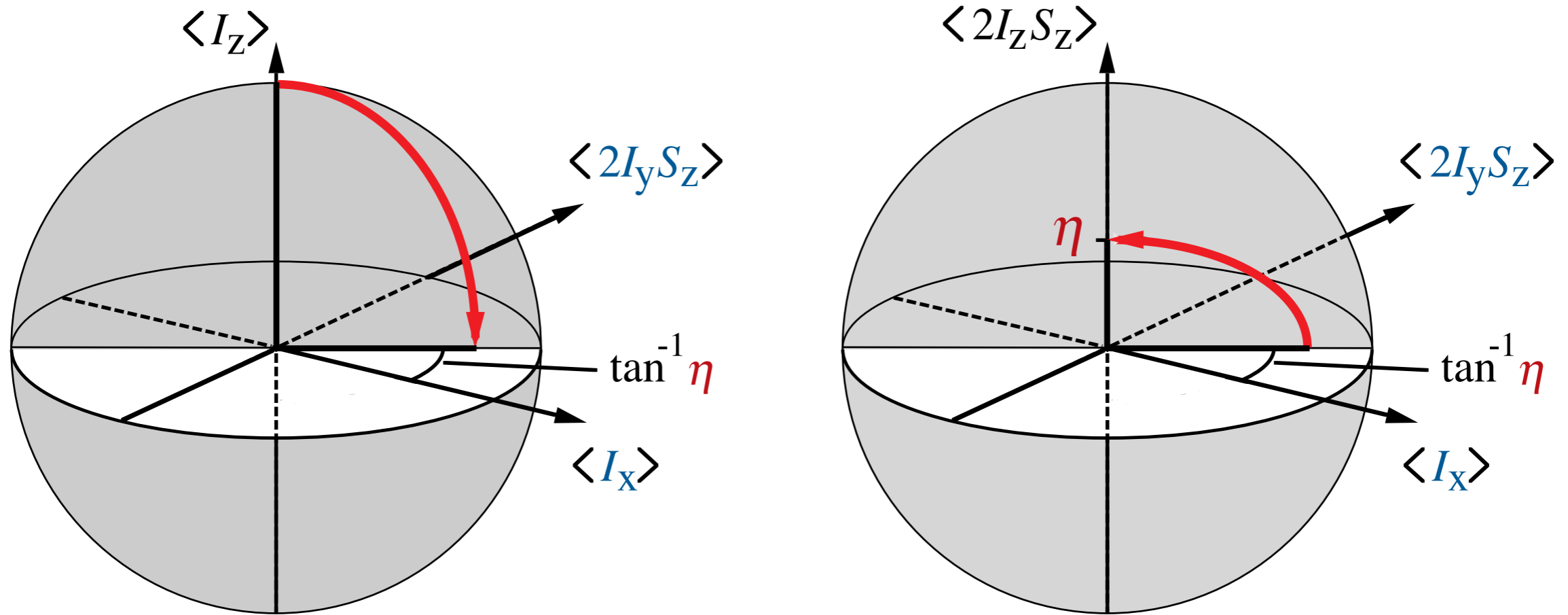
Conventional transfer (INEPT)

$$I_x \xrightarrow{J} 2I_yS_z$$

Relaxation-optimized transfer (ROPE)

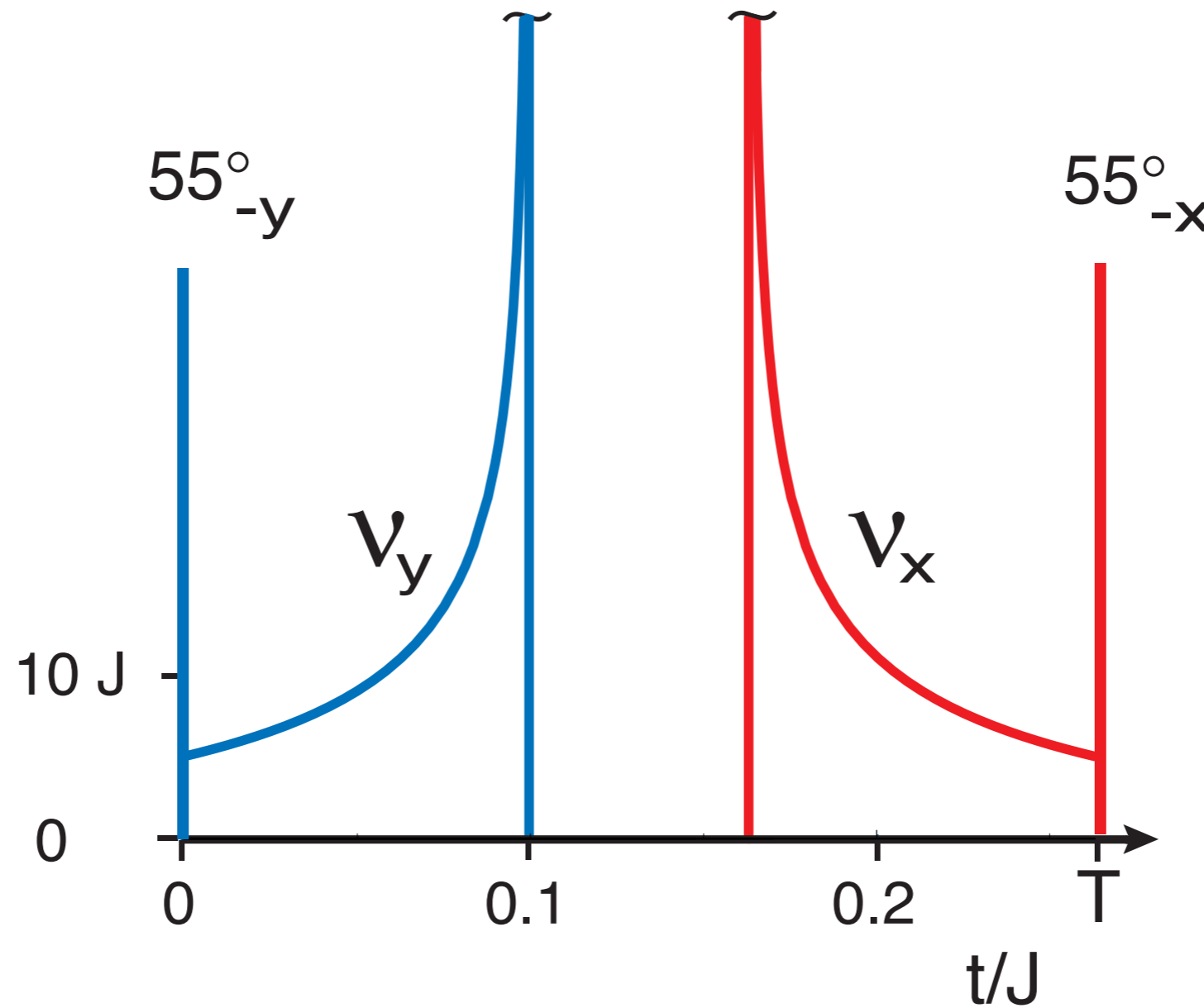


ROPE Trajectory



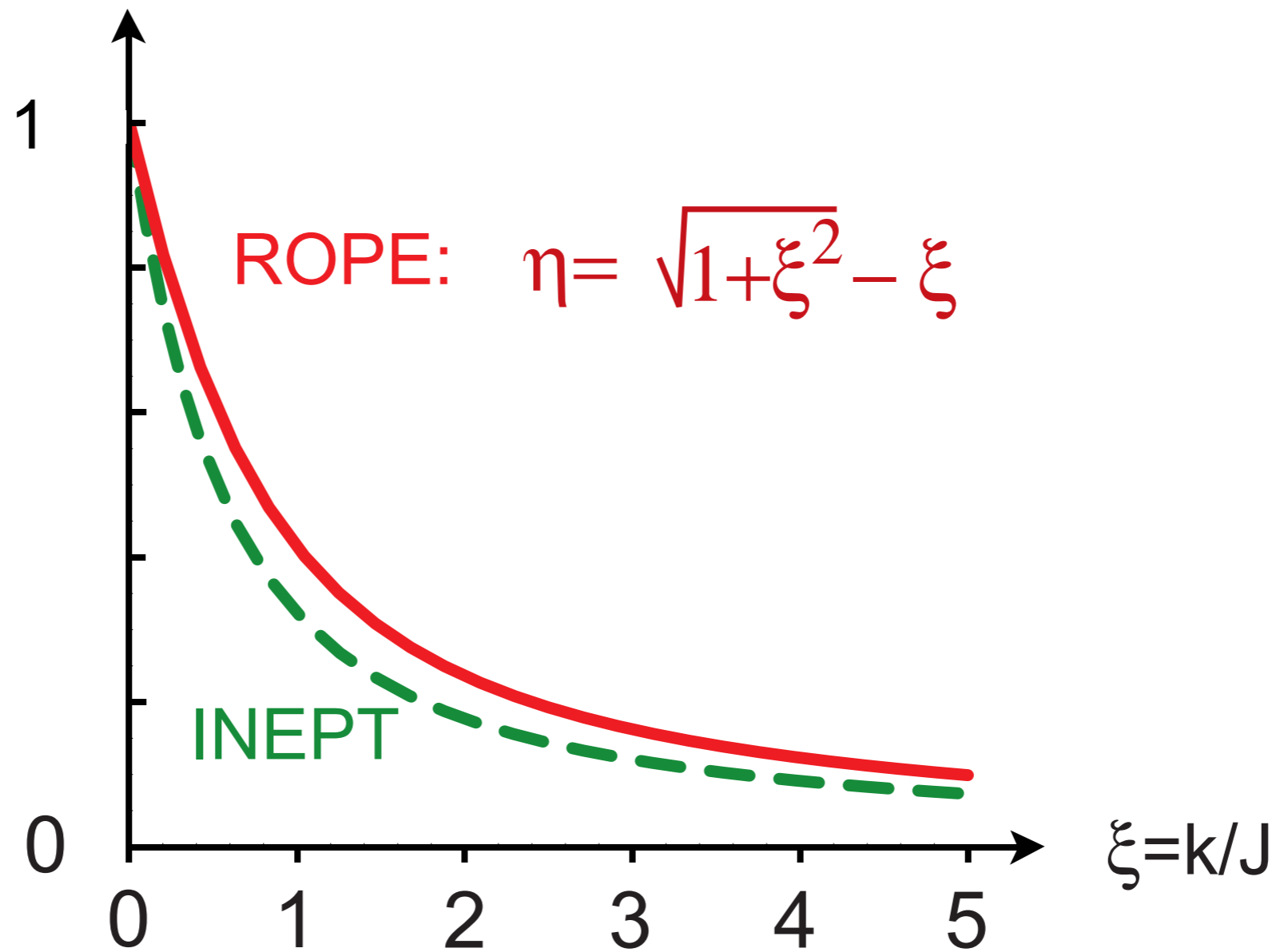
Optimal transfer efficiency $\eta = \sqrt{1 + \xi^2} - \xi$ $\xi = k/J$

ROPE Pulse Sequence

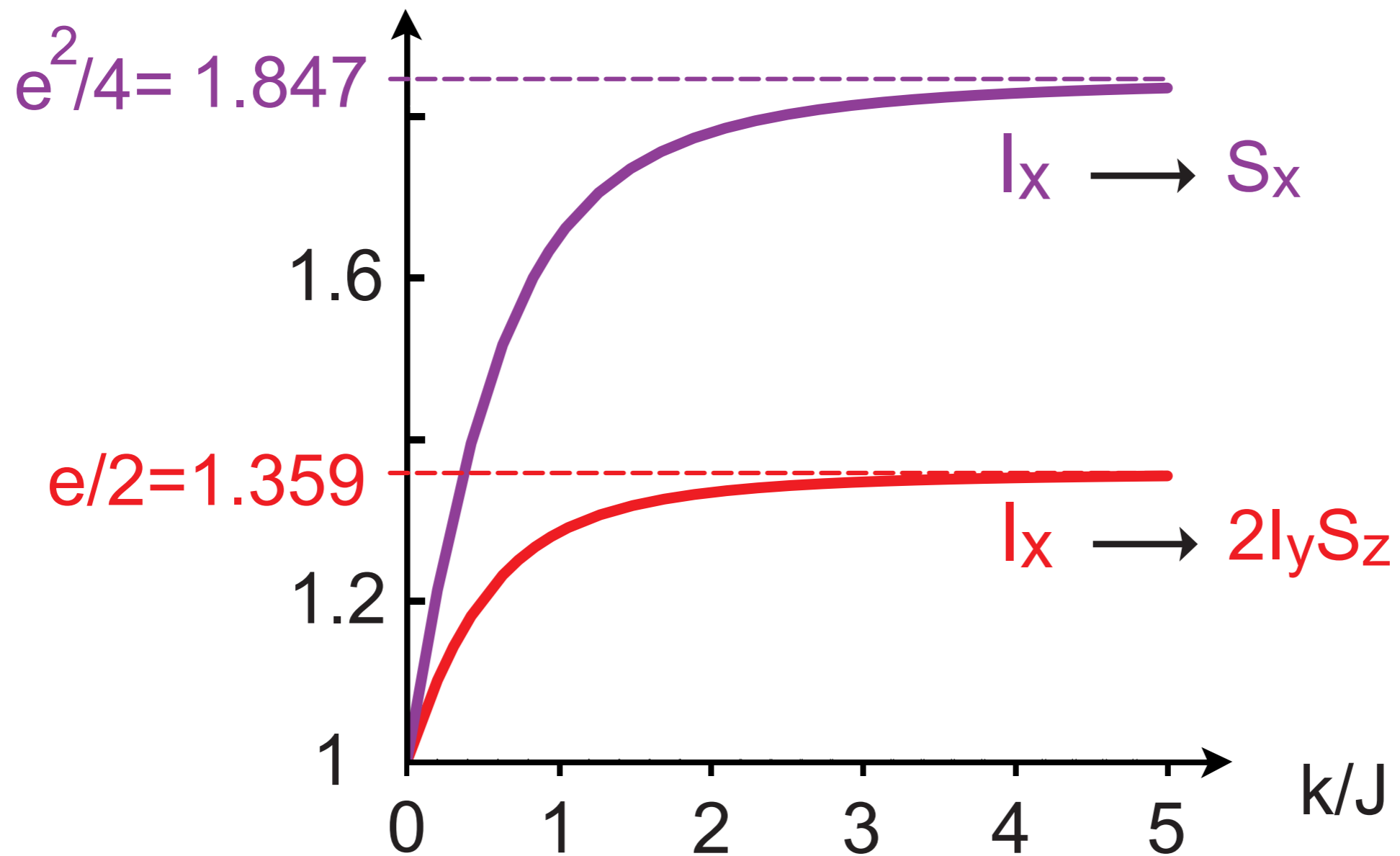


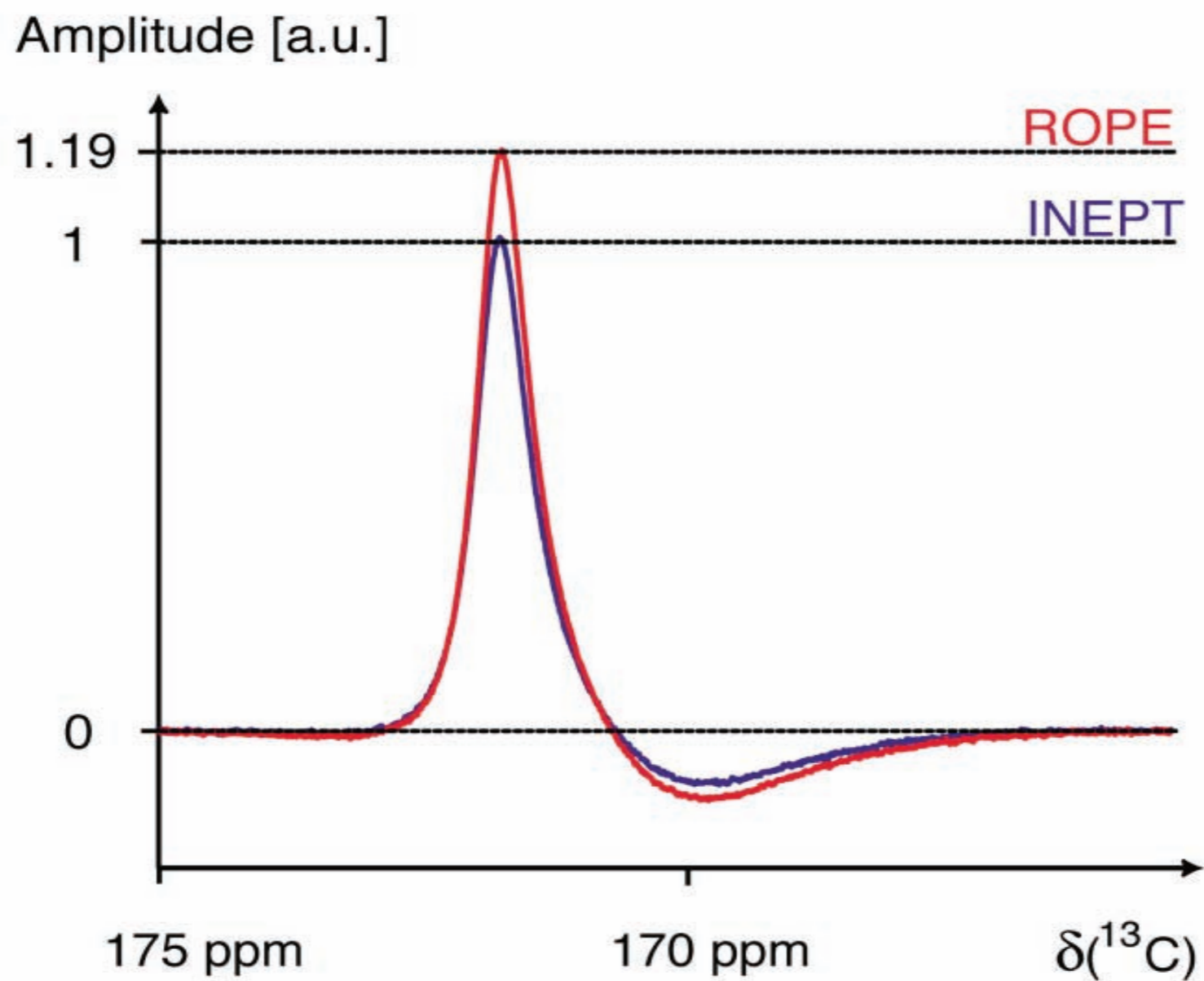
Khaneja et al., J. Magn. Reson. 162, 311 (2003)

Transfer-Efficiency



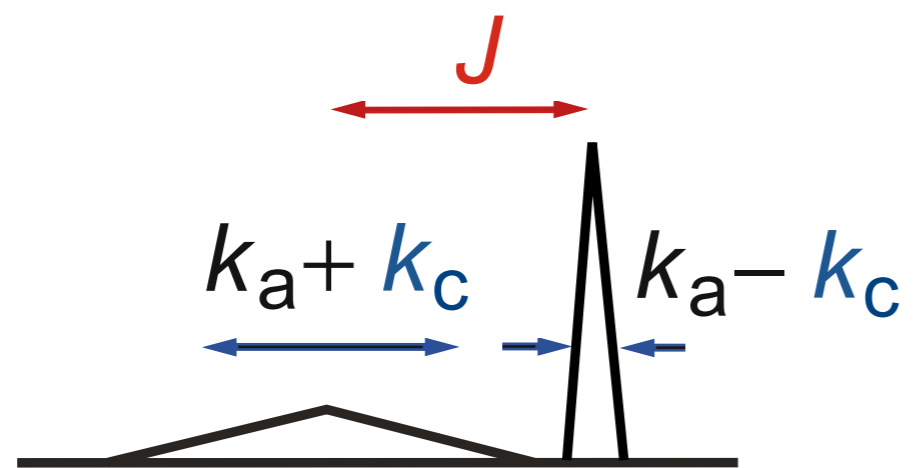
Gain (ROPE/INEPT)



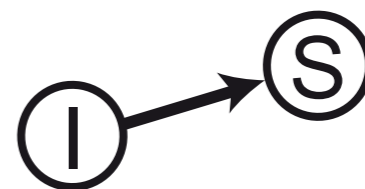


^{13}C -Formiate in 92% D_6 -Glycerol and 8% D_2O (T=250 K)

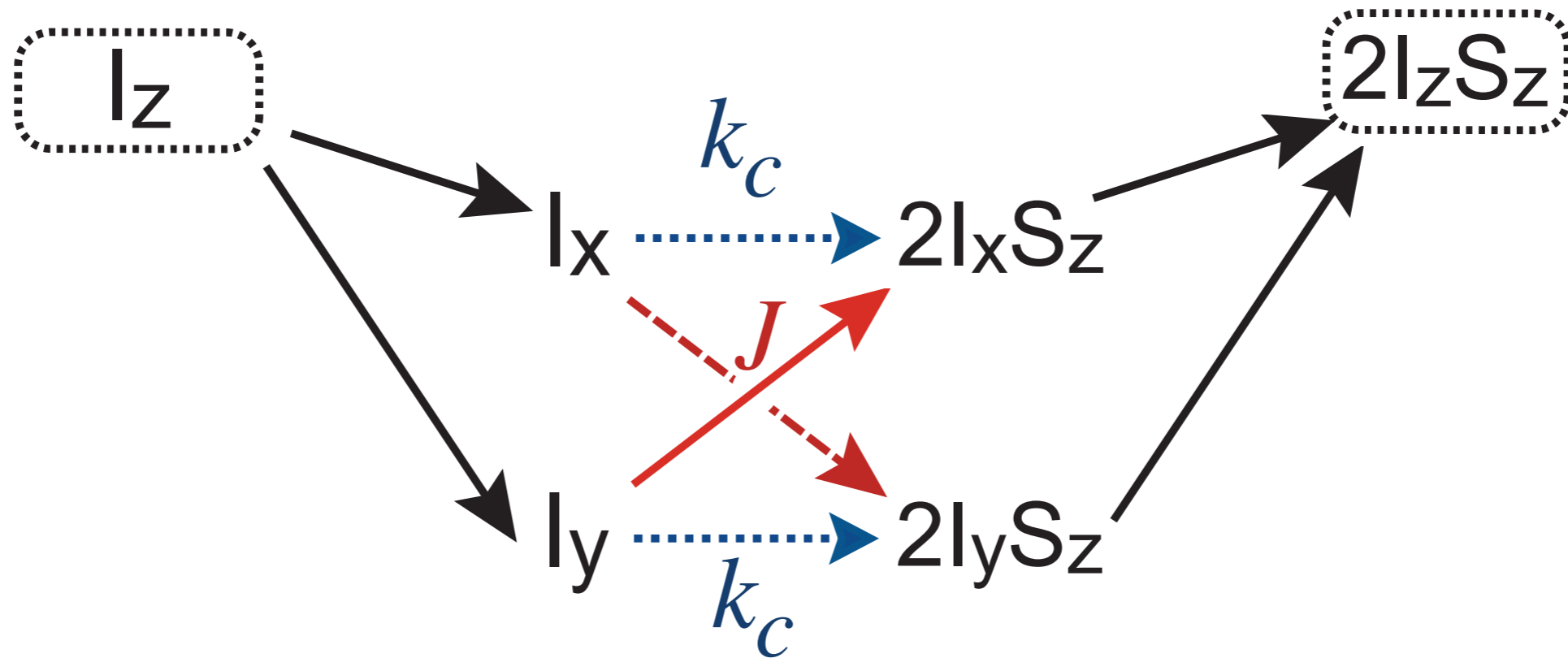
Khaneja et al., J. Magn. Reson. 162, 311 (2003)



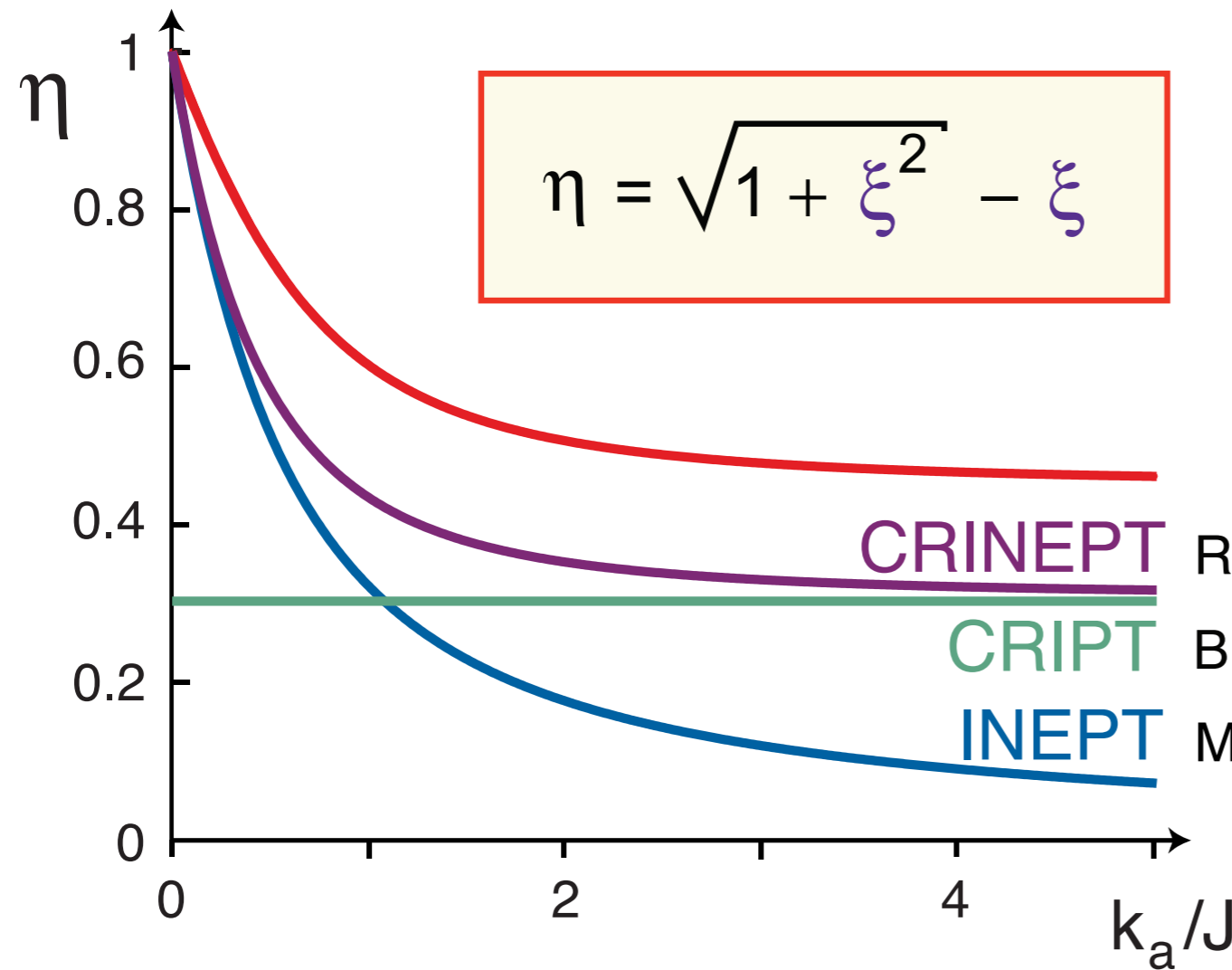
Multiplet of Spin I



Polarization Transfer in the Presence of Cross-Correlated Relaxation



Transfer Efficiency η for $k_c/k_a = 0.75$



$$\eta = \sqrt{1 + \xi^2} - \xi$$

with $\xi^2 = \frac{k_a^2 - k_c^2}{J^2 + k_c^2}$

CRINEPT Riek, Wider, Pervushin, Wüthrich (1999)

CRIPT Brüschweiler, Ernst (1992)

INEPT Morris, Freeman (1979)

maximum transfer efficiency:

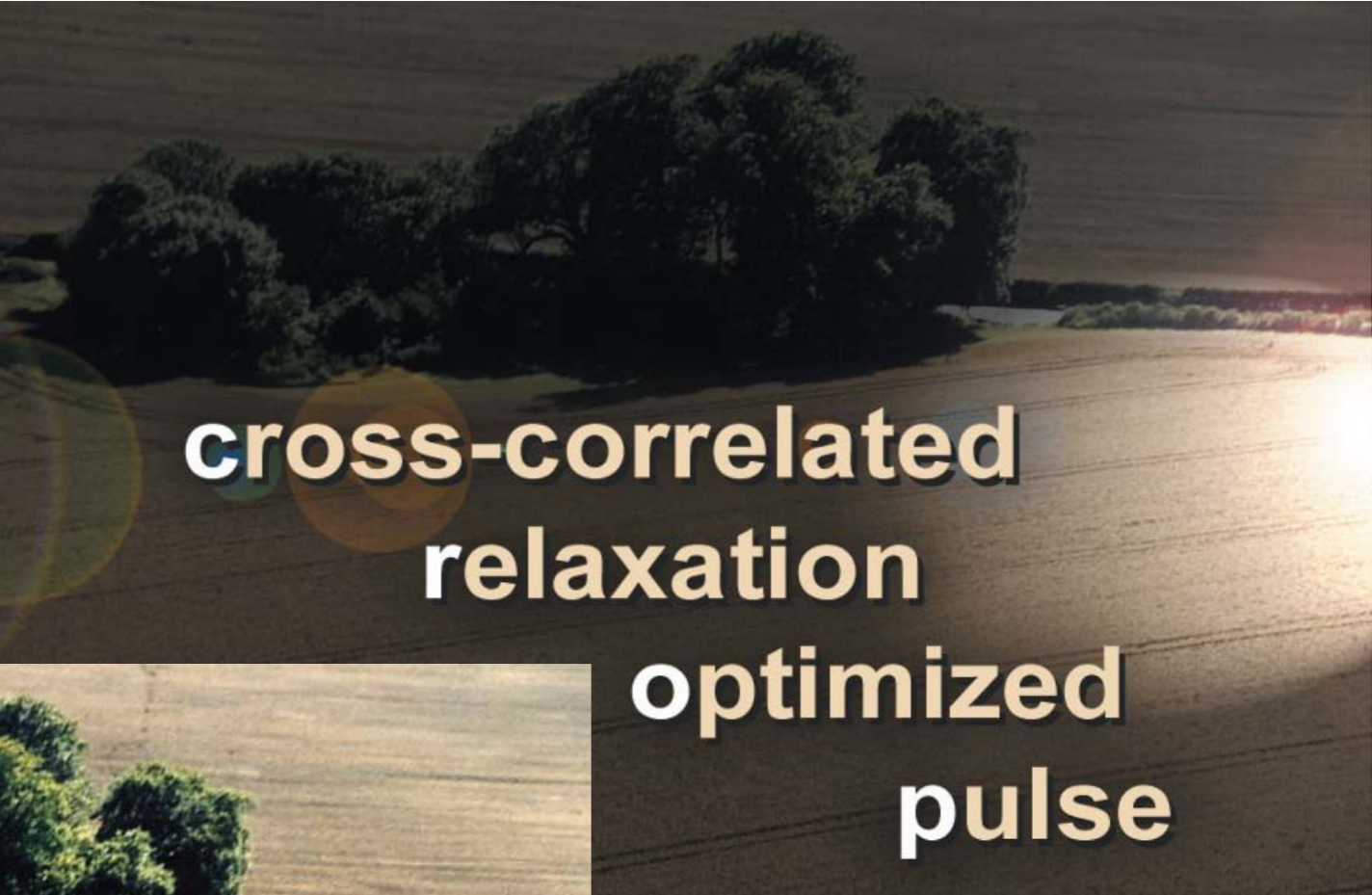
$$\eta = \sqrt{1 + \xi^2} - \xi$$

formal proof (based on principles of optimum control theory):

optimal return function $V(r_1, r_2)$

Hamilton-Jacobi-Bellman equation

$$\max_{u_1, u_2} \left[\frac{\partial V}{\partial r_1} \delta r_1 + \frac{\partial V}{\partial r_2} \delta r_2 \right] = 0$$



**cross-correlated
relaxation**

**optimized
pulse**

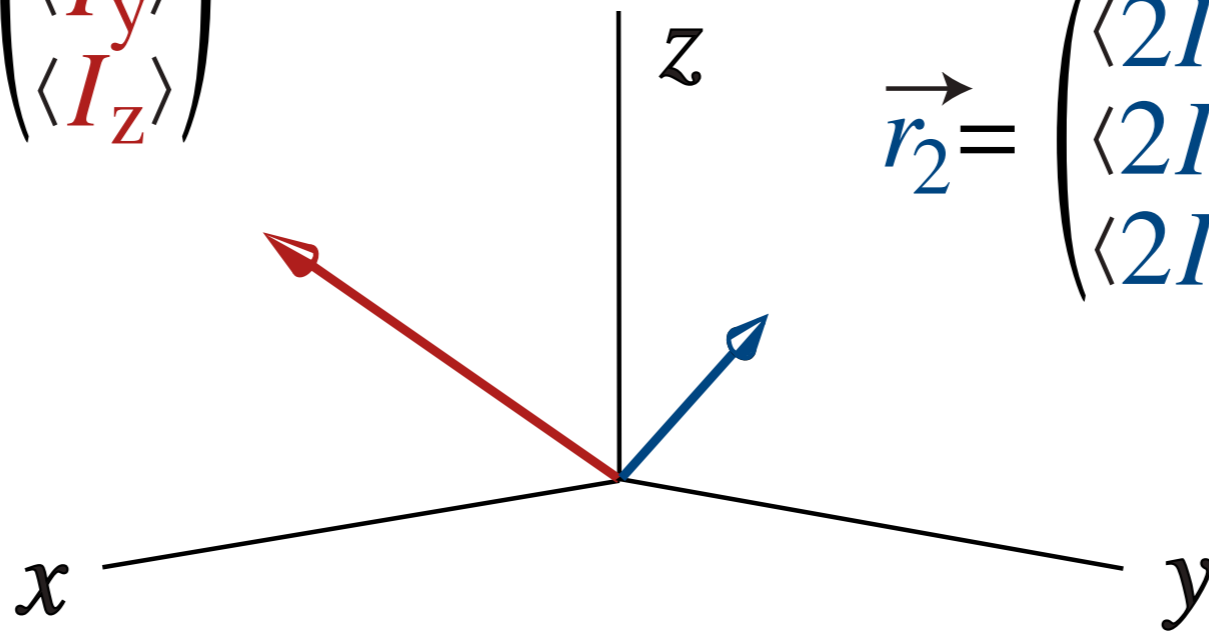


Khaneja, Luy, Glaser,
Proc. Natl. Acad. Sci (2003)

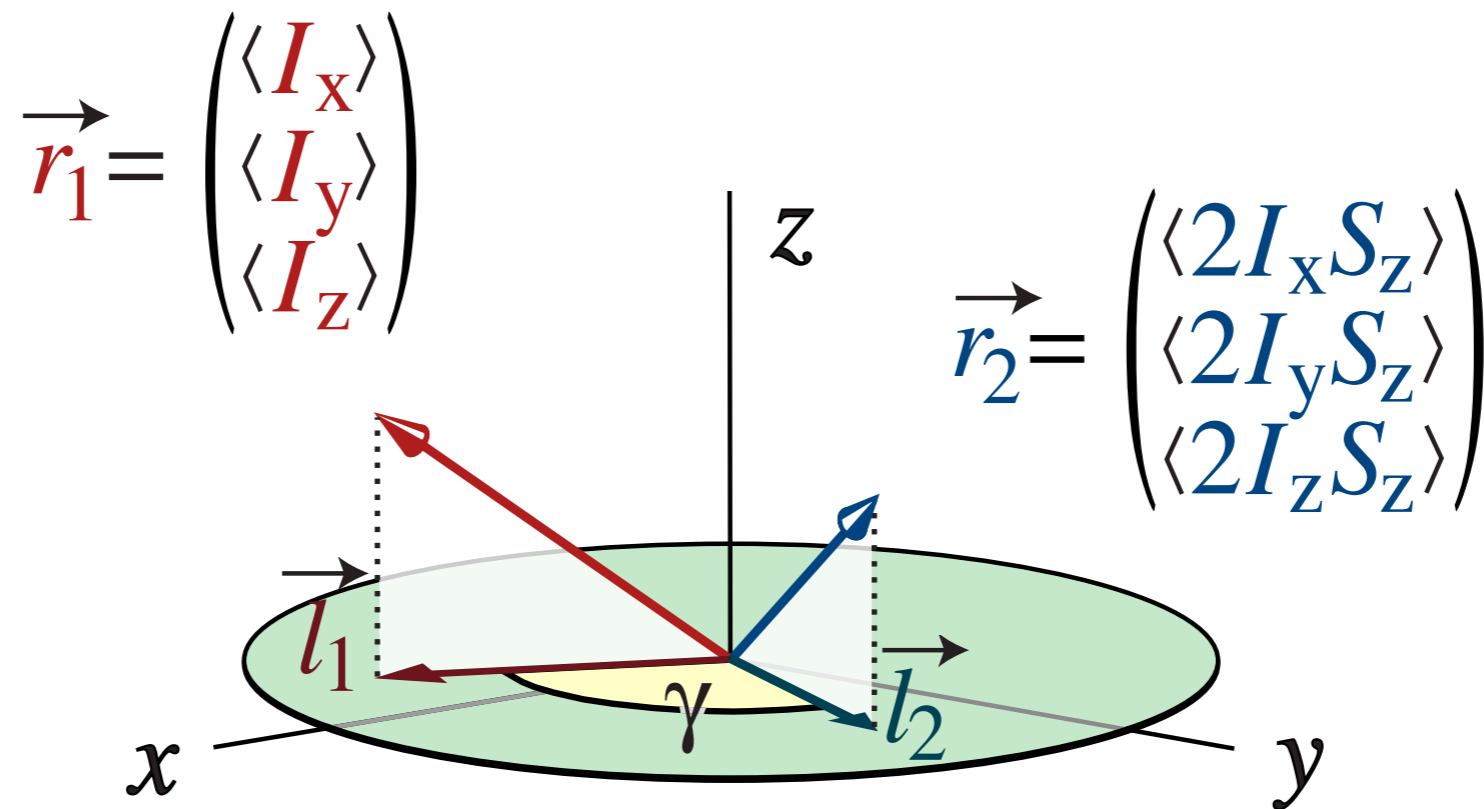
Representing in-phase and antiphase magnetization vectors \vec{r}_1 and \vec{r}_2 in a common frame of reference

$$\vec{r}_1 = \begin{pmatrix} \langle I_x \rangle \\ \langle I_y \rangle \\ \langle I_z \rangle \end{pmatrix}$$

$$\vec{r}_2 = \begin{pmatrix} \langle 2I_x S_z \rangle \\ \langle 2I_y S_z \rangle \\ \langle 2I_z S_z \rangle \end{pmatrix}$$



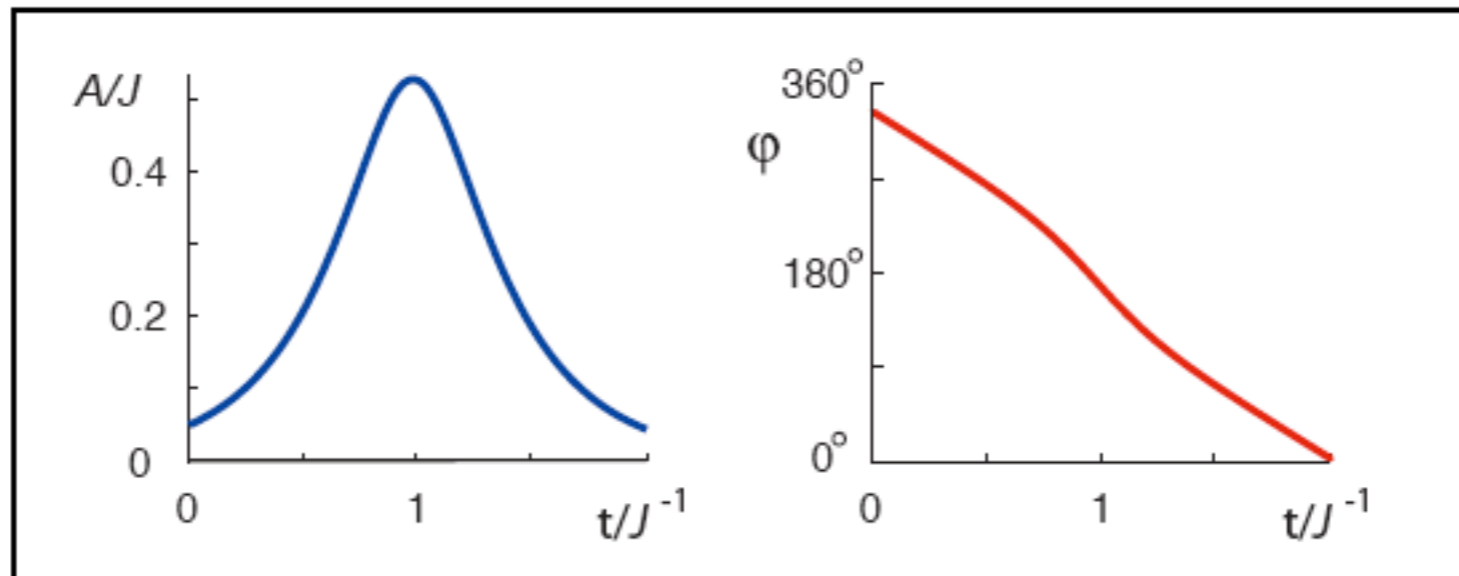
Optimal trajectory preserves ratio $\frac{l_2}{l_1} = \eta$ and angle γ



Optimal pulse shape for $k_c/k_a = 0.75$

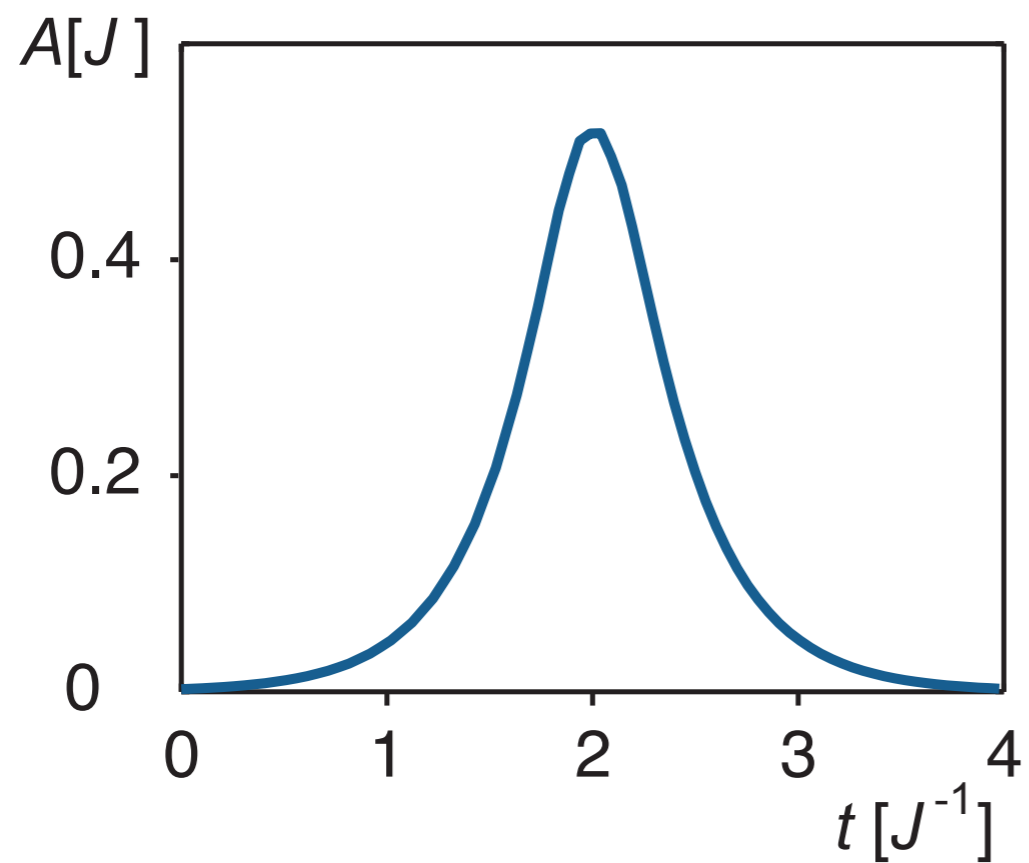
Amplitude

Phase

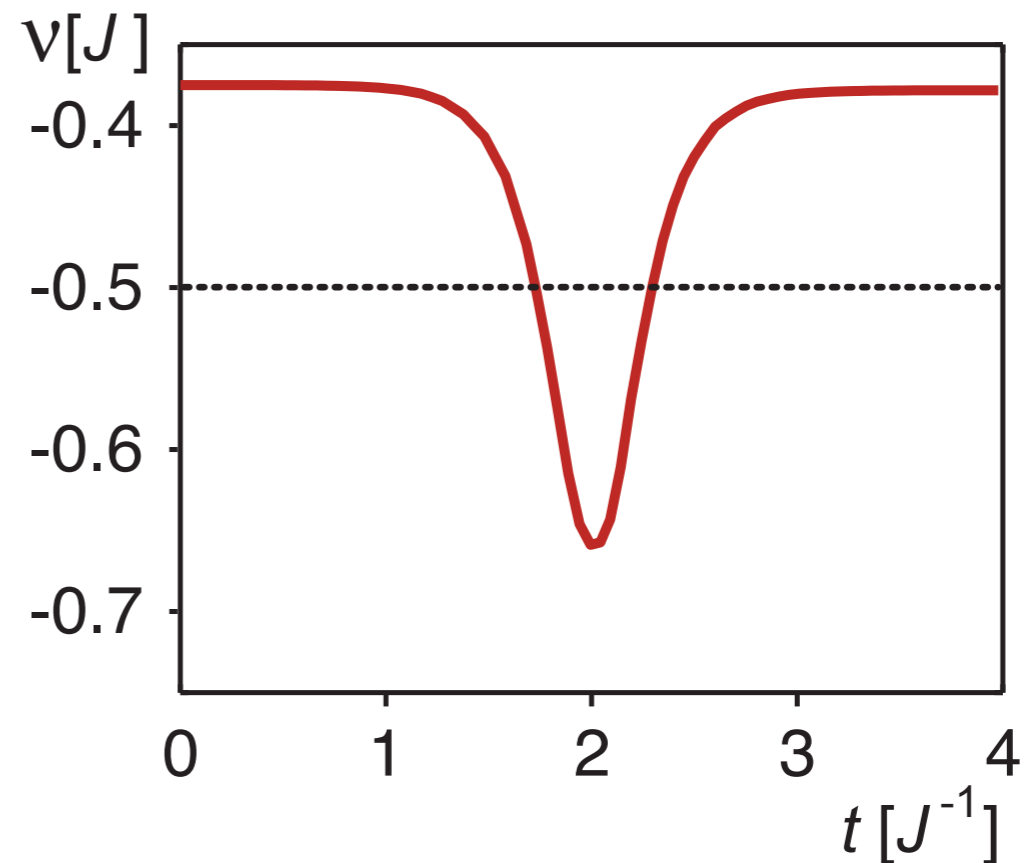


CROP (cross-correlated relaxation optimized pulse)

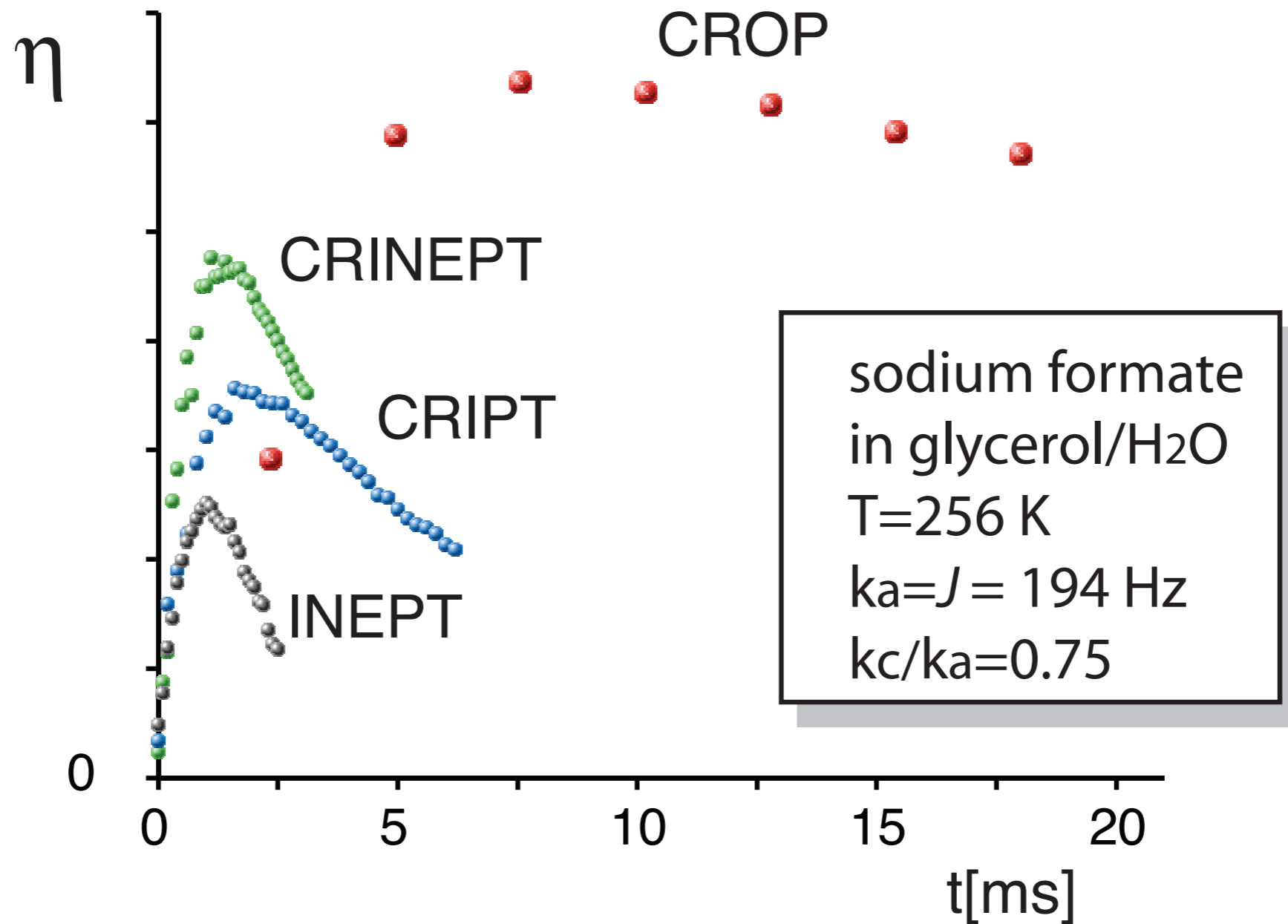
Amplitude



Frequency



Experimental Transfer Functions

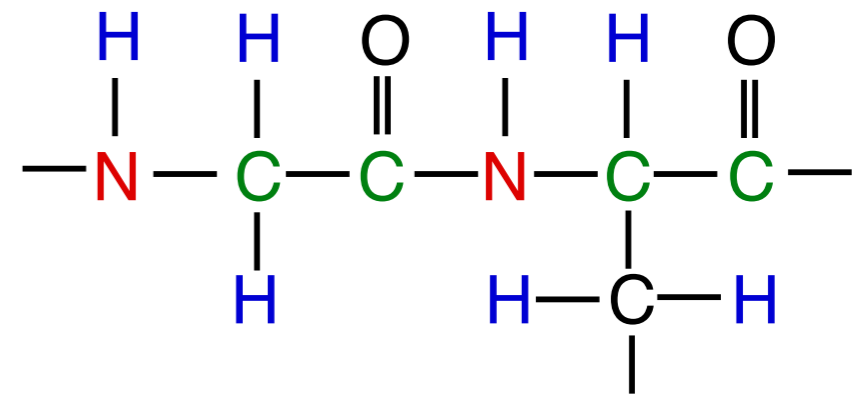


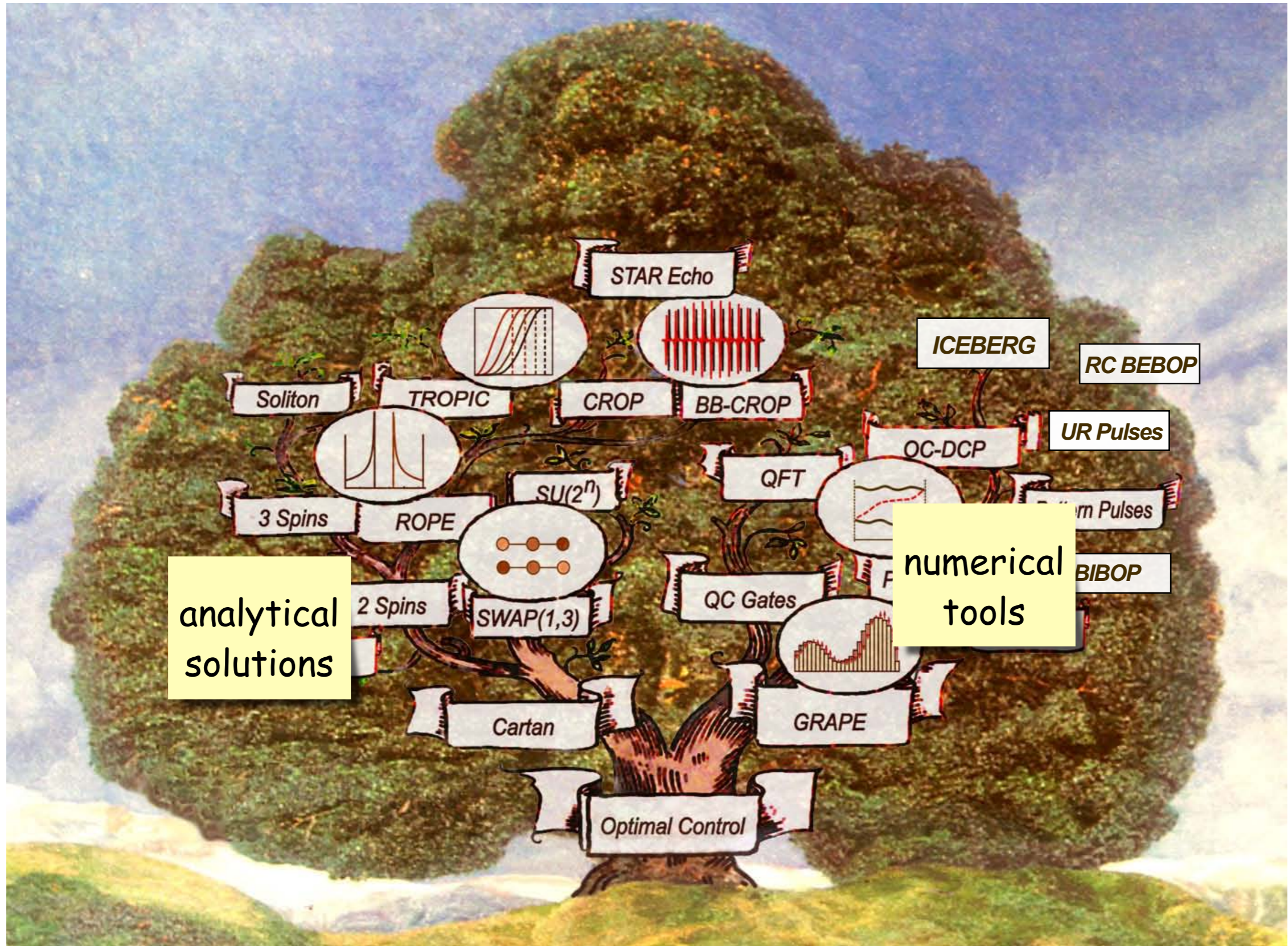
Relaxation-optimized heteronuclear transfer of polarization and spin order

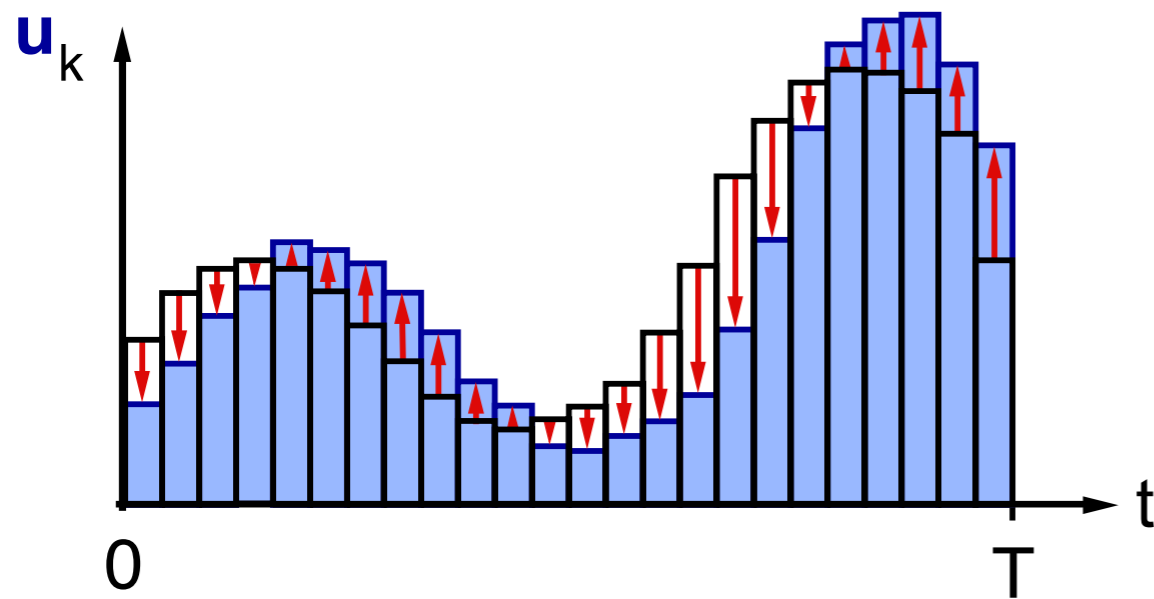
- in two-spin systems
 - without cross-correlated relaxation (ROPE)
 - with cross-correlated relaxation (CROP)
 - in spin chains (SPORTS ROPE)
 - simple case: no cross-correlated relaxation
 - same transverse relaxation rates,
 - no cross-corr. relaxation
 - analytical upper limit for spin order transfer
 - only one smooth pulse, approx. Gaussian
 - more efficient than concatenated INEPT
- solutions for general chains

Challenges for biological applications

- offset compensation
- experimental imperfections
- homonuclear couplings
- branched coupling networks







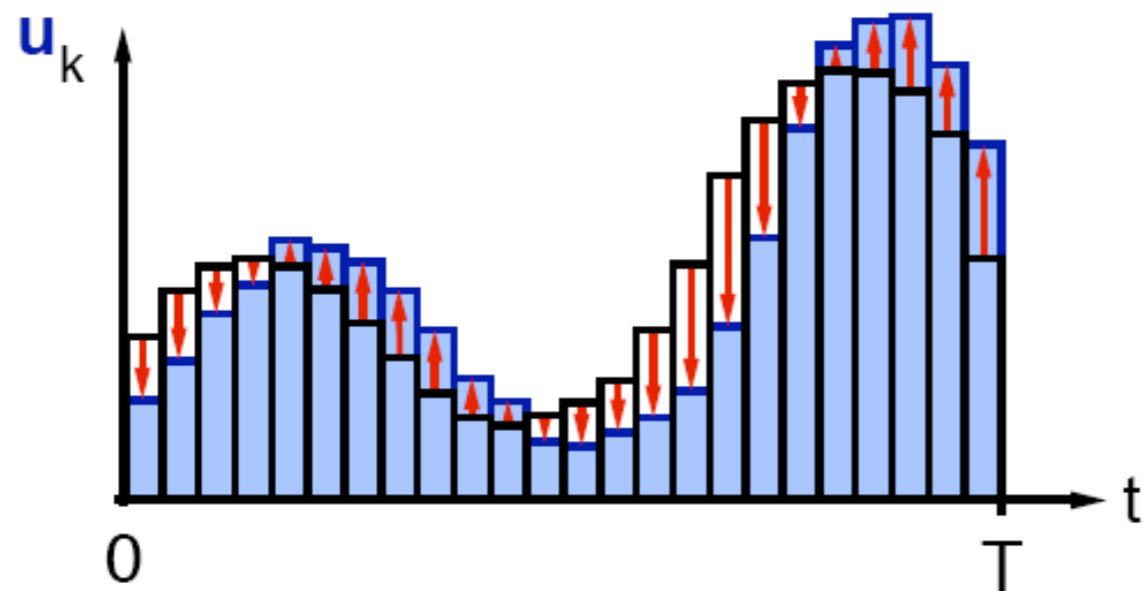
desired transfer: $A \longrightarrow C$

performance: $\langle C | \rho(T) \rangle$

$$\rho(0) = A$$

$$\lambda(T) = C$$

$$\mathbf{u}_k(t) \longrightarrow \mathbf{u}_k(t) + \varepsilon \langle \lambda(t) | [-i H_k, \rho(t)] \rangle$$



desired propagator: U_F

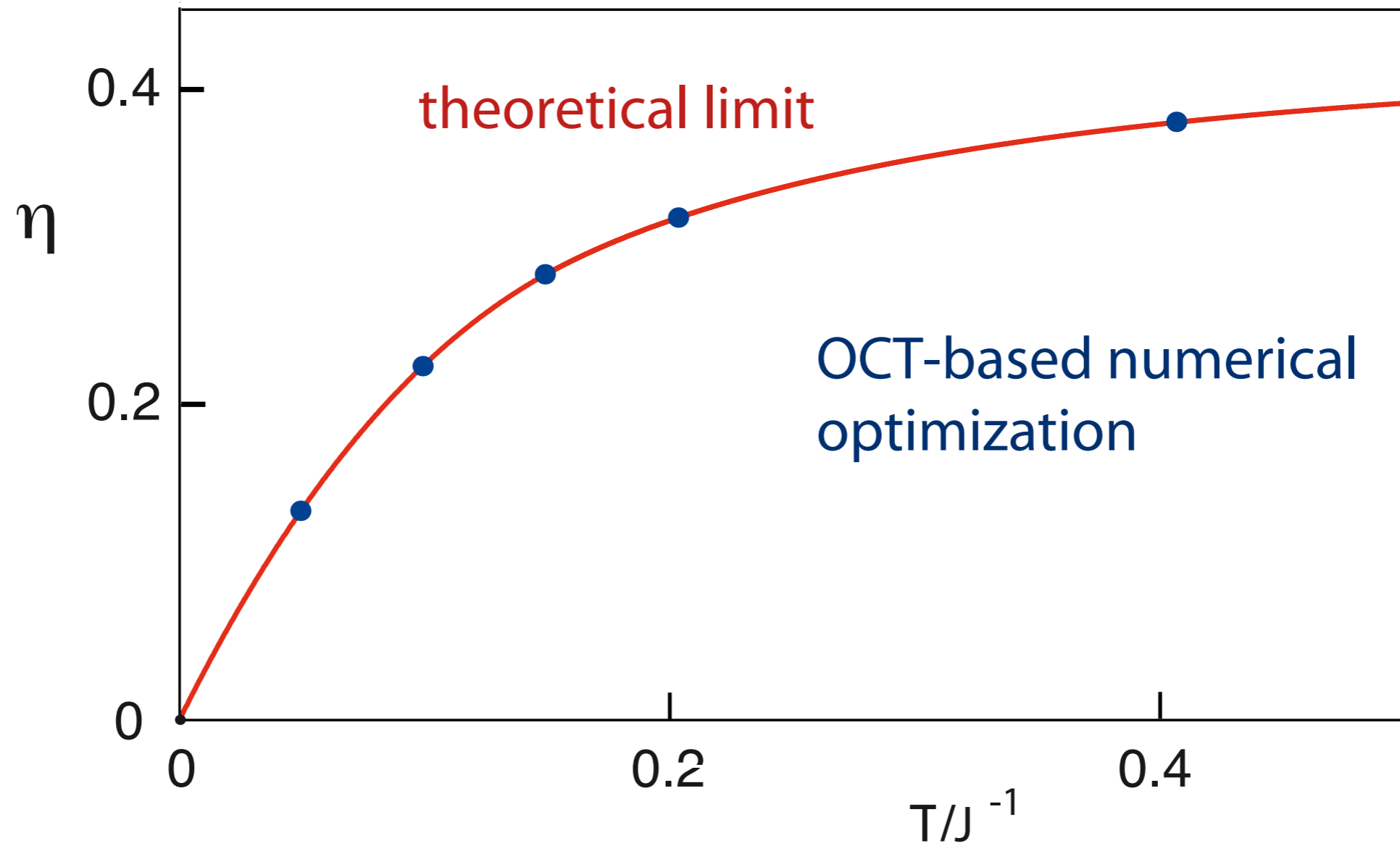
performance: $|\langle U_F | U(T) \rangle|^2$

$$U(0) = \mathbf{1}$$

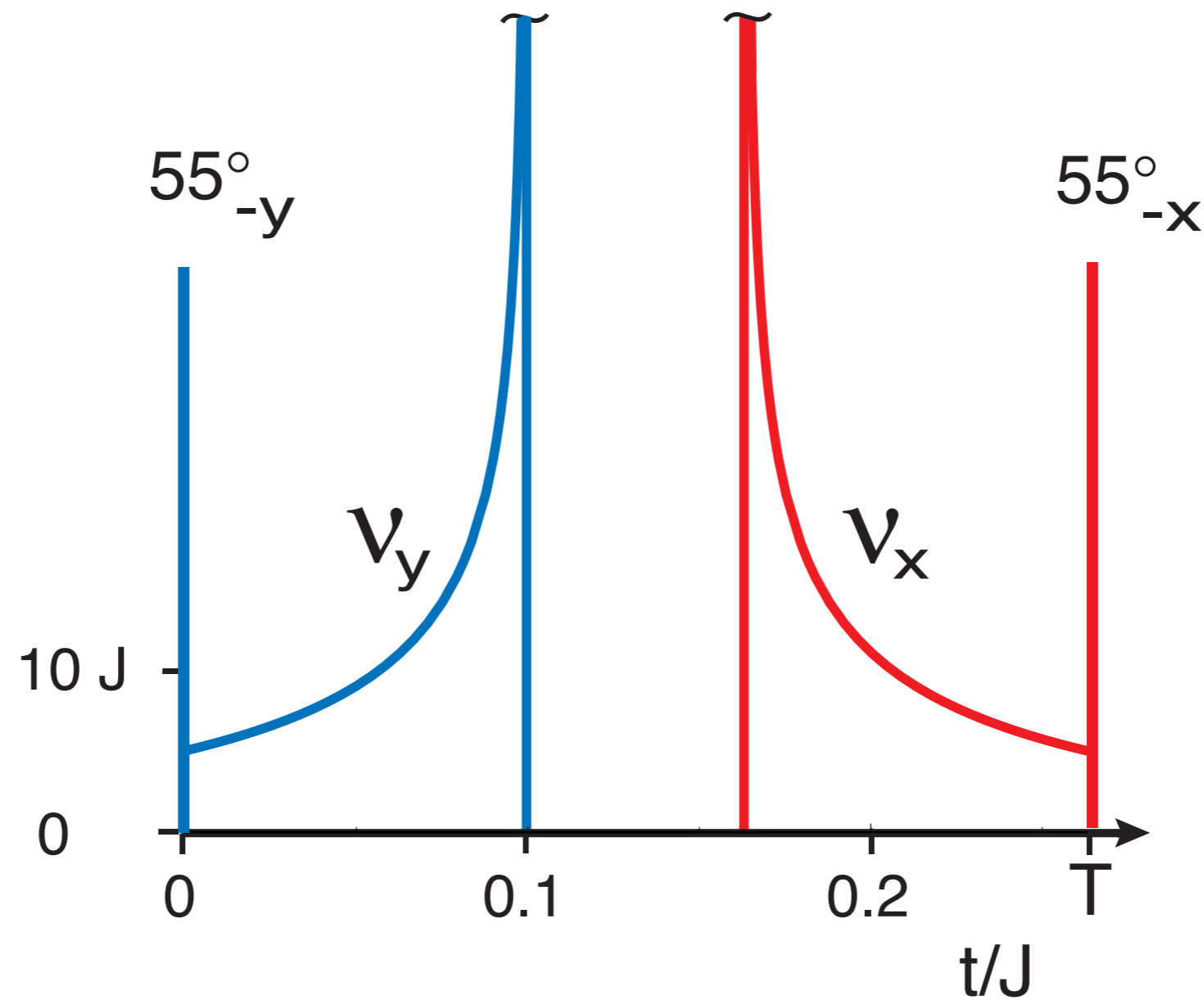
$$P(T) = U_F$$

$$\mathbf{u}_k(t) \longrightarrow \mathbf{u}_k(t) + \varepsilon \operatorname{Re} \left\{ \langle P(t) | -i H_k U(t) \rangle \langle U(t) | P(t) \rangle \right\}$$

relaxation-optimized coherence transfer

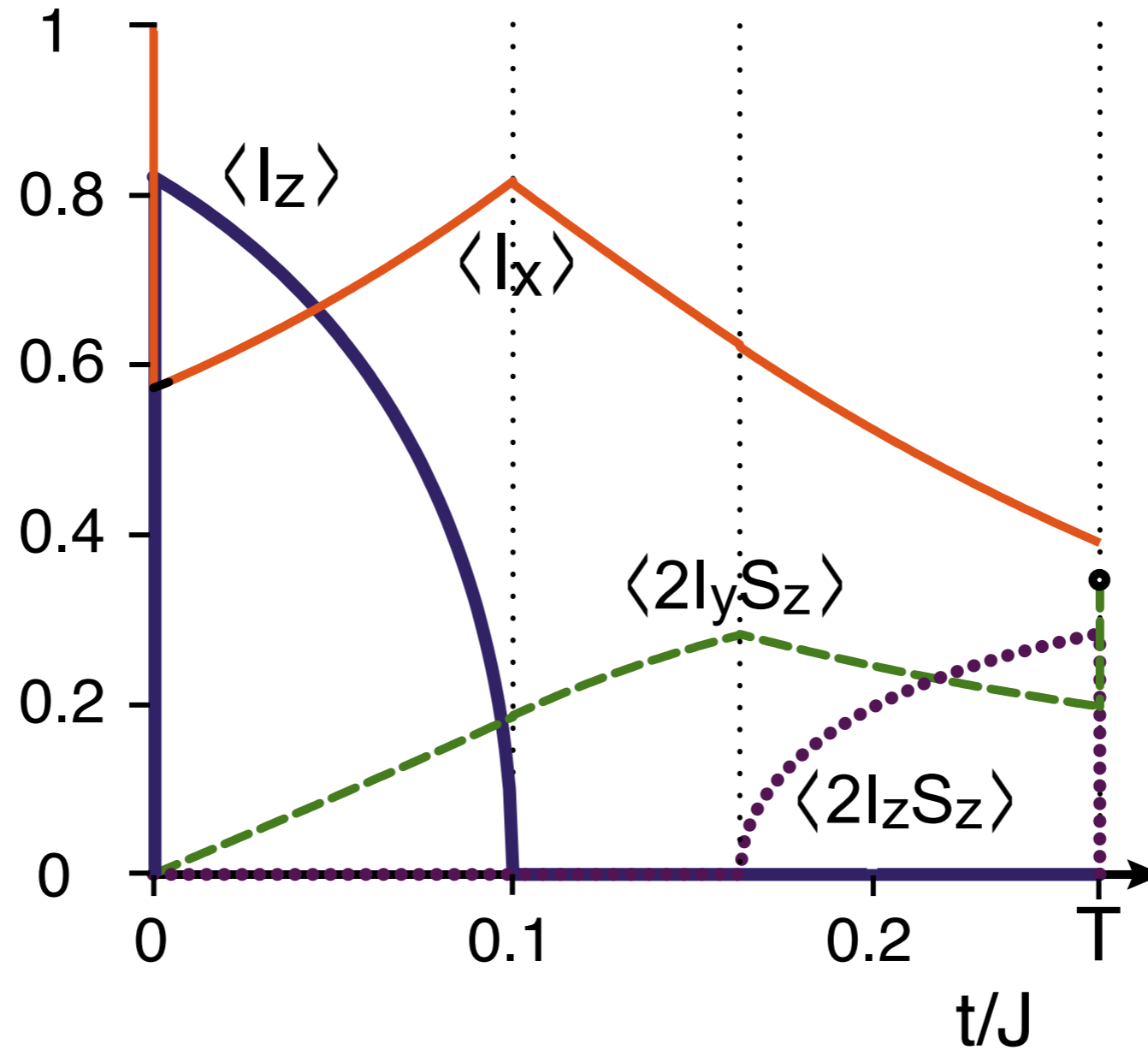


ROPE Pulse Sequence (finite time T)

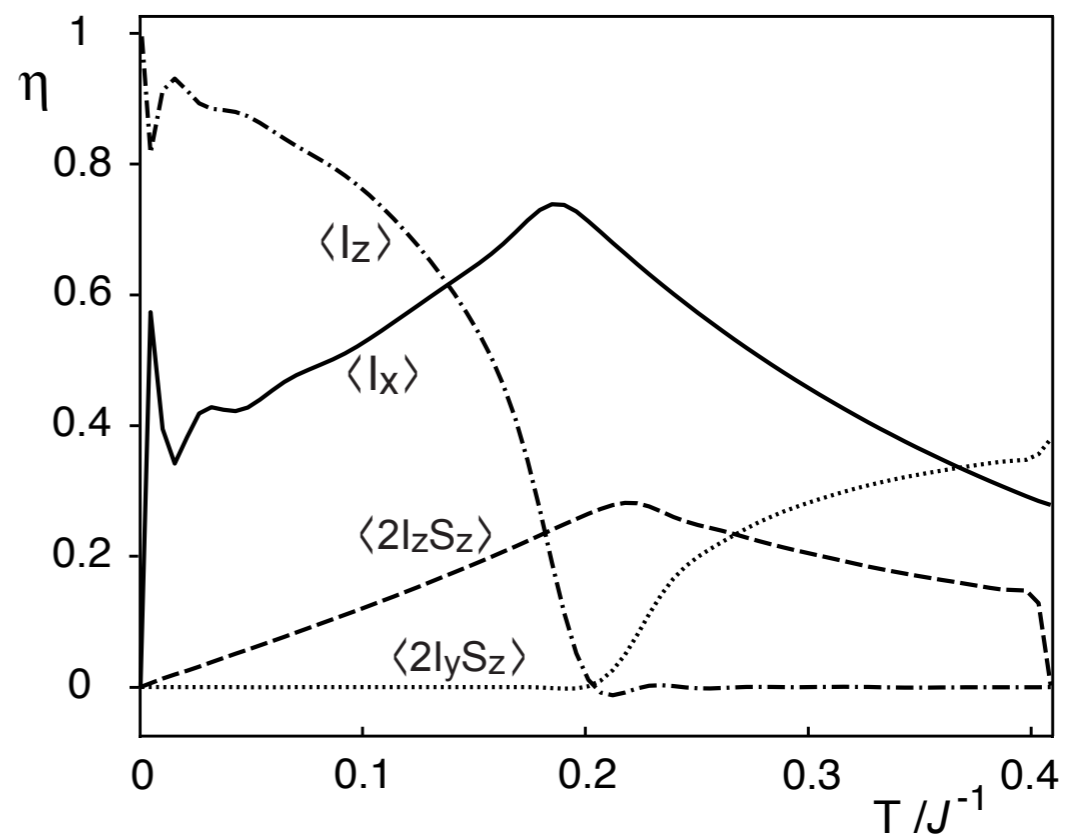
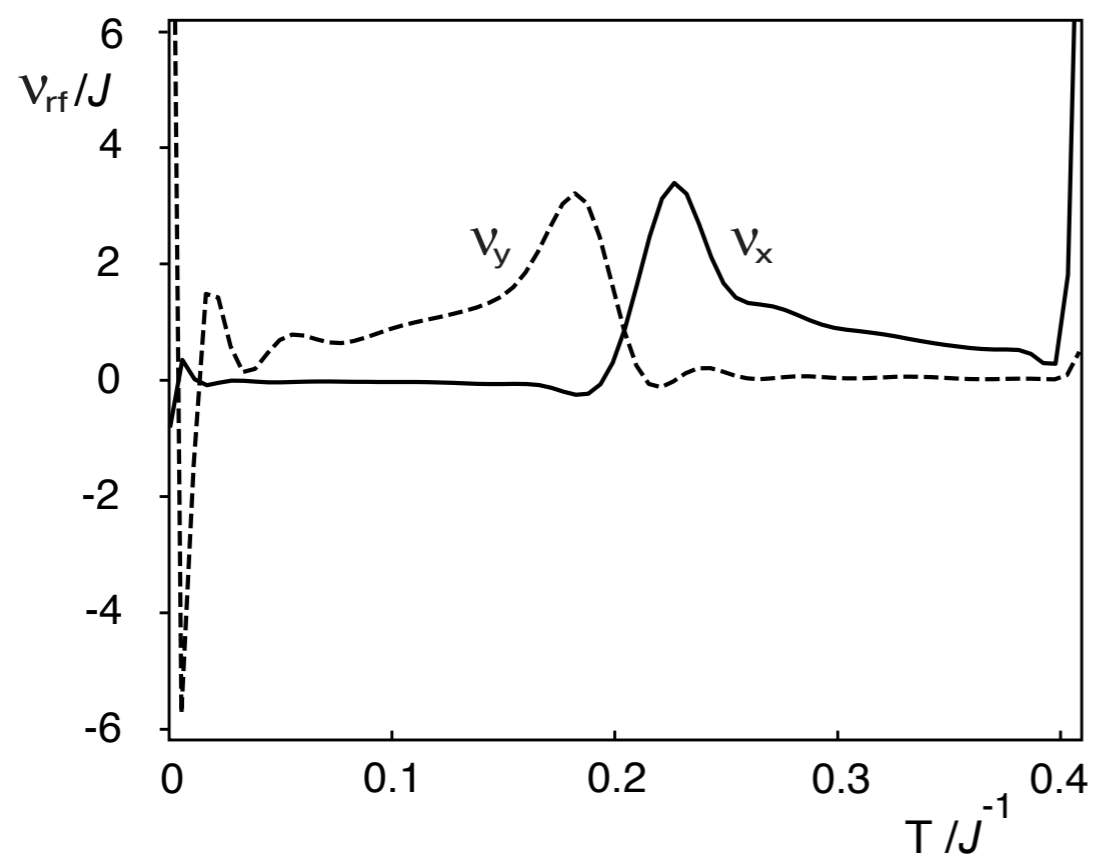


Khaneja et al., J. Magn. Reson. 162, 311 (2003)

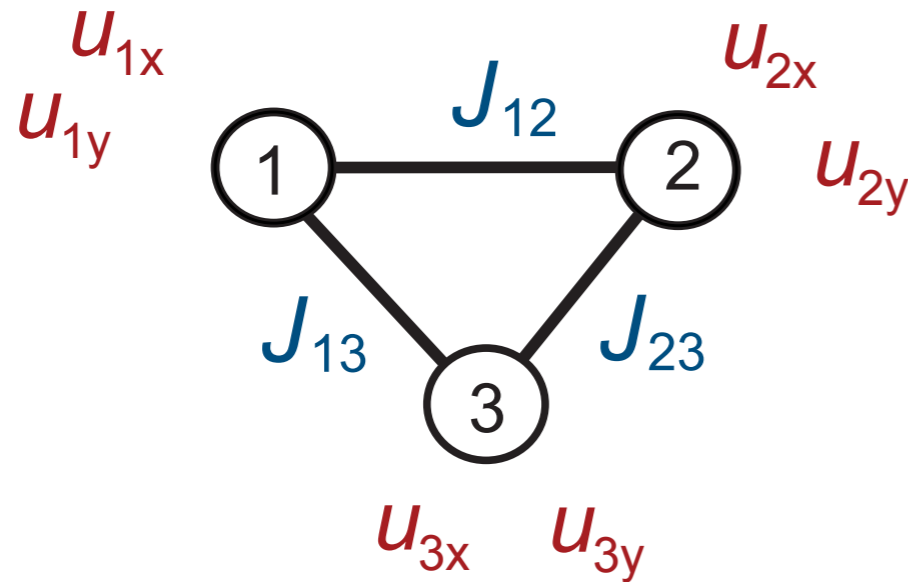
ROPE Trajectory (finite time T)



Numerically optimized ROPE sequences



Polarization transfer in homonuclear three spin systems



$$I_{1z} \longrightarrow I_{2z}$$

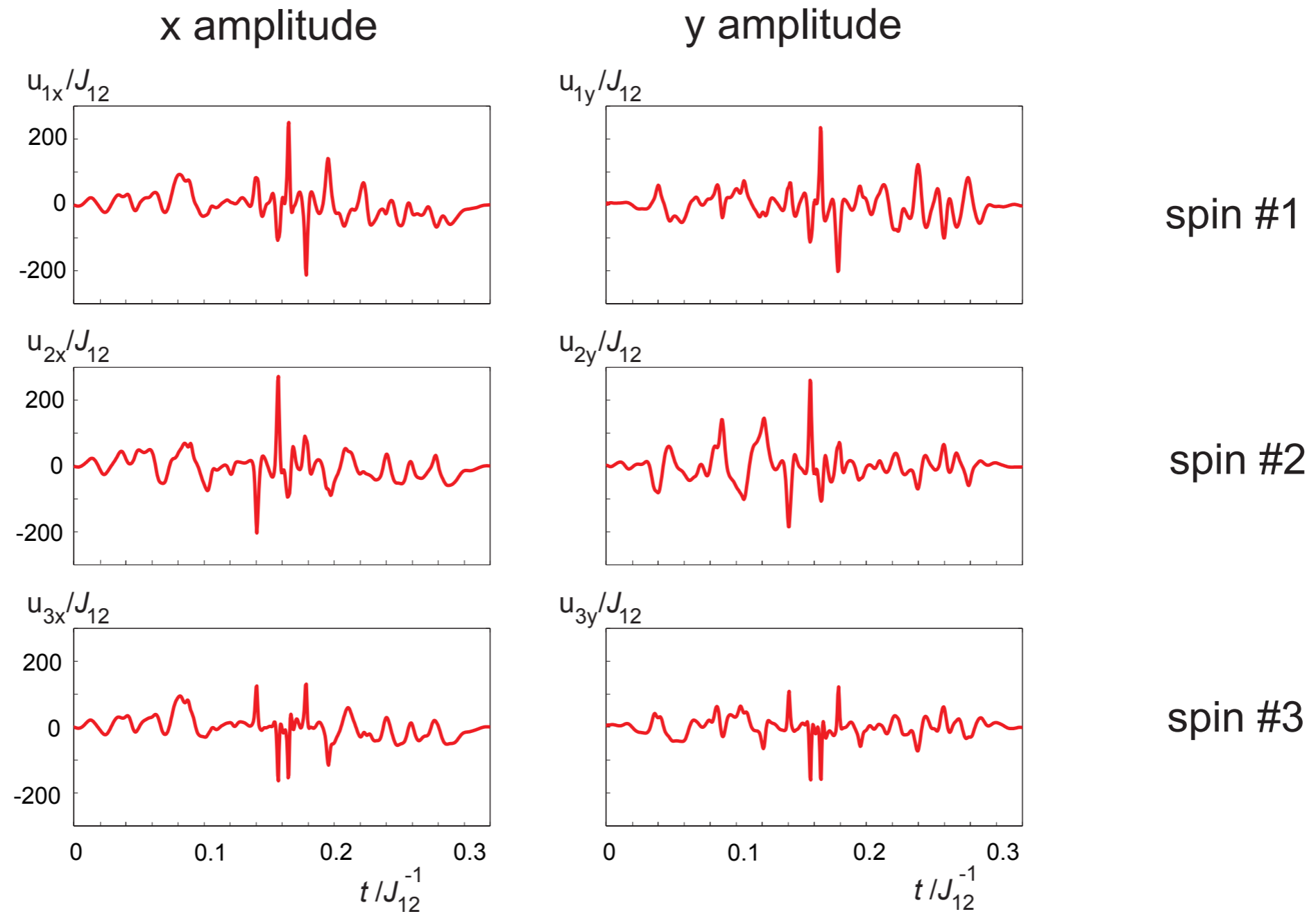
isotropic (Heisenberg) couplings

$$\sum_{m<n} 2\pi J_{mn} (I_{mx} I_{nx} + I_{my} I_{ny} + I_{mz} I_{nz})$$

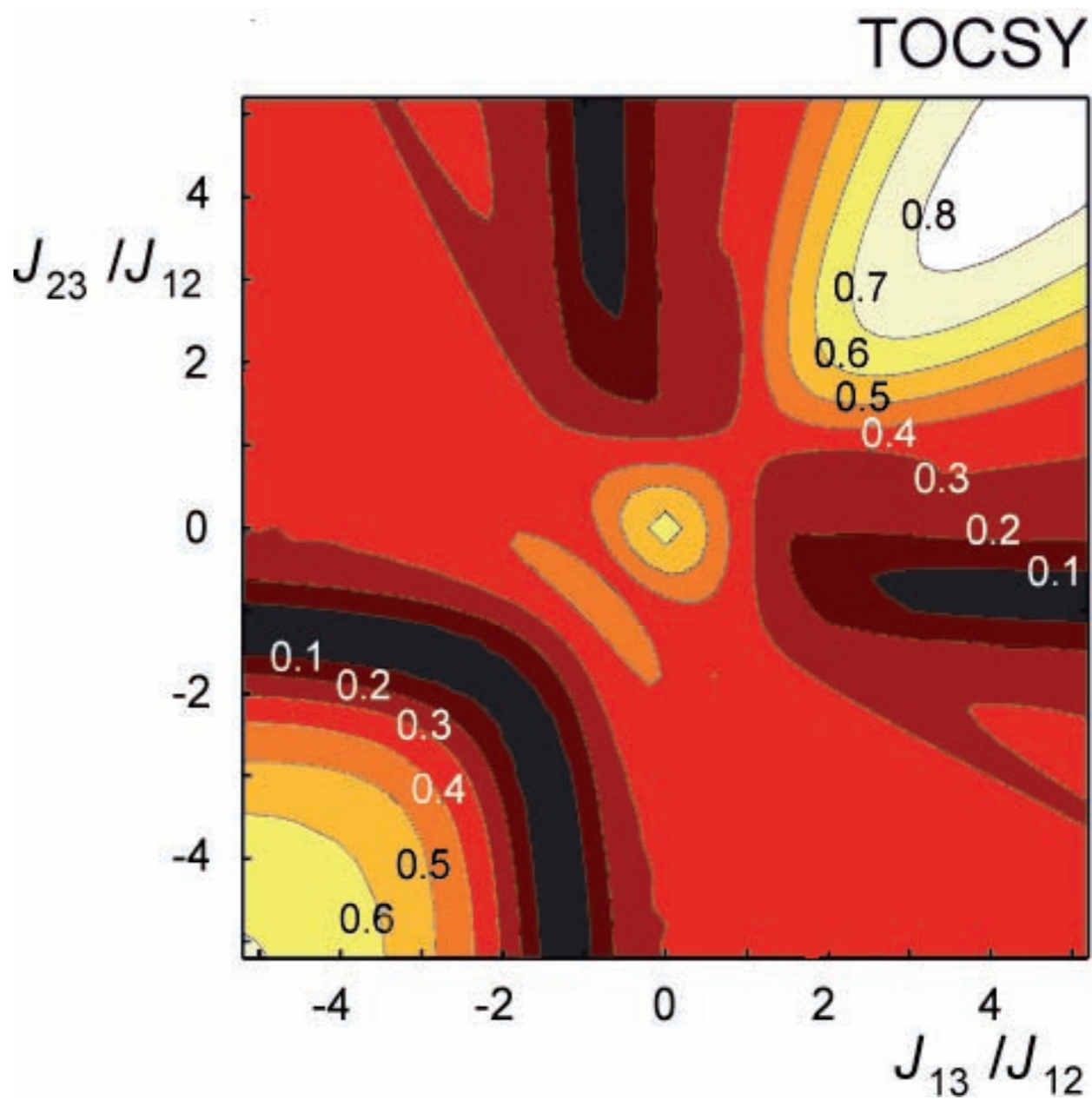
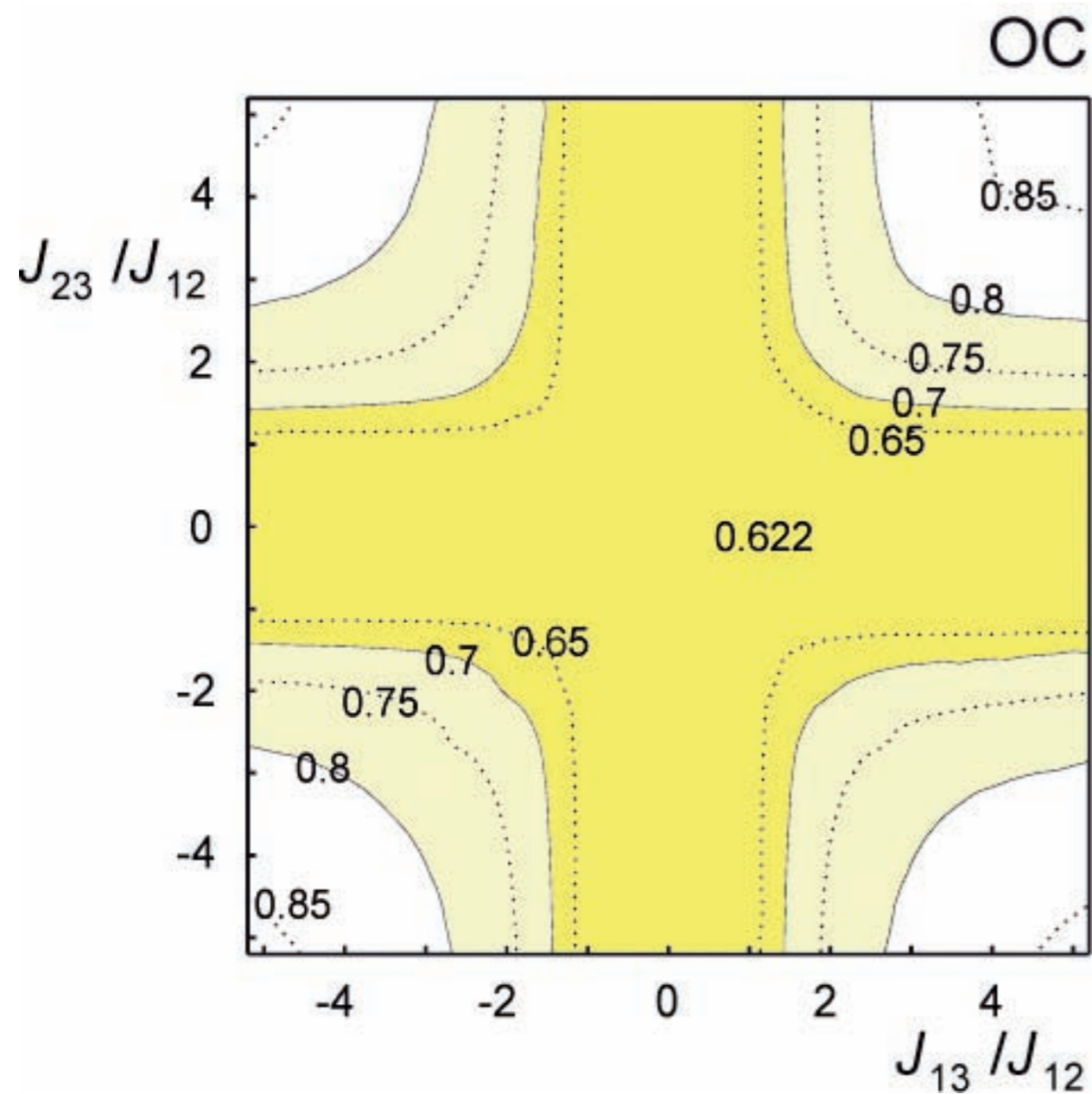
idealized setting: fast, selective pulses (six control amplitudes)

$$2\pi \sum_{m=1}^3 \{ u_{mx}(t) I_{mx} + u_{my}(t) I_{my} \}$$

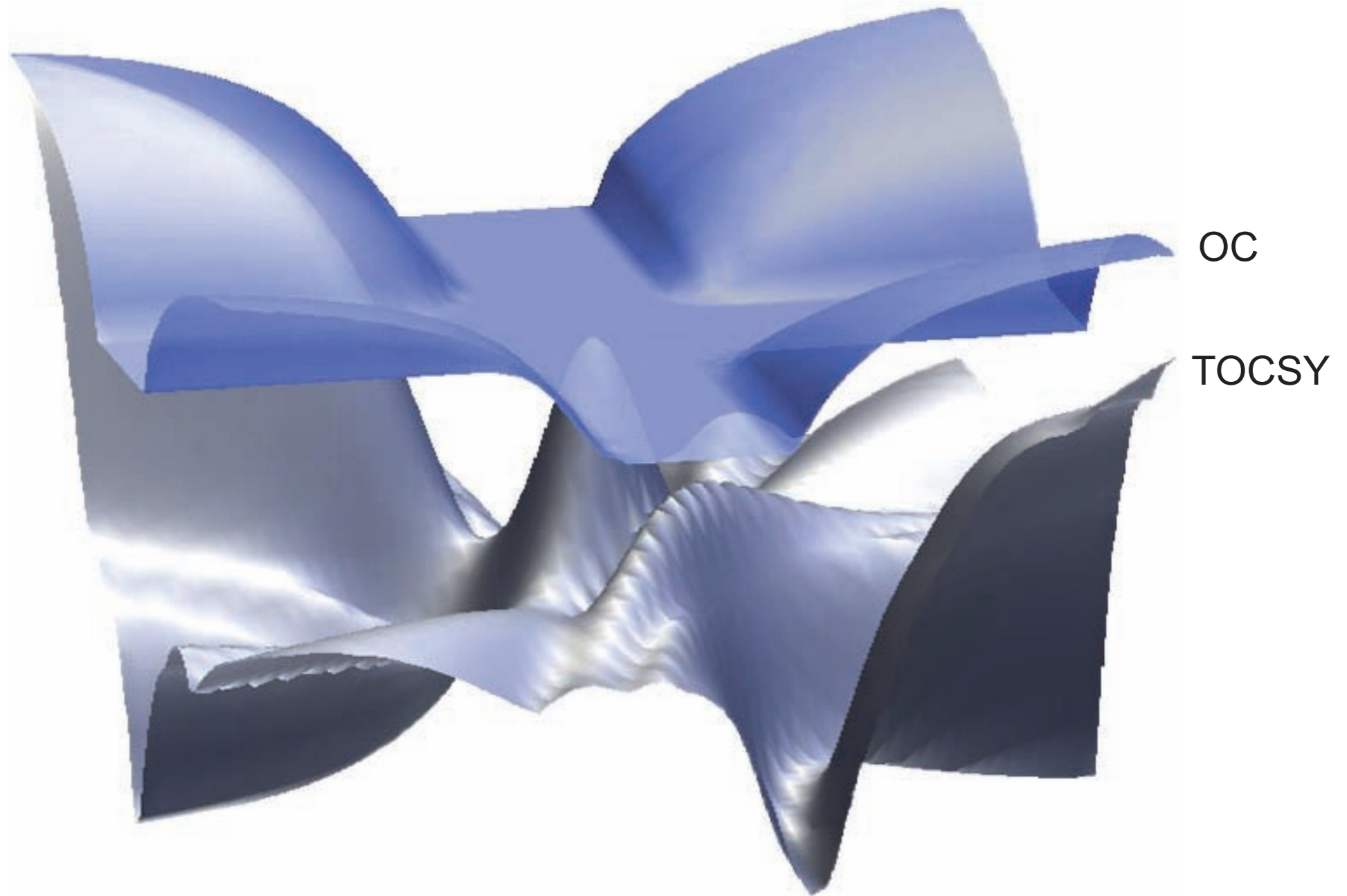
Optimized controls (radio frequency amplitudes)



Transfer efficiency as a function of relative coupling constants J_{13}/J_{12} and J_{23}/J_{12}



Transfer efficiency as a function of relative coupling constants J_{13}/J_{12} and J_{23}/J_{12}



Remark: The conventional (so-called TOCSY) experiment is equivalent to the transfer of energy between three coupled pendulums,

see

R. Marx, S. J. Glaser,
"Spins Swing Like Pendulums Do: An Exact Classical Model for
TOCSY Transfer in Systems of Three Isotropically Coupled Spins $1/2$ ",
J. Magn. Reson. 164, 338-342 (2003).



Polarization transfer in homonuclear three spin systems

isotropic (Heisenberg) couplings

$$\sum_{m<n} 2\pi J_{mn} (I_{mx} I_{nx} + I_{my} I_{ny} + I_{mz} I_{nz})$$

idealized control: fast, selective pulses (six control amplitudes)

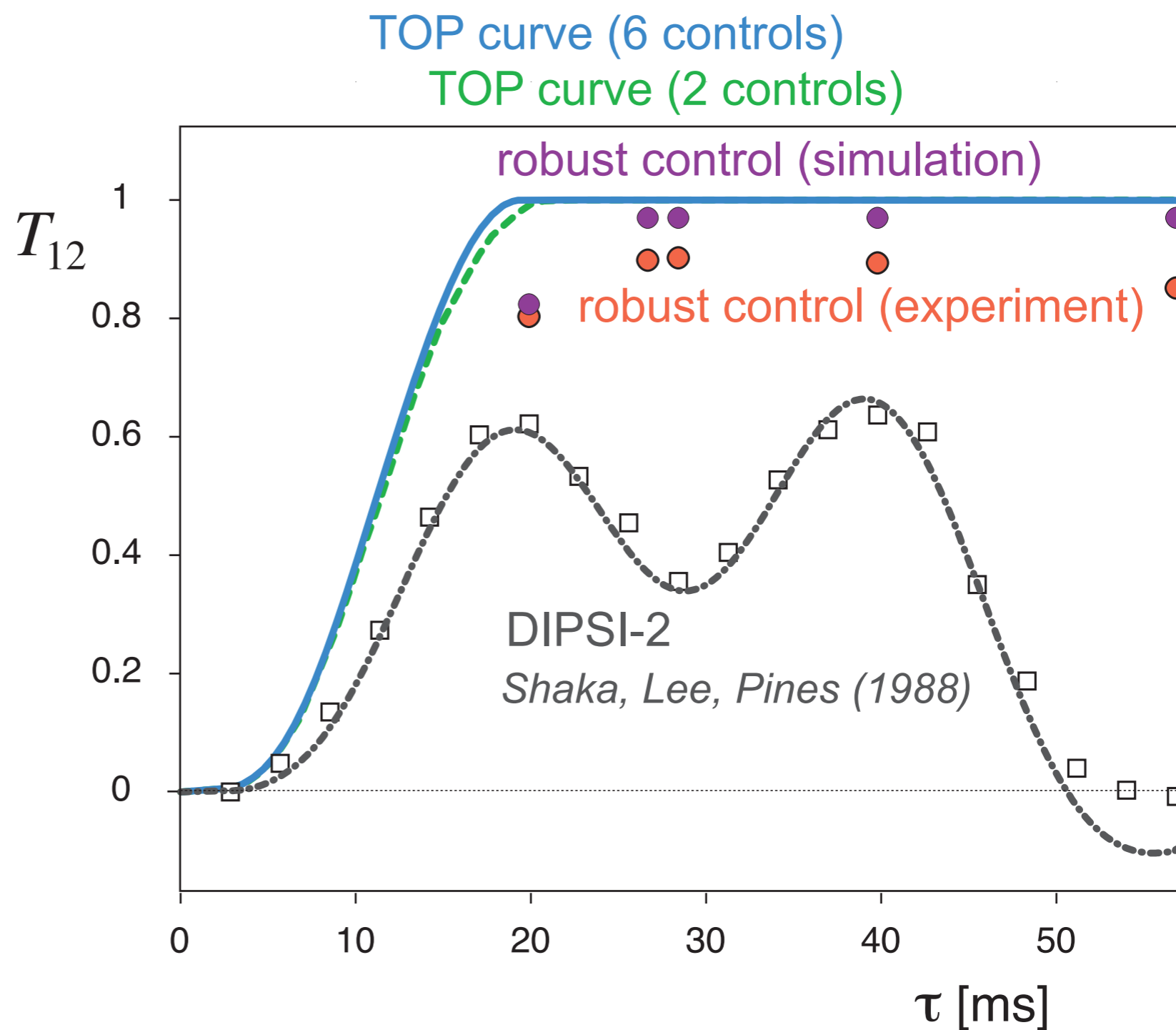
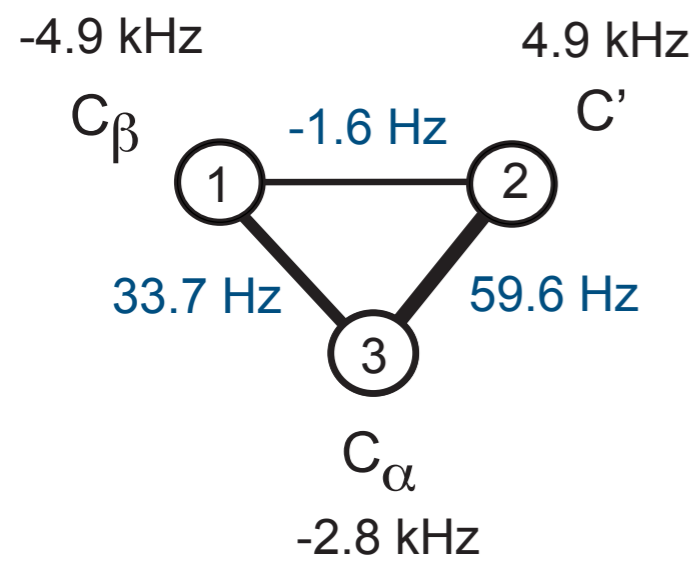
$$2\pi \sum_{m=1}^3 \{ u_{mx}(t) I_{mx} + u_{my}(t) I_{my} \}$$

realistic control: non-selective pulses (two control amplitudes)

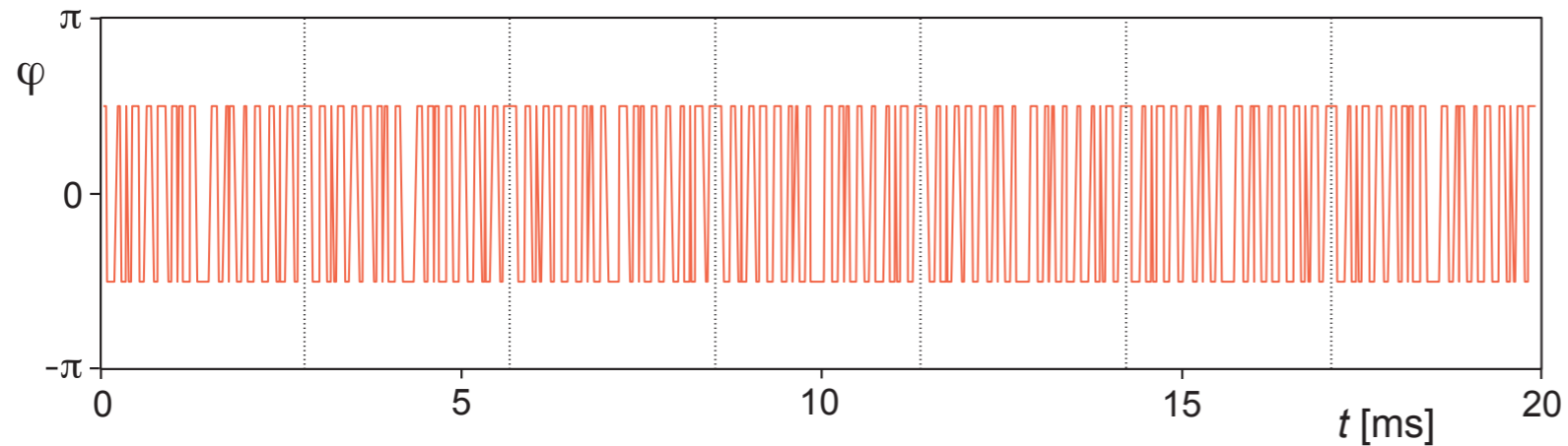
$$2\pi u_x(t) \sum_{m=1}^3 I_{mx} + 2\pi u_y(t) \sum_{m=1}^3 I_{my}$$

and (constant) offset terms $2\pi \sum_{m=1}^3 \nu_{mz} I_{mz}$

Homonuclear three spin model system: ^{13}C labelled alanine

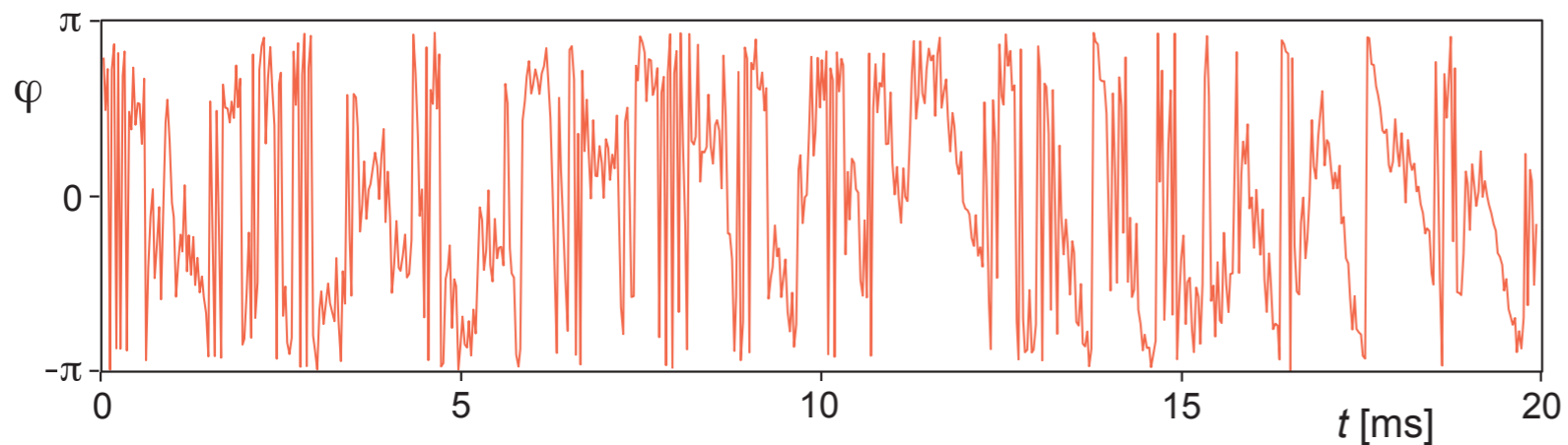


Conventional TOCSY sequence, e.g. DIPSI-2

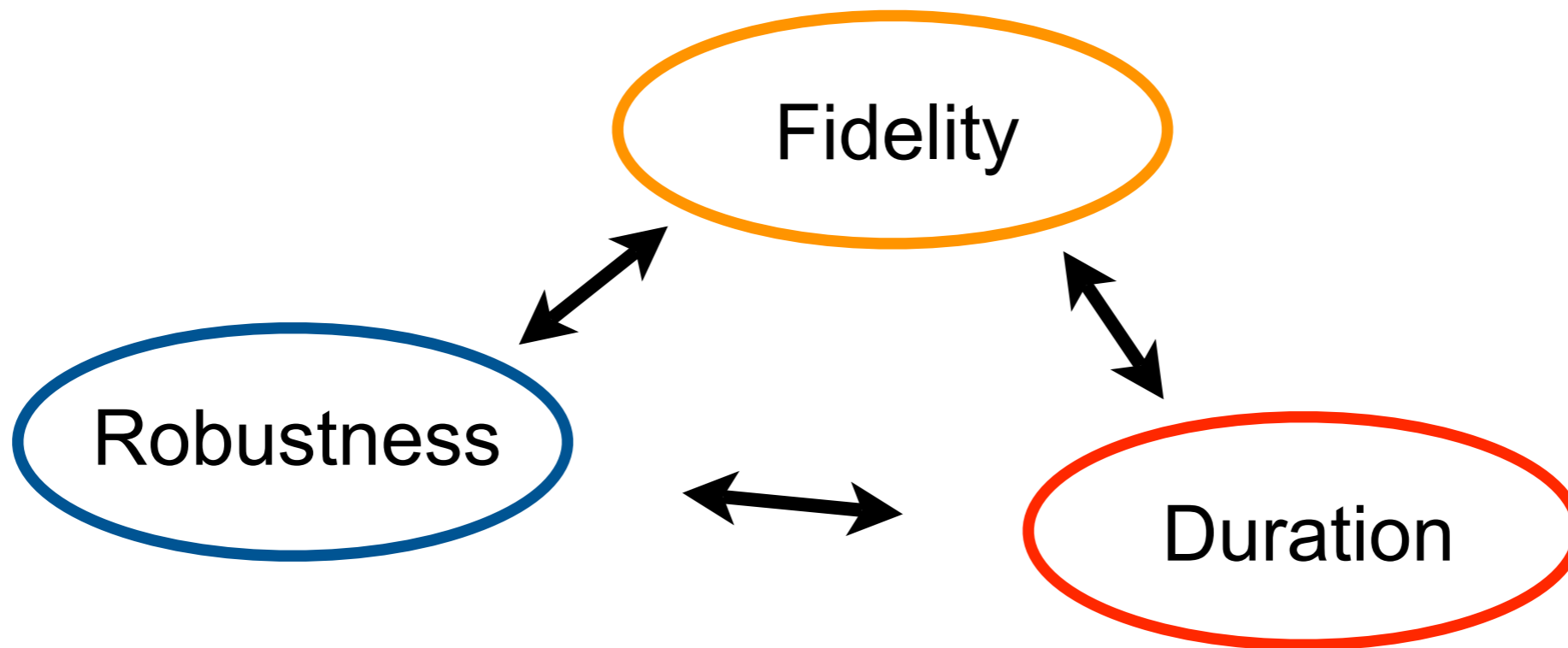


Shaka et al. (1988)

Robust phase-modulated polarization transfer sequences:



Neves et al. (2006)



Low power heteronuclear decoupling

$$\mathcal{H}(t) = \mathcal{H}_{off}^I + \mathcal{H}_{off}^S + \mathcal{H}_J^{IS} + \mathcal{H}_{rf}^S(t)$$

$$\mathcal{H}_{off}^I = 2\pi\nu_I I_z$$

$$\mathcal{H}_{off}^S = 2\pi\nu_S S_z$$

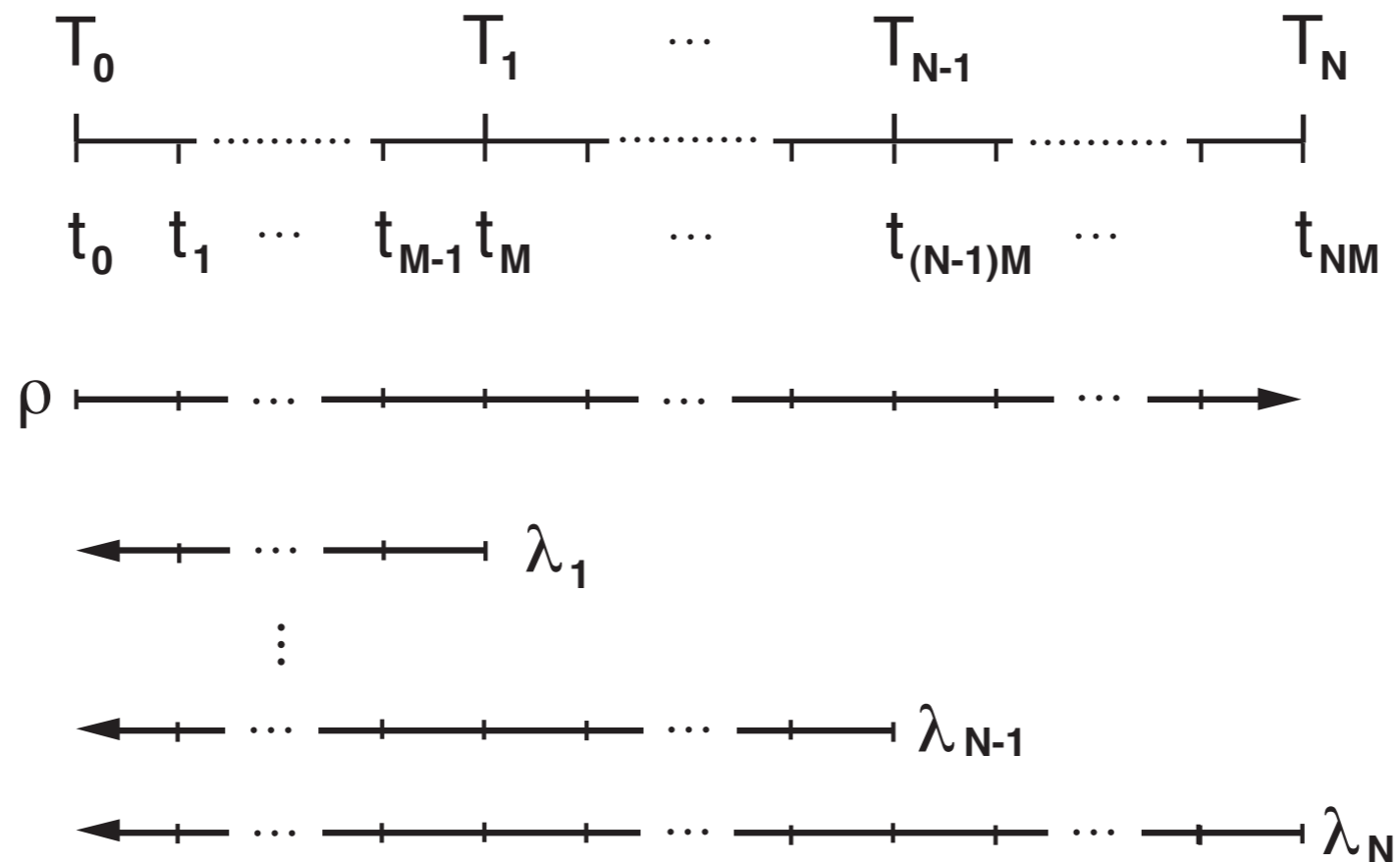
$$\mathcal{H}_J^{IS} = 2\pi J S_z I_z$$

$$\mathcal{H}_{rf}^S(t) = 2\pi \epsilon \{u_x(t) S_x + u_y(t) S_y\}$$

Low power heteronuclear decoupling

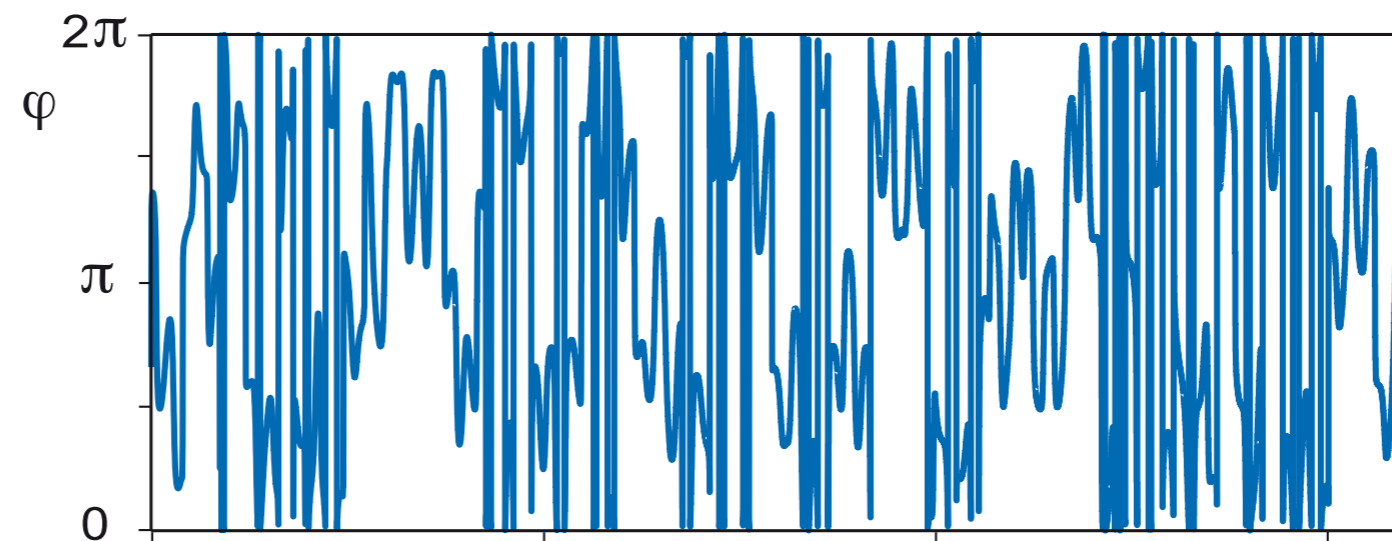
$$s_k = \langle I_x \rangle (T_k)$$

$$S_0(\epsilon, \nu_S) = \frac{1}{N+1} \sum_{k=0}^N s_k(\epsilon, \nu_S) \quad \Phi = \frac{1}{N_\epsilon N_\nu} \sum_{p=1}^{N_\epsilon} \sum_{q=1}^{N_\nu} \phi(\epsilon^{(p)}, \nu_S^{(q)})$$

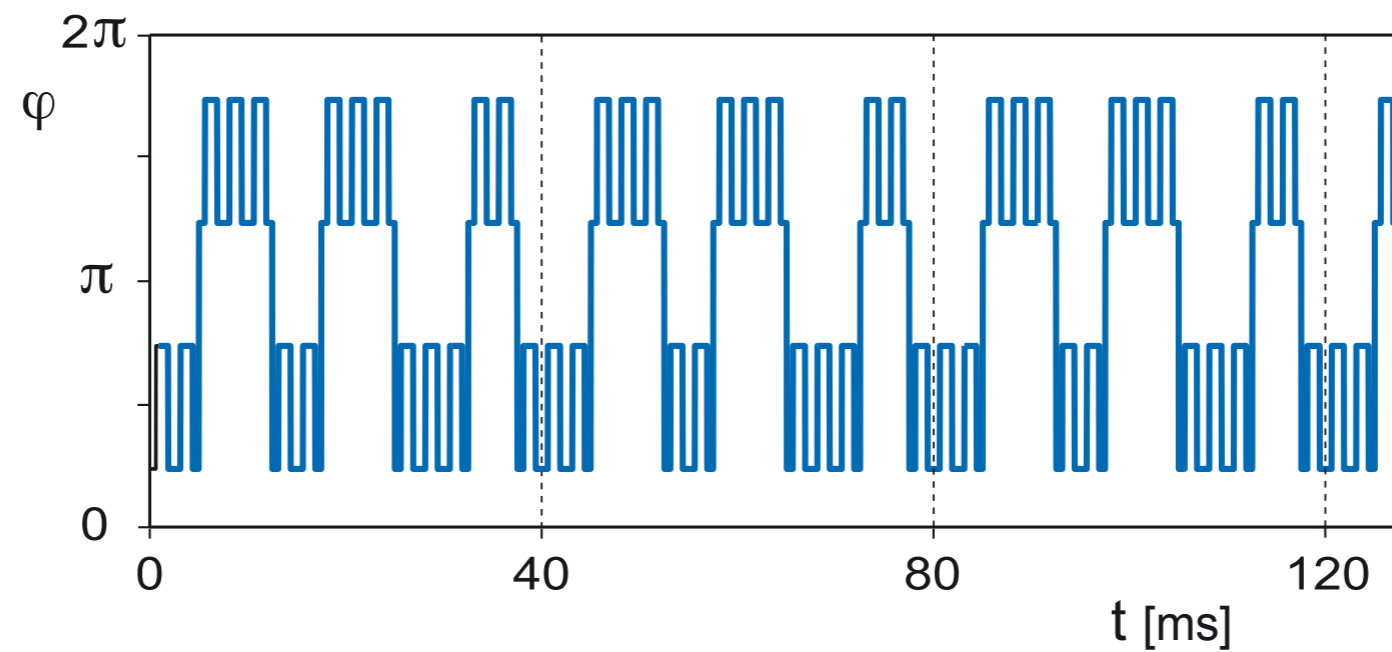


Low power heteronuclear decoupling

nominal rf amplitude: 400 Hz



TRACK-1



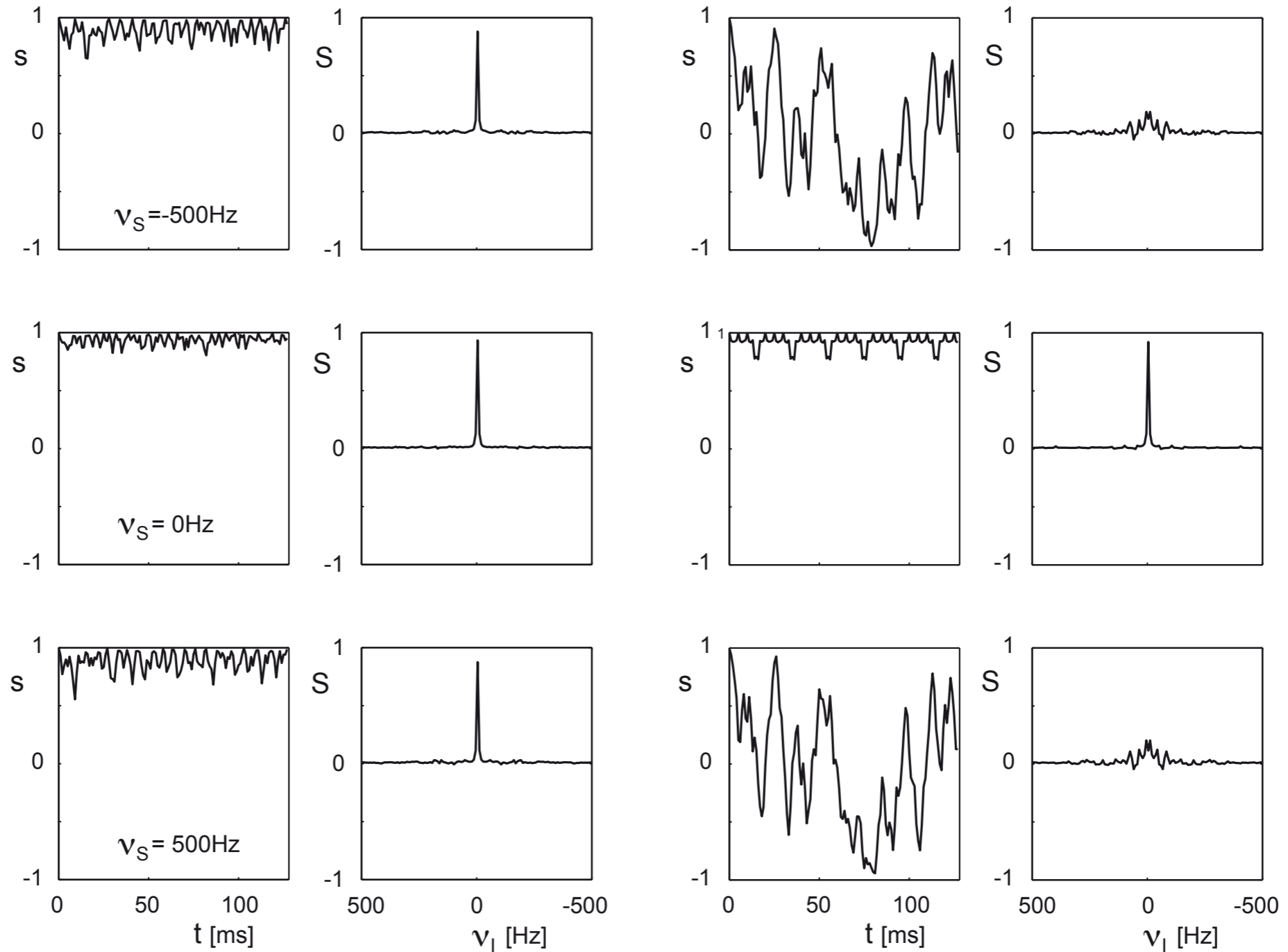
MLEV-16

Low power heteronuclear decoupling

nominal rf amplitude: 400 Hz

TRACK-1

MLEV-16

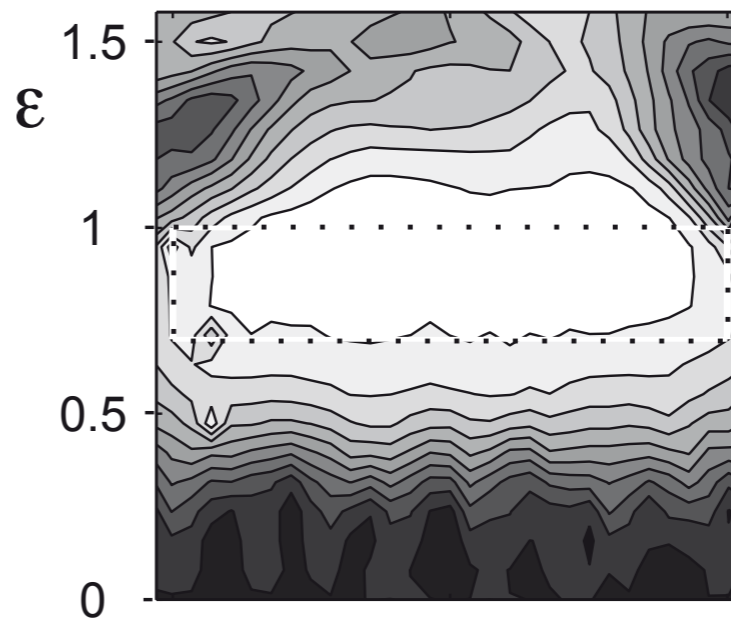
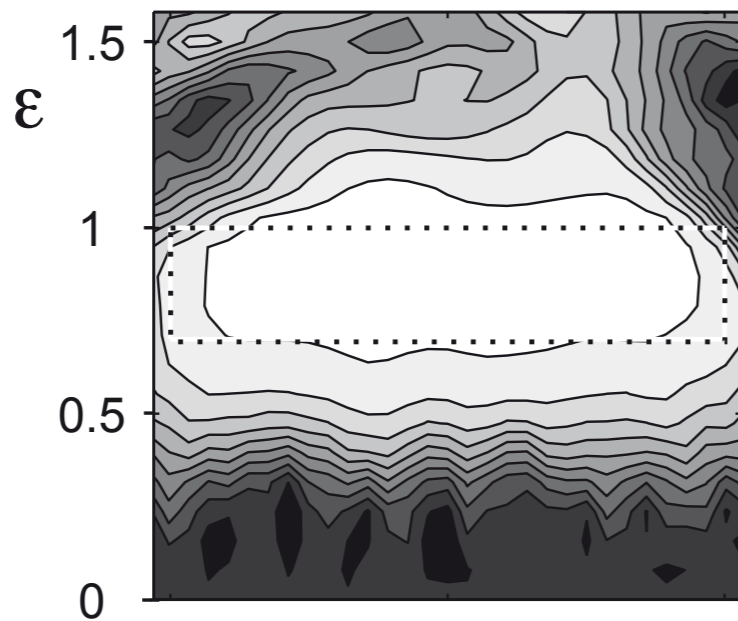


Low power heteronuclear decoupling

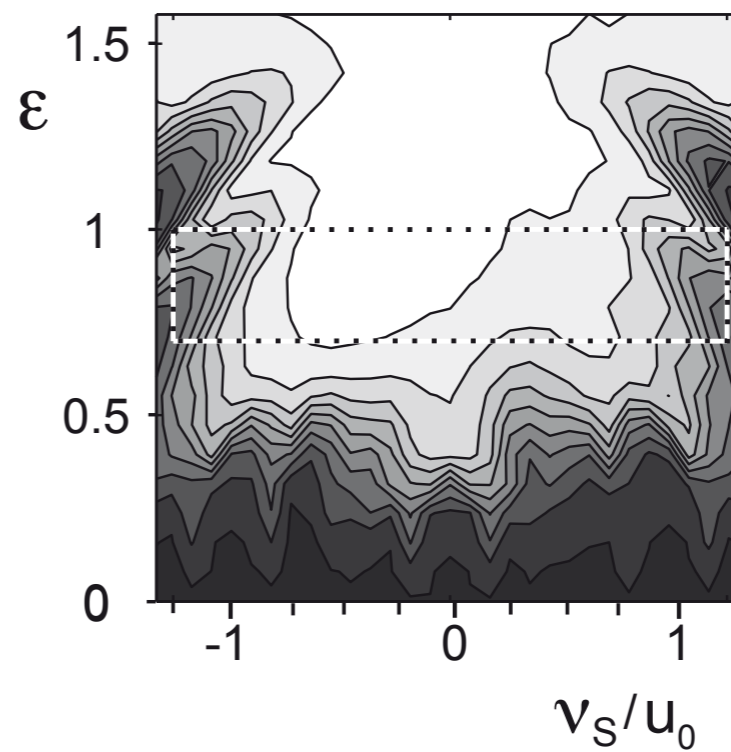
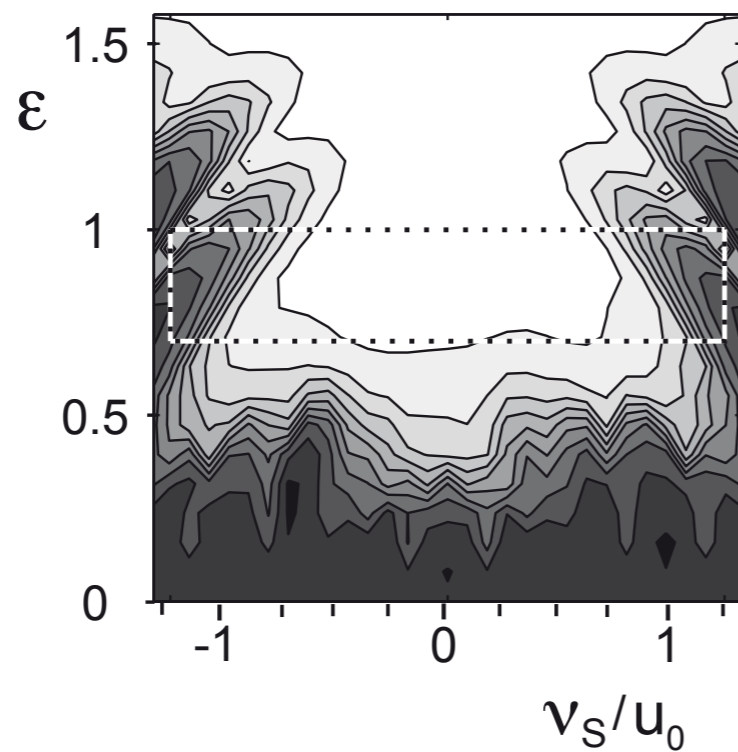
Simulation

Experiment

TRACK-1



MLEV-16



Low power heteronuclear decoupling

